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On the Polytopal Generalization of Sperner's Lemma

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Abstract

We introduce and prove Sperner's lemma, the well known combinatorial analogue of the Brouwer fixed point theorem, and then attempt to gain a better understanding of the polytopal generalization of Sperner's lemma conjectured in Atanassov (1996) and proven in De Loera et al. (2002). After explaining the polytopal generalization and providing examples, we present a new, simpler proof of a slightly weaker result that helps us better understand the result and why it is correct. Some ideas for how to generalize this proof to the complete result are discussed. In the last two chapters we provide a brief introduction to the basics of matroid theory before generalizing a matroid generalization of Sperner's lemma proven in Lovász (1980) to polytopes. At the end we present some partial progress towards proving the polytopal generalization of Sperner's lemma using this matroid generalization.

Contents

List of Figures

Chapter 1

Introducing Sperner's Lemma

In this thesis we will be attempting to develop an understanding of the polytopal generalization of Sperner's lemma and an intuition for why it is true. We attempt to do this by way of providing clear exposition explaining the result; a new, simpler partial proof of the result; and exploring the connection between this result and the concept of matroids. However, before we begin any of that we need to start at the basics—we need to understand the original lemma that this generalization is based on. Sperner's lemma is an elegant little theorem that is a combinatorial analogue of the Brouwer fixed point theorem. In fact, it is well known that Sperner's lemma, the Brouwer fixed point theorem, and the Knaster-Kuratowski-Mazurkiewicz (KKM) lemma are all equivalent results, where the Brouwer fixed point theorem is a result in topology, Sperner's lemma is a result in combinatorics, and the KKM lemma is a set covering result. Sperner's lemma is also well known for being used in the proof of the Kakutani fixed-point theorem—the result used by John Nash in his description of Nash equilibria. In this chapter we provide the foundation for the following chapters by introducing Sperner's lemma and providing intuition for why it is true in the form of two different proofs.

1.1 Sperner's Lemma in Two Dimensions

We'll start with the statement of Sperner's lemma in two dimensions and then explain each piece of terminology needed to understand the result.

Theorem 1.1 (Sperner's Lemma)**.** *A Sperner labeled triangulation of a triangle must have an odd number of full cells. In particular there must be at least one.*

To understand what this means, we need to define the terms triangulation, Sperner labeling, and full cells. Let's get started!

A *triangulation* of a polygon is a decomposition of the polygon into a set of triangles with non-intersecting interiors such that the faces of two triangles intersect at a face common to both of them, or not at all. We call the triangles in the decomposition *cells*. Sperner's lemma relies only on triangulations of the triangle, so we will not consider triangulations of other polygons for now. We can see examples of triangulations of a triangle in Figure 1.1, while Figure 1.2 shows examples of decompositions of triangles that are not triangulations.

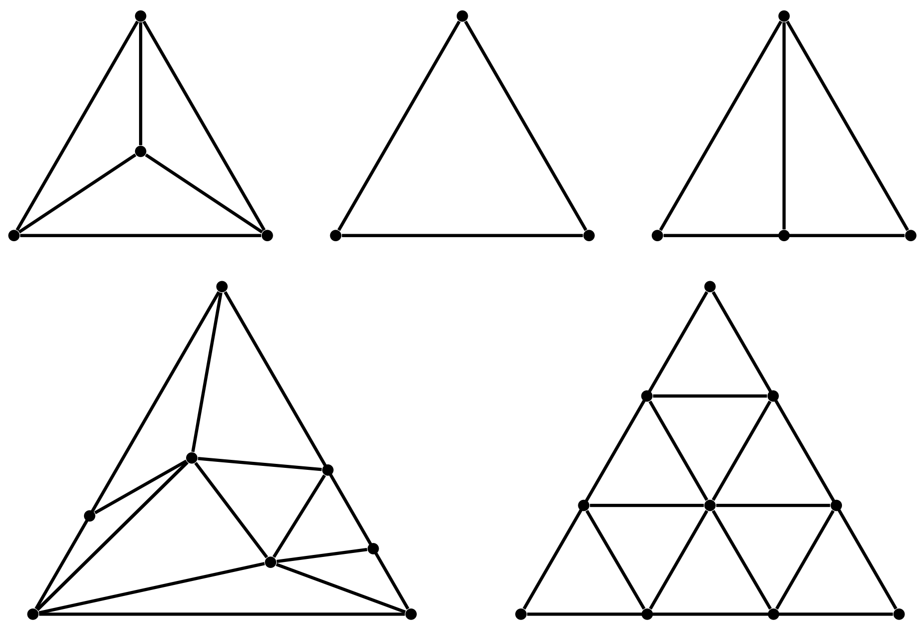


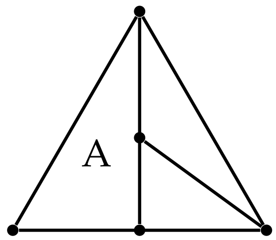
Figure 1.1 Examples of triangulations of a triangle

A *Sperner labeling* of a triangulation of a triangle ABC is an assignment of the labels $\{1, 2, 3\}$ to the vertices in the triangulation such that:

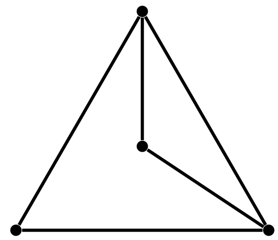
1. Each of the three vertices A, B, and C have distinct labels.

2. Any vertex on side $\overline{XY}$ for $X, Y \in \{A, B, C\}$ is labeled $L(X)$ or $L(Y)$ where $L(x)$ is the label of vertex x.

Notice that the definition places no restrictions on the labels of the vertices on the interior of the triangle. Figure 1.3 explains these labeling requirements



a. vertex is on the face of A



b. not all cells are triangles

Figure 1.2 Examples of decompositions that are not triangulations

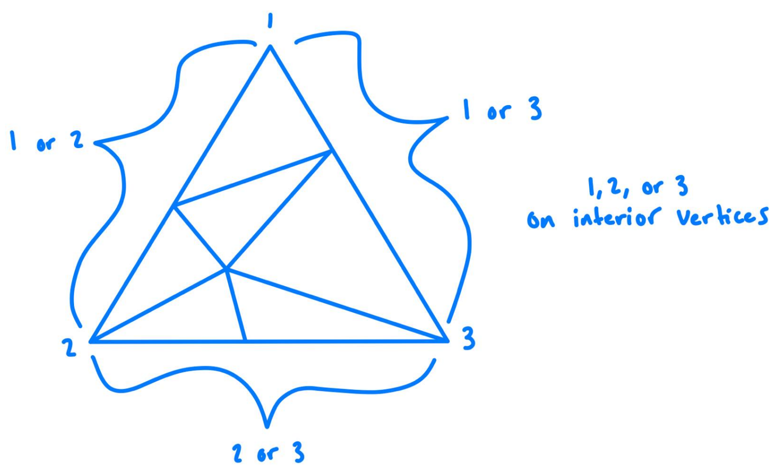


Figure 1.3 Sperner labeling rules for triangulated triangles

visually. Lastly, we call a cell in the triangulation a *full cell* if its label set (the labels on its vertices) is $\{1, 2, 3\}$. Figure 1.5 shows some examples of Sperner labeled triangulations with the full cells highlighted.

So Sperner's lemma tells us that when we have such a labeling, we must have an odd number of full cells, and since zero isn't odd, there must be at least one full cell. Figure 1.6 show's that changing the labels on the interior vertices in Figure 1.5 still results in an odd number of full cells—exactly what we would expect given this result. Now that we understand the result, let's prove it! The proof we present here follows the proof presented in Su (1999). Before we start, we need to introduce the concept of a path through the triangulation. The notion of a path is important for both this proof and proofs presented later in this thesis. To explain this notion, we are borrowing an excellent analogy presented in Su (1999).

Think of the triangle as a "house" and of the cells in the triangulation as "rooms" in the house. The $(1,3)$-edges (those whose endpoints are labeled $\{1,3\}$) both on the interior and on the boundary are the "doors" between the rooms. Note that if a room is not a full cell, it must have 0 or 2 doors, so if you enter such a room, you are able to leave it through a different door. This is because a room with at least one door must either be a full cell, or have a repeated label, which must give rise to another door. Thus, if you are walking through the house, each time going through a different door, then you will eventually exit the house (go through a door on the boundary), or reach a full cell. Note that if you do exit the house, the door must have been on the $(1,3)$-side of the triangle since there are no $(1,3)$-edges, and thus no doors, on any other side of the triangle.

A *path* is a sequence of cells that you would pass through by starting either at a door on the boundary or in a full cell, and then going through doors until you can no longer continue. Note that this means that both the first and last cells in a path must be either a full cell or a cell with one of its doors on the boundary. We do not care about the direction of paths, so we will consider a path and its reverse as the same path. Note that instead of defining $(1,3)$-edges as the doors, we could have defined any other type of edge with two labels as the doors ($(1,2)$-edges or $(2,3)$-edges). Because of this, we will sometimes specify the type of edge when discussing paths, so for example, in this case we would refer to the paths as $(1,3)$-paths. See figure 1.4 for an example of what paths look like. We now prove Theorem 1.1.

Proof. Consider the $(1,3)$ side of the triangle. Since we have a Sperner labeling, every vertex on that side must then be labeled 1 or 3. We claim that this side of the triangle must have an odd number of $(1,3)$ edges. This is because as we move along the side from one endpoint to the other, there must be an odd number of label changes (from 1 to 3 or 3 to 1) in order to get different labels at the endpoints. Each of these $(1,3)$-edges is associated with a $(1,3)$-path. Recall that endpoints of paths are either edges on the boundary or edges of full cells. This means that an even number of $(1,3)$-edges on the boundary are connected to each other by paths. Since there are an odd number of $(1,3)$-edges on the boundary, there must then be an odd number of $(1,3)$-edges whose associated path terminates at a full cell. Figure 1.7 shows how you can find all such full cells constructively.

At this point we have proven that there are an odd number of full cells

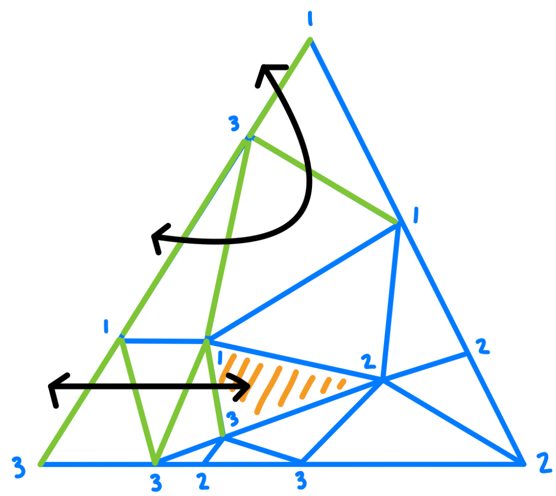


Figure 1.4 The $(1,3)$-paths in a Sperner labeled triangulation: $(1,3)$-edges are marked in green, full cells are marked in orange, and the $(1,3)$-paths are shown as black arrows.

who connect to the boundary by paths, so we are almost done. However, we must also prove that there cannot be an odd number of full cells whose associated paths do not terminate at the boundary, since then we would have an even number in total. However, this is easy to see since if they do not terminate at the boundary, they must terminate at another full cell, and so, all such full cells come in pairs. As there are an even number of full cells that do not connect to the boundary by paths, and an odd number of full cells that do, we can conclude that there must be an odd number of full cells in the Sperner labeled triangulation. □

This proof not only guarantees the existence of an odd number of full cells, but also provides an algorithm for actually finding an odd number of the full cells in the triangulation, though not necessarily all the full cells, as the last part of the proof illustrates. For this reason this proof is considered a constructive proof of Sperner's lemma. In section 1.2, we present an alternative proof of Sperner's lemma that is nonconstructive.

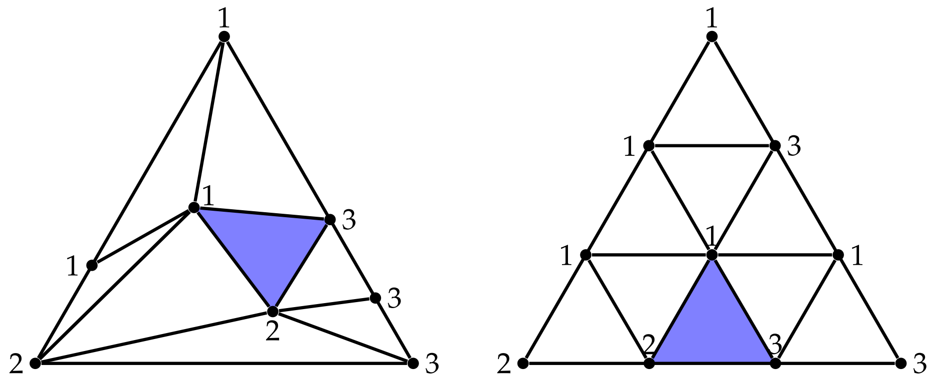


Figure 1.5 Sperner labeled triangulations with full cells highlighted

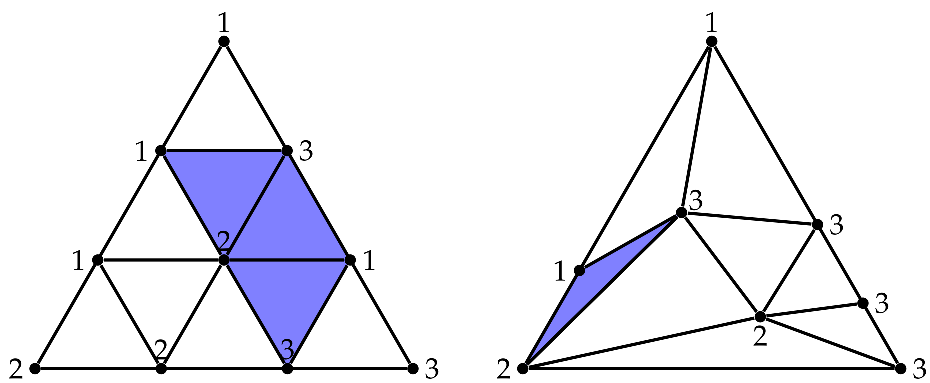
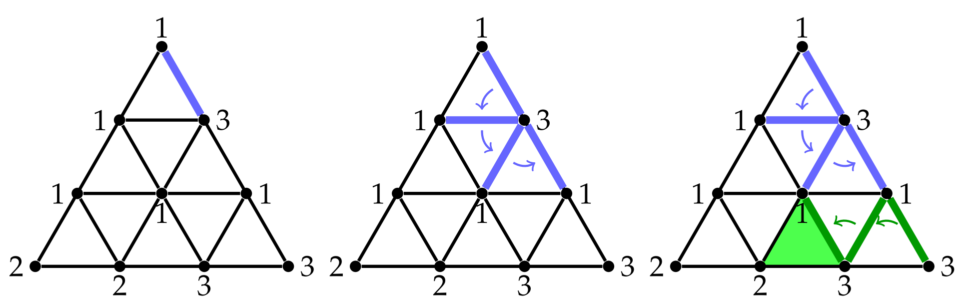


Figure 1.6 Full cells after changing interior labels in Figure 1.5



a. Choose a random $(1, 3)$ edge

b. Follow the path of $(1, 3)$ edges

c. Repeat for remaining $(1, 3)$ edges on side

Figure 1.7 Algorithm for proving the existence of an odd number of full cells

1.2 Sperner’s Lemma in Arbitrary Dimensions

Now that we have seen Sperner’s lemma in two dimensions, we can relatively easily generalize to d dimensions. The first step in doing so is answering the question "how do we generalize triangles into higher dimensions?" For this we need the notion of a simplex. A *d-simplex* is a d-dimensional polytope which is the convex hull of $d + 1$ points. In case you don’t know, a *convex hull* of a set of points is the intersection of all convex polytopes containing those points. For example,

1. the convex hull of two points is a line
2. the convex hull of three noncollinear points is a triangle
3. the convex hull of four points in the plane, with no three being collinear, is a quadrilateral
4. the convex hull of four affinely independent points is a tetrahedron

So a 0-simplex is a point, a 1-simplex is a line, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron (see Figure 1.8). At any higher dimension we can not visualize simplices, but the idea stays the same. A *k-face* of a d-simplex is the convex hull of any $k + 1$ of the $d + 1$ points making up the simplex. So for example a 2-face of a simplex is a triangle, and a 3-face of a simplex is a tetrahedron. For convenience, we refer to the $(d - 1)$-faces of a d-simplex (or of any d dimensional polytope) as facets. So the facets of a tetrahedron are triangles, the facets of a cube are squares, and the facets of a triangle are edges.

a. a 0-simplex **b.** a 1-simplex **c.** a 2-simplex **d.** a 3-simplex

Figure 1.8 Examples of simplices in different dimensions

The equivalent of a triangulation in higher dimension is a simplicial subdivision, but we will often still refer to it as a triangulation for convenience. Just as we previously decomposed a polygon into a set of triangles, we now decompose a d-dimensional polytope into a set of d-simplices. A *simplicial subdivision* of a d-dimensional polytope is a decomposition of the polytope into a set of d-simplices with non-intersecting interiors such that the faces of two simplices intersect at a face common to both of them, or not at all.

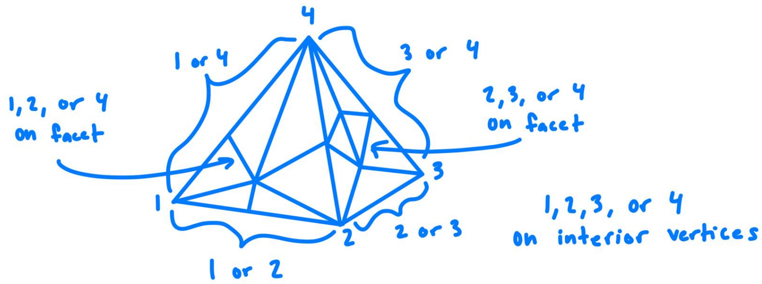


Figure 1.9 Sperner labeling rules for triangulated tetrahedrons

We can now define a Sperner labeling for a simplicial subdivision of a d-simplex analogously with how we defined a Sperner labeling for a triangulation of a triangle. A Sperner labeling of a d-simplex $A_1A_2\cdots A_{d+1}$ is an assignment of the labels $\{1, 2, \ldots, d+1\}$ to the vertices in the simplicial subdivision such that:

1. Each of the vertices A_i has a distinct label.

2. Any vertex on a k-face $B_1B_2\cdots B_{k+1}$ of the d-simplex (i.e. $B_i \in \{A_1, A_2, \ldots, A_{d+1}\}$ and $B_i \neq B_j$ if $i \neq j$) must have a label in the set $\{L(B_1), L(B_2), \ldots, L(B_{k+1})\}$ where $L(x)$ is the label of vertex x.

Once again notice that this definition places no restrictions on labels of vertices on the interior of the d-simplex. Also note that when we let $d = 2$, this definition is identical to the one given in the previous section. Figure 1.9 explains these labeling requirements for three dimensions ($d = 3$). A d-simplex in the Sperner labeled simplicial subdivision is called a *full cell* if its label set is $\{1, 2, \ldots, d+1\}$.

We can now state Sperner's lemma in full generality.

Theorem 1.2 (Sperner's Lemma)**.** *A Sperner labeled simplicial subdivision of a d-simplex must have an odd number of full cells. In particular there must be at least one.*

It is straightforward to generalize the constructive proof from earlier to arbitrary dimension using induction. The main adjustment we must make is to our definition of paths. Instead of our doors being a certain kind of edge, we now have them be a certain kind of $(d-1)$-simplex. In three dimensions this means that triangles within the simplicial subdivision are our doors. For example, we might say have $(1, 2, 3)$-paths where $(1, 2, 3)$-faces are our doors, or $(1,3,4)$-paths where $(1,3,4)$-faces are our doors. All of our reasoning about doors and paths from before remains correct, and so we can reason about them in exactly the same way as before. We can now present the proof of Sperner's lemma for arbitrary dimension, once again following the proof presented in Su (1999).

Proof. We proceed using induction. Our base case is when $d = 2$ and is simply Theorem 1.1. Now we assume that Theorem 1.2 holds for $d < n$ and show that it holds for $d = n$.

Consider a Sperner labeled simplicial subdivision of an n-simplex. Now consider the $(1,2,\ldots,n)$-facet of the n-simplex. Since the triangulation is Sperner labeled, when we consider this facet as a triangulated $n-1$-simplex, it is also Sperner labeled. Thus, by the inductive hypothesis it must have an odd number of full cells on it, which would mean $(n-1)$-simplices that are labeled $(1,2,\ldots,n)$. Just as in the proof of Sperner's lemma in two dimensions, we consider the $(1,2,\ldots,n)$-paths associated with each of these simplices, Since each path connects either two of these simplices to each other, or connects one of these simplices to a full cell, we know that there are an even number of these simplices that connect to each other, and thus that there are an odd number remaining which connect to a full cell. Now note that every full cell which does not connect to one of the $(1,2,\ldots,n)$-simplices on the boundary must connect to another full cell, so there must be an even number of such full cells. As there are an odd number of full cells that connect to one of the $(1,2,\ldots,n)$-simplices on the boundary and an even number that do not, we conclude that there must be an odd number of full cells in total, which completes the inductive proof. □

This proof, like the proof of Sperner's lemma in two dimensions, is constructive since it provides us with an algorithm for finding an odd number of full cells. To do so, we start by finding an odd number of full cells for any one of the 2-simplices on the boundary. Then we use those to find an odd number of full cells for the 3-simplex containing that 2-simplex as described in the above proof. If we repeat this process, each time moving up one dimension, we eventually arrive at an odd number of full cells of the Sperner labeled triangulation of the d-simplex.

However, there also exists a non-constructive proof of this result that relies on a parity argument, which we now present. A short sketch of the proof is laid out in Su (1999), and that is what the below proof is based off of.

Proof. Let N be the number of $(d-1)$-simplices with labels $\{1,2,\ldots,d\}$, and

M be the number of full cells. We will prove this result by showing that the parity of M and N must be the same, that is $N \equiv M \pmod 2$, and that N is odd.

In order to show this, we will use induction. Our base case is the two-dimensional case. Imagine placing pebbles in each cell of the Sperner labeled triangulation of the triangle equal to the number of $(1, 2)$ edges on the boundary of that cell. So a full cell would have 1 pebble and other cells would have 0 or 2 pebbles, see Figure 1.10. Another way to count the pebbles would be to observe that $(1, 2)$-edges on the interior of the triangulation contribute 2 pebbles, one for each triangle that has the $(1, 2)$-edge as a facet, and $(1, 2)$-edges on the boundary of the triangulation contribute 1 pebble. Thus, we see that

$$\text{\# full cells} \equiv \text{\# pebbles} \equiv \text{\# } (1,2) \text{ edges on boundary} \pmod 2$$

We know from the proof of Theorem 1.1 that in a Sperner labeled triangulation of a triangle there must be an odd number of $(1, 2)$ edges on the boundary, and so we conclude that there must be an odd number of full cells as well. Our inductive hypothesis is that there are an odd number of full cells in a Sperner labeled simplicial subdivision of a k-simplex. Now we will prove that this holds for a $(k+1)$-simplex. Place pebbles in each cell of the simplicial subdivision equal to the number of $(1, 2, \ldots, k + 1)$ k-simplices it has on its boundary. The remaining steps follow exactly from the steps taken in the base case, and so we can conclude that

$$\text{\# full cells} \equiv \text{\# } (1, 2, \ldots, k+1) \text{ } k\text{-simplices on boundary} \pmod 2$$

The $(1, 2, \ldots, k+1)$ k-simplices on the boundary are all in the k-face spanned by the vertices of S labeled $1, 2, \ldots k + 1$. This k-face is Sperner labeled, so using our inductive hypothesis, there must be an odd number of $(1, 2, \ldots, k+1)$ k-simplices in its simplicial subdivision. Finally, we conclude that since there is an odd number of $(1, 2, \ldots, k + 1)$ k-simplices on the boundary of our $(k + 1)$-simplex, that there must be an odd number of full cells as well. This completes our induction and proves the desired result. □

We have now seen two complete proofs of Sperner's lemma in arbitrary dimension. The goal of this chapter was to explain Sperner's lemma and provide intuition for why this result is true. In the next chapter we introduce a generalization of Sperner's lemma, known as the polytopal generalization of Sperner's lemma. This generalization tells us something about the number

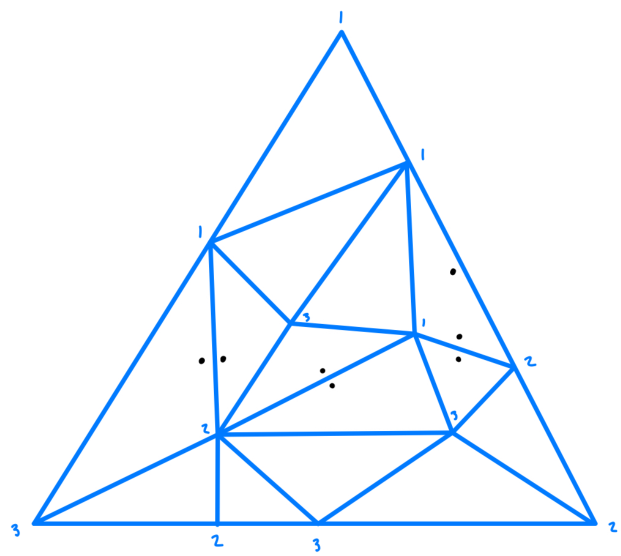


Figure 1.10 A Sperner labeled triangulation of a triangle with pebbles placed in cells depending on the number of $(1,2)$ edges on their boundary

of full cells in Sperner labeled triangulations of convex polytopes, as opposed to only simplices. As you might realize, our current definition of Sperner labelings and full cells do not make sense for any convex polytope, so we will also see how we generalize those appropriately. The understanding that we have built up in this chapter as well as many of the notions we have defined, especially the notion of paths, will help us understand and prove this generalization.

Chapter 2

The Polytopal Generalization of Sperner's Lemma

Much as its name suggests, the polytopal generalization of Sperner's lemma is a result very similar to Sperner's lemma, but instead of only considering triangulations of simplices, it considers triangulations of all convex polytopes, which are generalizations of polygons and polyehdra for arbitrary dimension. This generalization was originally conjectured in Atanassov (1996), where it was also proven for the $d = 2$ case, i.e. for triangulations of polygons. The result was later proven for all dimensions by De Loera et al. (2002). Their paper presents two different proofs of this result, one constructive and one non-constructive. However, both of these results rely on tricky arguments that lose some of the elegance of the proof of Sperner's lemma. One of the primary goals of this thesis was to produce a simpler proof of this result that allows for a more intuitive understanding of the result and more closely parallels the constructive proof of Sperner's lemma presented in Chapter 1. While this goal was not completely attained, reasonable progress has been made in this direction. In this chapter we will present a relatively simple proof of this result where one key assumption has been added, namely that the polytope has at least one simplicial facet. We will also discuss directions one might try for generalizing this proof to the complete result. However, to start off, we must first state the polytopal generalization of Sperner's lemma and work to understand what it is saying.

Theorem 2.1. *Any Sperner labeled triangulation of a convex (n, d)-polytope must contain at least $n - d$ non-degenerate full cells.*

As we are now dealing with polytopes instead of simplices, our old

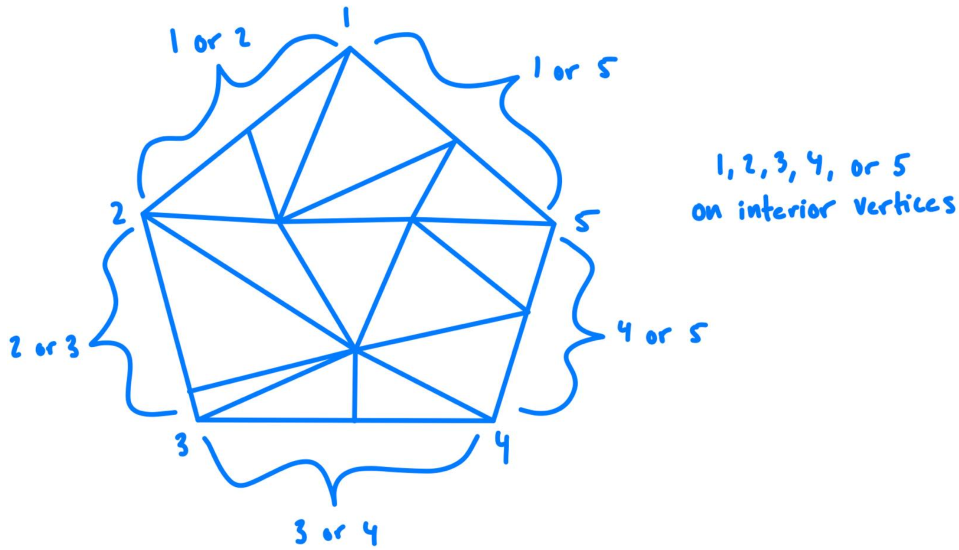


Figure 2.1 Sperner labeling rules for triangulated pentagons

definitions for Sperner labelings and full cells are no longer sufficient. We also need to define what an "(n, d)-polytope" is and explain what it means for a full cell to be non-degenerate. We will tackle these one at a time, starting from the easiest. A *(n, d)-polytope* is a polytope with n vertices in d dimensions. For example, a $(4, 2)$-polytope is a quadrilateral in the plane. We define Sperner labelings very similarly to how they were defined for Sperner's lemma. Let P be a triangulated (n, d)-polytope with vertices $\{A_1, A_2, \ldots, A_n\}$. A labeling of the vertices in the triangulation using the labels $\{1, 2, \ldots, n\}$ is called a Sperner labeling if

1. Each vertex of P has a unique label.
2. Each vertex on a k-face of P, $B_1B_2 \cdots B_{k+1}$, where $B_i \in \{A_1, A_2, \ldots, A_n\}$ and $B_i \neq B_j$ if $i \neq j$, has a label in the set $\{L(B_1), L(B_2), \ldots, L(B_{k+1})\}$, where $L(x)$ is the label of vertex x.

Figure 2.1 explains these labeling requirements in the case of $d = 2$ and $n = 5$ (i.e. for a pentagon). Since it is possible that there are more than $d + 1$ labels (when $n > d + 1$), we now define a *full cell* as any cell that has no repeating labels. We can not yet explain what it means for a full cell to be degenerate (that requires some additional explanation - see section x). For now, we just need to know that degenerate full cells are a subset of full cells and that full cells cannot be degenerate in two dimensions.

Now that we have defined everything, we can mostly understand what the polytopal generalization of Sperner's lemma is telling us. It is worth noting that unlike Sperner's lemma, there are no guarantees about the parity of the number of full cells. Instead, it gives a guaranteed lower bound on the number of full cells. For example, for a Sperner labeled triangulated pentagon, Theorem 2.1 guarantees that we have at least $5 - 2 = 3$ full cells. Figure 2.2 shows an example of a Sperner labeled pentagon with its full cells highlighted. If we apply Theorem 2.1 to a Sperner labeled d-simplex, we have $n = d + 1$, so we are guaranteed at least $(d + 1) - d = 1$ full cells, which is the same lower bound that Sperner's lemma provides. This is why this result is considered a generalization of Sperner's lemma

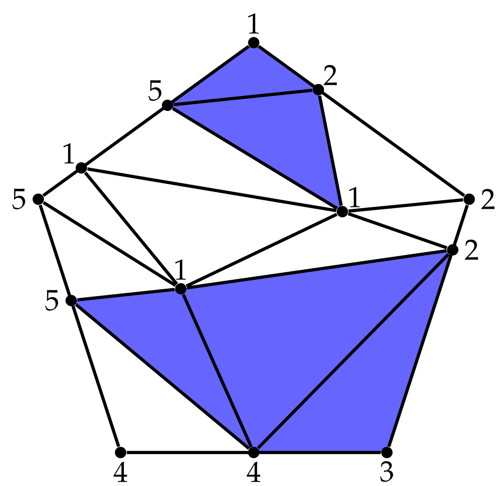


Figure 2.2 A Sperner labeled triangulated pentagon. Observe that in this case $n = 5$ and $d = 2$ so we are guaranteed $n - d = 3$ full cells. As we can see, we have 5 full cells (highlighted in blue).

2.1 A Proof in Two Dimensions

We noted above that the proof we present in this chapter requires the additional assumption that the polytope have at least one simplicial facet. In two dimensions, every facet is an edge, and thus simplicial. This means that our proof does prove the full result for the case of $d = 2$. However, while this may be true, it is by no means the simplest proof available for $d = 2$. While attempting to find a proof for the general result, the hardest step was usually finding a way to generalize a proof that worked in two dimensions to three dimensions and above. Here we will prevent one proof that we came up with that works for $d = 2$ but breaks down when $d \geq 3$. As we noted above, full cells in two dimensions cannot be degenerate (this is in fact due to every facet being simplicial, as we shall see later), so the result we are proving here is:

Proposition 2.2. *A Sperner labeled polygon has at least $n - 2$ full cells.*

Proof. Let P be a Sperner labeled n-gon. Consider an arbitrary facet of P, it must have two labels which we will refer to as a and b. From the proof of Theorem 1.1, we know that there must be an odd number of (a, b)-edges on that facet, and thus that an odd number of full cells are connected to this side by paths. In particular, we know there must be at least one such full cell. This full cell is connected by an (a, b)-path to the facet, so its labels must be (a, b, X), where X can be any label that is neither a nor b. Since we were considering an arbitrary facet, this means that each facet connects to at least one full cell. If each facet connects to a unique full cell, as they do in Figure 2.3, then we have n full cells and we are done.

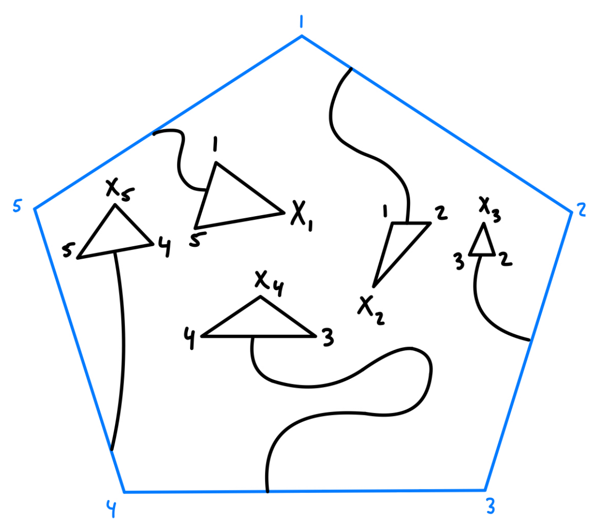


Figure 2.3 A Sperner labeled pentagon with each side connecting to at least one unique full cell. Note that $X_1, \ldots, X_5 \in \{1, 2, 3, 4, 5\}$ such that all five cells are full cells. The black lines represent the paths connecting each side to its full cell.

However, if this is not the case then at least two facets connect to the same full cell. Without loss of generality assume that one of the two facets is the (a, b)-facet, so the full cell must be labeled (a, b, X). Thus, the only facets that could also be connected to this full cell are the (a, X)-facet and the (b, X)-facet, if such facets exist. However, the only facet other than the (a, b)-facet that has the label a is one of the two facets adjacent to the (a, b) facet, and the only facet other than the (a, b)-facet that has the label b is the other facet adjacent to the (a, b)-facet. This means that the two facets connecting to the same full cell must be adjacent. Without loss of generality, we will assume that the (b, c)-facet is adjacent to the (a, b)-facet and is the other facet connecting to the full cell, which we now know is labeled (a, b, c).

The next step in the proof is to show that given the existence of an (a, b, c)-full cell, A, connected to the adjacent facets (a, b) and (b, c), we can construct a new Sperner labeled polygon with $n - 1$ vertices such that every full cell in the new polygon corresponds to a full cell in the original polygon but with no full cell corresponding to A.

To do so, start by constructing a sequence of adjacent edges that

1. starts at the vertex of P labeled a and follows the edges on the (a, b)-facet until it reaches the first edge in the path to the full cell A

2. then follows the (a, b)-edges in the path until it reaches the (a, b, c)-full cell A

3. crosses the (a, c)-edge of A

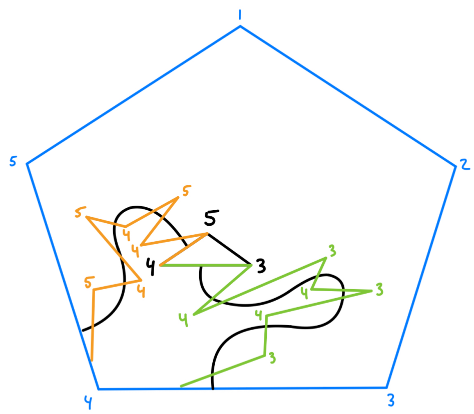
4. then follows the (b, c)-edges in the path from A to the (b, c)-facet

5. finally follows the (b, c)-edges on the (b, c)-facet until it reaches the vertex of P labeled c.

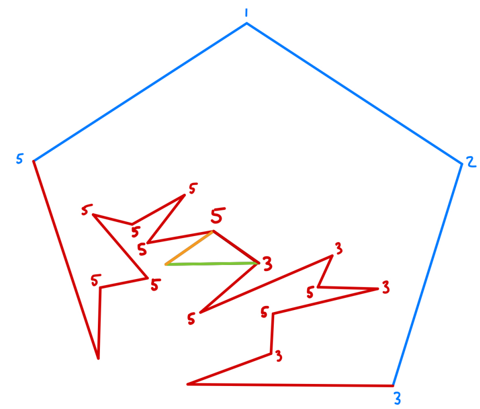
After constructing this sequence of edges, replace the label on every vertex in the triangulation labeled b with the label a. The process of constructing this sequence of edges from the paths and replacing the labels is shown in Figures 2.4a and 2.4b. Observe that at this point, we have a sequence of edges going from the vertex of P labeled a to the vertex of P labeled c such that the label of every vertex in the sequence is a or c. We can imagine "straightening out" this sequence of edges into a straight line between the two vertices to get a $(n - 1)$-gon that

1. is a subset of the original triangulation, so every full cell in this new $(n - 1)$-gon must correspond to a full cell of the original n-gon

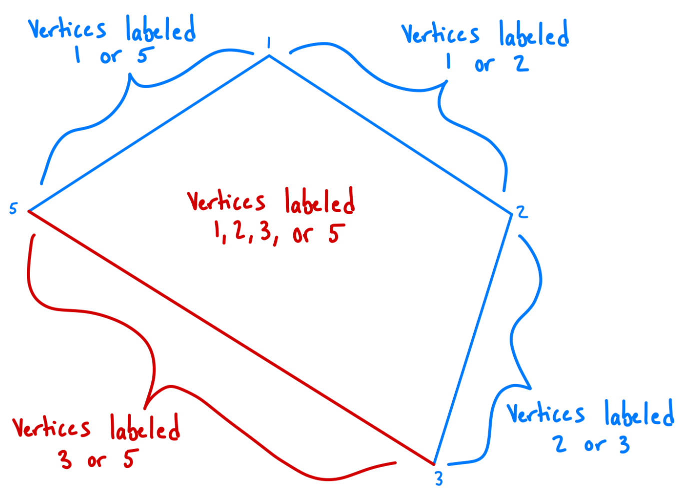
2. is Sperner labeled



a. A Sperner labeled pentagon with two sides connecting to the same full cell.



b. Replace each 4 in the Sperner labeling from **(a)** with a 5 and cut out the area enclosed by the two paths (shown in red).



c. Redraw the figure in **(b)** as a quadrilateral where the red path is one of the sides. This new quadrilateral is Sperner labeled.

Figure 2.4 Algorithm for going from n-gon to $(n-1)$-gon in the proof of the polytopal generalization of Sperner's lemma in two dimensions.

3. does not contain the full cell A.

This is the step between Figures 2.4b and 2.4c. The reason every full cell in the new Sperner labeled polygon corresponds to a full cell in the original polygon is that if you undo the relabeling of the vertices, the cell must still have no repeating labels.

We can now finish the proof with relative ease by applying induction. The base case is $n = 3$ and holds as a direct consequence of Sperner's lemma. Our inductive hypothesis is that Proposition 2.2 holds for all $n < k$. Now, consider a Sperner labeled k-gon. Following the algorithm described above, we can find a full cell A and a $(k-1)$-gon with the above-mentioned properties. By the inductive hypothesis, this $(k-1)$-gon has at least $k-3$ full cells, and by the construction, each of them corresponds to a full cell of the k-gon that is not A. Those full cells plus A brings us up to a total of at least $k-2$ full cells. By induction, this result holds for all $n > 3$, completing the proof. □

One question we might ask is: why is this result harder to prove in three dimensions and above? One reason is that paths connecting to full cells no longer partition the polygon in a clear way, which is something we relied on in the above proof. It is possible for two paths to be linked together, even though they do not intersect or interact in any way (imagine two links in a chain). The other reason is that in three and higher dimensions, the facets are no longer forced to be simplices, which introduces degenerate full cells which cause additional complications for proofs. In the next section we will finally define what it means for a full cell to be degenerate, as well as present the proof of the polytopal generalization of Sperner's lemma in the case where at least one facet is simplicial.

2.2 A Partial Proof of the Polytopal Generalization of Sperner's Lemma

As we mentioned earlier, one of the primary goals of this thesis was to produce a simpler proof of the polytopal generalization of Sperner's lemma. In particular, we hoped for a proof that more closely parallels the constructive proof of Sperner's lemma. We did not succeed at producing such a proof, but we do have a simpler proof of a slightly weaker result.

Proposition 2.3. *Any Sperner labeled triangulation of a convex (n, d)-polytope with at least one simplicial facet must contain at least $n - d$ non-degenerate full cells.*

For a proof of the stronger result, Theorem 2.1, refer to De Loera et al. (2002). As we shall see, we have found a proof that is, at least intuitively, quite simple. We believe that there may be a way to simplify the proof further by finding a replacement for the part of the proof that relies on results from Euclidean degree theory. We now present our proof. It is easiest to understand most of the ideas in this proof (other than perhaps degenerate full cells) by considering the two-dimensional case, and for that reason, the figures will be a running example in two dimensions.

To prove this result we must first define several concepts. Let P be a (n, d)-polytope with Sperner labeled triangulation T. We define a piecewise linear map $f : P \to P$ that maps each vertex of T to the vertex of P that shares the same label, and is linear on each d-simplex of T.

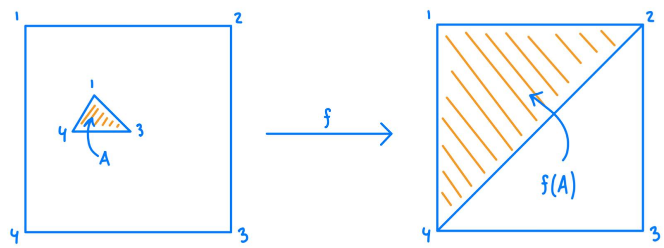


Figure 2.5 An example of the piecewise linear map f mapping a full cell in a square back into the square.

The piecewise linear map f allows us to define the concept of a degenerate full cell. We say a full cell, F, is degenerate if $f(F) \subset \partial P$. One way to think about this is that a full cell is degenerate if all of its labels come from a single facet of P. If we make the assumption that all the vertices of P are affinely independent, we can also think of degenerate full cells as full cells whose image under f encloses no volume. Earlier we said that any polytope with only simplicial facets can not have degenerate full cells, and now we can understand why that is. If every facet is simplicial, then every facet has exactly d labels, so it is impossible for all $d + 1$ labels of a full cell to come from the same facet.

The paths between full cells and fully labeled $(d-1)$-simplices on the boundary give us a way to partition the full cells of the triangulation into "connected components", which we will call networks. A *network* is the smallest set of full cells such that no full cells outside the network are reachable by path-following, along with the d-simplices that make up all paths reachable from these full cells. We require that it be the smallest set so that the union of two networks will not be considered a network.

Similarly, a *non-degenerate network* is the same as a network, except that

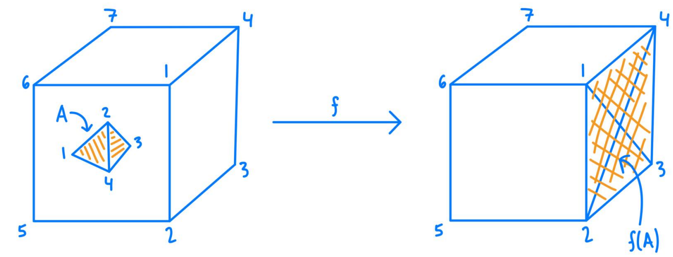


Figure 2.6 An example of a degenerate full cell. The full cell A gets mapped to the boundary of the cube by the piecewise linear map f.

it contains no degenerate full cells, and thus if a non-degenerate full cell is only reachable by path-following through a degenerate full cell, it would not be included in the non-degenerate network. It might be helpful to note that each network is the union of some (possibly empty set of) non-degenerate networks and (possibly empty set of) degenerate full cells, along with the d-simplices that make up paths that are part of no non-degenerate network (because the path is either (a) between two degenerate full cells or (b) between a degenerate full cell and a fully labeled $(d-1)$-simplex on the boundary). Figure 2.8 provides intuition for how a network can be made up of several non-degenerate networks.

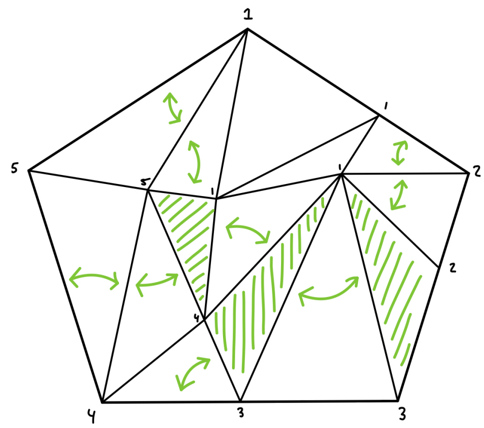
The next notion we need to define is that of a *full cell complex generated by a (non-degenerate) network*, which we will write FC$(\cdot)$. Let H be a network. Then FC(H) will be the simplicial complex constructed by taking copies of the full cells of H and quotienting together each pair of facets connected by a path in H. Figure 2.9 shows examples of full cell complexes.

From now on, we will abuse our notation somewhat by letting f be a piecewise linear map from $FC(H)$ to P instead of only being a map from P to P. This is convenient to do since the behavior of the map is the same in both cases. Figure 2.10 shows the images of some full cell complexes under the piecewise linear map f.

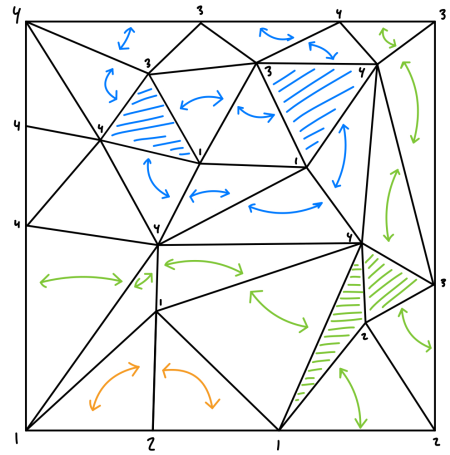
Our approach to proving Proposition 2.3 will involve showing that there is a non-degenerate network whose label set contains all n labels. This is important due to following lemma.

Lemma 2.4. *If the label set of a network contains n unique labels, then it contains at least $n-d$ full cells.*

Proof. In a network, every full cell is connected to at least one other full cell by a path. Also note that it is impossible to partition the full cells in a



a. A Sperner labeled pentagon with only one network, with paths shown as green arrows.



b. A Sperner labeled square with two networks. The paths connecting the full cells in each network are shown as colored arrows.

Figure 2.7 Two Sperner labeled polygons with networks shown.

network into nonempty sets A and B such that no full cell in A is connected by a path to a full cell in B, because this violates the minimality requirement for networks.

Now observe that if two full cells are connected by a path, they must share at least d labels. Since each full cell has $d+1$ unique labels, we can conclude that two connected full cells have at most $d+2$ unique labels. Starting at a full cell, we can follow one of the paths to a new full cell. We can repeat this path-following step, reaching a new full cell by following a path from one of the previously visited full cells, until no full cells are left unvisited in the network. At each step the label set of the visited full cells increases by at most one. Since the starting full cell has $d+1$ unique labels, this means that after $k-1$ steps we have visited k full cells and have at most $d+k$ unique labels. Thus, if a network contains n unique labels, it will take a minimum of $n-d-1$ steps to reach every label at least once, at which point we will have visited $n-d$ full cells. We conclude that any network with n unique full cells must have at least $n-d$ full cells. □

The general structure of the proof is as follows. We will show that $f(\mathrm{FC}(H))$ covers every point in $P \setminus \partial P$ the same number of times up to parity. Then, we will show that every point in $P \setminus \partial P$ must be covered an

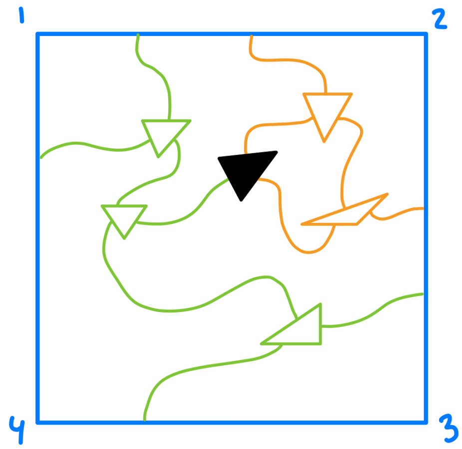
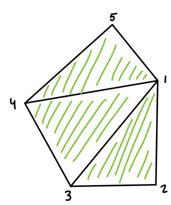
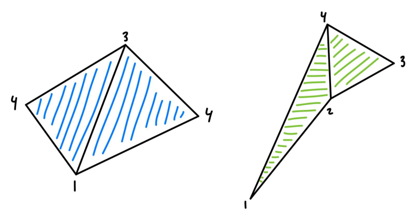


Figure 2.8 A representation of a network made up of two non-degenerate networks. The networks are colored green and orange and the degenerate full cell is colored black. **Note: degenerate full cells do not exist in two dimensions. This diagram is only supposed to help provide intuition for what is meant by a network being made of multiple non-degenerate networks.**



a. Full cell complex generated by Sperner labeled pentagon in Figure 2.7a.



b. Full cell complexes generated by Sperner labeled square in Figure 2.7b.

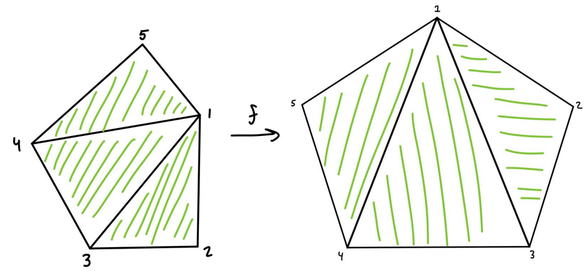
Figure 2.9 Example of full cell complexes generated by networks.

odd number of times, which in turn means that there must be at least one network that covers each point in $P \setminus \partial P$ an odd number of times. Such a network will be seen to have all n labels in its label set, and then Lemma 2.4 will guarantee the existence of $n - d$ full cells, completing the proof.

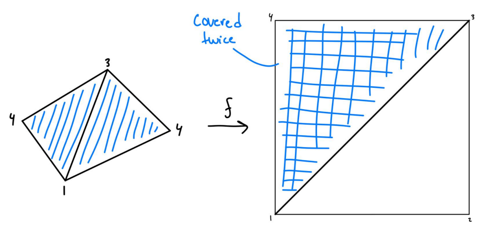
Thus, the first step we need to take is to prove the following lemma

Lemma 2.5. *Let H be a network. Every open set in $P \setminus \partial P$ is covered the same number of times up to parity by $f(FC(H))$.*

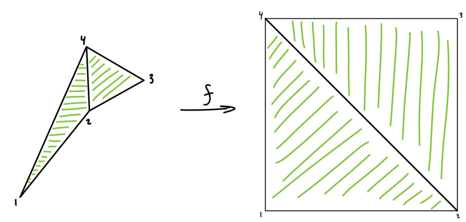
We can see that this holds true in our examples in Figure 2.10. Proving this will require us to take a quick detour into topological degree theory.



a. Image of full cell complex in Figure 2.9a under the piecewise linear map f.



b. Image of full cell complex in Figure 2.9b under the piecewise linear map f.



c. Image of full cell complex in Figure 2.9b under the piecewise linear map f.

Figure 2.10 Images of full cell complexes under the piecewise linear map f.

Intuitively, The *degree of a continuous mapping at the point*, x, written $d(g, x)$ is the number of times that a point in the image of the mapping is covered **accounting for orientation**. For example, if a point in the image has two elements in its preimage (we would say its covered twice) then the degree at that point would be 0 or 2. This depends on whether the orientation of the covering is the same at each of the preimages, in which case the degree is 2, or if the orientations are opposite, in which case the degree is 0. The orientation of the mapping at a point x is $\text{sgn}(|J(x)|)$ where $\text{sgn}(\cdot)$ is the sign function and $|J(x)|$ is the determinant of the Jacobian. Since orientation is only ever positive or negative, the parity of the number of times a point is covered and its degree must be the same. So if a point is covered an odd number of times, it must have odd degree, and if it is covered an even number of times, it must have even degree.

The theorem we need is the following.

Theorem 2.6 (Outerelo et al. (2009) Proposition IV.2.3)**.** *Let $m \in \mathbb{Z}^+$ and $D \subset \mathbb{R}^m$ be a bounded open set. Also let $f : \overline{D} \to \mathbb{R}^m$ be a continuous mapping.*

Then the degree $d(f, \cdot)$ is constant on every connected component of $\mathbb{R}^m \setminus f(\partial D)$.

No proof of this theorem will be presented here, but Outerelo et al. (2009) contains an excellent introduction to Euclidean degree theory along with a relatively accessible proof of this theorem. We now prove Lemma 2.5.

Proof. We will prove this in two steps. The first will be to show that the boundary of a full cell complex is mapped into the boundary of P by the piecewise linear map f. We then use that along with 2.6 to prove the lemma.

Part I: $f(\partial FC(H)) \subseteq \partial P$

The boundary of a full cell complex is comprised of facets of full cells. Based on how we defined the full cell complex, these facets are exactly the ones whose associated path in T connects to a facet of a degenerate full cell or to a $(d-1)$-simplex on ∂P. Note that if two $(d-1)$-simplices are connected by a path in T, they must have exactly the same labels (by the definition of a path), and thus the same image under f. If A is a facet of a degenerate full cell F, then using the definition of a degenerate full cell we see that $f(A) \subset f(F) \subset \partial P$. If A is a $(d-1)$ simplex on a facet, B, of P, then its label set must be a subset of the label set of the vertices of B. Thus, $f(A) \subset B \subset \partial P$. Putting this all together, we see that each facet on the boundary of $\mathrm{FC}(H)$ must map into ∂P under f, and therefore that $f(\partial \mathrm{FC}(H)) \subseteq \partial P$.

Part II: Showing that every open set in $P \setminus \partial P$ is covered the same number of times up to parity

Since $P \subset \mathbb{R}^d$, $\mathrm{FC}(H) \subset \mathbb{R}^d$ and is closed, and $f : \mathrm{FC}(H) \to P$ is continuous, we can apply Lemma 2.6. Thus, every connected component of $P \setminus f(\partial(\mathrm{FC}(H)^\circ))$ is covered the same number of times up to parity by $f(\mathrm{FC}(H))$. From Part I we know that $f(\partial \mathrm{FC}(H)) \subseteq \partial P$ so every connected component of $P \setminus \partial P$ is covered the same number of times up to parity by $f(\mathrm{FC}(H))$. Since $P \setminus \partial P$ is connected, every open set in $P \setminus \partial P$ is covered the same number of times up to parity by $f(\mathrm{FC}(H))$. □

Lemma 2.7. *Let T be a Sperner labeled triangulation of the polytope with at least one simplicial facet, P, and let $\{H_1, H_2, \ldots, H_k\}$ be the set of all networks in T. Then every open set in $P \setminus \partial P$ is covered an odd number of times by $\bigcup_{i=1}^{k} f(FC(H_i))$.*

Proof. Let F_Δ be the simplicial facet of P, and let $\{C_1, C_2, \ldots, C_k\}$ be the $(d-1)$-simplices on F_Δ with all unique labels.

Due to Sperner's Lemma, we know that k is odd. Now observe that the path connected to each C_i terminates at either a full cell of T or some C_j

where $i \neq j$. This means that the $(d-1)$-simplices that do not connect to a full cell must pair up. Thus, there must be an odd number of full cells that are connected to F_Δ directly by a path. Each other full cell in T that has a facet whose image under the piecewise linear map f is F_Δ must then be connected by a path to another such full cell—that is to say, they also pair up. So in total, there are an odd number of full cells with a facet which maps to F_Δ.

Now consider every d-simplex made up of vertices of P that has F_Δ as a facet, and call the intersection of them O_Δ. An example of this in two dimensions is shown in Figure 2.11. Note that O_Δ has non-empty interior and that the image of a full cell covers F_Δ under the piecewise linear map if and only if it also covers all of O_Δ. Since an odd number of full cells cover F_Δ, it must be that O_Δ is also covered an odd number of times.

Lemma 2.5 tells us that each full cell complex covers everything in $P \setminus \partial P$ the same number of times up to parity, so it must be that $\bigcup_{i=1}^{k} f(\mathrm{FC}(H_i))$ does as well. We have shown that one part of $P \setminus \partial P$ is covered an odd number of times by $\bigcup_{i=1}^{k} f(\mathrm{FC}(H_i))$, namely, O_Δ, so we conclude that all of $P \setminus \partial P$ is covered an odd number of times.

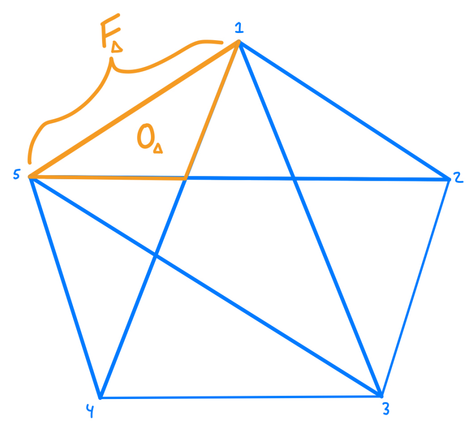


Figure 2.11 An example of an O_Δ set in two dimensions.

□

Using Lemma 2.7 and Lemma 2.5, we can now present a short proof of Proposition 2.3.

Proof. Using Lemma 2.5 we know that every open set in $P \setminus \partial P$ is covered the same number of times up to parity by the full cell complex of a network. Since the total covering must be odd, by Lemma 2.7, there must exist at least one network, H, such that $f(\mathrm{FC}(H))$ covers every open set in $P \setminus \partial P$ an odd number of times. However, note that a degenerate full cell covers $P \setminus \partial P$ zero times, so the full cell complex of each non-degenerate network that makes up H must also cover all of $P \setminus \partial P$ the same number of times up to parity. Thus, there must be a non-degenerate network in H that covers $P \setminus \partial P$ an odd number of times. We can conclude that this network must cover every part of $P \setminus \partial P$ at least once, and thus must have all n unique labels in its label set. Lemma 2.4 guarantees that this network has at least $n - d$ full cells. Since this network is non-degenerate, all of these full cells must be non-degenerate, which completes the proof.

□

2.3 Future Work

While we did not succeed at producing a simpler proof of the polytopal generalization of Sperner's lemma without an additional assumption, we do have some ideas for how the proof we presented can be generalized to prove the complete result. The rough idea is to use induction to show that each facet is covered an odd number of times by the full cells, and then show that this means that there is a volume enclosing set near a facet that is covered an odd number of times. This would replace the part of our proof that relies on the existence of a simplicial face, and the rest of the proof works the same way. The main difficulty with this approach is that the full cells connected by paths to a specific facet might terminate at degenerate full cells, so even though there may be an odd number of full cells connected by paths to the facet, it is unclear how to show that there is not only an even number of them covering a volume enclosing set near the facet (if there are an odd number of degenerate full cells). However, we do have one idea for how we might solve this issue. We are relatively confident that degenerate full cells cover the facet they map onto under the piecewise linear map f an even number of times, but have not proven this. If this is the case, then if we can show that each facet is covered an odd number of times in total, then there must be an odd number of non-degenerate full cells that cover it, and thus cover a volume enclosing set near it. While this seems like a very promising direction, we did not have time to flesh out all the details and turn it into a complete proof.

Chapter 3

Matroids

Before we introduce the concept of matroids, we should say something about why we are discussing them at all in this thesis. The thing that brought our attention to matroids was the paper Lovász (1980) which proves a generalization of Sperner's lemma using matroids. The proof presented in that paper has a construction that initially seemed similar to a part of the constructive proof in De Loera et al. (2002). For this reason, we thought we might be able to find some connection between these two results, or find a new proof of the polytopal generalization of Sperner's lemma using the matroid generalization of Sperner's lemma. While this did not end up being the case, we found some interesting partial results connecting matroids and the polytopal generalization of Sperner's lemma, as well as proving a matroid version of the polytopal generalization of Sperner's lemma. Before we present any of this work, we must first introduce the concept of matroids, which is the purpose of this chapter.

Matroids attempt to generalize the notion of linear independence from linear algebra to a more abstract setting. So instead of only considering independence in vector spaces, that is, linear independence, Matroids allow us to define a notion of independence on arbitrary sets. In this chapter we will be covering the basics of matroid theory that are needed to understand the ideas and results in the following chapters. It is important to note that this is not a comprehensive overview of the foundations of matroid theory. Everything in this section is based on Oxley (2003), and if you would like to learn more about matroids that is a great place to start. Matroids were first introduced in 1935 and have been an active area of research since the 1950s. They play an important role in combinatorial optimization and have applications in other fields as well. For example, matroids have applications in electric network theory and statics, see Recski (2013).

3.1 A Motivating Example

Instead of jumping right into the definition of matroids, we'll explore the idea of linear independence from linear algebra in order to motivate a definition for matroids.

Let's start by considering the dependence relations for the set of vectors

$$G = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

One way of encoding all the information relating to dependence is to enumerate every subset of G that is linearly independent (in this case we are assuming that we are in a vector field over $\mathbb{R}$):

$$\emptyset, \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \cdots, \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \cdots, \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Let's call this set I. This set contains every piece of information relating to the dependence relations of the elements of the set G. For example, given any subset, A, of the set G, we can check whether its elements are linearly independent by checking whether A is in I. If it is not, we can conclude that A is linearly dependent.

If we want to develop a notion of independence for objects other than vectors, we might try to encode the information in the same way, that is, by enumerating every set of those objects that we consider independent. However, we want this new notion of independence to "feel" like linear independence, so we do not want to be able to specify just any family of sets to be our linearly independent sets. What might this mean? Well, we probably would not want to allow the set $\{a, b, c\}$ to be independent without the set $\{a, b\}$ being independent.

So want to try to capture the essential properties of linear independence and require that our new notion of independence also have these properties. To figure out which properties these are, let's consider the properties of the set I above. We've already alluded to one property we care about, which is that I is *hereditary*, that is, every subset of a linearly independent set is linearly independent. The set I has another property fundamental to the idea of linear independence, though it's probably not one that is immediately apparent: If A and B are linearly independent and $|A| > |B|$, then there is an element $a \in A \setminus B$ such that $B \cup \{a\}$ is independent.

Let's consider a concrete example to build intuition for why this property holds. Consider the following linearly independent sets of vectors in $\mathbb{R}^3$

$$A = \left\{ \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix} \right\}, \qquad B = \left\{ \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix} \right\}$$

Since both sets are linearly independent and $|A| > |B|$, we claim that there must be a vector in A that we can append to B such that the new set is linearly independent. Why must this be the case? Well, any two vectors in $\mathbb{R}^3$ span a plane, and we know that it is impossible for three linearly independent vectors in $\mathbb{R}^3$ to be in the same plane (otherwise they would be linearly dependent). Thus, at least one of the three vectors cannot be in the plane spanned by the two vectors in the other set. The vector not in the plane is exactly the vector that we can append to the set to form a linearly independent set with three elements. In our example, the plane spanned by B is the xy-plane and there is exactly one vector in A not in the xy-plane, the vector $\begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix}$ (see Figure 3.1). Appending this vector to B, we get the linearly independent set

$$B \cup \{a\} = \left\{ \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix} \right\}$$

By generalizing this line of reasoning to arbitrary dimensions, we see why the property of linear independence we have been considering must hold in general. In some sense, this property captures the idea of dimensions in vector spaces.

The two properties of linear independence we have considered together characterize the essential properties of linear independence that we need in order to define an independence relation that "feels" like linear independence. So we can now define a notion of independence on any set, G, by providing a set I of subsets of G, which we will refer to as the independent sets, with the following requirements:

1. The set I is nonempty.

2. The set I is hereditary, that is, every subset of an independent set is independent.

3. If A and B are independent and $|A| > |B|$, then there is an element $a \in A \setminus B$ such that $B \cup \{a\}$ is independent.

The first requirement is necessary so that at least the empty set is always independent, and the other two are the properties we observed linear independence to have. By requiring these properties, we have assured that this new notion of independence behaves at least somewhat like the notion of linear independence that we are familiar with. Now that we have considered how we might construct a notion of independence for general sets, let's look at the definition of a matroid.

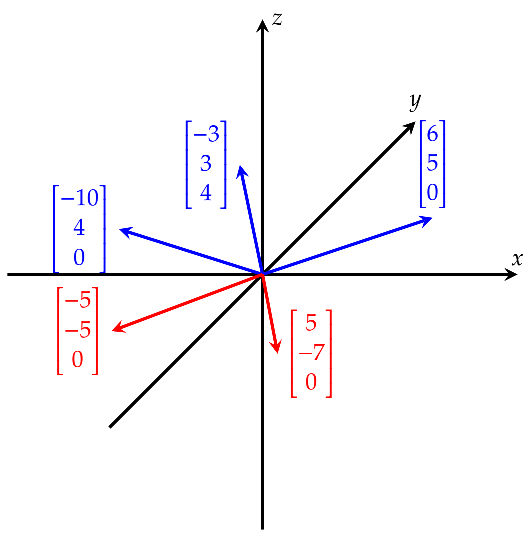


Figure 3.1 The three linearly independent blue vectors can't all be in the span of the two linearly independent red vectors

3.2 Definition of a Matroid

A *matroid* is a pair (G, I), where G is a finite set called the *ground set*, and I is a family of subsets of G, called the *independent sets*, with the following properties:

1. The set I is nonempty.

2. (*hereditary property*) Every subset of an independent set is independent.
3. (*augmentation property*) If A and B are independent and $|A| > |B|$, then there is an element $a \in A \setminus B$ such that $B \cup \{a\}$ is independent.

This definition should look very familiar as it is the exact definition we arrived at in Section 3.1. Observe that the pair (G, I) of sets from earlier, where

$$G = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and I was all the linearly independent subsets of G, satisfies these properties, so we have already seen one example of a matroid. Let's consider a more abstract matroid now that we are armed with the technical definition.

Example 3.1. Let $G = \{a, b, c\}$ and $I = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Then we have a matroid (G, I).

Notice how, in order to define our matroid in Example 3.1, we needed to list out the entirety of the set I. This can quickly become unwieldy as the size of the ground set and the number of independent sets increases. However, there is another way to uniquely specify a matroid, and that is by specifying its maximal independent sets. Once we have the maximal independent sets, we can find all the sets in I by finding all subsets of the maximal independent sets. We might ask why its true that for any matroid (G, I), we can specify the same matroid with only its maximal independent sets. This is because we know that (1) every subset of a maximal independent set must be independent by the hereditary property, and (2) every independent subset must be a subset of some maximal independent set due to the augmentation property. The augmentation property implies (2) because given some maximal independent set A and some smaller independent set B, we must be able to augment B with elements of A to form a set B^* such that $|A| = |B^*|$, and thus this B^* is a maximal independent set containing B. Thus, enumerating every subset of the maximal independent sets is guaranteed to get us exactly the independent sets of the matroid.

Example 3.2. The maximal independent sets of the matroid from Example 3.1 are $\{\{a, b\}, \{a, c\}\}$. Observe that these uniquely specify the matroid since $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ is the set of all subsets of the maximal independent sets, and is also exactly the set I of the matroid.

Since the maximal independent subsets fully define the matroid, we give them a special name: we call each of them a *basis* of the matroid and call all of them together the *bases* of the matroid. Rephrasing our above observation, we can say that *every matroid is uniquely specified by its bases*. However, we cannot specify just any family of sets to be the bases of a matroid. There are some requirements on what can constitute the bases for a matroid, induced by the definition of matroid:

1. The bases set is nonempty
2. Every basis has the same cardinality
3. (*basis exchange property*) If A and B are bases, for every element $a \in A$, there must exist an element $b \in B$ such that $(A \setminus \{a\}) \cup \{b\}$ is a basis

Let's consider these one at a time and see how they come about from the definition of a matroid.

1. Since the set I must be nonempty, the bases set must be nonempty.
2. If we had two bases A and B such that $|A| > |B|$, then by the augmentation property we would be able to find a new independent set $B^* \supsetneq B$, which contradicts the maximality of B (since B is a basis). Therefore, all bases must have the same cardinality.
3. Let A and B are bases. For every element $a \in A$, the set $A \setminus \{a\}$ must be independent by the hereditary property. Since $|B| > |A \setminus \{a\}|$, the augmentation property guarantees us that there must be an element $b \in B$ such that $(A \setminus \{a\}) \cup \{b\}$.

Since every basis of a matroid has the same cardinality, we call the cardinality of the basis sets the *rank* of the matroid. Note that a rank r matroid cannot have any independent set with more than r elements.

Example 3.3. The rank of the matroid from Example 3.1 is 2 since its bases are $\{a, b\}$ and $\{a, c\}$, both of which have cardinality 2.

Example 3.4. The rank of the matroid from Section 3.1 is 3 since one of the basis elements is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

Example 3.5. We cannot specify a matroid with ground set $\{a, b, c, d\}$ and bases $\{\{a, b\}, \{c, d\}\}$ since the family of sets

$$I = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}\}$$

does not satisfy the requirements for a matroid. For example, $|\{a\}| < |\{c, d\}|$ but both $\{a, c\}$ and $\{a, d\}$ are dependent so we have not satisfied the augmentation property.

3.3 Flats of a Matroid

There is one more idea from matroid theory that we will need to understand for the result in the next chapter, and that is the idea of a flat of a matroid. However, before we explain what a flat is we must generalize our idea of rank from the previous section.

We define a rank function, $\text{rank}(A)$, on subsets, A, of the ground set, to be the cardinality of the largest independent subsets of A.

Example 3.6. The rank of a matroid equals $\text{rank}(G)$, where G is the ground set.

Example 3.7. Using the matroid from Example 3.1, $\text{rank}(\{a\}) = 1$, $\text{rank}(\{a, b\}) = 2$, and $\text{rank}(\{b, c\}) = 1$.

The closure, $\text{cl}(A)$, of a subset, $A \subseteq G$, is the maximal set A^* such that $A \subseteq A^*$ and $\text{rank}(A) = \text{rank}(A^*)$. One way to think of this is that for each element in the ground set, you check whether you could use it and the elements in A to create an independent set larger than any independent set in A, and if you can't, then this element is in the closure of A.

Example 3.8. Continuing Example 3.7, $\text{cl}(\{a\}) = \{a\}$, $\text{cl}(\{a, b\}) = \{a, b, c\}$, and $\text{cl}(\{b, c\}) = \{b, c\}$.

Now we can finally define a flat of a matroid. A *flat* of a matroid is any subset, A, of the ground set, such that $A = \text{cl}(A)$. Note that since the closure function is idempotent (i.e. $\text{cl}(\text{cl}(A)) = \text{cl}(A)$), the closure of any set is a flat of the matroid. For this reason we say that the *flat generated by the set A* is the closure of A.

Example 3.9. We can see from Example 3.8 that $\{a\}$ and $\{b, c\}$ are flats of the matroid and that the flat generated by the set $\{a, b\}$ is the set $\{a, b, c\}$.

Chapter 4

Connecting Matroids and Sperner's Lemma

Now that we have explored both Sperner's Lemma and matroids, we are ready to put them together and explore the connections between them. The main motivation for believing that a deeper connection exists comes from a generalization of Sperner's Lemma where we label the vertices in the simplicial subdivision with elements of a matroid. This generalization was proven by Dr. László Lovász in Lovász (1980). In this chapter we present this result as well as generalize it to polytopes. We then discuss some partial results hinting at connections between matroids and the number of full cells in a Sperner labeling.

4.1 The Matroid Generalization of Sperner's Lemma

We can now finally state the matroid generalization of Sperner's Lemma from Lovász (1980).

Theorem 4.1 (Lovász). *Let S be a d-simplex, K a simplicial subdivision of S, and M a rank $d+1$ matroid. Given a labeling of the vertices in K by elements of M such that*

1. *The label set of the vertex-set of S, $L(V(S))$, is independent in M.*
2. *For each face of S, the labels of the vertices of K on the face are in the flat generated by the labels of the vertices of that face.*

Then K has a simplex whose label set forms a basis.

The big idea here is that this result is almost exactly Sperner's Lemma, but instead of using $d+1$ integers as labels, we use the elements of a matroid with rank $d+1$ as labels, and our conception of a full cell changes to a simplex whose vertices are all independent, that is, they form a basis of the matroid. The two requirements on the labeling are almost identical as well, with the main difference being that instead of requiring vertices on a boundary face to be labeled from the label set of the vertices defining that boundary face, we require them to be labeled from the flat generated by that label set.

Example 4.2. We define a matroid M with ground set $\{a,b,c,d,e\}$ and bases $\{\{a,b,c\},\{a,b,e\},\{a,c,d\},\{a,c,e\},\{b,c,d\},\{a,d,e\},\{b,d,e\},\{c,d,e\}\}$. We will label the vertices of our triangle with labels a, b, and c. Then we will be able to label the vertices on the sides of the triangle with elements from the flat generated by $\{a,b\}$, $\{b,c\}$, and $\{a,c\}$, depending on the endpoints of that side. The flat generated by $\{a,b\}$ is $\{a,b,d\}$, the flat generated by $\{b,c\}$ is $\{b,c,e\}$, and the flat generated by $\{a,c\}$ is $\{a,c\}$. If we label a triangulation of a triangle in the way described here, we are guaranteed to have a basis cell by Theorem 4.1. Figure 4.1 is an example of such a labeled triangulation, and as we can see, it has at least one basis cell (basis cells marked with green).

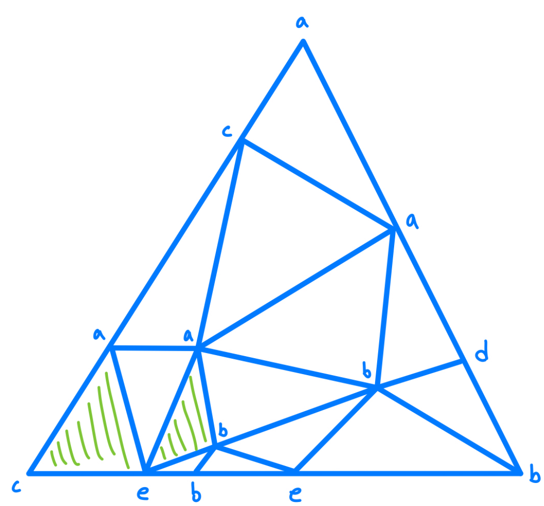


Figure 4.1 A triangulation of triangle appropriately labeled with elements of matroid M from Example 4.2 for the requirements of Theorem 4.1. The basis cells are marked with green.

This result is considered a generalization of Sperner's Lemma because we can define a matroid such that this result reduces to Sperner's Lemma. To do so, define a rank $(d+1)$-matroid with ground set $G = \{0, 1, 2, \dots, d\}$ and bases set $\{G\}$. Then label any simplicial subdivision of a d-simplex with elements of this matroid following the requirements in Theorem 4.1. Note that this labeling is also a Sperner labeling. Due to this theorem, we know that there is a simplex in the simplicial subdivision whose label set forms a basis, however, since the only basis is G, we know that there must be some simplex whose label set is $\{0, 1, 2, \dots, d\}$. Thus, this result implies that a Sperner labeled simplicial subdivision is guaranteed to have a full cell.

We will now present a simplified proof of Theorem 4.1 inspired by the proof in Lovász (1980). Lovász' proof is nonconstructive and follows a similar line of reasoning to the pebble counting argument presented in Section 1.2. We have taken the main idea from his proof but chosen to make use of Sperner's Lemma directly instead of repeating the counting argument that he relies on. Interestingly, the proof shown here can be considered either constructive or nonconstructive, depending on whether you rely on the constructive or non-constructive proof of Sperner's Lemma. This is because when we apply Sperner's Lemma, we can use it either to guarantee the existence of a full cell or to actually find a full cell. First, we prove a simple lemma about matroids.

Lemma 4.3. *Let A and B be independent sets of a matroid such that $A \cup B$ is independent. Then the flat generated by A and the flat generated by B are disjoint.*

Proof. If $A = \mathrm{cl}(A)$ the result is straightforward, since for every element in $a \in A$, the independence of $A \cup B$ along with the hereditary property of independent sets guarantees that $B \cup \{a\}$ is an independent set, and thus, that $a \notin \mathrm{cl}(B)$.

Now assume that $A \subsetneq \mathrm{cl}(A)$. Choose $c \in \mathrm{cl}(A) \setminus A$. Let $a \in A$ and let $A' = (A \cup \{c\}) \setminus \{a\}$. Since $|A \cup B| > |A'|$, the augmentation property of independent sets tells us that there must exist a $B' \subset (A \cup B) \setminus A' = B \cup \{a\}$ such that $A' \cup B'$ is independent and $|A' \cup B'| = |A \cup B|$. Note that A' and B' are disjoint and $|A'| = |A|$, so it must be that $|B'| = |B|$. By the definition of $\mathrm{cl}(A)$, $A' \cup a = A \cup c$ cannot be independent. Thus, B' cannot contain a, for if it did, it would contradict the independence of $A' \cup B'$, which means that $B' = B$. Finally, we see that $A' \cup B$ is independent, so $c \notin \mathrm{cl}(B)$, and thus the flat generated by A and the flat generated by B are disjoint. □

Now we prove Theorem 4.1.

Proof. Let $A = \{a_0, a_1, \ldots, a_d\}$ be a basis of M, and define a sequence of flats $F_0, F_1, \ldots, F_d$ where we define F_i as the flat generated by the set $\{a_0, a_1, \ldots, a_i\}$. Note that $F_i \subsetneq F_{i+1}$ since the independence of $\{a_0, a_1, \ldots, a_i, a_{i+1}\}$ implies that $a_{i+1} \notin F_i$.

Now define a new labeling of K in the following manner. Given a vertex with label m in the original labeling, label it i, where i is chosen such that $m \in F_i \setminus F_{i-1}$. We claim that this new labeling is a Sperner labeling.

First observe that since $V(S)$ is independent, it's impossible for two labels in the label set of $V(S)$ to both be in the flat F_i without either of them being in the flat F_{i-1}. Therefore, each of the vertices of S has a distinct label in the new labeling. Given a vertex v on a k-face $S_{n_1}S_{n_2}\cdots S_{n_{k+1}}$ of S (where each n_i is a unique element in $\{0, \ldots, d\}$) we know that $L(v)$ is in the flat generated by $\{L(S_{n_1}), L(S_{n_2}), \ldots, L(S_{n_{k+1}})\}$. Thus, by Lemma 4.3, $L(v)$ is not in the closure of the label set of any subset of $V(S)$ disjoint with $\{L(S_{n_1}), L(S_{n_2}), \ldots, L(S_{n_{k+1}})\}$. This is equivalent to saying that if $L(v) \in F_i \setminus F_{i-1}$, then there must exist some $1 \leq j \leq k+1$ such that $L(S_{n_j}) \in F_i \setminus F_{i-1}$, which in turn means that in the new labeling, v will have the same label as one of the vertices making up the k-face it is a part of. Thus the new labeling is a Sperner labeling.

By Sperner's Lemma, there must be a full cell in K, which means that it has label set $\{0, 1, \ldots, d\}$. Returning to the original labeling, this cell must then have labels $\{x_0, x_1, \ldots, x_d\}$ where $x_0 \in F_0$ and $x_i \in F_i \setminus F_{i-1}$. It's straightforward to confirm that this set is independent—$\{x_0\}$ is independent since x_0 is in the flat F_0, and then if we imagine adding one element at a time we know that each new set is independent due to how we constructed the flats F_i. Finally, we conclude that there exists a simplex in K whose vertices form a basis. □

4.2 The Matroid Generalization of the Polytopal Generalization of Sperner's Lemma

One of the benefits of how we proved the matroid generalization of Sperner's Lemma, is that we can use a very similar proof to prove a matroid generalization of the polytopal generalization of Sperner's Lemma.

Theorem 4.4. *Let P be a convex (n, d)-polytope, K a simplicial subdivision of P, and M a rank n matroid. Given a labeling of the vertices in K by elements of M such that*

1. *The label set of the vertex-set of P, $L(V(P))$, is independent in M.*
2. *For each face of P, the labels of the vertices of K on the face are in the flat generated by the labels of the vertices of that face.*

Then K has $n - d$ simplices whose label sets are independent sets.

Proof. Define a sequence of flats just as we did in the proof of Theorem 4.1 and then create a new labeling of the simplicial subdivision using that sequence to get a Sperner labeling of the polytope. By the polytopal generalization of Sperner's lemma, Theorem 2.1, we are guaranteed that there are at least $n - d$ non-degenerate full cells. By the construction of the new labeling, a non-degenerate full cell corresponds to a cell whose label set is an independent set in the original labeling, completing the proof. □

4.3 Some Partial Progress Proving the Polytopal Generalization of Sperner's Lemma using the Matroid Generalization of Sperner's Lemma

We will now discuss the work we have done attempting to prove the polytopal generalization of Sperner's lemma using the matroid generalization of Sperner's lemma. It is probably not clear how this might be done, but that is because the statement of the matroid generalization of Sperner's Lemma in Lovász' paper is different from the one we presented in Section 4.1. The main theorem he proves in his paper is actually

Theorem 4.5. *Let K be a simplicial complex which is also a d-dimensional manifold. Assume that the vertices K are labeled with elements of a rank $d + 1$ matroid M. Then if K contains a d-simplex for which the labels on its vertices form a basis, then there must be at least one other such d-simplex.*

Theorem 4.1 is then stated as a corollary to this result. Proving this is very similar to our proof of Theorem 4.1—you either construct a path of $(d - 1)$-simplices from the d-simplex you know about until you reach another one, or you use a counting argument similar to the one in the proof of Theorem 1.2. If you would like to see a written out proof, refer to Lovász (1980).

Given this result, it is worth asking how we get Theorem 4.1 as a corollary, as Lovász states it without proof. One reasonable interpretation is that you would use the same proof with some minor alterations to prove the corollary, and this, as we have discussed above, does work. However, there is also a way to prove Theorem 4.1 by directly applying Theorem 4.5. We will demonstrate this by an example that will make abundantly clear how a formal proof would go. Let M be a matroid with ground set $\{a, b, c, d\}$ and bases $\{\{a, b, c\}, \{a, b, d\}\}$ Consider the triangulation of a triangle labeled with elements of the matroid M in Figure 4.2.

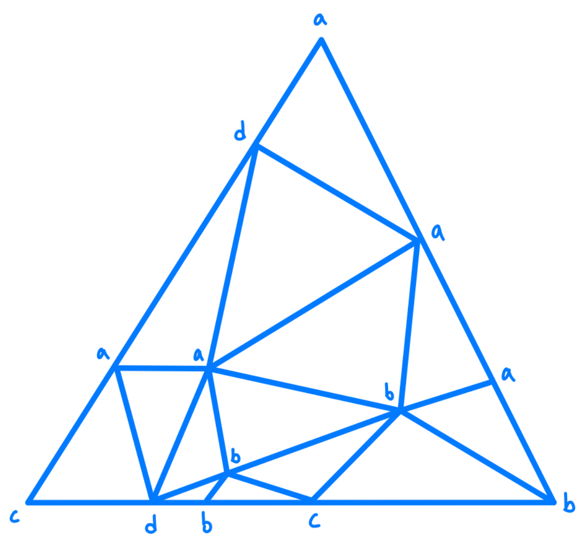


Figure 4.2 A matroid labeled triangulation of a triangle for which we'd like to prove the existence of a basis cell using Theorem 4.5.

We want to prove that this triangulation must have a triangle for which the labels of its vertices form a basis using Theorem 4.5 instead of by inspection. One way that we can do this is by constructing another triangle with a labeled triangulation such that

1. the vertices on the boundaries and the labels of those vertices align with the boundaries of the labeled triangulation in Figure 4.2

2. there is only one simplex whose vertex labels form a basis

Then we can quotient them together to form a 2-manifold and apply Theorem 4.5 to conclude that there must be a second triangle whose vertex labels form a basis. Since we know that there is only one in the second triangle (by construction), the second one must be in our original triangle, which is what we had set out to prove. Figure 4.3 is an example of a labeled triangulation of a triangle that satisfies the above requirements with respect to the labeled triangulation in Figure 4.2. Therefore, we have proven that the triangulation in Figure 4.2 must have a basis cell. It should be clear that we can make an equivalent construction for any labeling of any triangulation of a triangle, so this proves Theorem 4.1.

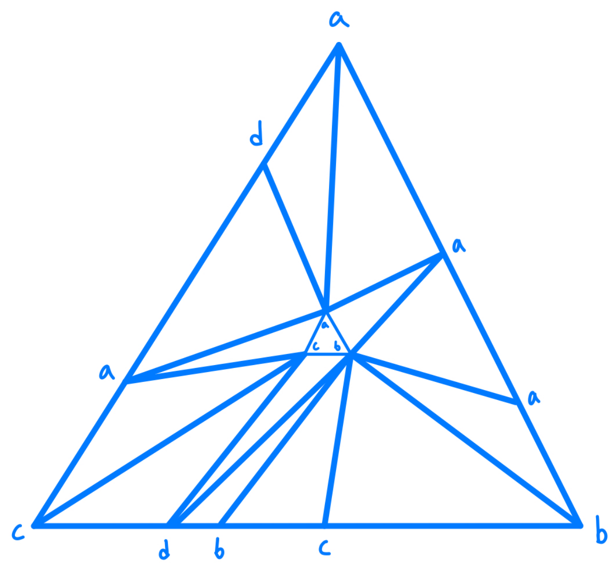


Figure 4.3 A triangle whose boundary aligns with the boundary of the triangle in Figure 4.2 and has only one basis cell.

Now let's return to the original question, can we do this to prove the polytopal generalization of Sperner's Lemma? We don't have a definite answer to this, but it seems reasonable that we could make a construction similar to the one in Figure 4.3 for any convex polygon, and possibly for any convex polytope, but this time with a matroid whose ground set is $\{1, 2, \ldots, d + 1\}$ and only basis being the ground set (this makes it so that a basis cell is equivalent to a full cell of Sperner's Lemma).

However, making such a construction becomes a little more complicated for polygons and even more complicated for polytopes. For polygons, it is easy to draw a triangulation that works, but isn't immediately clear how to guarantee that there will be only one basis simplex, since the copy of the polygon on the interior needs to be triangulated as well. Also, we now need to repeat this argument multiple times, each time with a different matroid so that we can guarantee the existence of not only one full cell, but $n-2$ full cells. For polytopes, we not only need to solve the problems we are facing with polygons, but it also becomes difficult to define a simplicial subdivision that works with a smaller copy of the polytope in the interior—in fact, we are not certain that it is possible.

For now lets focus on polygons. Let's consider the case of a square to get a better idea of what we were saying above. Consider the construction shown in Figure 4.4.

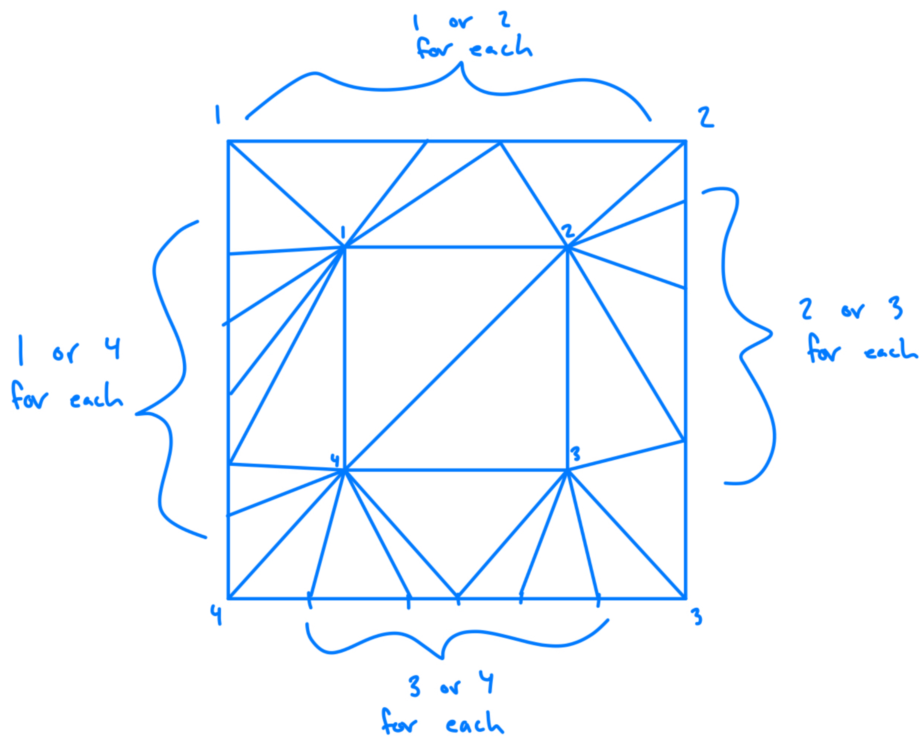


Figure 4.4 By quotienting the boundary of this square with another square and using Theorem 4.5 we can prove that the other square must have a basis cell (full cell)

We could quotient this square with another labeled, triangulated square (assuming the boundaries aligned, but it is easy to fix if they do not) and then apply Theorem 4.5 to guarantee the existence of a full cell in the other triangulated square, but only if only one of $\{1, 2, 4\}$ and $\{2, 3, 4\}$ are bases of the matroid. If they are both bases of the matroid, then we do not satisfy the condition of only one basis cell existing in the triangulation, and then when we apply Theorem 4.5 we are not guaranteed that the other basis cell we find is in the other triangulated square and not in the same one. In this case however, it is easy to define a matroid with the desired properties. We need the ground set to be $\{1, 2, 3, 4\}$ just as before, but now let our bases be $\{1, 2, 4\}$ and $\{1, 2, 3\}$. Then using this matroid and applying Theorem 4.5, we guarantee the existence of a full cell with labels $\{1, 2, 4\}$ or $\{1, 2, 3\}$ in the other triangulated square. We can now define another matroid, again with ground set $\{1, 2, 3, 4\}$, but this time with bases $\{2, 3, 4\}$ and $\{1, 3, 4\}$. Applying the theorem again we guarantee the existence of a full cell with labels $\{2, 3, 4\}$ or $\{1, 3, 4\}$. Thus, we have guaranteed the existence of 2 unique full cells in every triangulation of a square. Note that this is what the polytopal generalization of Sperner's Lemma guarantees as well since $n = 4$ and $d = 2$ so $n - d = 2$.

The important step in the above example is that we split up all sets of size 3 with elements $\{1, 2, 3, 4\}$ into two disjoint sets such that each set satisfied the requirements for being the bases of a matroid (it is straightforward to check that they did in the above example). While it is easy to generalize the construction we used for the square to any convex polygon with n vertices, is not clear that we can define $n - 2$ rank 3 matroids with ground set $\{1, 2, \ldots, n\}$ such that their bases are disjoint. If we could, that would provide a proof of the polytopal generalization of Sperner's Lemma for two dimensions.

In the higher dimensional case, we still have the problem of needing to define $n-d$ matroids with disjoint bases that we faced in the two-dimensional case, but now we have the additional difficulty of it not being as easy to construct a simplicial subdivision of a d-dimensional convex polytope that behaves the same way as the construction used above. If we could define such a construction, and prove that we can define the needed $n - d$ matroids, that would provide a proof for the polytopal generalization of Sperner's Lemma in all dimensions.

Future Work

There are several possible avenues for making progress on this problem. The first and probably easiest would be to prove that there is a way to define $n - 2$ rank 3 matroids with ground set $\{1, 2, \ldots, n\}$ such that their bases are disjoint for all $n > 3$. With this we would be able to write out a full proof of the polytopal generalization of Sperner's Lemma in two dimensions. The next step would be to generalize this to $n - d$ rank $d + 1$ matroid with ground set $\{1, 2, \ldots, n\}$ such that all their bases are disjoint for all $n > 3$. This would be a necessary piece of the puzzle for proving the polytopal generalization of Sperner's Lemma using matroids.

A different avenue of work that is also necessary would be attempting to construct simplicial subdivisions for all d-dimensional convex polytopes with n vertices such that there are very few full cells (and no more than one copy of each). If we had this, we would hopefully be able to combine it with the different matroids to prove the polytopal generalization of Sperner's Lemma.

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