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Some Theoretical and Computational Aspects of the Odd Lindley Fréchet Distribution

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Abstract

In this article, we study an extension of the Fréchet model by using the the odd Lindley-G family of distributions, which was introduced by [17]. Its some statistical properties such as quantile function, density shapes, moments, generating functions and order statistics are obtained. We estimate its parameters by maximum likelihood method. The Monte Carlo simulation is used for assessing the performance of the maximum likelihood method. The usefulness of the odd Lindley Fréchet model is illustrated by means of three real data sets.

Keywords: Fréchet Distribution; Odd Lindley-G family; Odd Lindley Fréchet distribution.

Öz

Oransal Lindley Fréchet Dağılımının Bazı Teorik ve Hesaplamalı Yönleri

Bu çalışmada, Fréchet modelinin genişletilmiş bir versiyonu [17] tarafından önerilen oransal Lindley dağılım ailesi kullanılarak çalışılmıştır. Bu modele ait kuantil fonksiyonu, yoğunluk biçimi, momentler, üreten fonksiyon ve sıra istatistikleri gibi istatistiksel özellikleri elde edilmiştir. Model parametrelerinin en çok olabilirlik tahminleri elde edildi. En çok olabilirlik parametre tahminleri için bir simülasyon çalışılması verilmiştir. Önerilen modelin gerçek veri seti üzerindeki uygunluğu için üç veri analizi yapılmıştır.

Anahtar Sözcükler: Fréchet dağılımı; Oransal Lindley-G ailesi; Odd Lindley Fréchet dağılımı.

1. Introduction

The Fréchet (Fr) distribution [8] is one of the probability distributions used to model extreme events. It is special case of the generalized extreme value distribution and is also known as inverse Weibull or extreme value type II distributions. It was developed within the general extreme value theory, which deals with the stochastic behaviour of the maximum and the minimum of independent and identically distributed random variables. The probability density function (pdf) and cumulative distribution function (cdf) of Fr distribution are respectively given by (for $x>0$),

$$g(x;a,b)=ba^{b}x^{-(b+1)}\exp\left\{-\left(\frac{a}{x}\right)^{b}\right\} \tag{1}$$

and

$$G(x;a,b)=\exp\left\{-\left(\frac{a}{x}\right)^{b}\right\}, \tag{2}$$

where a>0 is a scale parameter and b>0 is a shape parameter. There are wider application areas of the Fr distribution such as reliability, lifetime, hydrology, earthquakes and floods in the literature. For more details about the Fr distribution and its applications, see [10]. Sometimes, the distributional properties of the ordinary Fr distribution may not be adequate to model real data. For this reason, to increase its modelling flexibility and application areas, some generalizations of the Fr distribution studied in the literature such as exponentiated Fr [15], beta Fr [14], transmuted Fr [12] Marshall Olkin Fr [11], transmuted Marshall-Olkin Fr [1], transmuted exponentiated generalized Fréchet [18], Kumaraswamy Marshall-Olkin Fr [2], Weibull Fr [3], beta transmuted Fr [4] and Kumaraswamy transmuted Marshall-Olkin Fr [19], exponentiated Weibull Fr [6] distributions.

On the other hand, the odd Lindley-G (OL-G) family of the distributions was introduced by Silva et al. [17]. The pdf and cdf of this family are respectively given by,

$$f(x;\lambda,\xi)=\frac{\lambda^{2}}{1+\lambda}\frac{g(x;\xi)}{\bar{G}(x;\xi)^{3}}\exp\left\{-\lambda\left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)\right\} \tag{3}$$

and

$$F(x;\lambda,\xi)=1-\frac{\lambda+\bar{G}(x;\xi)}{(1+\lambda)\bar{G}(x;\xi)}\exp\left\{-\lambda\left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)\right\}, \tag{4}$$

where $G(x;\xi)$ is the baseline cdf with ξ parameter vector, $\bar{G}(x;\xi)=1-G(x;\xi)$ is the survival function of the baseline model and λ>0 is the scale parameter. In this way, the baseline model is extended. The authors show that the member of this family can be more flexible than baseline model in terms of data modelling.

The aim of this paper is to study the extension of the Fr distribution with the structure of the OL-G family. Hence, we will increase model flexibility of the Fr on different areas. With this aim, by using (1)-(4), we obtain the pdf and cdf of the odd Lindley Fr (OLFr) distribution with following equations

$$f(x;\lambda,a,b)=\frac{\lambda^{2}}{1+\lambda}\frac{ba^{b}x^{-(b+1)}e^{-\left(\frac{a}{x}\right)^{b}}}{\left(1-e^{-\left(\frac{a}{x}\right)^{b}}\right)^{3}}\exp\left\{-\lambda\left(\frac{e^{-\left(\frac{a}{x}\right)^{b}}}{1-e^{-\left(\frac{a}{x}\right)^{b}}}\right)\right\},x>0 \tag{5}$$

and

$$F(x;\lambda,a,b)=1-\frac{1+\lambda-e^{-\left(\frac{a}{x}\right)^b}}{(1+\lambda)\left(1-e^{-\left(\frac{a}{x}\right)^b}\right)}\exp\left\{-\lambda\left(\frac{e^{-\left(\frac{a}{x}\right)^b}}{1-e^{-\left(\frac{a}{x}\right)^b}}\right)\right\}, x>0, \quad (6)$$

respectively.

Its hazard rate function (hrf) can be easily obtained from the well-known relationship $h(x;\lambda,a,b)=f(x;\lambda,a,b)/\left[1-F(x;\lambda,a,b)\right]$. We denote a random variable X having pdf (5) and cdf (6) by $X\sim OLFr(\lambda,a,b)$. The OLFr density function (5) can be expressed as an infinite mixture of exponentiated-Fr density functions, by using the power series for the exponential function, we have

$$f(x;\lambda,a,b)=\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{k+2}}{k!(1+\lambda)}\underbrace{a^b x^{-b-1}e^{-\left(\frac{a}{x}\right)^b}}_{g(x;a,b)}\underbrace{e^{-k\left(\frac{a}{x}\right)^b}}_{G(x;a,b)^k}\underbrace{\left(1-e^{-\left(\frac{a}{x}\right)^b}\right)^{-k-3}}_{A}.$$

Then by using generalized binomial expansion for the quantity A, we can write

$$f(x;\lambda,a,b)=\sum_{m,k=0}^{\infty}\mathbf{\Psi}_{m,k}\pi_{m+k+1}(x), \quad (7)$$

where $\mathbf{\Psi}_{m,k}=\frac{(-1)^k\lambda^{k+2}\Gamma(m+k+3)}{m!k!(\lambda+1)(m+k+1)\Gamma(k+3)}$, $\pi_{m+k+1}(x)=(m+k+1)\underbrace{a^b x^{-b-1}e^{-\left(\frac{a}{x}\right)^b}}_{g(x;a,b)}\underbrace{e^{-(m+k)\left(\frac{a}{x}\right)^b}}_{G(x;a,b)^{m+k}}$ is the Fr density with scale parameter $a(m+k+1)^{1/b}$ and shape parameter b, and $\Gamma(\cdot)$ is the complete gamma function. Similarly,

$$F(x;\lambda,a,b)=\sum_{m,k=0}^{\infty}\mathbf{\Psi}_{m,k}\Pi_{m+k+1}(x),$$

where $\Pi_{m+k+1}(x)=\underbrace{e^{-(m+k+1)\left(\frac{a}{x}\right)^b}}_{G(x;a,b)^{m+k+1}}$ is the Fr cdf with scale parameter $a(m+k+1)^{1/b}$ and shape parameter b.

The rest of the paper is outlined as follows. In Section 2, we derive some mathematical properties for the OLFr model including quantile function, shapes, moments, generating function, residual life and reversed residual life functions and order statistics and their moments are introduced at the end of the section. Maximum likelihood estimation of the model parameters is addressed in Section 3. In section 4, simulation results to assess the performance of the proposed maximum likelihood estimation procedure are discussed. We provide the applications to real data sets to illustrate the applications and importance of the OLFr model in Section 5. Finally, we offer some concluding remarks in Section 6.

2. Some Properties

In this section, we provide some statistical properties of the OLFr distribution.

2.1. The Quantile Function

The quantile function (qf) of OLFr distribution is given as follows: if u has a uniform random number on uniform distribution with interval (0, 1), then uth quantile, denoted by x_u, of the OLFr distribution is solution of the following nonlinear equation

$$(1-u)(1+\lambda)\frac{1-e^{-\left(\frac{a}{x_u}\right)^b}}{1+\lambda-e^{-\left(\frac{a}{x_u}\right)^b}}\exp\left\{\lambda\left(\frac{e^{-\left(\frac{a}{x_u}\right)^b}}{1-e^{-\left(\frac{a}{x_u}\right)^b}}\right)\right\}=0, 0<u<1.$$

According to Silva et al. [17], we can also write for x_u following equation

$$x_u=a\left[-\log\left(1+\lambda\left[1+W_{-1}\left\{(1+\lambda)(u-1)e^{-1-\lambda}\right\}\right]^{-1}\right)\right]^{-1/b},$$

where W_{-1} denotes negative branch of Lambert W function. Hence, U has a uniform random variable on $U(0,1)$, then X_U has OLFr random variable.

2.2. Shapes

The shapes of the pdf and hrf may be described analytically. Let $g_{Fr}(x;a,b)$, $G_{Fr}(x;a,b)$ and $h_{Fr}(x;a,b)$ are the pdf, cdf and hrf of the ordinary Fréchet distribution respectively. The critical points of the OLFr pdf are the roots of the equation:

$$-\frac{b+1}{x}+ba^b x^{-b-1}-\lambda\frac{g_{Fr}(x;a,b)}{\left[1-G_{Fr}(x;a,b)\right]^2}+3h_{Fr}(x;a,b)=0.$$

The critical points of the OLFr hrf are the roots of the equation:

$$\frac{b\left(\frac{a}{x}\right)^b-b-1}{x}+\frac{g_{Fr}(x;a,b)}{\lambda+G_{Fr}(x;a,b)}+2\frac{h_{Fr}(x;a,b)}{1-G_{Fr}(x;a,b)}=0.$$

Since above equations are of complex structure, we can see the pdf and hrf shapes of the OLFr distribution graphically by using software packages such as MAPLE, R, MATHEMATICA. In this way, we can also determine the local maximums and minimums and inflexion points of the distribution (see Figure 1). From Figure 1, we can see that the pdf of the OLFr distribution are decreasing and uni-modal shaped. At the same time, the hrf shapes of the OLFr distribution can be decreasing, increasing and upside-down bathtub shaped.

2.3. Moments

The r^{th} ordinary moment of X is given by from equation (7)

$$\mu'_r=E(X^r)=\sum_{m,k=0}^{\infty}\boldsymbol{\Psi}_{m,k}\int_0^{\infty}x^r\pi_{m+k+1}(x)dx,$$

then we obtain

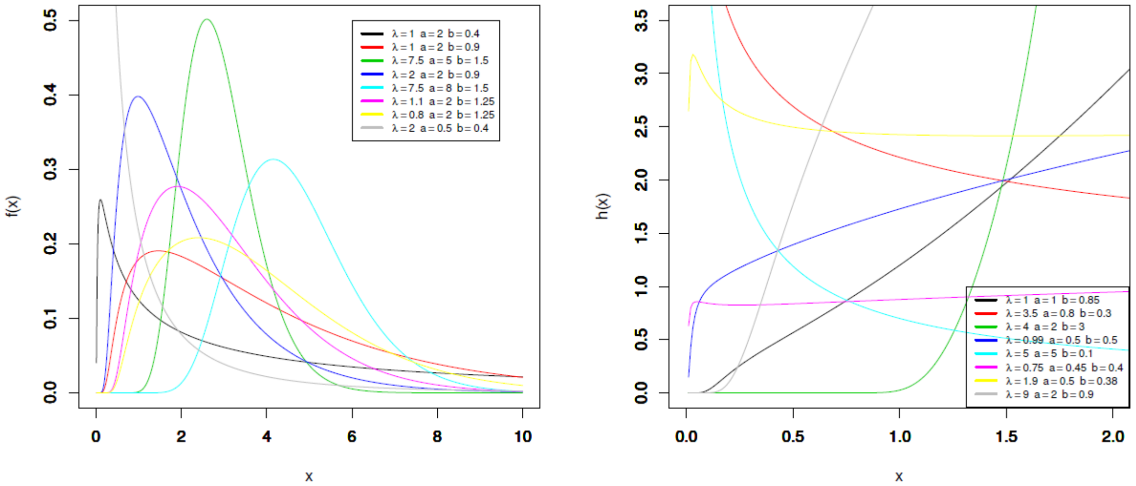

Figure 1. Plots of the OLFr pdf and hrf for some parameter values values.

$$\mu_r' = E(X^r) = \sum_{m,k=0}^{\infty} a^r \mathbf{\Psi}_{m,k} \left(m+k+1\right)^{r/b} \Gamma\left(1-\tfrac{r}{b}\right), \forall r < b. \qquad (8)$$

When r=1 in (8), we have the mean of X. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The r^{th} central moment of X, say μ_r , is

$$\mu_r = E\left[X - \mu_1'\right]^r = \sum_{h=0}^{r} (-1)^h \binom{r}{h} \mu_r' \left(\mu_1'\right)^{r-h}, r = 2,3,\ldots.$$

Hence, the skewness and kurtosis cofficients can be obtained from $\sqrt{\beta_1} = \mu_3 / \mu_2^{3/2}$ and $\beta_2 = \mu_4 / \mu_2^2$ respectively. The values for mean, variance, skewness and kurtosis have been numerically calculated for selected values of model parameters in Table 1. We can say that the OLFr model can be useful for various data modelling in terms of skewness and kurtosis.

Table 1. Mean, variance, coefficients of skewness and kurtosis for different values of parameters**.**

(λ, a, b)	μ_1'	μ_2	$\sqrt{\beta_1}$	β_2
0.5, 0.5, 0.5	0.3392	4.5793	19.3769	779.1696
1, 1, 1	0.3174	0.4211	2.9935	15.6687
1, 0.5, 0.5	0.2609	0.9526	11.0092	250.9296
0.5, 0.5, 2	0.0894	0.0526	2.7047	10.1487
0.1, 0.1, 1	0.0080	0.0090	21.4260	682.5193
5, 5, 5	3.0327	3.8616	-0.8346	1.8286
100, 100, 5	70.9618	111.5811	-5.9099	39.9747
100, 100, 0.5	4.2387	4.0673	0.8048	4.4273

2.4. Generating function

By using the power series for the exponential function, the moment generating function, $M_X(t)=E\left(e^{tX}\right)$, of X can be derived from equation (7) as

$$M_X(t)=\sum_{m,k=0}^{\infty}\mathbf{\Psi}_{m,k}\sum_{r=0}^{\infty}\tfrac{t^r}{r!}\mu_r^{'}=\sum_{m,k,r=0}^{\infty}a^r\tfrac{t^r}{r!}\mathbf{\Psi}_{m,k}(m+k+1)^{r/b}\,\Gamma\left(1-\tfrac{r}{b}\right),\forall r<b.$$

2.5. Order Statistics

Let $X_1,X_2,\ldots,X_n$ be a random sample from the OLFr distribution and let $X_{1:n},X_{2:n},\ldots,X_{2:n}$ be the corresponding order statistics. The pdf of ith order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x)=\frac{f(x)}{\mathrm{B}(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F(x)^{j+i-1}, \tag{9}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function. Substituting (5)-(7) in equation (9) and by using binomial expansions, the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=\sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i}\mathbf{\Psi}_{j,m,p}\,\pi_{j+m+p}(x),$$

where

$$\mathbf{\Psi}_{j,m,p}=\sum_{k=0}^{i-1}\frac{(-1)^{k+m}\lambda^{j+m+2}(1+\lambda)^{-j-1}}{m!\,\mathrm{B}(i,n-i+1)(j+m+p+1)}\binom{j+m+p}{j+m}\binom{k+n-1}{j}\binom{i-1}{k}.$$

Further, the q^{th} moment of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^q\right)=\sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i}a^q\mathbf{\Psi}_{j,m,p}(j+m+p)^{q/b}\,\Gamma\left(1-\tfrac{q}{b}\right),\ \forall q<b.$$

Hence, the L-moments of X based on the q^{th} moment of $X_{i:n}$ is given by

$$\delta_r=\frac{1}{r}\sum_{d=0}^{r-1}(-1)^d\binom{r-1}{d}E\left(X_{r-d:r}\right),\ r\geq 1.$$

3. Maximum likelihood estimations of the model parameters

Here, we estimate the parameters of the OLFr distribution by the method of maximum likelihood estimation (MLE). Let $X_1,X_2,\ldots,X_n$ be a random sample from the OLFr distribution with observed values $x_1, x_2, \ldots, x_n$, and $\boldsymbol{\Psi} = (\lambda, a, b)^T$ be vector of the model parameters. The log-likelihood function of $\boldsymbol{\Psi}$ may be expressed as

$$\ell = 2n\log\lambda + n\log\alpha - n\log(1+\lambda) + n\log b + nb\log a - (b+1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log s_i$$
$$-3\sum_{i=1}^{n}\log(1-s_i) - \lambda\sum_{i=1}^{n}\frac{s_i}{1-s_i},$$

where $s_i = \exp\left\{-\left(\frac{a}{x_i}\right)^b\right\}$. The components of the score vector, $\left(U_\lambda = \frac{\partial\ell}{\partial\lambda}, U_a = \frac{\partial\ell}{\partial a}, U_b = \frac{\partial\ell}{\partial b}\right)^T$, are given by

$$U_\lambda = \frac{2n}{\lambda} - \frac{n}{1+\lambda} - \sum_{i=1}^{n}\frac{s_i}{1-s_i},$$

$$U_a = \frac{nb}{a} + \sum_{i=1}^{n}\frac{m_i}{s_i} + 3\sum_{i=1}^{n}\frac{m_i}{1-s_i} - \sum_{i=1}^{n}\frac{m_i}{(1-s_i)^2}$$

$$U_b = \frac{n}{b} + n\log a - \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\frac{z_i}{s_i} + 3\sum_{i=1}^{n}\frac{z_i}{1-s_i} - \lambda\sum_{i=1}^{n}\frac{z_i}{(1-z_i)^2},$$

where $m_i = -ba^{b-1}x_i^{-b}s_i$ and $z_i = -s_i\left(\frac{a}{x_i}\right)^b\log\left(\frac{a}{x_i}\right)$. By setting the non-linear system of equations $U_\lambda = U_a = U_b = 0$ and solving them simultaneously, the MLE of parameters are obtained. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms.

4. Simulation Study

In this section, we apply the simulation study to see the performance of the MLEs of the OLFr distribution with respect to sample size *n*. We generated *N*=1000 samples of size *n*=20, 21,… , 120 from OLFr distribution with *a*=1, *b*=0.5 and $\lambda = 9$ by using inverse transform method. The MLEs, say $\left(\hat{\lambda}_i, \hat{a}_i, \hat{b}_i\right)$ for *i*=1,2,...,*N*, have been obtained by using CG routine in R programme. Further, we calculate emprical mean, standard deviations (sd), bias and mean square error of the estimate (MSE) for MLEs. The bias and MSE are calculated by (for $h = \lambda, a, b$) $Bias_{\hat{h}} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i - h\right)$ and $MSE_{\hat{h}} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i - h\right)^2$.

The results are presented in Figure 2. From Figure 2, we can say that for three parameters the empirical means are very close to true parameter values and they are quite stable. Moreover, the bias, MSE and sd decrease as sample size increases for all cases. The MSEs are very close to 0 especially for λ parameter.

5. Applications

In this section, we provide three applications to three real data sets to prove the importance and flexibility of the OLFr distribution. The real data sets are related to lifetime, breaking stress and flood exceedances areas. We compare the fit of the OLFr with competitve models namely: Marshall-Olkin Fréchet (MO-Fr)

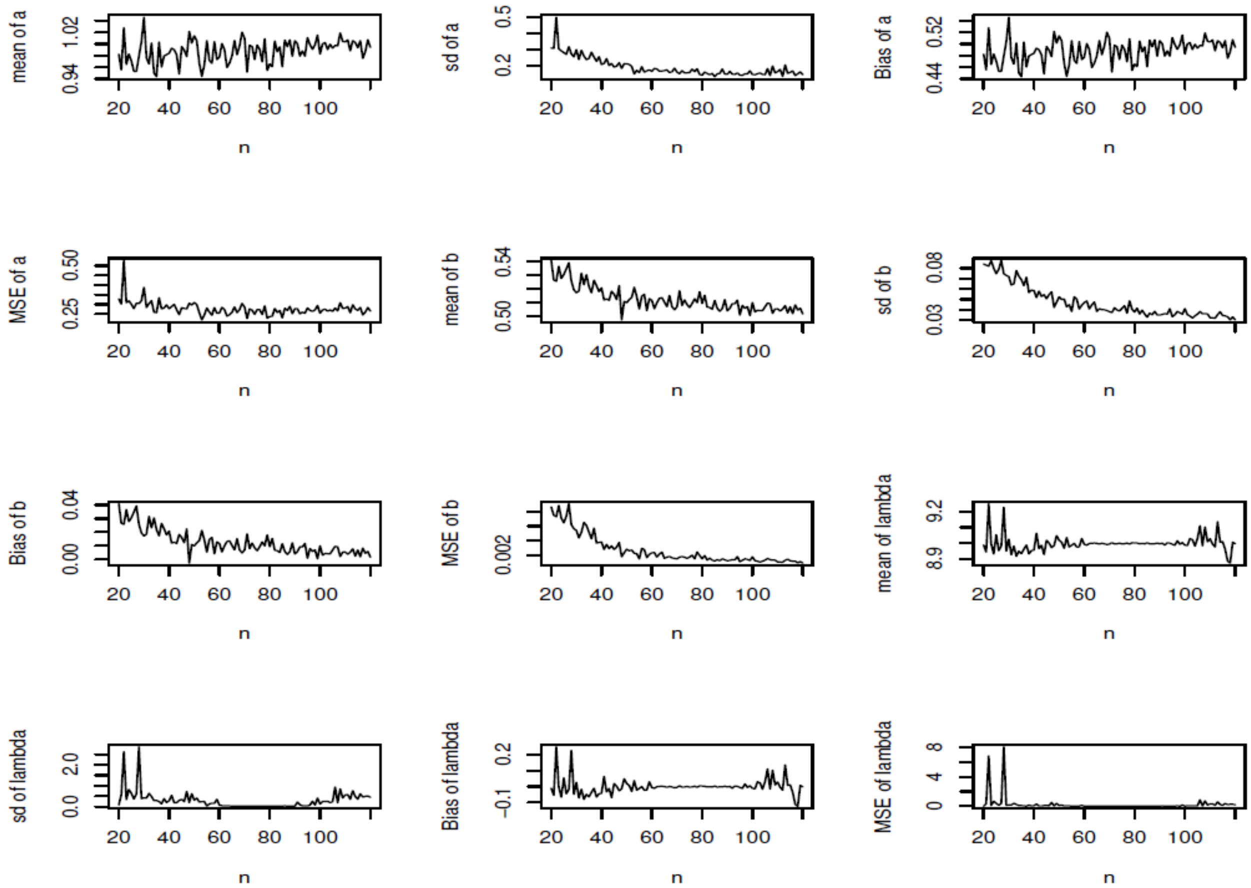

Figure 2. Plots of the emrical mean, sd, biases and MSE of a, b, λ versus *n*.

[11], Exponentiated Generalized Fréchet (EG-Fr) [7], Beta exponential Fréchet (BE-Fr) [13] and Fr distributions. The cdfs of these distributions are, respectively, given by (for x>0 and $\alpha, \beta, \lambda, a, b > 0$):

$$F_{MO-Fr}(x; \alpha, a, b) = \left[\alpha + (1-\alpha)\exp\left\{-\left(\tfrac{a}{x}\right)^b\right\}\right]^{-1} \exp\left\{-\left(\tfrac{a}{x}\right)^b\right\},$$

$$F_{EG-Fr}(x; \alpha, a, b) = \left[1 - \left(1 - \exp\left\{-\left(\tfrac{a}{x}\right)^b\right\}\right)^{\alpha}\right]^{\beta} \exp\left\{-\left(\tfrac{a}{x}\right)^b\right\}$$

and

$$F_{BE-Fr}(x;\alpha,a,b)=\left[1-\left(1-\exp\left\{-\left(\tfrac{a}{x}\right)^{b}\right\}\right)^{\alpha}\right]^{\beta}\exp\left\{-\left(\tfrac{a}{x}\right)^{b}\right\},$$

where $B(\cdot,\cdot)$ is the complete beta function and $B(\cdot;\cdot,\cdot)$ is the incomplete beta function.

The first real data set represents the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia. These data have been analyzed by [9]. The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

The second real data set is the exceedances of flood peaks (m^3/s) of the Wheaton River in Canada. The data consist of 72 exceedances for the years 1958–1984 and have been studied by [5]. The data are: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0.

The third data set refers to breaking stress of carbon fibres (in Gba) [16] and consists of 100 observations: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

In order to compare the distributions, we consider the measures of goodness-of-fit including the estimated log-likelihood value $\hat{\ell}$, Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and consistent Akaike information criterion (CAIC). We estimate the unknown parameters of each distribution by the method of maximum likelihood estimation. The required computations are obtained by using the "maxLik" sub-routine in R-software. When searching the best fit among others to data, the distribution with the smallest AIC, CAIC, BIC and HQIC values and the biggest log-likelihood is chosen.

The results of these applications are listed in Tables 2 and 3. These results show that the OLFr distribution has the lowest AIC, CAIC, BIC and HQIC and, has the biggest estimated log-likelihood among all the fitted models. Hence, it could be chosen as the best model under these criteria. Also, we sketched the plots of the estimated pdf's and cdf's for each fitted distributions in Figures 3-5. We can see that the OLFr distribution provides a good fit and can be used as a competitive model to the other considered models from these Figures.

4. Conclusions

In this article, we studied an extended version of the Fréchet model using the the odd Lindley-G family of distributions. We obtain some of its mathematical properties. We estimate its parameters by maximum likelihood method. We used the Monte Carlo simulation for assessing the performance of the maximum likelihood method. The usefulness of the OLFr model is illustrated by means of three real data sets. We hope that the OLFr model will attract wider applications in areas such as life testing, earthquakes, horse racing, floods, rainfall, queues in supermarkets, wind speeds and sea waves. As a future work we will consider the bivariate and the multivariate extensions of OLFr distribution.

References

[1] A.Z. Afify, G. G. Hamedani, I. Ghosh, M. E. Mead, 2015, The transmuted Marshall-Olkin Fréchet distribution: properties and applications, *Int. J. Statist. Probab*., 4, 132-184.

[2] A. Z. Afify, H. M. Yousof, G. M. Cordeiro, M. Ahmad, 2016a, The Kumaraswamy Marshall-Olkin Fréchet distribution with applications, *Journal of ISOSS*, 2, 1-18.

[3] A. Z. Afify, H. M. Yousof, G. M. Cordeiro, R. R. Pescim, G. R. Aryal, 2017, The Weibull Fréchet distribution and its applications, *Journal of Applied Statistics*, 43, 2608-2626.

[4] A. Z. Afify, H. M. Yousof, S. Nadarajah, 2017, The beta transmuted-H family of distributions: properties and applications, *Statistics and its Inference*, 10, 505-520.

[5] V. Choulakian, M.A. Stephens, 2001, Goodness-of-fit for the generalized Pareto distribution. Technometrics, 43, 478–484.

[6] G. M. Cordeiro, A. Z. Afify, H. M. Yousof, R. R. Pescim, G. R. Aryal, 2017, The exponentiated Weibull-H family of distributions: Theory and Applications, *Mediterranean Journal of Mathematics*, 14, 155.

[7] G. M. Cordeiro, E. M. Ortega, D. C. Cunha, 2013, The exponentiated generalized class of distributions, Journal of Data Science, 11, 1-27.

[8] M. Fréchet, 1927, Sur la loi de probabilité de lécart maximum, *Ann. de la Soc. polonaisede Math*, 6, 93-116.

[9] P. Feigl, M. Zelen, 1965, Estimation of exponential survival probabilities with concomitant information, *Biometrics*, 826-838.

[10] S. Kotz, S. Nadarajah, 2000, *Extreme value distributions: Theory and applications*. Imperial College Press, London.

[11] E. Krishna, K. K. Jose, T. Alice, M. M. Ristic, 2013, The Marshall-Olkin Fréchet distribution, *Communications in Statistics-Theory and Methods*, 42, 4091-4107.

[12] M. R. Mahmoud, R. M. Mandouh, 2013, On the transmuted Fréchet distribution, *Journal of Applied Sciences Research*, 9, 5553-5561.

[13] M. E. Mead, A. Z. Afify, G. G. Hamedani, I. Ghosh, 2017, The beta exponential Fréchet distribution with applications, *Austrian Journal of Statistics*, 46, 41-63.

[14] S. Nadarajah, A. K. Gupta, 2004,. The beta Fréchet distribution, *Far East Journal of Theoretical Statistics*, 14, 15-24.

[15] S. Nadarajah, S. Kotz, 2003, The exponentiated Fréchet distribution, *InterStat. Available online at http://interstat.statjournals.net/YEAR/2003/abstracts/0312001.php*

[16] M. D. Nichols, W. J. Padgett, 2006, A bootstrap control chart for Weibull percentiles, *Quality and reliability engineering international*, 22, 141-151.

[17] F G. Silva, A. Percontini, E. de Brito, M. W. Ramos, R. Venancio, G. M. Cordeiro, 2017, The odd Lindley-G family of distributions, *Austrian Journal of Statistics*, 46, 65-87.

[18] H. M. Yousof, A. Z. Afify, M. Alizadeh, N. S. Butt, G. G. Hamedani, M. M. Ali, 2015, The transmuted exponentiated generalized-G family of distributions, *Pak. J. Stat. Oper. Res*., 11, 441-464.

[19] H. M. Yousof, A. Z. Afify, A. N. Ebraheim, G. G. Hamedani, N. S. Butt, 2016, On six-parameter Fréchet distribution: properties and applications, *Pak. J. Stat. Oper. Res.*, 12, 281-299.

Table 2. MLEs, standard errors of the estimates (in parentheses) and $\hat{\ell}$ values.

Data Set	Model	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$-\hat{\ell}$
I	OLFr	1.1187 (0.4813)	9.2564 (5.7354)	0.4952 (0.0751)	-	-	152.6437
	MO-Fr	-	1.9094 (1.3114)	0.9873 (0.1507)	8.0535 (7.3946)	-	154.6892
	EG-Fr	-	0.1123 (0.0006)	3.0407 (0.0028)	0.2126 (0.0033)	15.0406 (6.4526)	156.1730
	BE-Fr	1.7819 (0.0425)	80.6087 (0.7739)	68.9589 (0.5534)	34.2210 (3.7045)	0.0576 (0.0068)	153.9517
II	OLFr	0.3928 (0.2975)	0.7395 (0.8446)	0.6086 (0.0580)	-	-	251.7202
	MO-Fr	-	0.1234 (0.0495)	1.1946 (0.1074)	117.8414 (4.2575)	-	257.7168
	BE-Fr	191.8994 (1.2417)	202.8226 (1.2138)	0.0376 (0.0028)	159.6263 (0.2027)	2.0530 (0.0357)	256.5862
	Fr	-	2.8790 (0.5580)	0.6521 (0.0538)			267.0189
III	OLFr	461.9339 (5.9925)	200.2515 (4.9354)	0.4297 (0.0047)	-	-	142.7755
	MO-Fr	-	233.0111 (0.4150)	0.4609 (0.0124)	0.00028 (0.0001)	-	154.4639
	EG-Fr	-	0.1681 (0.0088)	5.5545 (0.0079)	0.3079 (0.0176)	60.7849 (13.7714)	174.8004
	Fr	-	1.8915 (0.1137)	1.7690 (0.1118)			173.1440

Table 3. Information criteria results**.**

Data Set	Model	AIC	BIC	CAIC	HQIC
I	OLFr	311.2875	312.1151	315.7770	312.7981
	MO-Fr	315.3784	316.2060	319.8680	316.8890
	EG-Fr	320.3459	321.7745	326.3319	322.3600
	BE-Fr	317.9035	320.1257	325.3860	320.4211
II	OLFr	509.4404	509.7933	516.2704	512.1594
	MO-Fr	521.4337	521.7866	528.2637	524.1527
	BE-Fr	523.1725	524.0816	534.5558	527.7042
	Fr	538.0378	538.2117	542.5912	539.8505
III	OLFr	291.5511	291.8011	299.3666	294.7142
	MO-Fr	314.9277	315.1777	322.7432	318.0908
	EG-Fr	357.6007	358.0218	368.0214	361.8181
	Fr	350.2879	350.4116	355.4982	352.3966

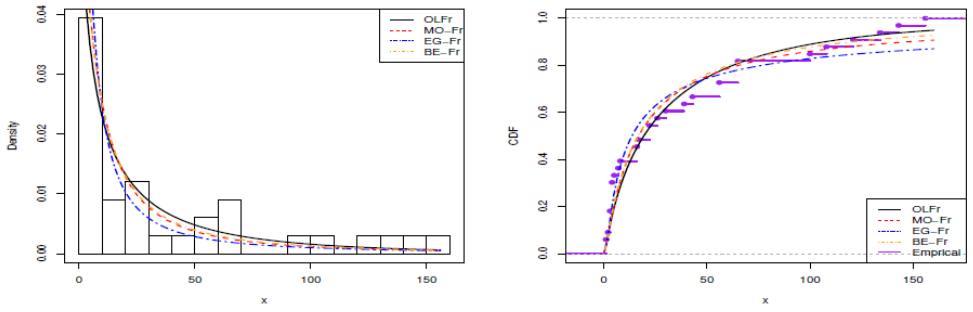

Figure 3. The fitted pdfs (left) and cdfs (right) for the first data set.

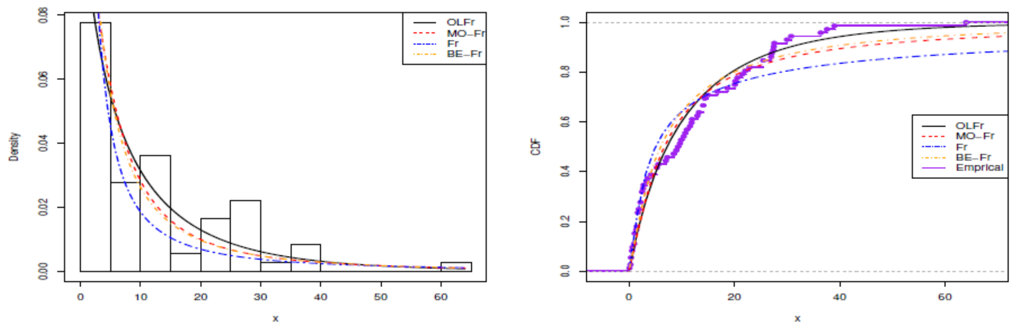

Figure 4. The fitted pdfs (left) and cdfs (right) for the second data set.

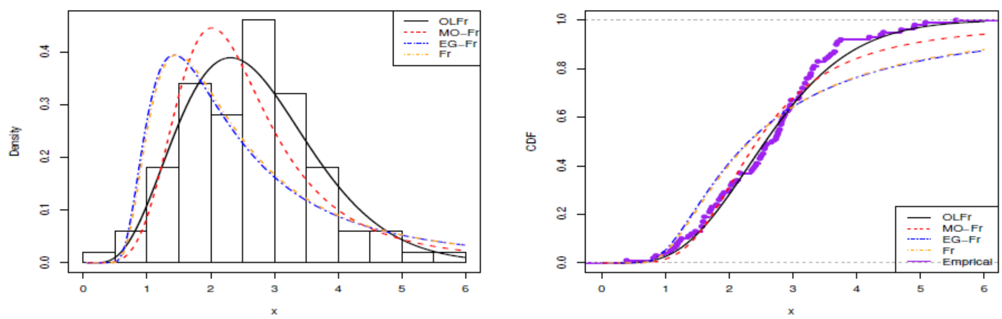

Figure 5. The fitted pdfs (left) and cdfs (right) for the third data set.