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The mean-field limit for systems of self-propelled agents with "topological interaction" cannot be obtained by means of the usual Dobrushin approach. We get results by adapting to the multidimensional case the techniques developed by Trocheris in 1986 to treat the Vlasov–Poisson equation in one dimension.

1. Introduction

Many interesting physical systems can be described at the microscopic level using particle dynamics and at the mesoscopic level using kinetic equations. In the broad field of two-body interactions, the link between these two regimes is mathematically well-understood in the case of the mean-field limit, i.e., when the density of the particles grows roughly in proportion to their number N, the mean free path vanishes as $1/N$ and the interaction intensity scales with $1/N$. In this limit, each particle feels the interaction with the others as a mean.

A rigorous mathematical proof of this result can be completed in the case of two-body interactions with sufficiently regular potentials. This classical achievement was obtained independently by several authors in the 1970s (see [7; 17; 30]) and its explanation is particularly clear in Dobrushin's argument [17], where the result follows from noticing that the empirical measure associated with the particle system is a weak solution of the mean-field equation; the proof follows by showing the weak continuity, with respect to the initial datum, of the weak solutions.

Though the theory for regular pairwise interactions is sufficiently well-understood, going beyond it and considering singular potentials is a harder task. This is the case with the three-dimensional Vlasov–Poisson equation, which is the most important equation of plasma physics, based on the choice of the Coulomb potential, and of galactic dynamics, based on the choice of the Newton potential. In this equation, the potential $1/r$ is singular at the origin and does not belong to any L^p space. Although the mean-field limit for the Vlasov–Poisson equation remains an open problem, there has been important progress in recent years; see [24; 25], where

the mean-field limit is proven for potentials with singularities "weaker than $1/r$", and also [28; 29]. In the case of the one-dimensional Vlasov–Poisson equation, the problem has been solved in [31; 32], and with a simpler proof in [23], the force being discontinuous, but not divergent.

The mean-field limit is a case of propagation of chaos, i.e., the j-particle distributions become factorized in the limit. This property is the key for obtaining a kinetic description of the particle dynamics (see, for instance, [8] and [13; 19; 26] for some reviews on this point of view).

In recent years, the conceptual and mathematical apparatus of kinetic equations has been used in the study of self-propelled particle systems of a biological nature; in particular, it's been used in the study of the motion of swarms and other animals. Starting with the pioneering paper [33], several models have been proposed to explain the evolution of these systems. In the simplest models [15; 14; 33], a bird is modeled as a self-propelling particle that interacts with its neighbors. The interaction is such that neighboring birds tend to align their velocities. For many of these models, the mean-field limit has been used to obtain a kinetic description of the dynamics (see, for instance, [3; 9; 10; 11; 20; 21]).

A few years ago, supported by observational data [1; 2; 12], "topological" models for interaction were introduced: an agent reacts to the presence of another not according to the distance between them, but according to the proximity ranking (see (1-1), (1-2), and (1-3) for a rigorous formulation). These models come out of the case of two-body interactions, and present various problems in their kinetic treatment. In particular, the solutions of the kinetic equation are not weakly continuous with respect to the initial datum, and there are also some difficulties in defining the particle motion.

We prove a result on the mean-field limit for topological models. We focus our attention on the topological Cucker–Smale model, but with the same ideas it is possible to consider more general cases. A first result in this direction has been proved in [22], for a smoothed version of the model in which the weak continuity in the initial datum is recovered. We also mention that a kinetic Boltzmann equation for a stochastic particle model with rank-based interaction has been obtained in [16] by using the BBGKY hierarchy.

We formulate the problem and summarize our results. A Cucker–Smale type model for the motion of N agents, in the mean-field scaling, is the system

$$\begin{cases} \dot{X}_i(t) = V_i(t), \\ \dot{V}_i(t) = \dfrac{1}{N} \sum_{j=1}^{N} p_{ij}(V_j(t) - V_i(t)), \end{cases} \tag{1-1}$$

where the "communication weights" $\{p_{ij}\}_{i,j=1}^{N}$ are positive functions that take into

account the interactions between agents. In classical models, p_{ij} depends only on the distance $|X_i - X_j|$ between the agents. In topological models, the weights depend on the positions of the agents by their rank:

$$p_{ij} := K(M(X_i, |X_i - X_j|)), \tag{1-2}$$

where $K : [0, 1] \to \mathbb{R}^+$ and, for $r > 0$, the function

$$M(X_i, r) := \frac{1}{N} \sum_{k=1}^{N} \mathcal{X}\{|X_k - X_i| \le r\} \tag{1-3}$$

counts the number of agents at distance less than or equal to r from X_i, normalized with N. Note that in this case, p_{ij} is a stepwise function of the positions of all of the agents. In the sequel, we assume that K is a positive decreasing function that is Lipschitz continuous, and such that $\int_0^1 K(z) \, dz = \gamma$.

In the mean-field limit $N \to +\infty$, the one-agent distribution function $f_t = f(t, x, v)$ is expected to satisfy the equation

$$\partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (W[Sf_t, f_t](x, v) f_t) = 0, \tag{1-4}$$

where $Sf_t(x) := \int f_t(x, v) \, dv$ is the spatial distribution and where, given a probability density f in $\mathbb{R}^d \times \mathbb{R}^d$ and a probability density ρ in $\mathbb{R}^d$,

$$W[\rho, f](x, v) := \int K(M[\rho](x, |x - y|)) \, (w - v) f(y, w) \, dy \, dw, \tag{1-5}$$

with

$$M[\rho](x, r) := \int_{|x' - x| \le r} \rho(x') \, dx'. \tag{1-6}$$

A weak formulation of this equation is given, requiring that the solution f_t fulfills

$$\int \alpha(x, v) \, df_t(x, v) = \int \alpha(X_t(x, v), V_t(x, v)) \, df_0(x, v)$$

for any $\alpha \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, where f_0 is the initial probability measure and where $(X_t(x, v), V_t(x, v))$ is the flow defined by

$$\begin{cases} \dot{X}_t(x, v) = V_t(x, v), \\ \dot{V}_t(t, x, v) = W[Sf_t, f_t](X_t(x, v), V_t(x, v)), \\ X_0(x, v) = x, \quad V_0(x, v) = v. \end{cases} \tag{1-7}$$

In other words, f_t is the push-forward of f_0 along the flow generated by the velocity field, determined by f_t itself.

It is easy to verify that the empirical measure

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_i^N(t)} \delta_{V_i^N(t)}$$

associated with the solution of (1-1), (1-2) and (1-3) is a weak solution of (1-4). Namely, $M[S\mu_t^N](X, r)$ is exactly $M(X, r)$ defined in (1-3). (From now on we use the more complete notation $M[S\mu_t^N](X, r)$.) Thus, we can rewrite the agent evolution in (1-1) as

$$\begin{cases} \dot{X}_i^N(t) = V_i^N(t), \\ \dot{V}_i^N(t) = W[S\mu_t^N, \mu_t^N](X_i^N(t), V_i^N(t)). \end{cases} \quad (1\text{-}8)$$

In the Dobrushin approach to the mean-field limit, the result is achieved from this fact and from the weak continuity, with respect to the initial datum, of the weak solutions of (1-4). We cannot use this approach in presence of topological interaction, since in general the solutions of (1-7) are not weakly continuous with respect to the initial datum (see Section 3). We can overcome this difficulty if the solution of (1-4) has a bounded density. To obtain our result, we adapt the ideas used in [32] for the derivation of the one-dimensional Vlasov equation in presence of discontinuity of the force. In particular, we prove that:

(Theorem 3.4) The N-particle dynamics is well-defined, except for a set of measure zero.

(Theorem 4.3) If f_0 is bounded, there exists a unique weak solution f_t of the topological Cucker–Smale equation, and it is bounded.

(Theorem 5.2) If μ_t^N solves (1-8) and $\mu_0^N \rightharpoonup \mu_0$, then $\mu_t^N \rightharpoonup f_t$.

We divide the work as follows. In Section 2 we discuss some properties of the "discrepancy distance", the main tool for dealing with topological interactions. In Section 3 we discuss existence, uniqueness and regularity of the agent dynamics (1-8), proving Theorem 3.4. In Section 4 we discuss the existence, uniqueness and regularity of the weak solutions of the mean-field equation (1-4) with bounded initial datum, proving Theorem 4.3. In Section 5 we prove Theorem 5.2.

2. Distances and weak convergence

We recall that the 1-Wasserstein distance $\mathscr{W}$ of two probability measures on $\mathbb{R}^d$ can be defined by duality using Lipschitz functions. Denoting by $\mathrm{Lip}(\phi)$ the Lipschitz constant of ϕ, we can write

$$\mathscr{W}(\rho_1, \rho_2) = \sup_{\substack{\phi \in C_b(\mathbb{R}^d) \\ \mathrm{Lip}(\phi) \leq 1}} \int \phi(\mathrm{d}\rho_1 - \mathrm{d}\rho_2) = \sup_{\substack{\phi \in C_b^1(\mathbb{R}^d) \\ \|\nabla\phi\|_\infty \leq 1}} \int \phi(\mathrm{d}\rho_1 - \mathrm{d}\rho_2).$$

The counter of the number of particles in (1-6) is not continuous with respect to $\mathscr{W}$, so we work with the weaker topology induced by another distance, the discrepancy, defined as

$$\mathscr{D}(\rho_1, \rho_2) := \sup_{x, r > 0} \left| \int_{B_r(x)} d\rho_1 - \int_{B_r(x)} d\rho_2 \right|.$$

Here and after, we denote by $B_r(x)$ the closed ball of center x and radius r in $\mathbb{R}^d$; we will also write B_R for $B_R(0)$. The discrepancy distance is mostly used in one dimension to quantify the uniformity of sequence of points (see [18; 27]), but its multidimensional version is cited in [30], in the contest of kinetic limits.

By definition, the following proposition holds.

Proposition 2.1 (Lipschitzianity of M with respect to $\mathscr{D}$). *Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^d$. Then for any $x \in \mathbb{R}^d$ and $r > 0$,*

$$|M[\rho_1](x, r) - M[\rho_2](x, r)| \leq \mathscr{D}(\rho_1, \rho_2).$$

We can define $\mathscr{D}$ in terms of regular functions. Let X be the set $C_b^1([0, +\infty); \mathbb{R})$, and define

$$\|\phi\|_X := \int_0^{+\infty} |\phi'(r)| \, dr.$$

Then

$$\mathscr{D}(\rho_1, \rho_2) = \sup_{\substack{\phi \in X \\ \|\phi\|_X \leq 1}} \sup_x \int \phi(|x - y|)(d\rho_1(y) - d\rho_2(y)).$$

This assertion is an easy consequence of the following lemma.

Lemma 2.2. *Let g_1 and g_2 be two probability measures on $[0, +\infty)$. Then*

$$\sup_{r \geq 0} \left| \int_{[0,r]} dg_1 - \int_{[0,r]} dg_2 \right| = \sup_{\substack{\phi \in X \\ \|\phi\|_X \leq 1}} \int_0^{+\infty} \phi(dg_1 - dg_2). \tag{2-1}$$

Proof. Fix $r > 0$. There exists $\phi_{r,\varepsilon} \in X$ with $\|\phi_{r,\varepsilon}\|_X = 1$ such that $\phi_{r,\varepsilon}(s) = 1$ if $0 \leq s \leq r$ and $\phi_{r,\varepsilon}(s) = 0$ if $s \geq r + \varepsilon$. For any measure g,

$$\lim_{\varepsilon \to 0} \int_0^{+\infty} (\phi_{r,\varepsilon}(s) - \mathcal{X}\{s \in [0, r]\}) \, dg(s) = 0;$$

hence

$$\int_{[0,r]} (dg_1 - dg_2) = \lim_{\varepsilon \to 0} \int_0^{+\infty} \phi_{r,\varepsilon}(dg_1 - dg_2) \leq \sup_{\substack{\phi \in X \\ \|\phi\|_X \leq 1}} \int_0^{+\infty} \phi(dg_1 - dg_2).$$

To prove the opposite inequality, we denote by G_1 and G_2 the distribution functions

of g_1 and g_2:

$$G_i(r) := \int_{[0,r]} \mathrm{d}g_i.$$

Integrating by parts,

$$\int_0^{+\infty} \phi(\mathrm{d}g_1 - \mathrm{d}g_2) = -\int_0^{+\infty} \phi'(r)(G_1(r) - G_2(r))\,\mathrm{d}r \le \|\phi\|_X \|G_1 - G_2\|_\infty.$$

We conclude the proof by noticing that $\|G_1 - G_2\|_\infty$ is exactly the left-hand-side of (2-1).   □

For our purposes, we need the equivalence of $\mathscr{D}$ and $\mathscr{W}$ in the case in which one of the two measures has bounded density. We note that in the general case the equivalence is false, as can be easily checked by considering two Dirac measures δ_{x_1} and δ_{x_2}; $\mathscr{W}$ vanishes when $|x_1 - x_2| \to 0$, while $\mathscr{D}$ equals one whenever $x_1 \neq x_2$. Nevertheless, using the covering principles as in [4; 5; 6], for measures on a compact set, it can be proved the continuity of the Wasserstein distance $\mathscr{W}$ with respect to the discrepancy distance $\mathscr{D}$. For the sake of completeness, we give a proof in the appendix, although this property is not really necessary for our results.

In the definition of $\mathscr{D}$, we will choose functions $\phi \in C_b([0, +\infty), \mathbb{R})$, with first derivative continuous up to a finite number of jumps. With abuse of notation, we keep calling this set of functions X. Let us expose some technical properties.

Given $\phi \in X$, we define some useful regularizations, $\phi^\pm$, ϕ_ε and ψ_ε, with $\varepsilon > 0$, as follows. Denoting by $\tilde{\phi}$ the function

$$\tilde{\phi}(r) := \int_0^r |\phi'(s)|\,\mathrm{d}s,$$

we define

$$\phi^\pm(r) := \begin{cases} \frac{1}{2}(\tilde{\phi}(r) \pm \phi(r)) & \text{if } r \ge 0, \\ \pm\frac{1}{2}\phi(0) & \text{if } r < 0, \end{cases}$$

and

$$\phi_\varepsilon(r) := \phi^+(r + \varepsilon) - \phi^-(r - \varepsilon). \tag{2-2}$$

Finally, fixing a regular mollifier η supported in $(0, 1)$, we define

$$\psi_\varepsilon(r) := \int_0^\varepsilon \eta_\varepsilon(s)\phi^+(r + s)\,\mathrm{d}s - \int_0^\varepsilon \eta_\varepsilon(s)\phi^-(r - s)\,\mathrm{d}s. \tag{2-3}$$

where $\eta_\varepsilon(s) := \varepsilon^{-1}\eta(s/\varepsilon)$.

We summarize the properties of these regularizations in the following lemma, where we indicate with c any constant which does not depends on ϕ and ε.

Lemma 2.3. (i) $\phi^\pm$ *are not decreasing. Moreover,*

$$\int_0^{+\infty} (\phi^\pm)'(r) \ dr \le \|\phi\|_X \tag{2-4}$$

and $\phi(r) = \phi^+(r) - \phi^-(r)$ *for* $r \ge 0$.

(ii) $\phi_\varepsilon \in X$, $\phi(r) \le \phi_\varepsilon(r)$ *and*

$$\int_0^{+\infty} (\phi_\varepsilon(r) - \phi(r)) \ dr \le 2\varepsilon \|\phi\|_X. \tag{2-5}$$

(iii) $\psi_\varepsilon(r) \ge \phi(r)$. *Moreover,* ψ_ε *is a* C^1 *function in* X,

$$\|(\psi_\varepsilon)'\|_\infty \le \frac{2}{\varepsilon} \|\eta\|_\infty \|\phi\|_X \tag{2-6}$$

and

$$\int_0^{+\infty} |\psi_\varepsilon(r) - \phi(r)| \ dr \le c\varepsilon \|\phi\|_X. \tag{2-7}$$

Proof. The proof is elementary; we only describe how to get the bounds in (ii) and (iii). Since $\phi = \phi^+ - \phi^-$, we rewrite the left-hand side of (2-5) as

$$\int_0^{+\infty} (\phi^+(r+\varepsilon) - \phi^+(r)) + (\phi^-(r) - \phi^-(r-\varepsilon)) \ dr$$
$$= \int_0^{+\infty} \left(\int_0^\varepsilon ((\phi^+)'(r+\xi) + (\phi^-)'(r-\xi)) \, d\xi \right) dr \le 2\varepsilon \|\phi\|_X.$$

The estimate in (2-6) is immediate. For (2-7), we rewrite $\psi_\varepsilon(r) - \phi(r)$ as

$$\int_0^1 \eta(s)(\phi^+(r+\varepsilon s) - \phi^+(r) + \phi^-(r) - \phi^-(r-\varepsilon s)) \ ds$$
$$= \varepsilon \int_0^1 s\eta(s) \left(\int_0^1 (\phi^+)'(r+\varepsilon s\xi) \ d\xi + \int_0^1 (\phi^-)'(r-\varepsilon s\xi) \ d\xi \right) ds.$$

We conclude by integrating in r, switching the order of integration and using (2-4). $\qquad\square$

Now we can prove the following proposition.

Proposition 2.4. *Let* ρ *and* ν *be two probability measures on* $\mathbb{R}^d$ *with support in a ball* B_R *and such that* $\rho \in L^\infty(\mathbb{R}^d)$. *Then*

$$\mathscr{D}(\nu, \rho) \le C(\|\rho\|_\infty, R)\sqrt{\mathscr{W}(\nu, \rho)},$$

where C *is a constant that depends on the dimension* d, *as well as on* $\|\rho\|_\infty$ *and on* R.

Proof. Let ϕ be in X and consider ψ_ε as in (2-3). Fixing $x \in \mathbb{R}^d$, let Φ and Ψ_ε be the functions

$$\Phi(y) := \phi(|x - y|) \quad \text{and} \quad \Psi_\varepsilon(y) := \psi_\varepsilon(|x - y|).$$

Then, from (iii) of Lemma 2.3,

$$\int \Phi \, d\nu - \int \Phi \, d\rho \leq \int \Psi_\varepsilon \, d\nu - \int \Phi \, d\rho = \int \Psi_\varepsilon \, d(\nu - \rho) + \int (\Psi_\varepsilon - \Phi) \, d\rho.$$

From (2-6) of Lemma 2.3, the first term is bounded by $(c/\varepsilon)\|\phi\|_X \mathscr{W}(\nu, \rho)$. Regarding the second term, denoting by σ_r the uniform measure on $\partial B_r(x)$, we have

$$\int (\Psi_\varepsilon - \Phi) \, d\rho \leq \|\rho\|_\infty \int_0^{+\infty} dr \, (\psi_\varepsilon(r) - \phi(r)) \int_{\partial B_r(x)} \mathcal{X}\{z \in B_R\} \sigma(dz)$$

$$\leq c\varepsilon R^{d-1} \|\phi\|_X \|\rho\|_\infty, \tag{2-8}$$

where in the last inequality we have used (2-7). Optimizing on ε and passing to the supremum in ϕ, we get the proof. $\qquad\square$

Note that if μ^N is an empirical measure and ν a probability measure that does not give mass to the atoms of μ^N, $\mathscr{D}(\mu^N, \rho) \geq 1/N$. With this constraint, the discrepancy between two empirical measures is "small" if the measures are close in the sense specified in the following proposition.

Proposition 2.5. *Let*

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \quad \text{and} \quad \nu^N = \frac{1}{N} \sum_{i=1}^N \delta_{y_i}$$

be two empirical measures on $\mathbb{R}^d$, and take $\delta > 0$ such that $|x_i - y_i| \leq \delta$ for all $i = 1, \ldots, N$. Then for any probability measure $\rho \in L^\infty(\mathbb{R}^d)$ supported on a ball B_R,

$$\mathscr{D}(\mu^N, \nu^N) \leq cR^{d-1}\delta\|\rho\|_\infty + c\mathscr{D}(\mu^N, \rho).$$

Proof. Given $\phi \in X$ with $\|\phi\|_X \leq 1$, we construct ϕ_δ as in (2-2) and, fixing $x \in \mathbb{R}^d$, we consider $\Phi(y) := \phi(|x - y|)$ and $\Phi_\delta(y) := \phi_\delta(|x - y|)$.

Since $|x - x_i| - \delta \leq |x - y_i| \leq |x - x_i| + \delta$, we have that

$$\Phi(y_i) = \phi^+(|x - y_i|) - \phi^-(|x - y_i|) \leq \Phi_\delta(x_i).$$

Then

$$\int \Phi \, d(\nu^N - \mu^N) \leq \int (\Phi_\delta - \Phi) \, d\mu^N = \int (\Phi_\delta - \Phi) \, d(\mu^N - \rho) + \int (\Phi_\delta - \Phi) \, d\rho.$$

Since $(\phi_\delta - \phi) \in X$, the first term is bounded by $c\mathscr{D}(\mu^N, \rho)$. Using (2-5) and reasoning as in (2-8) we estimate the second term with $c\delta R^{d-1}\|\rho\|_\infty$. $\qquad\square$

3. Agent dynamics

One of the difficulties in handling (1-8) is that the dynamic is not continuous with respect to the initial datum. For instance, consider three agents $\{X_i\}_{i=1}^3$ on a line, such that

$$X_1(0) = -1, \quad X_2(0) = \varepsilon, \quad X_3(0) = 1, \tag{3-1}$$
$$V_1(0) = -1, \quad V_2(0) = 0, \quad V_3(0) = 1,$$

with $\varepsilon \in (-1, 1) \setminus \{0\}$. Then $p_{i,j} = M(X_i, |X_i - X_j|)$ takes the values $1/3, 2/3, 1$. Suppose for simplicity that $K(2/3) = 3$ and $K(1) = 0$. Then the equations for V_1 and V_3 become

$$\begin{cases} \dot{V}_1(t) = V_2(t) - V_1(t), \\ \dot{V}_3(t) = V_2(t) - V_3(t), \end{cases}$$

while

$$\dot{V}_2(t) = \begin{cases} V_3(t) - V_2(t) & \text{if } \varepsilon \in (0, 1), \\ V_1(t) - V_2(t) & \text{if } \varepsilon \in (-1, 0). \end{cases}$$

It follows that

$$\begin{cases} V_1(t) = -(1 + e^{-2t})/2, \\ V_2(t) = -(1 - e^{-2t})/2, \\ V_3(t) = (-1 + 4e^{-t} - e^{-2t})/2, \end{cases}$$

if $\varepsilon \in (-1, 0)$, while

$$\begin{cases} V_1(t) = -(-1 + 4e^{-t} - e^{-2t})/2, \\ V_2(t) = (1 - e^{-2t})/2, \\ V_3(t) = (1 + e^{-2t})/2, \end{cases}$$

if $\varepsilon \in (0, 1)$, so that $\{X_i(t), V_i(t)\}_{i=1}^3$ is discontinuous at $\varepsilon = 0$. Note that the discontinuity of the trajectories in the phase space is easily translated in the weak discontinuity of the empirical measure at time t, with respect to the initial measure.

This discontinuity reflects the fact that, for data as in (3-1) with $\varepsilon = 0$, there is not a unique way to define the dynamics. Nevertheless, we can prove that the system (1-8) is well-posed for almost all initial data. To do so, let us define some subsets of the phase space

$$\{(X, V) := (x_1, \ldots, x_N, v_1, \ldots, v_N) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}\},$$

where $d \geq 1$ is the dimension of the configuration space of the agents.

Definitions. • $\mathcal{R}$ is the set of "the regular points", i.e., the set of points (X, V) such that for each triad of different indices it holds that $|x_i - x_k| \neq |x_j - x_k|$.

• $\mathcal{S}$ is the "isorank" manifold, i.e., the set of points (X, V) such that there exists a triad of different indices i, j, k for which $|x_i - x_k| = |x_j - x_k|$, i.e., the agents i and j have the same rank with respect to the agent k.

• $\mathcal{S}_r$ is the set of the "regular points" of the isorank manifold, i.e., the subset of points $(X, V) \in \mathcal{S}$ such that if $|x_i - x_k| = |x_j - x_k|$ then x_i, x_j, x_k are different and $(v_i - v_k) \cdot \hat{n}_{ik} \neq (v_j - v_k) \cdot \hat{n}_{jk}$, where $\hat{n}_{ab} := (x_a - x_b)/|x_a - x_b|$.

We can define the dynamics locally in time, not only for initial data in $\mathcal{R}$, but also in $\mathcal{S}_r$. Namely, if initially the agents i and j have the same rank with respect to the agent k, we can redefine the force exerted on the agent k accordingly to the velocities: if $(v_i - v_k) \cdot \hat{n}_{ik} > (v_j - v_k) \cdot \hat{n}_{jk}$ we evaluate the rank as if $|x_i - x_k| > |x_j - x_k|$ for $t > 0$ and as if $|x_i - x_k| < |x_j - x_k|$ for $t < 0$. In other words, the different speeds of change of the distances among the agents allow the dynamics to leave $\mathcal{S}$ instantaneously.

We discuss the existence of the dynamics, so redefined.

Lemma 3.1. *If $(X, V) \in \mathcal{R} \cup \mathcal{S}_r$, there exists $\tau > 0$ such that the system (1-8) has a unique solution for $t \in (-\tau, \tau)$, with initial datum (X, V). Moreover the solution is locally Lipschitz in t and in (X, V).*

We omit the proof.

In $\mathcal{R}$ the solution is regular, so we can compute the determinant of the Jacobian of the flow $J(t) := J(X, V, t)$. It satisfies the equation

$$\frac{d}{dt} J(t) = -\left(\frac{d}{N} \sum_{i,j:i \neq j} p_{ij} \right) J(t) = -dN \, \gamma_N \, J(t), \tag{3-2}$$

where

$$\gamma_N := \frac{1}{N} \sum_{n=2}^{N} K(n/N).$$

Thus, volumes of the phase space are shrunk in time at a constant rate, therefore their measure cannot vanish in finite time. This implies the following fact, of which we omit the proof.

Lemma 3.2. *The subset of initial data $(X, V) \in \mathcal{R}$ such that the trajectory, at a first time in the future or in the past, intersects $\mathcal{S} \setminus \mathcal{S}_r$ has Lebesgue measure zero. Namely, $\mathcal{S} \setminus \mathcal{S}_r$ has dimension $2Nd - 2$.*

This lemma guarantees that, except for a subset of Lebesgue measure zero, we can prolong the dynamics with initial data in $\mathcal{R}$ also after a crossing in $\mathcal{S}$. To define the dynamics for all times, we need to control the number of crossings.

Lemma 3.3. *The subset of initial data $(X, V) \in \mathcal{R}$ such that the trajectory intersects $\mathcal{S}_r$ infinitely many times in finite time has Lebesgue measure zero.*

Proof. Fix $T > 0$ and suppose $(X, V) \in \mathcal{R}$ such that the solution $(X^N(t), V^N(t)) = (X_1(t), \ldots, X_N(t), V_1(t), \ldots, V_N(t))$ with initial data (X, V) intersects $\mathcal{S}_r$ a finite number of times in $[0, T - \varepsilon)$ and infinitely many times in $[0, T)$. The number

of particles is finite, so we can assume that there exists a triad of indices such that $|X_i - X_k| = |X_j - X_k|$ infinitely many times. Since the velocities V_i are bounded by a constant (as seen from simple considerations; see also Theorem 3.4), it follows from this equation that $|X_i - X_k|$ and $|X_j - X_k|$ are C^1 functions, with time derivatives uniformly Lipschitz, if $|X_i - X_k|$ and $|X_j - X_k|$ remain far from 0. Then as $t \to T$, either $|X_i - X_k| \to 0$ or $(V_i - V_k) \cdot \hat{n}_{ik}$ and $(V_j - V_k) \cdot \hat{n}_{jk}$ converge to the same limit. In both the cases, the trajectory reaches $\mathcal{S}$ at a point that is not in $\mathcal{S}_r$. We conclude the proof observing that the initial point with these properties lives in a subset of dimension $2Nd - 1$. $\qquad \square$

From these lemmas and other few considerations, we obtain the next result.

Theorem 3.4. *Except for a set of measure zero, given $(X, V) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, there exists a unique global solution*

$$(X^N(t, X, V), V^N(t, X, V)) \in C^1(\mathbb{R}^+, \mathbb{R}^{2dN}) \times C(\mathbb{R}^+, \mathbb{R}^{2dN})$$

with initial datum (X, V).
Moreover, given $R_x > 0$ and $R_v > 0$, we have that

$$|X_i(t)| \leq R_x + t R_v, \quad |V_i(t)| \leq R_v$$

for any i, if $|x_i| \leq R_x$ and $|v_i| \leq R_v$. Therefore, $V_i(t, X, V)$ has Lipschitz constant bounded by $2R_v K(0)$.

Proof. The proof follows easily from Lemma 3.1, Lemma 3.2 and Lemma 3.3.
The *a priori* bound on the support follows from (3-2) and by noticing that

$$\frac{\mathrm{d}}{\mathrm{d}t}|V_i(t)|^2 = -2 \sum_{j \neq i} p_{ij}(|V_i(t)|^2 - V_i(t) \cdot V_j(t))$$

is null or negative if $|V_i|^2$ is maximum in i. $\qquad \square$

4. The mean-field equation in L^∞

In this section, we show how to get an existence and uniqueness result for bounded weak solutions of (1-4). We start by stating some elementary facts.

Lemma 4.1. *Let $\rho \in L^\infty(\mathbb{R}^d)$ be a probability density. Then:*

(i) *Given $r_1, r_2 > 0$,*

$$|M[\rho](x, r_1) - M[\rho](x, r_2)| \leq c\|\rho\|_\infty |r_1^d - r_2^d|.$$

(ii) *Given $x_1, x_2 \in \mathbb{R}^d$ and $r > 0$,*

$$|M[\rho](x_1, r) - M[\rho](x_2, r)| \leq c\|\rho\|_\infty r^{d-1}|x_1 - x_2|.$$

Proof. The proof of the first assertion is immediate. For the second, we use the following splitting

$$
\mathcal{X}\{|x_1 - y| < r\} - \mathcal{X}\{|x_2 - y| < r\}
$$
$$
= \mathcal{X}\{|x_1 - y| < r\}\mathcal{X}\{|x_2 - y| \geq r\} - \mathcal{X}\{|x_2 - y| < r\}\mathcal{X}\{|x_1 - y| \geq r\}
$$

and we note that, if $|x_1 - x_2| \geq r$,

$$
\int_{|x_1 - y| < r} \mathcal{X}\{|x_2 - y| \geq r\} \, dy \leq cr^d \leq cr^{d-1}|x_1 - x_2|,
$$

while, if $|x_1 - x_2| < r$,

$$
\int_{|x_1 - y| < r} \mathcal{X}\{|x_2 - y| \geq r\} \, dy \leq \int \mathcal{X}\{r - |x_1 - x_2| < |x_1 - y| < r\} \, dy
$$
$$
= cr^d (1 - (1 - |x_1 - x_2|/r)^d) \leq cdr^{d-1}|x_1 - x_2|. \quad \square
$$

In the following, we denote by $\mathcal{B}_r$ the closed ball of center 0 and radius r in $L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and by $C_w([0, +\infty); L^\infty(\mathbb{R}^d \times \mathbb{R}^d))$ the set of families of bounded probability densities $\{f_t\}_{t \geq 0}$ which are weakly continuous in time in the sense of measures.

Lemma 4.2. *Let $\{f_t\}_{t \geq 0}$ be a family of probability densities for which there is a continuous nondecreasing function $r(t)$ such that $\{f_t\} \in C_w([0, +\infty); \mathcal{B}_{r(t)})$. Suppose that*

$$
\text{supp}(f_t) \subset B_{R_x(t)} \times B_{R_v(t)}, \tag{4-1}
$$

where $R_v(t)$ and $R_x(t)$ are two continuous nondecreasing functions. Then, for any initial datum $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, there exists a unique global solution of (1-7).

Proof. From classical Cauchy–Lipschitz theory, we need only check that $W[Sf_t, f_t]$ is bounded on compact sets, locally Lipschitz and continuous in t.

Recalling (1-5), the boundedness on compact sets follows from

$$
|W[Sf_t, f_t](x, v)| \leq \|K\|_\infty (R_v(t) + |v|).
$$

Since from (i) and (ii) of Lemma 4.1,

$$
|M[Sf_t](x_1, |x_1 - y|) - M[Sf_t](x_2, |x_2 - y|)| \leq
$$
$$
c\|Sf_t\|_\infty (|x_1| + |x_2| + |y|)^{d-1}|x_1 - x_2|,
$$

we have that, if (x_1, v_1) and (x_2, v_2) belong to a compact subset of $\mathbb{R}^d \times \mathbb{R}^d$,

$$
|W[Sf_t, f_t](x_1, v_1) - W[Sf_t, f_t](x_2, v_2)| \leq C(|x_1 - x_2| + |v_1 - v_2|),
$$

where C depends on R_x, R_v and the diameter of the compact set.

In order to prove that $W[Sf_t, f_t](x, v)$ is continuous in t, we first observe that $\mathscr{W}(Sf_t, Sf_s) \leq \mathscr{W}(f_t, f_s)$ and that, from the Lipschitzianity of K and Propositions 2.1 and 2.4, $K(M[Sf_t](x, |x - y|))$ is continuous in t. Since $K(M[Sf_t](x, |x - y|))$ is Lipschitz in y, also

$$\int K(M[Sf_t](x, |x - y|))(v - w)(f_t(y, w) - f_s(y, w)) \, dy \, dw$$

vanishes when $\mathscr{W}(f_t, f_s) \to 0$. $\square$

Now we can prove the main theorem of this section.

Theorem 4.3. *Let* $f_0(x, v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ *be a probability density such that* $\operatorname{supp}(f_0) \subset B_{R_x} \times B_{R_v}$. *Given* $T > 0$, *there exists a unique weak solution* $f \in C_w([0, T]; L^\infty(\mathbb{R}^d \times \mathbb{R}^d))$ *of the topological Cucker–Smale equation. Moreover*

$$\operatorname{supp}(f_t) \subset B_{R_x + t R_v} \times B_{R_v}. \tag{4-2}$$

Proof. We first note that, if the solution exists, (4-2) follows from an argument similar to the one used in the discrete case (see Theorem 3.4).

We now prove the existence. As in Lemma 4.2, consider a family of probability densities $\{g_t\}_{t \geq 0} \in C_w([0, T]; \mathcal{B}_M)$, with $M := \|f_0\|_\infty e^{d\gamma T}$ and such that (4-1) holds with $R_x(t) = R_x + t R_v$ and $R_v(t) = R_v$. The push-forward of f_0 along the flow generated by g_t, denoted by $\tilde{g}_t$, is weakly continuous in t and uniformly continuous in g_t, with $t \in [0, T]$. Moreover, the determinant of the Jacobian of the flow $J(t) := J(t, x, v)$ satisfies

$$\frac{d}{dt} J(t) = -J(t) d\gamma.$$

So the push-forward $\tilde{g}_t$ is bounded by $\|f_0\|_\infty e^{d\gamma t}$.

With a standard construction we can prove that, for T sufficiently small, the map $\{g_t\} \mapsto \{\tilde{g}_t\}$ is a contraction in $C_w([0, T]; \mathcal{B}_M)$, with the distance defined by the supremum on time of the Wasserstein distance; in this way we prove local existence and uniqueness. Using the *a priori* estimate on the supremum and on the support, we get the global result. $\square$

5. The mean-field limit

In this section we prove the main result regarding the mean-field limit for the topological Cucker–Smale equation. In the sequel, f_t is the fixed global solution of (1-7) as in Theorem 4.3, with initial datum f_0, and μ_t^N is the global solution of (1-8) in the sense of Theorem 3.4, with initial datum

$$\mu_0^N = \frac{1}{N} \sum_{i=0}^{N} \delta_{x_i} \delta_{v_i}.$$

We assume that f_0 and μ_0^N are supported in $B_{R_x} \times B_{R_v}$. Fixing T, we indicate by $C(T)$ any constant that depends only on T, R_x, R_v and $\|f_0\|_\infty$.

In order to get the result, we compare the N-agent dynamics with the "intermediate" dynamics given by

$$\begin{cases} \dot{X}_i^f(t) = V_i^f(t), \\ \dot{V}_i^f(t) = W[Sf_t, v_t^N](X_i^f, V_i^f), \end{cases} \tag{5-1}$$

where

$$v_t^N := \frac{1}{N} \sum_{k=1}^N \delta_{X_k^f(t)} \delta_{V_k^f(t)}$$

is the empirical measure. The initial datum is $v_0^N = \mu_0^N$, i.e.,

$$\{(X_i^f(0), V_i^f(0))\}_{i=1}^N = \{(x_i, v_i)\}_{i=1}^N.$$

Proposition 5.1. *Given $T > 0$, it holds that*

(i) *For $t \in [0, T]$,*

$$\mathscr{W}(f_t, v_t^N) \le C(T)\mathscr{W}(f_0, \mu_0^N). \tag{5-2}$$

(ii) *For $t \in [0, T]$, the distance*

$$\delta(t) := \max_{i=1,\dots,N}(|X_i^f(t) - X_i^N(t)| + |V_i^f(t) - V_i^N(t)|)$$

satisfies

$$\delta(t) \le C(T)\sqrt{\mathscr{W}(f_0, \mu_0^N)}. \tag{5-3}$$

Proof. Since f_t is bounded, $K(M[Sf_t](x, |x - y|))$ is locally Lipschitz in x and y (see (i) and (ii) of Lemma 4.1), and then $W[Sf_t, v](x, v)$ is weakly continuous in v, in the sense that

$$\sup_{x,v} |W[S_f, v_1](x, v) - W[S_f, v_2](x, v)| \le C(T)\mathscr{W}(v_1, v_2).$$

It is straightforward to prove that the solution v_t of the system

$$\begin{cases} \dot{X}_t = V_t, \\ \dot{V}_t = W[Sf_t, v_t](X_t, V_t), \\ v_t = \text{push-forward of } v_0 \text{ along the flow } (X_t, V_t), \end{cases}$$

is continuous in $\mathscr{W}$ with respect to the initial datum v_0. Taking $v_0 = f_0$ and $v_0 = \mu_0^N$, we get the proof of (i).

In order to estimate $\delta(t)$, we need to evaluate, for $0 \le s \le t$ and for $i = 1, \dots, N$, the difference $|\dot{V}_i^f(s) - \dot{V}_i^N(s)|$ given by

$$|W[Sf_s, v_s^N](X_i^f, V_i^f) - W[S\mu_s^N, \mu_s^N](X_i^N, V_i^N)|.$$

We estimate this quantity with the sum of three terms:

(a)$|W[Sf_s, v_s^N](X_i^f, V_i^f) - W[Sf_s, v_s^N](X_i^N, V_i^N)|,$

(b)$|W[Sf_s, v_s^N](X_i^N, V_i^N) - W[Sf_s, \mu_s^N](X_i^N, V_i^N)|,$

(c)$|W[Sf_s, \mu_s^N](X_i^N, V_i^N) - W[S\mu_s^N, \mu_s^N](X_i^N, V_i^N)|.$

Since $K(M[Sf_s](x, |x - y|))$ is Lipschitz in x, from the definition of W it is easy to prove that (a) is bounded by

$$(c \operatorname{Lip}(K)\|Sf_s\|_\infty R_x^{d-1}(s)R_v + c\|K\|_\infty)\delta(s),$$

and that (b) is estimated by

$$c \operatorname{Lip}(K)\|Sf_s\|_\infty R_x^{d-1}(s)R_v\delta(s).$$

Note that $\|Sf_s\|_\infty \le cR_v^d\|f_s\|_\infty$. From Proposition 2.1, we have that (c) is bounded by

$$c \operatorname{Lip}(K)R_v\mathscr{D}(Sf_s, S\mu_s^N).$$

Since

$$\mathscr{D}(Sf_s, S\mu_s^N) \le \mathscr{D}(Sf_s, Sv_s^N) + \mathscr{D}(Sv_s^N, S\mu_s^N),$$

by Proposition 2.5 with $\rho = Sf_s$, $\mu^N = Sv_s^N$ and $v^N = S\mu_s^N$, we get

$$\mathscr{D}(Sv_s^N, S\mu_s^N) \le c\delta(s) + c\mathscr{D}(Sf_s, Sv_s^N).$$

Writing $\delta(t)$ in terms of the time integral of $\delta(s)$ and the difference of the interaction terms, and using the Gronwall lemma, we readily get the estimate

$$\delta(t) \le C(T) \int_0^t \mathscr{D}(Sf_s, Sv_s^N) \, ds,$$

valid for $0 \le t \le T$. We conclude the proof by using Proposition 2.4, (5-2) and the fact that $\mathscr{W}(Sf_s, Sv_s^N) \le \mathscr{W}(f_s, v_s^N)$. □

Theorem 5.2. *Fix.$T > 0$. Let f_t be a solution of (1-7) as in Theorem 4.3 with initial datum f_0 and let μ_t^N be a solution of (1-8) in the sense of Theorem 3.4 with initial datum μ_0^N. Then for $0 \le t \le T$,*

$$\mathscr{W}(f_t, \mu_t^N) \le C(T) \max\{\mathscr{W}(f_0, \mu_0^N), \sqrt{\mathscr{W}(f_0, \mu_0^N)}\}.$$

Proof. By the triangle inequality,

$$\mathscr{W}(f_t, \mu_t^N) \le \mathscr{W}(f_t, v_t^N) + \mathscr{W}(v_t^N, \mu_t^N).$$

From (5-2), using (5-3) and the fact that $\mathscr{W}(v_t^N, \mu_t^N) \le \delta(t)$, we have proved the theorem. □

Appendix: Continuity of $\mathscr{W}$ with respect to $\mathscr{D}$

We now prove the continuity of the Wasserstein distance $\mathscr{W}$ with respect to the discrepancy distance $\mathscr{D}$ for compactly supported measures.

Consider two probability measures μ and ν, both with support in the ball B_R of $\mathbb{R}^d$. Fix $\varepsilon > 0$ and consider a Lipschitz test function ϕ; it is sufficient to consider ϕ with support of diameter less than cR, so that $\|\phi\|_\infty \leq cR$. Given such ϕ, take $\delta_1 > 0$ such that $\mathrm{Lip}(\phi)\delta_1 < \varepsilon$.

By the Besicovitch covering principle (see [4; 5; 6]), there exist N_ε disjoint closed balls $\{B_i\}_{i=1}^{N_\varepsilon}$ of radius at most δ_1 such that

$$\mu\Big(\bigcup_{i=1}^{N_\varepsilon} B_i\Big) \geq 1 - \varepsilon.$$

We estimate

$$\int \phi \, \mathrm{d}(\mu - \nu) = \int_{\mathbb{R}^d \setminus \cup B_i} \phi \, \mathrm{d}(\mu - \nu) + \int_{\cup B_i} \phi \, \mathrm{d}(\mu - \nu) =: A + B.$$

We have

$$A \leq \|\phi\|_\infty \Big(\mu\Big(\mathbb{R}^d \setminus \bigcup_{i=1}^{N_\varepsilon} B_i\Big) + \nu\Big(\mathbb{R}^d \setminus \bigcup_{i=1}^{N_\varepsilon} B_i\Big)\Big) \leq \|\phi\|_\infty(2\varepsilon + N_\varepsilon \mathscr{D}(\mu, \nu)),$$

while

$$B \leq \sum_{i=1}^{N_\varepsilon} \int_{B_i} (\sup \phi - \inf \phi) \, \mathrm{d}\nu + \sum_{i=1}^{N_\varepsilon} \int_{B_i} \sup \phi \, \mathrm{d}(\mu - \nu)$$
$$\leq 2 \mathrm{Lip}(\phi)\delta_1 + N_\varepsilon \|\phi\|_\infty \mathscr{D}(\mu, \nu).$$

Hence we obtain

$$\int \phi \, \mathrm{d}(\mu - \nu) \leq cR\varepsilon + cRN_\varepsilon \mathscr{D}(\mu, \nu).$$

Taking $\mathscr{D}(\mu, \nu) < \delta_2$ such that $N_\varepsilon \delta_2 < \varepsilon$, we have proved the result.

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