IMBALANCES OF BIPARTITE MULTITOURNAMENTS

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Abstract. A bipartite (a, b, p, q)-tournament is a bipartite tournament in which the parts of the tournament contain p, resp. q vertices and the vertices belonging to different parts of the tournament are connected with at least a and at most b arcs. The imbalance of a vertex is defined as the difference of its outdegree and indegree. In this paper existence criteria and construction algorithms are presented for bipartite $(0, b, p, q)$-tournaments having prescribed imbalance sequences and prescribed imbalance sets.

1. Introduction

An active research topic of graph theory is the characterization of different special graphs (as simple, oriented, bipartite, multipartite, signed, semicomplete, and football graphs, see e.g. [1, 5, 10, 12, 14, 15, 17, 18, 19, 22, 33, 35]),

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and different generalizations (as hypergraphs, hypertournaments, weighted graphs, see e.g. [21, 30, 31]) having prescribed degree properties.

The classical results, as the theorem published by Landau in 1953 [16], and the theorem of Erdős and Gallai published in 1960 [4] contained necessary and sufficient conditions for the existence of a tournament, respectively of a simple graph with prescribed parameters. Later also constructive results appeared as the Havel-Hakimi theorem [8, 9] on simple graphs and the construction algorithm for optimal (a, b, n)-tournaments [13].

The structure of the paper is as follows. Section 2 contains some preliminary results, while Section 3 deals with imbalances of $(0, \infty, p, q)$-tournaments. In Section 4 the reconstruction results of imbalance sequences are discussed, Section 5 is devoted to imbalance sets.

2. Preliminary notions and earlier results

Let $a,\ b$ and n be nonnegative integers ($b \geq a \geq 0,\ n \geq 1$), $\mathcal{T}(a, b.n)$ be the set of directed multigraphs $T = (V, E)$, where $|V| = n$, and elements of each pair of different vertices $u,\ v \in V$ are connected with at least a and at most b arcs [11]. $T \in \mathcal{T}(a, b, n)$ is called (a, b, n)-*tournament.* $(1, 1, n)$-tournaments are the usual tournaments, and $(0, 1, n)$-tournaments are also called oriented graphs or simple directed graphs [6]. The set $\mathcal{T}$ is defined by

$$\mathcal{T} = \bigcup_{b\geq 0,\ n\geq 1} \mathcal{T}(0, b, n).$$

According to this definition, $\mathcal{T}$ is the set of the finite directed loopless multigraphs.

For any vertex $v \in V$ let $d(v)^+$ and $d(v)^-$ denote the outdegree and indegree of x, respectively. Define $f(v) = d(v)^+ - d(v)^-$ as the imbalance of the vertex v. The imbalance sequence of $T \in \mathcal{T}$ is formed by listing the vertex imbalances of the vertices in nonincreasing or nondecreasing order.

The following result due to Avery [1] and Mubayi, Will and West [19] provides a necessary and sufficient condition for a nonincreasing sequence F of integers to be the imbalance sequence of a tournament $T \in \mathcal{T}(0, 1, n)$.

Theorem 2.1. *A nonincreasing sequence of integers $F = [f_1, \ldots, f_n]$ is an imbalance sequence of a tournament $T \in \mathcal{T}(0, 1, n)$ if and only if*

$$\sum_{i=1}^{k} f_i \leq k(n-k),$$

for $1 \leq k < n$ with equality when $k = n$.

Proof. See [1, 19]. ■

Arranging the sequence F in nondecreasing order, we have the following equivalent assertion.

Corollary 2.1. *A nondecreasing sequence of integers $F = [f_1, \ldots, f_n]$ is the imbalance sequence of a $(0,1,n)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i \geq k(k-n)$$

for $1 \leq k < n$, with equality when $k = n$.

The following theorem gives a characterization of imbalance sequences of $(0,b,n)$-tournaments [28].

Theorem 2.2. *If $b \geq 1$, then a nonincreasing sequence $F = [f_1, \ldots, f_n]$ of integers is the imbalance sequence of a $(0,b,n)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i \geq bk(n-k),$$

for $1 \leq k \leq n$ with equality when $k = n$.

Proof. See [28]. ■

In [28] also a construction algorithm of a $(0,b,n)$-tournament can be found. Some other results on imbalances of $(0,b,n)$-tournaments and their special cases can be found in [12, 20, 29, 34].

Reid in 1978 [32] introduced the concept of the score set of $(1,1,n)$-tournaments as the set of different scores (outdegrees) of the given tournament. At the same time he formulated the conjecture that for any set of nonnegative integers S there exists a tournament T having S as its score set. In the same paper he proved the conjecture for sets containing 1, 2, or 3 elements. Hager in 1986 [7] proved the conjecture for $|S| = 4$ and $|S| = 5$ and Yao in 1989 [36] published a proof of the whole conjecture.

There are some known results on the imbalance sets of $(0,1,n)$-tournaments (see e.g. [23, 26, 28]).

3. Imbalances in $(0, \infty, p, q)$-tournaments

Let a, b, p and q be nonnegative integers ($b \geq a \geq 0$, $p \geq 1$, $q \geq 1$), $\mathcal{B}(a,b,p,q)$ be the set of directed bipartite multigraphs $B = (U \cup V, E)$, where

$|U| = p$ and $|V| = q$, and the elements of each pair of vertices $u \in U$ and $v \in V$ are connected with at least a and at most b arcs. Then $B \in \mathcal{B}(a,b,p,q)$ is called (a,b,p,q)-*tournament.* $B \in \mathcal{B}(0,1,p,q)$ is an oriented bipartite graph and a $(1,1,p,q)$-tournament is a bipartite tournament.

According to this definition

$$\bigcup_{\substack{b\geq a\geq 0\\ p\geq 1,\ q\geq 1}} \mathcal{B} \tag{3.1}$$

is the set of finite directed bipartite multigraphs.

For any vertex $v \in U \cup V$ of $T \in \mathcal{B}(a,b,p,q)$ let $d(v)^+$ and $d(v)^-$ denote the outdegree and indegree of v, respectively. Define $f(u) = d(u)^+ - d(u)^-$ and $g(v) = d(v)^+ - d(v)^-$ as the imbalances of the vertex u for $u \in U$, resp. $v \in V$. Then the nonincreasing or nondecreasing sequences $F = [f_1, \ldots, f_p]$ and $G = [g_1, \ldots, g_q]$ are the imbalance sequences of the (a,b,p,q)-tournament $T = (U \cup V, E)$.

4. Reconstruction of imbalance sequences

This section starts with a necessary and sufficient condition for two sequences F and G to be imbalance sequences of some $(0,b,p,q)$-tournament. Then we deal with minimal reconstruction of imbalance sequences.

4.1. Existence of a realization of an imbalance sequence of a $(0,b,p,q)$-tournament

The following result is a combinatorial criterion for determining whether some prescribed sequences are realizable as imbalance sequences of a $(0,b,p,q)$-tournament. This is analogous to a result on degree sequences of simple graphs by Erdős and Gallai [4] and a result on bipartite tournaments due to Beineke and Moon [2].

Theorem 4.1. *Let b, p and q be positive integers. Two nonincreasing sequences $F = [f_1, \ldots, f_p]$ and $G = [g_1, \ldots, g_q]$ of integers are the imbalance sequences of some $(0,b,p,q)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j \leq bk(q-l) + bl(p-k) \tag{4.1}$$

for $1 \leq k \leq p$, $1 \leq l \leq q$, with equality when $k = p$ and $l = q$.

Proof. The necessity follows from the fact that a directed bipartite subgraph of a $(0, b, p, q)$-tournament induced by k vertices from the first part and l vertices from the second part has a sum of imbalances 0, and these vertices can gather at most $bk(q-l) + bl(p-k)$ imbalances from the remaining $(q-l)$ and $(p-k)$ vertices.

For sufficiency, assume that $F = [f_1, \ldots, f_p]$ and $G = [g_1, \ldots, g_q]$ are the sequences of integers in nonincreasing order satisfying conditions (4.1) but are not the imbalance sequences of any $(0, b, p, q)$-tournament. Let these sequences be chosen in such a way that p is the smallest possible and q is the smallest possible among the tournaments with the smallest p, and f_p is the least with that choice of p and q. We consider the following two cases.

Case (i). Suppose equality in (4.1) holds for some $k \le p$ and $l < q$, so that

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j = bk(q-l) + bl(p-k).$$

Consider the sequences

$$F' = [f'_i]_1^k = [f_1 - b(q-l), f_2 - b(q-l), \ldots, f_k - b(q-l)]$$

and

$$G' = [g'_j]_1^l = [g_1 - b(p-k), g_2 - b(p-k), \ldots, g_l - b(p-k)],$$

where for $1 \le i \le k$ and $1 \le j \le l$,

$$f'_i = f_i - b(q-l)$$

and

$$g'_j = g_j - b(p-k).$$

For $1 \le r < k$ and $1 \le s < l$, we have

$$\begin{aligned}\sum_{i=1}^{r} f'_i + \sum_{j=1}^{s} g'_j &= \sum_{i=1}^{r}[f_i - b(q-l)] + \sum_{j=1}^{s}[g_j - b(p-k)] = \\ &= \sum_{i=1}^{r} f_i + \sum_{j=1}^{s} g_j - rb(q-l) - sb(p-k) \le \\ &\le b[r(q-s) + s(p-r)] - rb(q-l) - sb(p-k) \le \\ &\le b[r(l-s) + s(k-r)]\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{k} f'_i + \sum_{j=1}^{l} g'_j &= \sum_{i=1}^{k}[f_i - b(q-l)] + \sum_{j=1}^{l}[g_j - b(p-k)] = \\ &= \sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j - kb(q-l) - lb(p-k) = \\ &= b[k(q-l) + l(p-k)] - b[k(q-l) + l(p-k)] = \\ &= 0.\end{aligned}$$

Thus the sequences $F' = [f'_i]_1^k$ and $G' = [g'_j]_1^l$ satisfy (4.1) and by the minimality of p and q, F' and G' are the imbalance sequences of some $(0,b,k,l)$-tournament $B'(U' \cup V', E')$.

Let

$$F'' = [f_{k+1} + bl, f_{k+2} + bl, \ldots, f_p + bl]$$

and

$$G'' = [g_{l+1} + bk, g_{l+2} + bk, \ldots, g_q + bk].$$

We have for $1 \le r \le p-k$ and $1 \le s \le q-l$,

$$\begin{aligned}\sum_{i=1}^{r}[f_{k+i} + bl] + \sum_{j=1}^{s}[g_{l+j} + bk] &= \sum_{i=1}^{r} f_{k+i} + \sum_{j=1}^{s} g_{l+j} + rbl + sbk = \\ &= \sum_{i=1}^{k+r} f_i + \sum_{j=1}^{l+s} g_j - \left(\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j\right) + rbl + sbk \le \\ &\le b(k+r)[q-(l+s)] + b(l+s)[p-(k+r)] - \\ &\qquad - b[k(q-l) + l(p-k)] - rbl - sbk \le \\ &\le b[r(q-l-s) + s(p-k-r)],\end{aligned}$$

with equality when $r = p-k$ and $s = q-l$. Therefore, by the minimality for p and q, the sequences F'' and G'' form the imbalance sequences of some $(0,b,p-k,q-l)$-tournament $B''(U'' \cup V'', E'')$.

Now construct a $(0,b,p,q)$-tournament $B(U \cup V, E)$ as follows.

Let $U = U' \cup U''$, $V = V' \cup V''$ and $U' \cap U'' = \phi$, $V' \cap V'' = \phi$ and arc set E containing those arcs which are between U' and V', and between U'' and V'', and b arcs from each vertex of U' to every vertex of V'', and b arcs from each vertex of V' to every vertex of U''. This is a contradiction.

Case (ii). Suppose that the strict inequality holds in (4.1) for all $k \neq p$ and $l \neq q$. That is,

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j < bk(q-l) + bl(p-k)$$

for $1 \leq k < p$, $1 \leq l < q$.

Let $F_1 = [f_1+1, f_2, \ldots, f_{p-1}, f_p-1]$ and $G_1 = [g_1, \ldots, g_q]$, so that F_1 and G_1 satisfy the conditions 4.1. Thus, by the minimality of f_p, the sequences F_1 and G_1 are the imbalances sequences of some $(0, b, p, q)$-tournament $B_1(U_1 \cup V_1)$. Let $f_{u_1} = f_1 + 1$ and $f_{u_p} = f_p + 1$. Since $f_{u_1} > f_{u_p} - 1$, therefore there exists a vertex $v \in V_1$ such that $u_1(0-0)v(1-0)u_p$, or $u_1(1-0)v(0-0)u_p$, or $u_p(1-0)v(1-0)u_1$, or $u_p(0-0)v(0-0)u_1$, in $D_1(U_1 \cup V_1, E_1)$ and if these are changed to $u_1(0-1)v(0-0)u_p$, or $u_1(0-0)v(0-1)u_p$, or $u_1(0-0)v(0-0)u_p$, or $u_1(0-1)v(0-1)u_p$ respectively, the result is a $(0, b, p, q)$-tournament with imbalance sequences F and G, which is a contradiction proving the result. ■

Since $(0, 1, p, q)$-tournaments (oriented graphs) are special (a, b, p, q)-tournaments, the following corollary of Theorem 4.1 gives a necessary and sufficient condition for nonincreasing sequences of integers to be imbalance sequences of some $(0, 1, p, q)$-tournament.

Corollary 4.1. *Two nonincreasing sequences* $F = [f_1, \ldots, f_p]$ *and* $G = [g_1, \ldots, g_q]$ *of integers are the imbalance sequences of some* $(0, 1, p, q)$*-tournament if and only if*

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j \leq k(q-l) + l(p-k), \tag{4.2}$$

for $1 \leq k \leq p$, $1 \leq l \leq q$ *with equality when* $k = p$ *and* $l = q$.

Proof. Let us substitute $b = 1$ into (4.1). ■

Another simple property of imbalance sequences of (a, b, p, q)-tournaments is

$$\sum_{i=1}^{p} f_i + \sum_{j=1}^{q} g_j = 0. \tag{4.3}$$

For arbitrary sequences of integer numbers F and G satisfying (4.3) one can find such a b that F and G are imbalance sequences of some $(0, b, p, q)$-tournament. We are interested in the minimal such b.

Let F_{max}, G_{max}, and z be defined as follows:

$$F_{max} = \max_{1 \le i \le p} |f_i|,$$

$$G_{max} = \max_{1 \le j \le p} |g_j|,$$

and

$$z = \max(F_{max}, G_{max}). \tag{4.4}$$

The following assertion gives lower and upper bound for b_{min}.

Lemma 4.1. *If $p \ge 1$ and $q \ge 1$, then*

$$\max\left(\left\lceil \frac{F_{\max}}{q} \right\rceil, \left\lceil \frac{G_{\max}}{p} \right\rceil\right) \le b_{\min} \le \max(F_{\max}, G_{\max}). \tag{4.5}$$

Proof. From one side it is easy to construct a $(0, z, p, q)$-tournament, where z is defined in (4.4), and from the other side even the uniform allocation of the degrees requires

$$b \ge \max\left(\left\lceil \frac{F_{max}}{q} \right\rceil, \left\lceil \frac{G_{max}}{p} \right\rceil\right). \tag{4.6}$$

■

We are interested in the least possible b allowing the realization of F and G.

4.2. Computation of b_{min} for a $(0, b, p, q)$-tournament

We are interested in the computation of the minimal value of b, satisfying (4.1). Using Theorem 4.1 we can compute b_{min}.

Let

$$\alpha(b, k, l) = \sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j$$

and

$$\beta(b, k, l) = bk(q - l) + bl(p - k)$$

for $1 \le i \le p$ and $1 \le j \le q$.

The following theorem allows quickly to compute b_{min}.

Theorem 4.2. *Two nonincreasing sequences* $F = [f_1, \ldots, f_p]$ *and* $G = [g_1, \ldots, g_q]$ *of integers are the imbalance sequences of some* $(0, b, p, q)$*-tournament* B *if and only if* $b \geq b_{min}$, *where*

$$b_{min} = \min_{1 \leq k \leq p, 1 \leq l \leq q} \{b \mid \alpha(b,k,l) \leq \beta(b,k,l)\}. \tag{4.7}$$

Proof. If $k = p$ and $l = q$, then both sides of (4.1) are equal to zero, otherwise the right side is positive and a multiple of b, therefore (4.7) holds, if b is sufficiently large. ■

The following program Minimal is based on Theorem 4.2. The pseudocode uses the conventions described in [3].

Input. p and q: the numbers of the elements in the prescribed imbalance sequences;
$F = [f_1, \ldots, f_p]$ and $G = [g_1, \ldots, g_q]$: given nonincreasing sequences of integers.

Output. b_{min}: the minimal number of allowed arcs between two vertices belonging to different parts of B.

Working variables. i, j: cycle variables;
S: actual sum of the imbalances;
$L = \alpha(b,k,l)$: the actual value of the left side of (4.1).

Minimal(p, q, F, G, b_{min})

```
S = 0
Fmax = max(|f_1|, |f_p|)
Gmax = max(|g_1|, |g_q|)
b_min = max(⌈Fmax/q⌉, ⌈Gmax/p⌉)
for i = 1 to p
      S = S + f_i
      L = S
      for j = 1 to q
            L = S + g_j
            b_min = max(b_min, ⌈(L/[i((q − j) + j(p − i) + j(p − i)]⌉
            if    b_min == max(Fmax, Gmax)
                  return b_min
return b_min
```

Minimal computes b_{min} in all cases in $O(pq)$ time.

5. Imbalance sets in bipartite multidigraphs

K. B. Reid in 1978 [32] introduced the concept of the score set of tournaments as the set of different scores (outdegrees) of a tournament. At the same time he formulated the conjecture that for any set of nonnegative integers S there exists a tournament T having S as its score set. At the same time he proved the conjecture for sets containing 1, 2, or 3 elements. Hager in 1986 [7] proved the conjecture for $|S| = 4$ and $|S| = 5$ and Yao [36] published a proof of the conjecture.

In an analogous manner we define the imbalance set of a bipartite multigraph $B = (U \cup V, E)$ as the union of the sets of different imbalances of the vertices in U and V.

5.1. Existence of a $(0, 1, p, p)$-tournament with prescribed imbalance sets

First we show the existence of a $(0, 1, p, q)$-tournament with given set of integers as imbalance sets.

Theorem 5.1. *Let* p, f_1, $\ldots$, f_p, g_1, $\ldots$, g_p *be positive integers and let* $F = [f_1, \ldots, f_p]$ *and* $Q = [-g_1, \ldots, -g_p]$, *where* $f_1 < \cdots < f_p$, $g_1 < \cdots < g_p$, *and* $(f_1, \ldots, f_p, g_1, \ldots, g_p) = t$. *Then there exists a* $(0, 1, p, p)$*-tournament with imbalance set* $F \cup G$.

Proof. Construct a $(0, 1, p, p)$-tournament $B(U \cup V, E)$ as follows. Let $U = U_1 \cup \cdots \cup U_p$, $V = V_1 \cup \cdots \cup V_p$ with $U_i \cap U_j = \emptyset$ $(i \neq j)$, $V_i \cap V_j = \emptyset$ $(i \neq j)$, $|U_i| = g_i$ for all i, $1 \leq i \leq p$ and $|V_j| = f_j$ for all j, $1 \leq j \leq p$. Let there be an arc from every vertex of U_i to each vertex of V_i for all i, $1 \leq i \leq p$, so that we obtain the $(0, 1, p, p)$-tournament $B(U \cup V, E)$ with the given imbalance sets of vertices as follows.

For $1 \leq i, j \leq p$, $f_u = |V_i| - 0 = f_i$, for all $u \in U_i$ and $g_v = 0 - |U_j| = -g_j$, for all $v \in V_j$.

Therefore, the imbalance set of $B(U \cup V, E)$ is $F \cup G$. ∎

5.2. Existence of a $(0, b, p, p)$-tournament with prescribed imbalance sets

Finally, we prove the existence of a $(0, b, p, p)$-tournament with prescribed sets of positive integers as its imbalance set.

Let $(f_1, \ldots, f_p, g_1, \ldots, g_p)$ denote the greatest common divisor of $f_1, \ldots, f_p, g_1, \ldots, g_p$.

Theorem 5.2. *Let b, p, $f_1, \ldots, f_p$, $g_1, \ldots, g_p$ be positive integers and let $F = [f_1, \ldots, f_p]$ and $G = [-g_1, \ldots, -g_p]$, where $f_1 < \cdots < f_p$, $g_1 < \cdots < g_p$, and $(f_1, \ldots, f_p, g_1, \ldots, g_p) = t \leq b$. Then there exists a $(0, b, p, p)$-tournament with imbalance set $F \cup G$.*

Proof. Since $(f_1, \ldots, f_p, g_1, \ldots, g_p) = t$, where $1 \leq t \leq b$, there exist positive integers $x_1, \ldots, x_p$, $y_1, \ldots, y_p$ with $x_1 < \cdots < x_p$, $y_1 < \cdots < y_p$ such that $f_i = tx_i$ for $1 \leq i \leq p$ and $g_j = ty_j$ for $1 \leq j \leq p$.

Construct a $(0, b, p, p)$-tournament $B(U \cup V, E)$ as follows. Let $U = U_1 \cup \cdots \cup U_p$, $V = V_1 \cup \cdots \cup V_p$ with $U_i \cap U_j = \emptyset$, $V_i \cap V_j = \emptyset$, $i \neq j$, $|U_i| = x_i$ for all i, $1 \leq i \leq p$, $|V_i| = x_i$ for all i, $1 \leq i \leq p$. Let there be t arcs directed from every vertex of U_i to each vertex of V_i for all i, $1 \leq i \leq p$, so that we obtain the $(0, b, p, p)$-tournament $B(U \cup V, E)$ with the imbalances of vertices as follows.

For $1 \leq i \leq p$,

$$f_u = t|V_i| - 0 = tx_i = f_i, \text{ for all } u \in U_i,$$

$$g_v = 0 - t|U_i| = -ty_1 = -g_1, \text{for all } v \in V_i.$$

Therefore the imbalance set of $B(U \cup V, E)$ is $F \cup G$. ■

An overview of the results on score sets can be found in [24, 32] and special results in [12, 23, 28, 34].

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