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ON THE RELATION OF NEGATIONS IN NELSON ALGEBRAS

A b s t r a c t. The aim of this paper is to investigate the relation between the strong and the "weak" or intuitionistic negation in Nelson algebras. To do this, we define the variety of Kleene algebras with intuitionistic negation and explore the Kalman's construction for pseudocomplemented distributive lattices. We also study the centered algebras of this variety.

Introduction

Non-classical negation has been investigated from different points of view, both in different logics and in their corresponding algebraic structures.

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A particular case of study is negation in the context of constructive logic (the journal Studia Logica, vol. 80 (2005), is dedicated to this subject).

The constructive character of intuitionistic disjunction is not shared by intuitionistic negation because in general $\neg(\alpha \wedge \beta)$ does not imply $\neg\alpha$ or $\neg\beta$. This motivated D. Nelson to present the constructive logic with strong negation (CLN) as an expansion of intuitionistic logic by a new negation symbol, which acts as an involutive negation $\sim$ (see [17]). The class of Nelson algebras, deeply studied in [2, 3, 4, 20, 21, 30], is the algebraic semantics of the propositional fragment of CLN.

Nelson algebras have an involutive negation $\sim$ (usually known as *strong* negation) and another negation (*intuitionistic*, as called in [29]) that can be defined through the weak implication $\to$ and the constant 0 by the rule

$$\neg x = x \to 0. \tag{1}$$

For the reader familiar with residuated lattices, Nelson algebras are termwise equivalent to Nelson residuated lattices, as shown in [4, 26, 27], where a strong implication $\Rightarrow$ can be defined in terms of the operations $\to$ and $\sim$ as

$$x \Rightarrow y := (x \to y) \wedge (\sim y \to \sim x).$$

In this context, we have that

$$\neg x = x \Rightarrow (x \Rightarrow 0),$$
$$\sim x = x \Rightarrow 0.$$

Strong and intuitionistic negations in Nelson algebras are negations playing different roles, and they also interact with each other. However, since $\neg$ is not seen as a primitive operation, the relation between both negations in the theory of Nelson algebras is somehow hidden. One of the purposes of the present paper is to shed some light on the roles of these negations in Nelson algebras and centered Nelson algebras.

Nelson algebras can be seen as Kleene algebras with an extra binary operation (see Section 1), the weak implication, which generalises relative pseudocomplementation. In [24], Sendlewski generalised results from Fidel and Vakarelov [12, 28] to show the deep connection between Nelson algebras and Heyting algebras.

In order to show a similar relation for a wider class of algebras, in [25] Sendlewski introduced the class $\mathcal{K}_\omega$ of Kleene algebras with weak pseudocomplementation (see also Section 1) to prove a topological connection between this class and pseudocomplemented distributive lattices. The class $\mathcal{K}_\omega$ encompasses the implication-free reducts of Nelson algebras with the additional operation $\neg$ as weak pseudocomplementation, and gives a partial answer to the question of the relationship between the strong and intuitionistic negations.

To obtain that result, we define KAN as a variety of Kleene algebras with an extra negation $\neg$, where the relation between the involutive negation $\sim$ and $\neg$ can be seen in the axioms defining KAN (this relation will be made clearer in Section 2.4). We show that the reduct of a Nelson algebra, with the negation as in (1), is a KAN-algebra. Roughly speaking, KAN-algebras have a weaker structure than Nelson algebras, just like pseudocomplemented distributive lattices have a weaker structure than Heyting algebras. We use the Kalman construction (see [15]) to obtain a categorical equivalence between the algebraic category KAN and the category PDLF whose objects are pairs of the form (L, F), where $L \in \mathsf{PDL}$ (the category of pseudocomplemented distributive lattices) and F is a Boolean filter of L.

In Section 2.3 we obtain that $\mathcal{K}_\omega = \mathsf{KAN}$. Our results show in a more immediate way than [25] that $\mathcal{K}_\omega$ is a variety and in Theorem 2.16 we give a categorical equivalence for KAN by following an alternative path to that given in [25, Theorem 4.4]. Although some lemmas in our work have counterparts in [25], the difference between the proofs of [25, Theorem 4.4] and Theorem 2.16 largely boils down to the fact that $\mathcal{K}_\omega$ is defined as a quasi-variety, while KAN is defined as a variety.

Furthermore, our approach facilitates the definition of quantifiers for KAN-algebras, thus allowing for a first-order study (see Corollary 4.7). We do this by considering monadic pseudocomplemented distributive lattices, which are pseudocomplemented distributive lattices endowed with a unary map that acts as an algebraic counterpart of the logical notion of an existential quantifier (see [13]). Given the relationship between KAN-algebras and pseudocomplemented distributive lattices, we are able to define weak quantifiers and weakly monadic KAN-algebras, and provide examples to show how these quantifiers behave.

Additionally, in this work we consider centered structures. In [8, Theorem 2.4], R. Cignoli proved that there exists an equivalence between the category of bounded distributive lattices and a particular full subcategory of centered Kleene algebras. Moreover, in [8, Theorem 3.14] he also proved that there exists an equivalence between the category of Heyting algebras and the category of centered Nelson algebras (see also [5, 14]). Centered structures have importance on their own and have been studied throughout the literature under different schemes. See for instance [5, 6, 7, 8, 10, 14, 22, 23].

We show that there exists a categorical equivalence between PDL and $\mathsf{KAN_c}$, the category whose objects are KAN-algebras endowed with a center, which can be seen as a generalisation of the one just mentioned. This equivalence is a particular case of Theorem 2.16, but it is also possible to give a more direct proof in terms of centered structures (see the proof of Lemma 3.3 and Remark 3.5 for details). Observe that an equivalence between both categories was also proved in [25, Corollary 4.6] in terms of the algebraic category whose objects are those algebras of $\mathcal{K}_\omega$ equipped with a constant c which is a fixpoint of the strong negation.

In summary, the main results of the paper are:

- If L is a pseudocomplemented distributive lattice and $F \subseteq L$ is a Boolean filter, then $\mathrm{K}(L, F) := \{(a, b) \in L \times L : a \wedge b = 0 \text{ and } a \vee b \in F\}$ is the universe of an algebra in KAN. Conversely, if $T \in \mathsf{KAN}$ then there exists a congruence θ with respect to $\wedge$, $\vee$ and $\neg$ such that T/θ is a pseudocomplemented distributive lattice and T^+/θ is a Boolean filter, where T^+ is defined as $\{x \in T : x \geq \sim x\}$. Both assignments can be extended to functors which determine a categorical equivalence (Theorem 2.16).

- If L is a pseudocomplemented distributive lattice, then $\mathrm{K}(L, L) = \{(a, b) \in L \times L : a \wedge b = 0\}$ is the universe of a centered KAN-algebra, with center $(0, 0)$. Conversely, if $T \in \mathsf{KAN_c}$, then T^+ is a pseudocomplemented distributive lattice. A categorical equivalence can be proved upon extending these assignments to functors (Theorem 3.6).

The paper is structured as follows. In Section 1 we recall some definitions and properties concerning pseudocomplemented distributive lattices and Kleene algebras. In Section 2 we introduce and study the variety KAN. We also define the category PDLF and we prove that there exists a categorical equivalence between PDLF and KAN (Theorem 2.16), and also give a first proof from that $\mathsf{KAN} = \mathcal{K}_\omega$ (Theorem 2.17). In Section 3 we show that there exists an equivalence between PDL and $\mathsf{KAN_c}$ (Theorem 3.6).

The construction of this categorical equivalence is not presented as a particular case of Theorem 2.16. We also prove that for every $L \in \mathsf{PDL}$ there exists a bijection between the congruences of L and the congruences of its associated centered KAN-algebra. In Section 4 we define the notion of quantifier in PDL as in the case of bounded distributive lattices (see [9]), we introduce a notion of quantifier in KAN and we study the relation between both constructions. Section 5 contains some additional results, such as that KAN-algebras are not necessarily isomorphic to reducts of Nelson algebras, and that the intuitionistic negation cannot be strengthened as a negation without becoming trivial. Moreover, we give a second proof that $\mathsf{KAN} = \mathcal{K}_\omega$ (corollaries 5.9 and 5.11).

1. Preliminaries and basic results

We assume the reader is familiar with bounded distributive lattices and Heyting algebras (see [1]).

A *pseudocomplemented distributive lattice* $(L, \wedge, \vee, \neg, 0, 1)$ is an algebra of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and for every $a, b \in L$ it holds that $a \wedge b = 0$ if and only if $a \leq \neg b$. This means that for every $a \in L$ there is a largest member of L which is disjoint with a, namely $\neg a$. The class of pseudocomplemented distributive lattices is a variety (see [1]) that we will denote PDL. Also note that in pseudocomplemented distributive lattices necessarily $1 = \neg 0$ and $0 = \neg 1$ hold.

Recall that if L is a lattice, a non-empty subset $F \subseteq L$ is said to be a *lattice filter* of L if F is an upset (i.e. for every $x, y \in L$, if $x \leq y$ and $x \in F$, then $y \in F$) and $x \wedge y \in F$ for all $x, y \in F$. The notion of *lattice ideal* is dually defined.

If L is an algebra and R an equivalence relation on L, we adopt the notation a/R for the equivalence class of a modulo R, and also L/R for the set of equivalence classes.

The definitions of Boolean filter and Boolean congruence on a given pseudocomplemented distributive lattice will be used throughout the paper, so we opt to introduce in the present section these definitions and the link between them.

Definition 1.1. Let $L \in \mathsf{PDL}$.

1) A lattice filter $F \subseteq L$ is called *Boolean* if it contains all dense elements, i.e. those $x \in L$ satisfying $\neg x = 0$.

2) We say that a congruence R on L is a *Boolean congruence* if L/R is a Boolean algebra, or equivalently, if $a \vee \neg a \in 1/R$ for every $a \in L$.

The following two lemmas are part of the folklore of the subject, and can be found for instance in [19]. However we opt to give a self contained proof of them.

Lemma 1.2. *Let $L \in \mathsf{PDL}$ and R a congruence on L. The following conditions are equivalent:*

1) *R is a Boolean congruence.*

2) *$1/R$ is a Boolean filter.*

Proof. Assume that R is a Boolean congruence and let $a \in L$ be such that $\neg a = 0$. Since $a = a \vee \neg a$ and $a \vee \neg a \in 1/R$, $a \in 1/R$. Conversely, suppose that $1/R$ is a Boolean filter. Since $\neg(a \vee \neg a) = \neg a \wedge \neg\neg a = 0$, $a \vee \neg a \in 1/R$. $\square$

Lemma 1.3. *Let $L \in \mathsf{PDL}$. If R is a Boolean congruence, then $1/R$ is a Boolean filter and if F is a Boolean filter, then the set*

$$H(F) = \{(a, b) \in L \times L : a \wedge f = b \wedge f \text{ for some } f \in F\}$$

is a Boolean congruence. Moreover, the assignments $R \mapsto 1/R$ and $F \mapsto H(F)$ define an order isomorphism between the poset of Boolean congruences of L and the poset of Boolean filters of L.

Proof. It follows from Lemma 1.2 that if R is a Boolean congruence, then $1/R$ is a Boolean filter. Now let F be a Boolean filter. We know that $H(F)$ is a congruence of the underlying lattice of L. In order to show that $H(F)$ preserves $\neg$, let $(x, y) \in H(F)$. Thus there exists $f \in F$ such that $x \wedge f = y \wedge f$. Since F is a Boolean filter, we have that $x \vee \neg x \in F$ and $y \vee \neg y \in F$. Hence $g = (x \vee \neg x) \wedge (y \vee \neg y) \wedge f \in F$. Straightforward computations show that $\neg x \wedge g = \neg y \wedge g = \neg x \wedge \neg y \wedge f$, so $(\neg x, \neg y) \in H(F)$.

Then $H(F)$ is a congruence on L. It is immediate that $F = 1/H(F)$, so it follows from Lemma 1.2 that $H(F)$ is a Boolean congruence.

To show that $F \mapsto H(F)$ is onto, let R be a Boolean congruence on L. We will show that $R = H(1/R)$. If $(x, y) \in H(1/R)$, then $x \wedge f = y \wedge f$ for some $f \in 1/R$. Hence $(x \wedge f, x) \in R$, $(y \wedge f, y) \in R$ and $(x \wedge f, y \wedge f) \in R$, so that $(x, y) \in R$ and thus $R \subseteq H(1/R)$. For the other inclusion, let $(x, y) \in R$, so $(x \vee \neg x, y \vee \neg x) \in R$. But $(x \vee \neg x, 1) \in R$, so $(y \vee \neg x, 1) \in R$. Similarly, we have that $(x \vee \neg y, 1) \in R$. Hence $((x \vee \neg y) \wedge (y \vee \neg x), 1) \in R$, i.e. $(x \vee \neg y) \wedge (y \vee \neg x) \in 1/R$. Let $f = (x \vee \neg y) \wedge (y \vee \neg x)$. Straightforward computations show that $x \wedge f = x \wedge y = y \wedge f$, so $(x, y) \in H(1/R)$. Then $R = H(1/R)$.

Let F, G be Boolean filters. We will prove that $F \subseteq G$ if and only if $H(F) \subseteq H(G)$. It is immediate that if $F \subseteq G$ then $H(F) \subseteq H(G)$. Conversely, assume that $H(F) \subseteq H(G)$. If $x \in F$, then $(x, 1) \in H(F) \subseteq H(G)$, i.e. $x \wedge g = g$ for some $g \in G$. Thus $x \in G$, so that $F \subseteq G$. Hence, the map $F \mapsto H(F)$ is an injection. Moreover, it is an order isomorphism. □

A *Kleene algebra* is an algebra $(T, \wedge, \vee, \sim, 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying that $(T, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\sim$ is an involution (i.e., $\sim\sim x = x$ for every $x \in T$) such that

1) $\sim(x \vee y) = \sim x \wedge \sim y$ and

2) $x \wedge \sim x \leq y \vee \sim y$

hold for every $x, y \in T$. A Kleene algebra is called *centered* if it has a center; that is, an element c such that $\mathrm{c} = \sim\mathrm{c}$. It is immediate that the center of a Kleene algebra, when it exists, is unique. We write BDL for the category of bounded distributive lattices, KA for the category of Kleene algebras and KA_{c} for the category of centered Kleene algebras. In all cases the morphisms are the corresponding algebra homomorphisms. It is interesting to note that if T and U are centered Kleene algebras and $f\colon T \to U$ is a morphism of Kleene algebras, then f preserves the center, i.e. $f(\mathrm{c}) = \mathrm{c}$.

Given an object $L \in \mathsf{BDL}$, the set

$$\mathrm{K}(L) \coloneqq \{(a, b) \in L \times L : a \wedge b = 0\}$$

endowed with operations

$$(a,b) \vee (d,e) \coloneqq (a \vee d, b \wedge e)$$
$$(a,b) \wedge (d,e) \coloneqq (a \wedge d, b \vee e)$$
$$\sim(a,b) \coloneqq (b,a)$$

and distinguished elements $(0,1)$ and $(1,0)$ is a Kleene algebra which is centered if in addition we set $(0,0)$ as the center. For a morphism $f\colon L \to M \in \mathsf{BDL}$, the map $\mathrm{K}(f)\colon \mathrm{K}(L) \to \mathrm{K}(M)$ defined by $\mathrm{K}(f)(a,b) = (f(a), f(b))$ is a morphism in KA (hence in $\mathsf{KA_c}$). Moreover, K is a functor from BDL to KA (and $\mathsf{KA_c}$).

Recall that a *Nelson algebra* [8, 30] is a Kleene algebra such that for each pair x, y there exists the binary operation $\to$ given by $x \to y := x \to_{\mathsf{HA}} (\sim x \vee y)$ (where $\to_{\mathsf{HA}}$ is the Heyting implication) and for every x, y, z it holds that $(x \wedge y) \to z = x \to (y \to z)$. Nelson algebras can be seen as algebras $(H, \wedge, \vee, \to, \sim, 0, 1)$ of type $(2,2,2,1,0,0)$. The class of Nelson algebras is a variety [2, 3, 21].

Fidel [12] and Vakarelov [28] proved independently that if $(H, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra, then $(\mathrm{K}(H), \wedge, \vee, \to, \sim, (0,1), (1,0))$ is a Nelson algebra, where

$$(a,b) \to (d,e) \coloneqq (a \to d, a \wedge e) \tag{2}$$

for pairs (a,b) and (d,e) in $\mathrm{K}(H)$ (note that we are using the notation $\to$ in two different senses). For more information about the subject see also [8, 24]. In particular, if H is a Heyting algebra and R is a Boolean congruence on H, then it follows from [24, Theorem 3.6] that the set $\{(a,b) \in \mathrm{K}(H) : (a \vee b, 1) \in R\}$ endowed with the operations defined on $\mathrm{K}(H)$ is a Nelson algebra.

In [25], Sendlewski defined quasi weakly pseudocomplemented and weakly pseudocomplemented Kleene algebras. A Kleene algebra T is said to be *quasi weakly pseudocomplemented* if for every $x \in T$ there exists the maximum of the set $\{y \in T : x \wedge y \leq \sim x\}$, which is denoted by $\neg x$. The element $\neg x$ is characterised by the following property: for every $y \in T$, $y \leq \neg x$ if and only if $x \wedge y \leq \sim x$. The unary operation $\neg$ so defined will be called a *quasi weak pseudocomplementation* and the associated algebra $(T, \wedge, \vee, \sim, \neg, 0, 1)$ of type $(2,2,1,1,0,0)$ a *Kleene algebra with a quasi weak pseudocomplementation*, or simply a qwp-Kleene algebra. A qwp-Kleene algebra will be called *Kleene algebra with a weak pseudocomplementation* or simply wp-Kleene algebra if $\neg(x \wedge y) = 1$ if and only if $\neg\neg x \leq \neg y$ for every x, y. The set of wp-Kleene algebras is denoted by $\mathcal{K}_\omega$. If $L \in \mathsf{PDL}$ and R is a Boolean congruence on L, then by [25, Lemma 2.3] that the set $\{(a, b) \in \mathrm{K}(L) : (a \vee b, 1) \in R\}$ endowed with the operations defined on $\mathrm{K}(L)$ is a qwp-Kleene algebra (moreover, it is a wp-Kleene algebra).

2. The variety KAN

In this section we will introduce and study the variety KAN. Moreover, will see that KAN is categorically equivalent to a certain category whose objects are pairs of the form (L, F), where $L \in \mathsf{PDL}$ and F is a Boolean filter of L. Finally, we will compare our variety with the class of wp-Kleene algebras introduced in [25].

2.1 Definition and basic properties

Definition 2.1. We define the variety KAN of Kleene algebras with intuitionistic negation as the variety of algebras $(T, \wedge, \vee, \sim, \neg, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that $(T, \wedge, \vee, \sim, 0, 1)$ is a Kleene algebra and the following conditions are satisfied for every $x, y \in T$:

(N1) $\neg(x \wedge \neg(x \wedge y)) = \neg(x \wedge \neg y)$,

(N2) $\neg(x \vee y) = \neg x \wedge \neg y$,

(N3) $x \wedge \sim x = x \wedge \neg x$,

(N4) $\sim x \leq \neg x$,

(N5) $\neg(x \wedge y) = \neg((\sim \neg x) \wedge y)$.

The members of the variety KAN will be called KAN-algebras.

If $(T, \wedge, \vee, \sim, \neg, 0, 1)$ is a KAN-algebra, an application of (N3) yields $\neg 1 = 1 \wedge \neg 1 = 1 \wedge \sim 1 = 1 \wedge 0 = 0$. Taking $x = 0$ in (N4) we obtain that $\neg 0 = 1$. In addition, if $x \leq y$, then $\neg y \leq \neg x$ by (N2).

In [11], a unary operator t defined on a bounded poset T is a negation if $t(0) = 1$, $t(1) = 0$ and $t(y) \leq t(x)$ whenever $x \leq y$. Therefore, the unary operation $\neg$ in the definition of KAN-algebras is a negation operator in this sense.

We call this operator intuitionistic negation following [29], given its relation with intuitionistic negation in Heyting algebras.

Example 2.2. Let $(T, \wedge, \vee, \sim, \to, 0, 1)$ be a Nelson algebra. Define the unary operation $\neg$ by $\neg x := x \to 0$. Taking into account the representation of Nelson algebras in terms of Heyting algebras due to Fidel [12] and Vakarelov [28], it can be proved that the algebra $(T, \wedge, \vee, \sim, \neg, 0, 1)$ is in KAN (the proof is similar to that of Lemma 2.4 below). In Section 5.1 we will prove that not every KAN-algebra is a reduct of a Nelson algebra.

Example 2.3. A Boolean algebra $(B, \wedge, \vee, \neg, 0, 1)$ with an additional unary operation $\sim$ defined as $\sim x := \neg x$ is a KAN-algebra. This is actually covered by the previous example, but we consider it separately as it will be of importance in Section 5.2.

Let H be a Heyting algebra. We can define a unary operation $\neg$ in $\mathrm{K}(H)$ as $\neg(a, b) := (a, b) \to (0, 1)$, where the operation $\to$ for $K(H)$ is (2) of Section 1. Clearly, $\neg(a, b) = (\neg a, a)$ (note that we are using the notation $\neg$ in two different senses). Motivated by this and the fact that Heyting algebras can be seen as pseudocomplemented distributive lattices, for $L \in \mathsf{PDL}$ we define the unary operation $\neg\colon \mathrm{K}(L) \to \mathrm{K}(L)$ by

$$\neg(a, b) := (\neg a, a).$$

Let $(L, \wedge, \vee, \neg, 0) \in \mathsf{PDL}$. We define

$$\mathbb{K}(L) = (\mathrm{K}(L), \wedge, \vee, \sim, \neg, (0, 1), (1, 0)).$$

Lemma 2.4. *If* $(L, \wedge, \vee, \neg, 0) \in \mathsf{PDL}$*, then* $\mathbb{K}(L) \in \mathsf{KAN}$*. Moreover,* $\mathbb{K}$ *can be extended to a functor from* PDL *to* KAN*.*

Proof. Since L is a bounded distributive lattice,

$$(\mathrm{K}(L), \wedge, \vee, \sim, (0, 1), (1, 0)) \in \mathsf{KA}.$$

We will prove conditions (N1)-(N5). Let $(a, b), (d, e) \in \mathrm{K}(L)$. Since $a \wedge \neg(a \wedge d) = a \wedge \neg d$,

$$\begin{aligned}\neg((a,b) \wedge \neg((a,b) \wedge (d,e))) &= \neg((a,b) \wedge (\neg(a \wedge d), a \wedge d)) \\ &= (\neg(a \wedge \neg(a \wedge d)), a \wedge \neg(a \wedge d)) \\ &= (\neg(a \wedge \neg d), a \wedge \neg d) \\ &= \neg(a \wedge \neg d, b \vee d) \\ &= \neg((a,b) \wedge (\neg d, d)) \\ &= \neg((a,b) \wedge \neg(d,e)),\end{aligned}$$

so (N1) holds. Condition (N2) follows from the identity $\neg(a \vee b) = \neg a \wedge \neg b$ in PDL:

$$\begin{aligned}\neg((a,b) \vee (d,e)) &= (\neg(a \vee d), a \vee d) \\ &= (\neg a \wedge \neg d, a \vee d) \\ &= (\neg a, a) \wedge (\neg d, d) \\ &= \neg(a,b) \wedge \neg(d,e).\end{aligned}$$

Condition (N3) can be proved in the following way:

$$\begin{aligned}(a,b) \wedge \sim (a,b) &= (0, b \vee a) \\ &= (a \wedge \neg a, b \vee a) \\ &= (a,b) \wedge (\neg a, a) \\ &= (a,b) \wedge \neg(a,b).\end{aligned}$$

In order to see condition (N4), note that $\sim (a,b) \leq \neg(a,b)$ if and only if $b \leq \neg a$ if and only if $a \wedge b = 0$ if and only if $(a,b) \in \mathrm{K}(L)$. Then $\sim (a,b) \leq \neg(a,b)$. Finally, we will show condition (N5):

$$\begin{aligned}\neg((a,b) \wedge (d,e)) &= (\neg(a \wedge d), a \wedge d) \\ &= \neg(a \wedge d, \neg a \vee e) \\ &= \neg((a, \neg a) \wedge (d,e)) \\ &= \neg(\sim (\neg a, a) \wedge (d,e)) \\ &= \neg(\sim \neg(a,b) \wedge (d,e)).\end{aligned}$$

Thus, $\mathbb{K}(L) \in \mathsf{KAN}$.

Let $f\colon L \to M$ be a morphism in PDL and let $(a,b) \in \mathrm{K}(L)$. We define $\mathbb{K}(f) = \mathrm{K}(f)$ as in the case of BDL and KA. Then

$$\begin{aligned}\mathbb{K}(f)(\neg(a,b)) &= \mathbb{K}(f)(\neg a, a)\\ &= (f(\neg a), f(a))\\ &= (\neg f(a), f(a))\\ &= \neg(f(a), f(b))\\ &= \neg\mathbb{K}(f)(a,b).\end{aligned}$$

Therefore $\mathbb{K}(f)$ is a morphism in KAN. □

We are able to show some additional properties of KAN-algebras.

Proposition 2.5. *Let $T \in \mathsf{KAN}$ and $x \in T$. Then*

(N6) $\neg(x\wedge \sim x) = \neg(x \wedge \neg x) = 1$,

(N7) $\sim\neg x \leq x \leq \neg \sim x$,

(N8) $\sim\neg x \leq \neg\neg x \leq \neg \sim x$ *and*

(N9) $\neg \sim \neg x = \neg x$.

Proof. Property (N6) follows from (N1), since

$$\neg(x \wedge \neg x) = \neg(x \wedge \neg(x \wedge 1)) = \neg(x \wedge \neg 1) = \neg 0 = 1.$$

Properties (N7) and (N8) follow from (N4) and the fact that $\sim\sim x = x$. (N9) is a consequence of (N5), as $\neg x = \neg(x\wedge 1) = \neg((\sim\neg x)\wedge 1) = \neg \sim \neg x$. □

The next lemma will be crucial for later results.

Lemma 2.6. *Let $T \in \mathsf{KAN}$ and $x, y \in T$. Then*

$$x \leq y \text{ if and only if } \sim\neg x \leq \sim\neg y \text{ and } \neg \sim x \leq \neg \sim y.$$

Proof. It is immediate that if $x \leq y$, then $\sim\neg x \leq \sim\neg y$ and $\neg \sim x \leq \neg \sim y$. Conversely, suppose that $\sim\neg x \leq \sim\neg y$ and $\neg \sim x \leq \neg \sim y$. Since T is a Kleene algebra, it holds that $x\wedge \sim x \leq y \vee \sim y$. Thus

$$[(x\wedge \sim x) \wedge \neg \sim x]\vee \sim\neg x \leq [(y\vee \sim y) \wedge \neg \sim y]\vee \sim\neg y.$$

We will show that the left-hand side of this inequality is x and the right-hand side is y. Indeed,

$$\begin{aligned}[(x\wedge \sim x) \wedge \neg \sim x]\vee \sim \neg x &= (x\wedge \sim x)\vee \sim \neg x \\ &= x \wedge (\sim x\vee \sim \neg x) \\ &= x\wedge \sim (x \wedge \neg x) \\ &= x\wedge \sim (x\wedge \sim x) \\ &= x \wedge (x\vee \sim x) \\ &= x\end{aligned}$$

and

$$\begin{aligned}[(y\vee \sim y) \wedge \neg \sim y]\vee \sim \neg y &= [y \vee (\sim y \wedge \neg \sim y)]\vee \sim \neg y \\ &= [y \vee (\sim y\wedge \sim\sim y)]\vee \sim \neg y \\ &= y\vee \sim \neg y \\ &= y.\end{aligned}$$

Hence $x \leq y$. □

Let $(T, \wedge, \vee, \sim, \neg, 0, 1)$ be a KAN-algebra and let $\theta \subseteq T^2$ be defined as

$$(x, y) \in \theta \text{ if and only if } \neg x = \neg y \tag{3}$$

θ is an equivalence relation which will be critical in proving a categorical equivalence for the class KAN. Recall that x/θ stands for the set $\{y \in T : (x, y) \in \theta\}$, while the set $\{x/\theta : x \in T\}$ is denoted by T/θ.

Lemma 2.7. *Let* $(T, \wedge, \vee, \sim, \neg, 0, 1) \in \mathsf{KAN}$. *Then* θ *as defined in* (3) *is a congruence with respect to* $\wedge$, $\vee$ *and* $\neg$.

Proof. Let $x, y, z \in T$ and assume that $(x, y) \in \theta$; i.e. $\neg x = \neg y$. It follows from (N2) that $\neg(x \vee z) = \neg(y \vee z)$, so $(x \vee z, y \vee z) \in \theta$. Besides it follows from (N5) that

$$\neg(x \wedge z) = \neg((\sim \neg x) \wedge z) = \neg((\sim \neg y) \wedge z) = \neg(y \wedge z).$$

Then $(x \wedge z, y \wedge z) \in \theta$. It is also immediate that $(\neg x, \neg y) \in \theta$. □

For each $T \in \mathsf{KAN}$, with an abuse of notation, $(T/\theta, \wedge, \vee, \neg, 0, 1)$ will denote a new algebraic structure whose operations $\wedge$, $\vee$ and $\neg$ are well defined due to Lemma 2.7 (0 means $0/\theta$ and 1 means $1/\theta$). Clearly, $(T/\theta, \wedge, \vee, 0, 1)$

is a bounded distributive lattice and the order $\leq$ in T/θ can be characterised as

$$x/\theta \leq y/\theta \text{ if and only if } \neg y \leq \neg x.$$

Indeed, if $x/\theta \leq y/\theta$, then $\neg y = \neg(x \vee y) = \neg x \wedge \neg y$, so $\neg y \leq \neg x$. Conversely, if $\neg y \leq \neg x$, then $\neg y = \neg y \wedge \neg x = \neg(x \vee y)$. Thus $x/\theta \leq y/\theta$.

Lemma 2.8. *If $T \in \mathsf{KAN}$, then $T/\theta \in \mathsf{PDL}$.*

Proof. As previously stated, it follows at once from Lemma 2.7 that T/θ is a bounded distributive lattice. We will show that $x/\theta \wedge y/\theta = 0/\theta$ if and only if $x/\theta \leq \neg y/\theta$. If $x/\theta \wedge y/\theta = 0/\theta$, then $\neg(x \wedge y) = 1$, so that

$$\begin{aligned} \neg x \wedge \neg\neg y &= \neg(x \wedge \neg(x \wedge y)) \wedge \neg\neg y \\ &= \neg(x \wedge \neg y) \wedge \neg\neg y \\ &= \neg((x \wedge \neg y) \vee \neg y) \\ &= \neg\neg y, \end{aligned}$$

by (N1) and (N2). Hence $\neg\neg y \leq \neg x$, which means that $x/\theta \leq \neg y/\theta$ by the comment preceding this lemma. Conversely, assume that $x/\theta \leq \neg y/\theta$. Since

$$x/\theta \wedge y/\theta \leq y/\theta \wedge \neg y/\theta,$$

$\neg(y \wedge \neg y) \leq \neg(x \wedge y)$. But $\neg(y \wedge \neg y) = 1$ by property (N6), so $\neg(x \wedge y) = 1$. Therefore, $x/\theta \wedge y/\theta = 0/\theta$. □

If $L \in \mathsf{PDL}$ and we consider $(a, b), (c, d)$ in the KAN-algebra $\mathbb{K}(L)$,

$$((a, b), (c, d)) \in \theta \text{ if and only if } \neg(a, b) = \neg(c, d) \text{ if and only if } a = c. \quad (4)$$

Therefore two pairs in $\mathrm{K}(L)$ are equivalent (modulo θ) if and only if their first coordinates coincide.

Lemma 2.9. *Let $L \in \mathsf{PDL}$. Then $g\colon \mathbb{K}(L)/\theta \to L$ given by*

$$g((a, b)/\theta) = a$$

is an isomorphism.

Proof. Observe that the operations $\wedge, \vee, \neg$ defined in the pairs of $\mathrm{K}(L)$ behave in the first coordinates exactly as the corresponding operations

$\wedge, \vee, \neg$ in L. This fact together with (4) and the fact that θ is a congruence imply that g is a well-defined morphism. It is injective because $g((a,b)/\theta) = g((c,d)/\theta)$ if and only if $a = c$ if and only if $(a,b)/\theta = (c,d)/\theta$. And it is also onto, for if $a \in L$, then $(a, \neg a) \in \mathrm{K}(L)$ is such that $g((a,\neg a)/\theta) = a$. □

Lemma 2.10. *Let $T \in \mathsf{KAN}$. Then the assignment $\rho\colon T \to \mathbb{K}(T/\theta)$ given by $\rho(x) = (x/\theta, (\sim x)/\theta)$ is an injective morphism.*

Proof. Let $x \in T$. By (N6) we have that $\neg(x \wedge \sim x) = 1$, so ρ is a well-defined map. In what follows we will prove that ρ is a morphism. It is immediate that ρ preserves bottom and top elements, as well as $\wedge$, $\vee$ and $\sim$. To show that ρ preserves $\neg$, observe that $x/\theta = (\sim \neg x)/\theta$ by (N9). Then

$$\begin{aligned}\rho(\neg x) &= ((\neg x)/\theta, (\sim \neg x)/\theta)\\ &= (\neg x/\theta, x/\theta)\\ &= \neg(x/\theta, (\sim x)/\theta)\\ &= \neg\rho(x),\end{aligned}$$

so ρ is a morphism in KAN.

In order to prove that ρ is injective, let $(x/\theta, (\sim x)/\theta) = (y/\theta, (\sim y)/\theta)$. Then $\neg x = \neg y$ and $\neg \sim x = \neg \sim y$. It follows from Lemma 2.6 that $x = y$. Hence ρ is an injective function. □

Let $L \in \mathsf{PDL}$. Subalgebras T of $\mathbb{K}(L)$ that satisfy $g(T/\theta) = L$ (that is, $T/\theta \cong L$) will be of special interest.

2.2 A categorical equivalence for KAN

As proved in Section 1, congruences associated with Boolean filters are exactly the ones where the quotient is a Boolean algebra. In the same way that pairs of Heyting algebras and Boolean filters characterise Nelson algebras (details in [24] via Boolean congruences, for a version via Boolean filters see [4, Theorem 2.3]), we will show that pseudocomplemented distributive lattices and Boolean filters characterise KAN-algebras.

First for $L \in \mathsf{PDL}$ and F a Boolean filter of L, we define the set

$$\mathrm{K}(L,F) \coloneqq \{(a,b) \in L \times L : a \wedge b = 0 \text{ and } a \vee b \in F\}.$$

If $L' \in \mathsf{PDL}$, F' is a Boolean filter of L' and $f\colon L \to L'$ is a morphism in PDL such that $f(F) \subseteq F'$, we define the function $\mathbb{K}(f)\colon \mathbb{K}(L,F) \to \mathbb{K}(L',F')$ by $\mathbb{K}(f) = (f(a), f(b))$. This is a well-defined function, since for $(a,b) \in \mathrm{K}(L,F)$, $a \wedge b = 0$ implies that $f(a) \wedge f(b) = f(a \wedge b) = f(0) = 0$, and also $f(a) \vee f(b) = f(a \vee b) \in F'$ because $a \vee b \in F$ and $f(F) \subseteq F'$.

Theorem 2.11. *If $L \in \mathsf{PDL}$ and F is a Boolean filter of L, then the set $\mathrm{K}(L,F)$ is the universe of a subalgebra of $\mathbb{K}(L)$, which we will denote $\mathbb{K}(L,F)$, and it is therefore in KAN. Moreover, $\mathbb{K}(L,F)/\theta \cong L$.*

Additionally, if $L' \in \mathsf{PDL}$, F' is a Boolean filter of L' and $f\colon L \to L'$ is a morphism in PDL such that $f(F) \subseteq F'$, then $\mathbb{K}(f)$ is a morphism in KAN.

Proof. It is immediate that $\mathrm{K}(L,F)$ is closed under $\wedge$, $\vee$ and $\sim$. Also, as $\neg(a,b) = (\neg a, a)$, we only have to show that $a \vee \neg a \in F$, but this follows from the fact that $\neg(a \vee \neg a) = 0$ in PDL. Hence $\mathbb{K}(L,F)$ is a subalgebra of $\mathbb{K}(L) \in \mathsf{KAN}$.

To show that $\mathbb{K}(L,F)/\theta \cong L$, we will see that $g(\mathrm{K}(L,F)) = L$, where g is the map defined in Lemma 2.9. Actually, we will prove that the restriction of g to $\mathrm{K}(L,F)$ is surjective. Let $a \in L$. Since $a \vee \neg a \in F$, $(a, \neg a) \in \mathrm{K}(L,F)$, so $g((a,\neg a)/\theta) = a$, which was our aim.

Now let $f\colon L \to L'$ be a morphism in PDL. Clearly, bottom and top elements, $\wedge$, $\vee$ and $\sim$ are preserved by $\mathbb{K}(f)$. To see that it preserves $\neg$, we compute

$$\begin{aligned}\mathbb{K}(f)(\neg(a,b)) &= (f(\neg a), f(a)) \\ &= (\neg f(a), f(a)) \\ &= \neg(f(a), f(b)) \\ &= \neg\mathbb{K}(f)(a,b).\end{aligned}$$

Therefore $\mathbb{K}(f)$ is a morphism in KAN. □

We denote by PDLF the category whose objects are pairs (L,F), where $L \in \mathsf{PDL}$ and F is a Boolean filter of L, and whose arrows $f\colon (L,F) \to (L',F')$ are morphisms $f\colon L \to L'$ such that $f(F) \subseteq F'$. Straightforward computations show that we have a functor $\mathbb{K}\colon \mathsf{PDLF} \to \mathsf{KAN}$.

Remark 2.12. If L is a distributive lattice, I an ideal of L and F a filter of L, one obtains the Kleene algebra (see [28])

$$\mathrm{K}(L,I,F) := \{(a,b) \in L \times L : a \wedge b \in I \text{ and } a \vee b \in F\}$$

with the operations defined as before. As in the Nelson case, the addition of the operation $\neg(a, b) = (\neg a, a)$ will result in a KAN-algebra only if $I = \{0\}$. Indeed, if $i \in I$, then clearly $(i, 1) \in \mathrm{K}(L, I, F)$ and

$$(i, 1) = (i, 1) \wedge \sim (i, 1) = (i, 1) \wedge \neg(i, 1) = (i, 1) \wedge (\neg i, i) = (0, 1),$$

so $i = 0$.

Notice that we are using the notation $\mathbb{K}$ for two different purposes, but it will always be clear from the context.

We will now proceed to prove the converse of Theorem 2.11, that is, that every KAN-algebra T takes the form $\mathbb{K}(L, F)$ for some $L \in \mathsf{PDL}$ and some Boolean filter $F \subseteq L$. It is clear that L will be T/θ. First we need to identify the Boolean filter.

For $T \in \mathsf{KAN}$ we define its *positive* and *negative parts* respectively as

$$T^+ = \{x \in T : x \geq \sim x\} \qquad \text{and} \qquad T^- = \{x \in T : x \leq \sim x\}.$$

Lemma 2.13. *Let $T \in \mathsf{KAN}$ and $x \in T$. Then*

1) *$x \in T^+$ if and only if $\neg \sim x = 1$.*

2) *$x \in T^-$ if and only if $\neg x = 1$.*

Proof. By Lemma 2.6, observe that $x \geq \sim x$ if and only if $\sim \neg x \geq \sim \neg \sim x$ and $\neg \sim x \geq \neg \sim\sim x = \neg x$, which holds if and only if $\neg \sim x \geq \neg x$. It is thus clear that $x \geq \sim x$ when $\neg \sim x = 1$. Conversely, if $x \geq \sim x$, then $\neg \sim x = \neg(x \wedge \sim x) = 1$ by (N6). Hence part 1) holds. Part 2) can be deduced upon noting that $x \in T^-$ if and only if $\sim x \in T^+$. □

Lemma 2.14. *Let $T \in \mathsf{KAN}$. Then the positive part T^+ is a lattice filter of T that contains all elements $x \in T$ that satisfy $\neg\neg x = 1$. Consequently, T^+/θ is a Boolean filter of T/θ.*

Proof. First we will show that T^+ is a filter. Clearly $1 \in T^+$, and if $x \in T^+$ and $y \in T$ satisfies $y \geq x$, then $\sim y \leq \sim x \leq x \leq y$, so $y \in T^+$. Now, if $x, y \in T^+$, we have by Lemma 2.13 that $\neg \sim x = \neg \sim y = 1$. Hence

$$\neg \sim (x \wedge y) = \neg(\sim x \vee \sim y) = \neg \sim x \wedge \neg \sim y = 1.$$

Thus $x \wedge y \in T^+$.

We know from property (N8) that $\neg\neg x \leq \neg \sim x$. So if $\neg\neg x = 1$, we get that $\neg \sim x = 1$. Then $x \in T^+$.

It is now clear that T^+/θ is a filter in T/θ. The only part that is not immediate is that $y/\theta \geq (x\vee \sim x)/\theta$ implies $y/\theta \in T^+/\theta$, but this follows from the fact that $y/\theta = (y \vee x\vee \sim x)/\theta$.

Now, if $\neg x/\theta = 0/\theta$ in T/θ, then $\neg\neg x = 1$ and thus $x/\theta \in T^+/\theta$. Therefore T^+/θ is a Boolean filter. $\square$

Theorem 2.15. *Let $T \in \mathsf{KAN}$. Then $T \cong \mathbb{K}(T/\theta, T^+/\theta)$. Moreover, if $T' \in \mathsf{KAN}$ and $f\colon T \to T'$ is a morphism in KAN, then $\hat{f}\colon T/\theta \to T'/\theta$ given by $\hat{f}(x/\theta) = f(x)/\theta$ is a morphism in PDL such that $\hat{f}(T^+/\theta) \subseteq (T')^+/\theta$.*

Proof. Consider the assignment $\rho\colon T \to \mathbb{K}(T/\theta, T^+/\theta)$ given by $\rho(x) = (x/\theta, (\sim x)/\theta)$ (although we called ρ the map given in Lemma 2.10, we think it is clear the framework considered in each case). Since $\neg(x\wedge \sim x) = 1$ by (N6) and $x\vee \sim x \in T^+$, we have that $x/\theta \wedge (\sim x)/\theta = (x \wedge (\sim x))/\theta = 0/\theta \in T/\theta$ and $x/\theta \vee (\sim x)/\theta = (x \vee (\sim x))/\theta \in T^+/\theta$. Thus ρ is a well-defined map.

The fact that ρ is an injective morphism in KAN follows from Lemma 2.10. Before checking surjectivity, observe that $x \in T^+$ if and only if $\neg \sim x \geq \neg x$, if and only if $\neg \sim x = 1$, if and only if $(\sim x)/\theta = 0/\theta$ and recall that $x/\theta \wedge y/\theta = 0/\theta$ if and only if $y/\theta \leq \neg x/\theta$, as $T/\theta \in \mathsf{PDL}$. Now, given $(x/\theta, y/\theta) \in \mathbb{K}(T/\theta, T^+/\theta)$, we have that $x/\theta \wedge y/\theta = 0/\theta$ and $x/\theta \vee y/\theta \in T^+/\theta$, so that $y/\theta \leq \neg x/\theta$ and $(x \vee y)/\theta = z/\theta$ with $(\sim z)/\theta = 0/\theta$. Hence, since $(\neg x/\theta, (\sim \neg x)/\theta))$ and $(z/\theta, (\sim z)/\theta)$ belong to $\rho(T)$ and $\rho(T)$ is a subalgebra of $\mathbb{K}(T/\theta) = (\mathrm{K}(T/\theta), \wedge, \vee, \sim, \neg, 0/\theta, 1/\theta)$, from

$$\begin{aligned}
\sim (\neg x/\theta, (\sim \neg x)/\theta) \vee \sim (z/\theta, (\sim z)/\theta) &= ((\sim \neg x)/\theta, \neg x/\theta) \vee ((\sim z)/\theta, z/\theta) \\
&= (x/\theta, \neg x/\theta) \vee (0/\theta, (x \vee y)/\theta) \\
&= (x/\theta, (\neg x \wedge (x \vee y))/\theta) \\
&= (x/\theta, (\neg x \wedge y))/\theta) \\
&= (x/\theta, y/\theta)
\end{aligned}$$

we deduce that $(x/\theta, y/\theta) \in \rho(T)$.

Finally, if $T' \in \mathsf{KAN}$ and $f\colon T \to T'$ is a morphism, then $\hat{f}\colon x/\theta \mapsto f(x)/\theta$ is clearly a pseudocomplemented lattice morphism, for θ is a congruence with respect to $\wedge$, $\vee$ and $\neg$. If $x \in T^+$, then $\neg \sim x = 1$, so $\neg \sim f(x) = 1 \in (T')^+$. Hence $\hat{f}(x/\theta) \in (T')^+/\theta$, which completes this proof. □

By previous results we have that if $T \in \mathsf{KAN}$, then $(T/\theta, T^+/\theta) \in \mathsf{PDLF}$. Also, if $f\colon T \to T'$ is a morphism in KAN, then $\hat{f}$ is a morphism in PDLF. Therefore it is immediate that the previous assignments define a functor from KAN to PDLF.

Theorem 2.16. *The functor* $\mathbb{K}\colon \mathsf{PDLF} \to \mathsf{KAN}$ *defines a categorical equivalence.*

Proof. Theorem 2.11 shows that $\mathbb{K}$ is well defined. It is also immediate that it is faithful. By Theorem 2.15 it is full and dense. Therefore $\mathbb{K}$ is an equivalence. □

2.3 Comparison with wp-Kleene algebras

In this subsection we prove that $\mathsf{KAN} = \mathcal{K}_\omega$, where $\mathcal{K}_\omega$ is the variety (and the algebraic category) of wp-Kleene algebras (see Section 1 and [25]).

Recall from Section 1 that a Kleene algebra equipped with an additional operator $\neg$ is in $\mathcal{K}_\omega$ if it satisfies the following properties:

$$y \leq \neg x \text{ if and only if } x \wedge y \leq \sim x,$$
$$\neg(x \wedge y) = 1 \text{ if and only if } \neg\neg x \leq \neg y.$$

If $L \in \mathsf{PDL}$ and R is a Boolean congruence on L, we define

$$\mathsf{KA}_R(L) := \{(a,b) \in \mathrm{K}(L) : (a \vee b, 1) \in R\}.$$

Now we prove the following result.

Theorem 2.17. $\mathsf{KAN} = \mathcal{K}_\omega$.

Proof. Let $T \in \mathcal{K}_\omega$. It follows from [25, Theorem 4.4] that there exist $L \in \mathsf{PDL}$ and a Boolean congruence R on L such that $T \cong \mathsf{KA}_R(L)$. Given that $1/R$ is a Boolean filter by Lemma 1.2, $\mathsf{KA}_R(L) = \mathrm{K}(L, 1/R) \in \mathsf{KAN}$.

Then $T \in \mathsf{KAN}$ and thus $\mathcal{K}_\omega \subseteq \mathsf{KAN}$. Conversely, let $T \in \mathsf{KAN}$, so it follows from Theorem 2.16 that there exist $L \in \mathsf{BDL}$ and a Boolean filter F such that $T \cong \mathrm{K}(L, F)$. It will be sufficient to prove that $\mathrm{K}(L, F) \in \mathcal{K}_\omega$. By Lemma 1.3 we have that $H(F)$ is a Boolean congruence. Since $F = 1/H(F)$, $T \cong \mathrm{K}(L, F) = \mathsf{KA}_{H(F)}(L) \in \mathcal{K}_\omega$, as desired. Then $\mathsf{KAN} \subseteq \mathcal{K}_\omega$, so $\mathsf{KAN} = \mathcal{K}_\omega$. □

We define the category $B_\omega^{Con_0}$ introduced in [25, Section 4]: the objects are pairs (L, R), where $L \in \mathsf{PDL}$ and R is a Boolean congruence on L. The morphisms are maps $f : (L, R) \to (M, S)$ such that $f : L \to M$ is a morphism in PDL and $R \subseteq f^{-1}(S)$ (which means that if $(x, y) \in R$, then $(f(x), f(y)) \in S$).

Remark 2.18. Straightforward computations based on Lemma 1.3 show that $B_\omega^{Con_0}$ and PDLF are isomorphic categories.

In [25, Theorem 4.4], it was proved that there exists a categorical equivalence between $\mathcal{K}_\omega$ and $B_\omega^{Con_0}$. In Theorem 2.16 we prove an analogous result with $\mathcal{K}_\omega$ replaced by KAN (both classes turn out to be equal by Theorem 2.17) and $B_\omega^{Con_0}$ by PDLF (which are isomorphic categories by Remark 2.18), by generalising the more natural relation between Heyting and Nelson algebras from [24]. In conclusion, while Theorem 2.16 can be seen as a variant of [25, Theorem 4.4] by changing Boolean congruences for Boolean filters, it also follows more naturally from the definition of the variety KAN, given that the aim of both papers (the present paper and [25]) differ.

2.4 Remarks about the relation between the strong and intuitionistic negation

The unary operation $\neg$ in a given KAN-algebra is a *weak pseudocomplement* by Sendlewski's condition

$$b \leq \neg a \text{ if and only if } a \wedge b \leq \sim a.$$

This already gives a partial answer to our question about the relationship between the negations $\neg$ and $\sim$. Properties (N1)-(N9) together with Lemma 2.6 give a more detailed answer. Moreover, Theorem 5.6 will show that while $\sim$ is an involution, the operator $\neg$ cannot have the extra properties one may expect from a negation without becoming trivial (in the sense that it coincides with $\sim$).

3. The variety $\mathsf{KAN}_{\mathrm{c}}$

Inspired by results due to J. Kalman related to lattices [15], R. Cignoli proved in [8] that the construction of J. Kalman can be extended to a functor K from the category of bounded distributive lattices to the category of Kleene algebras and this functor has a left adjoint [8, Theorem 1.7]. He also showed that there exists an equivalence between the category of bounded distributive lattices and the full subcategory of centered Kleene algebras whose objects satisfy a condition called interpolation property [8, Theorem 2.4]. Moreover, he also proved the fact that there exists an equivalence between the category of Heyting algebras and the category of centered Nelson algebras [8, Theorem 3.14].

Let $\mathsf{KAN}_{\mathrm{c}}$ be the variety whose members are the algebras $(T, \wedge, \vee, \sim, \neg, \mathrm{c}, 0, 1)$ of type $(2, 2, 1, 1, 0, 0, 0)$ such that $(T, \wedge, \vee, \sim, \neg, 0, 1)$ is a KAN-algebra and $\sim \mathrm{c} = \mathrm{c}$. We also write $\mathcal{K}_\omega^{\mathrm{c}}$ for the variety whose members are the algebras $(T, \wedge, \vee, \sim, \neg, 0, 1)$ of type $(2, 2, 1, 0, 0, 0)$ such that $(T, \wedge, \vee, \sim, \neg, 0, 1) \in \mathcal{K}_\omega$ and $\sim \mathrm{c} = \mathrm{c}$. It follows from Theorem 2.17 that $\mathcal{K}_\omega^{\mathrm{c}} = \mathsf{KAN}_{\mathrm{c}}$.

In this section we study properties of $\mathsf{KAN}_{\mathrm{c}}$ and we prove that PDL and $\mathsf{KAN}_{\mathrm{c}}$ are equivalent categories following an alternative path to that given in [25, Corollary 4.6].

3.1 Centered KAN-algebras

Recall from Section 1 that KA_{c} denotes the variety of centered Kleene algebras. We will recall the categorical equivalence between BDL and KA_{c} to obtain an equivalence between PDL and the category of centered KAN-algebras.

Let $(T, \wedge, \vee, \sim, \mathrm{c}, 0, 1) \in \mathsf{KA}_{\mathrm{c}}$. The set

$$\mathrm{C}(T) := \{x \in T : x \geq \mathrm{c}\}$$

is the universe of a subalgebra of $(T, \wedge, \vee, \mathrm{c}, 1)$ and $(\mathrm{C}(T), \wedge, \vee, \mathrm{c}, 1) \in \mathsf{BDL}$. For a morphism $g \colon T \to U \in \mathsf{KA}_{\mathrm{c}}$, the map $\mathrm{C}(g) \colon \mathrm{C}(T) \to \mathrm{C}(U)$, given by $\mathrm{C}(g)(x) = g(x)$, is a morphism in BDL. Therefore, we can define a functor C from KA_{c} to BDL.

Let $L \in \mathsf{BDL}$. The map $\alpha_L \colon L \to \mathrm{C}(\mathrm{K}(L))$ given by $\alpha_L(a) = (a, 0)$ is an isomorphism in BDL. If $T \in \mathsf{KA}_{\mathrm{c}}$, the map $\beta_T \colon T \to \mathrm{K}(\mathrm{C}(T))$ given by $\beta_T(x) = (x \vee \mathrm{c}, {\sim}x \vee \mathrm{c})$ is an injective morphism in KA_{c} which is surjective if and only if

$$\text{For } x, y \geq c, \text{ if } x \wedge y = \mathrm{c}, \text{ there exists } z \text{ such that } z \vee \mathrm{c} = x \text{ and } {\sim}z \vee \mathrm{c} = y \tag{CK}$$

(see [14]). Condition (CK) is not necessarily verified in every centered Kleene algebra (see for instance [5, Figure 1]). The functor K can then be seen as a functor from BDL to the full subcategory of KA_{c} whose objects satisfy (CK).

The following result is [5, Theorem 2.7] (see also [8, Theorem 2.4]).

Theorem 3.1. *The functors* K *and* C *establish a categorical equivalence between* BDL *and the full subcategory of* KA_{c} *whose objects satisfy* (CK)*, with natural isomorphisms* α *and* β*.*

As a reformulation of [8, Theorem 3.14] (see also [7]) we also have that the functors K and C establish a categorical equivalence between the category of Heyting algebras and the category of centered Nelson algebras with natural isomorphisms α and β.

It is easy to see that an algebra $(T, \wedge, \vee, \sim, \neg, \mathrm{c}, 0, 1)$ of type $(2, 2, 1, 1, 0, 0, 0)$ is in $\mathsf{KAN}_{\mathrm{c}}$ if and only if $(T, \wedge, \vee, \sim, \mathrm{c}, 0, 1) \in \mathsf{KA}_{\mathrm{c}}$ and $(T, \wedge, \vee, \sim, \neg, 0, 1)$ is a KAN-algebra. If $L \in \mathsf{PDL}$, we write $\mathbb{K}_c(L)$ for the algebra $\mathbb{K}(L)$ with center $(0, 0)$ in its signature. If $f \colon L \to L'$ is a morphism in PDL, we write $\mathbb{K}_c(f) \colon \mathbb{K}_c(L) \to \mathbb{K}_c(L')$ given by $\mathbb{K}_c(f) = \mathrm{K}(f)$. We write $\mathbb{K}_c$ for the functor from PDL to $\mathsf{KAN}_{\mathrm{c}}$.

Lemma 3.2. *Let* $T \in \mathsf{KAN}_{\mathrm{c}}$*. Then* $\neg \mathrm{c} = 1$*.*

Proof. Since $\sim \mathrm{c} = \mathrm{c}$, $\mathrm{c} \in T^-$. By Lemma 2.13 we have $\neg \mathrm{c} = 1$. □

Given $T \in \mathsf{KAN}$, the positive part T^+ is a sublattice which is upper bounded. In case $T \in \mathsf{KAN}_{\mathrm{c}}$, it can be proved the sets T^+ and $\{x \in T : x \geq \mathrm{c}\}$ coincide. Thus we will also write $\mathrm{C}(T)$ to denote the algebra $(T^+, \wedge, \vee, \neg_{\mathrm{c}}, \mathrm{c}, 1)$, where $\neg_c \colon T^+ \to T^+$ is defined by

$$\neg_{\mathrm{c}} x = \neg x \vee c.$$

Lemma 3.3. *If* $T \in \mathsf{KAN}_{\mathrm{c}}$*, then* $\mathrm{C}(T) \in \mathsf{PDL}$*.*

Proof. We only need to show that for $x, y \geq \mathrm{c}$,

$$x \wedge y = \mathrm{c} \text{ if and only if } x \leq \neg_{\mathrm{c}} y.$$

To do this, note first that if $w, z \in T^+$, then $w \leq z$ if and only if $\neg \sim w \leq \neg \sim z$ and $\sim \neg w \leq \sim \neg z$ by Lemma 2.6. But since the first one becomes trivial by Lemma 2.13, $w \leq z$ if and only if $\neg z \leq \neg w$.

Now, from Lemma 3.2 and the proof of Lemma 2.8,

$$x \wedge y = \mathrm{c} \text{ if and only if } \neg(x \wedge y) = \neg c = 1 = \neg 0 \text{ if and only if } \neg\neg y \leq \neg x.$$

The last inequality can be rewritten as $\neg(\neg y \vee \mathrm{c}) \leq \neg x$ and it is equivalent to

$$x \leq \neg y \vee \mathrm{c} = \neg_{\mathrm{c}} y$$

by our initial observation. □

The functor $\mathrm{C} \colon \mathsf{KA}_{\mathrm{c}} \to \mathsf{BDL}$ can be extended to a functor $\mathrm{C} \colon \mathsf{KAN}_{\mathrm{c}} \to \mathsf{PDL}$.

Note that if $T \in \mathsf{KAN}_{\mathrm{c}}$, then $\mathbb{K}_c(\mathrm{C}(T)) \in \mathsf{KAN}_{\mathrm{c}}$ with center (c, c).

Corollary 3.4. *Let* $T \in \mathsf{KAN}_{\mathrm{c}}$*. Then* $\beta_T \colon T \to \mathbb{K}_c(\mathrm{C}(T))$ *is an isomorphism.*

Proof. Recall that θ is the equivalence relation in T defined by (3), which is also a congruence with respect to $\wedge$, $\vee$ and $\neg$. Observe that $\mathrm{K}(T/\theta, T/\theta) = \mathbb{K}(T/\theta)$. It follows from the proof of Theorem 2.15 that the map $\rho \colon T \to \mathbb{K}(T/\theta)$ given by $\rho(x) = (x/\theta, (\sim x)/\theta)$ is an isomorphism in KAN, so it is also an isomorphism in $\mathsf{KAN}_{\mathrm{c}}$.

Let $f_\theta \colon \mathrm{C}(T) \to T/\theta$ be given by $f_\theta(x) = x/\theta$. We are going to show that f_θ is an isomorphism in PDL. Actually, we only need to show that f_θ is a bijection because f_θ is a morphism in PDL due to Lemma 2.7. If $f_\theta(x) = f_\theta(y)$ (i.e. $x/\theta = y/\theta$), then $\neg x = \neg y$, so $\sim \neg x = \sim \neg y$. Besides we have that $x, y \geq \mathrm{c}$. Then $\sim x, \sim y \leq \mathrm{c}$ and thus $\neg \sim x, \neg \sim y \geq \neg \mathrm{c}$. Then $\neg \sim x = \neg \sim y$, as $\neg \mathrm{c} = 1$ by Lemma 3.2. Hence $x = y$ by Lemma 2.6, so f_θ is injective. Now consider $x \in T$. Since $\neg \mathrm{c} = 1$, $f_\theta(x \vee \mathrm{c}) = x/\theta$, so f_θ is surjective. Then f_θ is an isomorphism in PDL, which implies that $\mathrm{K}(f_\theta) \colon \mathrm{K}(\mathrm{C}(T)) \to \mathrm{K}(T/\theta)$ is an isomorphism in KAN, so it is an isomorphism in $\mathsf{KAN_c}$. Also note that $\mathrm{K}(f_\theta)(x,y) = (x/\theta, y/\theta)$. Then the map $\gamma\colon \mathbb{K}_\mathrm{c}(T/\theta) \to \mathbb{K}_\mathrm{c}(\mathrm{C}(T))$ given by $\gamma(x/\theta, y/\theta) = (f_\theta^{-1}(x/\theta), f_\theta^{-1}(y/\theta))$ is an isomorphism in $\mathsf{KAN_c}$.

As consequence, the map $\gamma \circ \rho\colon T \to \mathrm{C}(\mathbb{K}_\mathrm{c}(T))$ is an isomorphism in $\mathsf{KAN_c}$. It is immediate that $\beta_T = \gamma \circ \rho$. Therefore β_T is an isomorphism in $\mathsf{KAN_c}$. □

Remark 3.5. Let $T \in \mathsf{KAN_c}$. In what follows we will show that there is an alternative proof of the fact that the map β_T is an isomorphism in $\mathsf{KAN_c}$. In order to see it, note that since β_T is a monomorphism of centered Kleene algebras, it suffices to show that β_T preserves the operation $\neg$ and that it is a surjective map (recall that β_T is a surjective map if and only if it satisfies condition (CK)). Throughout the proof we will use the equality $\neg\mathrm{c} = 1$ (see Lemma 3.2).

It is immediate that the map β_T preserves the operation $\neg$ if and only if for every $x \in T$ it holds the equality $\sim \neg x \vee \mathrm{c} = x \vee \mathrm{c}$. It follows from Lemma 2.6 that the previous equality is equivalent to the following two: $\neg \sim (\sim \neg x \vee \mathrm{c}) = \neg(\sim x \vee \mathrm{c})$ and $\sim \neg(\sim \neg x \vee \mathrm{c}) =\sim \neg(x \vee \mathrm{c})$. The first equality can be verified by using (N5) and that $\neg c = 1$. The second equation follows from (N9) and $\neg\mathrm{c} = 1$. So β_T preserves the operation $\neg$.

Finally, in order to show that β_T is a surjective map, we will see that condition (CK) is satisfied in T. Let $x, y \in T$ be such that $x \wedge y = \mathrm{c}$. Define the element $z = x \wedge \neg y$. Straightforward computations show that $z \vee \mathrm{c} = x$. Now we need to prove that $z \wedge \mathrm{c} =\sim y$, which is equivalent to showing that $c \wedge \neg y =\sim y$. From Lemma 2.6 this last equality is equivalent to $\neg \sim (\mathrm{c} \wedge \neg y) = \neg \sim\sim y$ and $\sim \neg(\mathrm{c} \wedge \neg y) =\sim \neg \sim y$. A direct computation based on (N2), (N9) and the fact that $\neg\mathrm{c} = 1$ shows that $\neg \sim (\mathrm{c} \wedge \neg y) = \neg \sim\sim y$ is satisfied. The validity of $\sim \neg(\mathrm{c} \wedge \neg y) =\sim \neg \sim y$ is a straightforward consequence of (N5) and $\neg\mathrm{c} = 1$. Therefore β_T is an isomorphism in $\mathsf{KAN_c}$.

It is easily seen that if $L \in \mathsf{PDL}$, then the map $\alpha_L\colon L \to \mathrm{C}(\mathbb{K}_c(L))$, which is an isomorphism in BDL, is also an isomorphism in PDL. Taking into account the results of this section and the categorical equivalence stated in Theorem 3.1, we obtain the following theorem.

Theorem 3.6. *The functors $\mathbb{K}_c$ and* C *establish a categorical equivalence between* PDL *and* $\mathsf{KAN_c}$ *with natural isomorphisms α and β.*

From this result, it is natural to ask what is the relation between the constructions of the categorical equivalences given in theorems 2.16 and 3.6. In what follows we study this relation.

Let T be in $\mathsf{KAN}_{\mathrm{c}}$. With an abuse of notation we also denote by T the c-free reduct of T, which is a KAN-algebra. Then we have two pseudocomplemented lattices that are obtained from T: $C(T)$ as previously defined and T/θ from Section 2. The following result shows that both pseudocomplemented distributive lattices are isomorphic.

Theorem 3.7. *Let $T \in \mathsf{KAN}_{\mathrm{c}}$. Then*

1) $T/\theta = T^{+}/\theta$,

2) $T/\theta \cong \mathrm{C}(T)$.

Proof. Claim 1) will follow after we prove that $T/\theta \subseteq T^{+}/\theta$. Let $x \in T$. Then $x \vee \mathrm{c} \in T^{+}$ and thus $\neg x = \neg(x \vee \mathrm{c})$ by Lemma 3.2. Hence $x/\theta = (x \vee \mathrm{c})/\theta$, so $T/\theta \subseteq T^{+}/\theta$. Claim 2) is in the proof of Corollary 3.4. □

Theorem 3.8. *Let T be a KAN-algebra and θ the congruence defined in (3). Assume that $T/\theta = T^{+}/\theta$ (as sets). Then there is an element $c \in T$ such that $(T, \wedge, \vee, \sim, \neg, c, 0, 1)$ is in $\mathsf{KAN}_{\mathrm{c}}$.*

Proof. It is enough to note that under the assumption $T/\theta = T^{+}/\theta$ we have $T \cong \mathrm{K}(T/\theta, T/\theta)$. If 0 denotes the class $0/\theta \in T^{+}/\theta = T/\theta$, then $(0,0) \in \mathrm{K}(T/\theta, T/\theta)$ and it is immediate that $(0,0)$ is a center. Thus T is isomorphic to the c-free reduct of the centered KAN-algebra $(\mathrm{K}(T/\theta, T/\theta), \wedge, \vee, \sim, \neg, (0,0), (0,1), (1,0))$ and the theorem follows. □

3.2 Congruences for algebras of $\mathsf{KAN}_{\mathrm{c}}$

We will write $\mathrm{Con}(A)$ for the lattice of congruences on an algebra A. Given $L \in \mathsf{PDL}$, we will prove that there is an order isomorphism between $\mathrm{Con}(L)$ and $\mathrm{Con}(\mathbb{K}_c(L))$. Similarly, we will show that if $T \in \mathsf{KAN}_{\mathrm{c}}$, then $\mathrm{Con}(T)$ and $\mathrm{Con}(\mathrm{C}(T))$ are order isomorphic.

If $L \in \mathsf{BDL}$ and R is a congruence on L, then the relation S_R defined as

$$((a,b),(d,e)) \in S_R \text{ if and only if } (a,d) \in R \text{ and } (b,e) \in R$$

is a congruence on $\mathrm{K}(L)$. If S is a congruence on $\mathrm{K}(L)$, then the relation R^S defined as

$$(a,b) \in R^S \text{ if and only if } ((a,0),(b,0)) \in S$$

is a congruence on L. In [5] (see also [7, Remark 7.12]), it was proved that if $L \in \mathsf{BDL}$, the assignments $R \mapsto S_R$ and $S \mapsto R^S$ define an order isomorphism between $\mathrm{Con}(L)$ and $\mathrm{Con}(\mathbb{K}_c(L))$. This result can be extended to the framework of pseudocomplemented distributive lattices.

Proposition 3.9. *Let $L \in \mathsf{PDL}$. The assignments $R \mapsto S_R$ and $S \mapsto R^S$ define an order isomorphism between $\mathrm{Con}(L)$ and $\mathrm{Con}(\mathbb{K}_c(L))$.*

Proof. We only need to prove that if $R \in \mathrm{Con}(L)$ and $S \in \mathrm{Con}(\mathbb{K}_c(L))$, then $S_R \in \mathrm{Con}(\mathbb{K}_c(L))$ and $R^S \in \mathrm{Con}(L)$.

If $R \in \mathrm{Con}(L)$ and $((a,b),(d,e)) \in S_R$, then $(a,d) \in R$ and $(b,e) \in R$. In particular, since $(\neg a, \neg d) \in R$, $((\neg a, a),(\neg d, d)) \in S_R$, which means that $(\neg(a,b), \neg(d,e)) \in S_R$. Thus $S_R \in \mathrm{Con}(\mathbb{K}_c(L))$.

Conversely, if $S \in \mathrm{Con}(\mathbb{K}_c(L))$ and $(a,b) \in R^S$, then $((a,0),(b,0)) \in S$. In particular, $(\neg(a,0), \neg(b,0)) = ((\neg a, a),(\neg b, b)) = ((\neg a, a) \vee (0,0), (\neg b, b) \vee (0,0)) = ((\neg a, 0),(\neg b, 0)) \in S$; that is, $(\neg a, \neg b) \in R^S$. Therefore $R^S \in \mathrm{Con}(L)$. □

For $T \in \mathsf{KAN_c}$ and $R \in \mathrm{Con}(T)$ we define the binary relation $\Gamma(R)$ as the restriction to $\mathrm{C}(T) \times \mathrm{C}(T)$ of R. For $S \in \mathrm{Con}(\mathrm{C}(T))$ we define the relation $\Sigma(S) \subseteq T \times T$ by

$$(x,y) \in \Sigma(S) \text{ if and only if } (x \vee \mathrm{c}, y \vee \mathrm{c}) \in S \text{ and } (\sim x \vee \mathrm{c}, \sim y \vee \mathrm{c}) \in S.$$

Proposition 3.10. *Let $T \in \mathsf{KAN_c}$. The assignments $R \mapsto \Gamma(R)$ and $S \mapsto \Sigma(S)$ define an order isomorphism between $\mathrm{Con}(T)$ and $\mathrm{Con}(\mathrm{C}(T))$.*

Proof. It is immediate that $\Gamma(R) \in \mathrm{Con}(\mathrm{C}(T))$ whenever $R \in \mathrm{Con}(T)$. Conversely, let $S \in \mathrm{Con}(\mathrm{C}(T))$. Operations $\wedge, \vee$ and $\sim$ are easily seen to be preserved by $\Sigma(S)$. We only need to prove that $\Sigma(S)$ is a congruence with respect to $\neg$. Let $(x,y) \in \Sigma(S)$, so that $(x \vee \mathrm{c}, y \vee \mathrm{c}) \in S$ and $(\sim x \vee \mathrm{c},$

$\sim y \vee \mathrm{c}) \in S$. In particular, $(\neg_{\mathrm{c}}(x \vee \mathrm{c}), \neg_{\mathrm{c}}(y \vee \mathrm{c})) = (\neg(x \vee \mathrm{c}) \vee \mathrm{c}, \neg(y \vee \mathrm{c}) \vee \mathrm{c}) \in S$. Since $\neg(x \vee \mathrm{c}) = \neg x$ and $\neg(y \vee \mathrm{c}) = \neg y$ by Lemma 3.2,

$$(\neg x \vee \mathrm{c}, \neg y \vee \mathrm{c}) \in S. \tag{5}$$

On the other hand, it follows from Theorem 3.6 that $T \cong \mathbb{K}_c(\mathrm{C}(T))$. Since the equation $z \vee \mathrm{c} = \sim (\neg z) \vee \mathrm{c}$ holds in $\mathbb{K}_c(\mathrm{C}(T))$, it holds in T. Thus

$$(\sim (\neg x) \vee \mathrm{c}, \sim (\neg y) \vee \mathrm{c}) = (x \vee \mathrm{c}, y \vee \mathrm{c}) \in S. \tag{6}$$

Hence $(\neg x, \neg y) \in \Sigma(S)$ by (5) and (6), meaning that $\Sigma(S) \in \mathrm{Con}(T)$. The rest of the proof follows from straightforward computations. □

4. Quantifiers

In this section we define quantifiers on PDL based on [9]. We also introduce a notion of quantifier on KAN, and we study the relation between quantifiers on L (for $L \in \mathsf{PDL}$) and quantifiers on $\mathbb{K}(L)$.

Since Halmos [13], the notion of quantifier has been generalised in many ways. For example, the definition of a Q-distributive lattice is given in [9] and in [18] a quantifier for De Morgan algebras is defined.

The following is [9, Definition 1.1].

Definition 4.1. Let $L \in \mathsf{BDL}$ and $\nabla \colon L \to L$ a unary map. We say that ∇ is a quantifier if it satisfies the following conditions for every $a, b \in L$:

(Q1) $\nabla 0 = 0$,

(Q2) $a \leq \nabla a$,

(Q3) $\nabla(a \wedge \nabla b) = \nabla a \wedge \nabla b$,

(Q4) $\nabla(a \vee b) = \nabla a \vee \nabla b$.

Conditions (Q1)-(Q4) were first introduced by Halmos [13] as an algebraic counterpart of the logical notion of an existential quantifier.

We have that $\nabla 1 = 1$, $\nabla\nabla a = \nabla a$ and $a \in \nabla(L)$ if and only if $a = \nabla a$. We also have that $\nabla(L)$ is a subalgebra of L. Moreover, if the pseudocomplement $\neg a$ exists and $a \in \nabla(L)$, then $\neg a = \nabla \neg a$.

The notion of quantifier on Heyting algebras is defined by some authors as in the case of bounded distributive lattices (see for instance [16]). Motivated by this fact we define a quantifier on pseudocomplemented distributive lattices as in the case of bounded distributive lattices. It is interesting to note that if $L \in \mathsf{PDL}$, then $\nabla(L)$ is a subalgebra of L (it can be deduced from the proof of [9, Proposition 1.2 (iii)]). The variety whose algebras are pseudocomplemented distributive lattices endowed with a quantifier will be called monadic pseudocomplemented distributive lattices (in a reference to Halmos' monadic Boolean algebras [13]) and denoted by MPDL.

Definition 4.2. Let $T \in \mathsf{KAN}$ and $\Omega\colon T \to T$ a unary map. We say that Ω is a *weak quantifier* if the following conditions are satisfied for every $x, y \in T$:

(W1) $\Omega 0 = 0$,

(W2) $\neg \Omega x \leq \neg x$,

(W3) $\neg \Omega(x \wedge \Omega y) = \neg(\Omega x \wedge \Omega y)$,

(W4) $\Omega(x \vee y) = \Omega x \vee \Omega y$,

(W5) $\neg \Omega x = \sim \Omega x$,

(W6) $\Omega x = \Omega \sim \neg x$.

WMKAN will stand for the variety of weakly-monadic KAN-algebras, i.e. KAN-algebras endowed with a weak quantifier.

The following lemma is straightforward from the definition.

Lemma 4.3. *If* $T \in \mathsf{KAN}$ *and* Ω *is a weak quantifier, then*

(W7) $\Omega 1 = 1$,

(W8) $x \leq y$ *implies* $\Omega x \leq \Omega y$,

(W9) $\Omega\Omega x = \Omega x$,

(W10) $x \in T^{-}$ *implies* $\Omega x = 0$.

For $L \in \mathsf{PDL}$ and a quantifier ∇ on L, we define $\Omega_\nabla \colon K(L) \to K(L)$ as

$$\Omega_\nabla(a, b) = (\nabla a, \neg\nabla a).$$

Notice that Ω_∇ is well defined because $\nabla a \wedge \neg\nabla a = 0$ for every $a \in L$. Moreover, if F is a Boolean filter on L, then $\nabla a \vee \neg\nabla a \in F$, so Ω_∇ is closed in the subalgebra $\mathrm{K}(L, F)$.

We write MPDLF for the category whose objects are pairs formed by an algebra $(L, \wedge, \vee, \neg, 0, \nabla) \in \mathsf{MPDL}$ and a Boolean filter F on L. The morphisms of MPDLF are the morphisms of PDLF which preserve the respective quantifiers.

Proposition 4.4. *Let $L \in \mathsf{PDL}$ and let ∇ be a quantifier on L. Then Ω_∇ is a weak quantifier on $\mathbb{K}(L)$ (and therefore on $\mathbb{K}(L, F)$ for every Boolean filter $F \subseteq L$). Moreover, $\mathbb{K}$ extends to a functor from* MPDLF *to* WMKAN.

Proof. First we prove that Ω_∇ is a weak quantifier. Let $(a, b), (d, e) \in \mathrm{K}(L)$.

(W1) $\Omega_\nabla(0, 1) = (\nabla 0, \neg\nabla 0) = (0, 1)$.

(W2) $\neg\Omega_\nabla(a, b) = \neg(\nabla a, \neg\nabla a) = (\neg\nabla a, \nabla a) \leq (\neg a, a) = \neg(a, b)$.

(W3)
$$\begin{aligned} \neg\Omega_\nabla((a, b) \wedge \Omega_\nabla(d, e)) &= \neg\Omega_\nabla(a \wedge \nabla d, b \vee \neg\nabla d) \\ &= \neg(\nabla(a \wedge \nabla d), \neg\nabla(a \wedge \nabla d)) \\ &= \neg(\nabla a \wedge \nabla d, \neg\nabla(a \wedge \nabla d)) \\ &= \neg((\nabla a, \neg\nabla a) \wedge (\nabla d, \neg\nabla d)) \\ &= \neg(\Omega_\nabla(a, b) \wedge \Omega_\nabla(d, e)). \end{aligned}$$

(W4)
$$\begin{aligned} \Omega_\nabla((a, b) \vee (d, e)) &= \Omega_\nabla(a \vee d, b \wedge e) \\ &= (\nabla(a \vee d), \neg\nabla(a \vee d)) \\ &= (\nabla a \vee \nabla d, \neg\nabla a \wedge \neg\nabla d) \\ &= \Omega_\nabla(a, b) \vee \Omega_\nabla(d, e). \end{aligned}$$

(W5) $\neg\Omega_\nabla(a, b) = \neg(\nabla a, \neg\nabla a) = (\neg\nabla a, \nabla a) =\sim \Omega_\nabla(a, b)$.

(W6) $\Omega_\nabla(a, b) = (\nabla a, \neg\nabla a) = \Omega_\nabla(a, \neg a) = \Omega_\nabla \sim \neg(a, b)$.

Let $f\colon (L,\nabla,F)\to(L',\nabla',F')$ be a morphism in MPDLF. Let $a,b\in \mathrm{K}(L)$. Then

$$\begin{aligned}\mathbb{K}(f)(\Omega_\nabla(a,b)) &= \mathbb{K}(f)(\nabla a,\neg\nabla a)\\ &= (f(\nabla a), f(\neg\nabla a))\\ &= (\nabla' f(a),\neg\nabla' f(a))\\ &= \Omega_{\nabla'}(f(a),f(b))\\ &= \Omega_{\nabla'}(\mathbb{K}(f)(a,b)),\end{aligned}$$

so $\mathbb{K}$ is functorial. ☐

On the other hand, given $T\in\mathsf{KAN}$ and a weak quantifier Ω on T, we try to recover a quantifier on T/θ as

$$\nabla_\Omega(x/\theta)=(\Omega x)/\theta.$$

Proposition 4.5. *Given $T\in\mathsf{KAN}$ and Ω a weak quantifier on T, ∇_Ω is a well-defined operator in T/θ and it is a quantifier. Moreover, if $f\colon(T,\Omega)\to(T',\Omega')$ is a morphism in WMKAN, then the morphism in PDLF $\hat f\colon T/\theta\to T'/\theta$ given by $\hat f(x/\theta)=f(x)/\theta$ sends ∇_Ω into $\nabla_{\Omega'}$.*

Proof. Recall that $x/\theta=y/\theta$ if and only if $\neg x=\neg y$. Therefore, $\sim\neg x=\sim\neg y$ and by (W6) we have $\Omega x=\Omega y$, so $\nabla_\Omega(x/\theta)=\nabla_\Omega(y/\theta)$. Thus, ∇_Ω is well defined. By (W1)-(W4), it is clear that ∇_Ω is a quantifier. Finally,

$$\begin{aligned}\hat f(\nabla_\Omega(x/\theta)) &= \hat f((\Omega x)/\theta)\\ &= f(\Omega x)/\theta\\ &= (\Omega' f(x))/\theta\\ &= \nabla_{\Omega'}((f(x))/\theta)\\ &= \nabla_{\Omega'}\hat f(x/\theta).\end{aligned}$$

☐

The maps Ω_∇ and ∇_Ω behave as inverses in the sense that if $(L,\nabla,F)\in\mathsf{MPDLF}$, then ∇_{Ω_∇} is a quantifier on $\mathbb{K}(L,F)/\theta\cong L$, and if $(T,\Omega)\in\mathsf{WMKAN}$, then Ω_{∇_Ω} is a weak quantifier on $\mathbb{K}(T/\theta,T^+/\theta)\cong T$. Moreover, recalling the natural isomorphisms

$$\begin{aligned}&g^{-1}\colon L\to \mathrm{K}(L,F)/\theta \quad \text{given by}\quad g^{-1}(a)=(a,\neg a)/\theta\\ &\rho\colon T\to K(T/\theta,T^+/\theta)\quad \text{given by}\quad \rho(x)=(x/\theta,(\sim x)/\theta)\end{aligned}$$

between the categories PDLF and KAN (see lemmas 2.9 and 2.10), we have:

Theorem 4.6.

1) *If* $(L, \nabla, F) \in \mathsf{MPDLF}$, *then* $g^{-1}(\nabla a) = \nabla_{\Omega_\nabla}(g^{-1}(a))$.

2) *If* $(T, \Omega) \in \mathsf{WMKAN}$, *then* $\rho(\Omega x) = \Omega_{\nabla_\Omega}(\rho(x))$.

Hence g^{-1} *and* ρ *are isomorphisms in the new categories.*

Proof. The first part is immediate by definition, as

$$\begin{aligned}\nabla_{(\Omega_\nabla)}(g^{-1}(a)) &= \nabla_{(\Omega_\nabla)}((a, \neg a)/\theta)\\ &= (\Omega_\nabla(a, \neg a))/\theta\\ &= (\nabla a, \neg \nabla a)/\theta\\ &= g^{-1}(\nabla a).\end{aligned}$$

For the second part, we also need (W5), as

$$\begin{aligned}\Omega_{(\nabla_\Omega)}(\rho(x)) &= \Omega_{(\nabla_\Omega)}(x/\theta, (\sim x)/\theta)\\ &= (\nabla_\Omega(x/\theta), \neg\nabla_\Omega(x/\theta))\\ &= ((\Omega x)/\theta, \neg(\Omega x)/\theta)\\ &= ((\Omega x)/\theta, (\neg \Omega x)/\theta)\\ &= ((\Omega x)/\theta, (\sim \Omega x)/\theta)\\ &= \rho(\Omega x).\end{aligned}$$

□

As a corollary, we have the following result.

Corollary 4.7. *There exists a categorical equivalence between* MPDLF *and* WMKAN.

Let $T \in \mathsf{KAN}_c$ and let Ω be a weak quantifier. If we define $\Omega_c(x) = \Omega x \vee c$, then for pairs we have $\Omega_c(a, b) = (\nabla a, 0)$, and besides the following hold:

T1. $x \vee c \leq \Omega_c(x \vee c)$,

T2. $\Omega_c((x \vee c) \wedge \Omega_c(y \vee c)) = \Omega_c(x \vee c) \wedge \Omega_c(y \vee c)$,

T3. $\Omega_c(x \vee y \vee c) = \Omega_c(x \vee c) \vee \Omega_c(y \vee c)$, T4. $\Omega_c(x \vee c) = \Omega_c x$,

T5. $\Omega_c(x \wedge c) = c$.

Let WMKAN_c be the variety of KAN-algebras with an additional operator satisfying T1-T5. We call these operators c-weak quantifiers. In this case we have the bijections $\nabla \mapsto (\nabla\cdot, 0)$ and $\Omega_c \mapsto \Omega_c|_{C(T)}$ between quantifiers and c-weak quantifiers. The isomorphisms α and β preserve these operations, and therefore we have a categorical equivalence between WMKAN_c and MPDL.

Example 4.8. Let L be the PDL-reduct of the Heyting algebra $\mathbf{B}_2 \times \mathbf{B}_2$, where $\mathbf{B}_2$ is the two-element Boolean algebra, and the lattice skeleton of the KAN-algebra $\mathbb{K}(L)$ (see Figure 1).

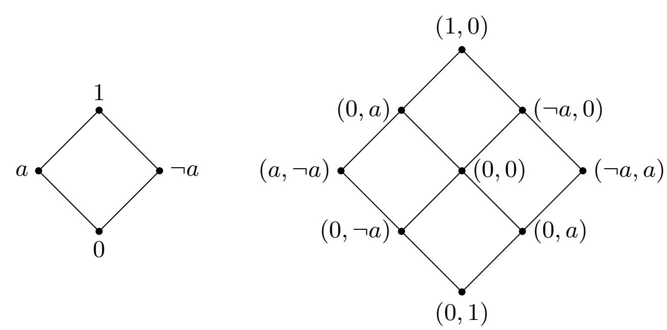

Figure 1: L and $\mathbb{K}(L)$ from Example 4.8.

It can be shown that the only two quantifiers on L are the trivial ones denoted by ∇_1 and ∇_2, and described in Table 1. The corresponding weak quantifiers on $\mathbb{K}(L)$ are shown in Table 2.

<table>
<tr><th>x</th><th>$\nabla_1 x$</th><th>$\nabla_2 x$</th></tr>
<tr><td>0</td><td colspan="2">0</td></tr>
<tr><td>1</td><td colspan="2">1</td></tr>
<tr><td>a</td><td>a</td><td>1</td></tr>
<tr><td>$\neg a$</td><td>$\neg a$</td><td>1</td></tr>
</table>

Table 1:

<table>
<tr><th>x</th><th>$\Omega_{\nabla_1} x$</th><th>$\Omega_{\nabla_2} x$</th></tr>
<tr><td>$(0,1)$</td><td colspan="2">$(0,1)$</td></tr>
<tr><td>$(0,a)$</td><td colspan="2">$(0,1)$</td></tr>
<tr><td>$(0,\neg a)$</td><td colspan="2">$(0,1)$</td></tr>
<tr><td>$(0,0)$</td><td colspan="2">$(0,1)$</td></tr>
<tr><td>$(1,0)$</td><td colspan="2">$(1,0)$</td></tr>
<tr><td>$(a,0)$</td><td>$(a,\neg a)$</td><td>$(1,0)$</td></tr>
<tr><td>$(\neg a,0)$</td><td>$(\neg a,a)$</td><td>$(1,0)$</td></tr>
<tr><td>$(a,\neg a)$</td><td>$(a,\neg a)$</td><td>$(1,0)$</td></tr>
<tr><td>$(\neg a,a)$</td><td>$(\neg a,a)$</td><td>$(1,0)$</td></tr>
</table>

Table 2:

Example 4.9. Let L now be the PDL-reduct of the Heyting algebra $\mathbf{B}_2 \oplus (\mathbf{B}_2 \times \mathbf{B}_2)$ (where $\oplus$ is the ordinal sum of posets). The diagrams corresponding to the lattice reducts of L and $\mathbb{K}(L)$ are sketched in Figure 2. Recall that in L we have that $\neg a = \neg b = \neg d = 0$. Quantifiers on L are described in Table 3 and the weak quantifiers on $\mathbb{K}(L)$ are shown in Table 4.

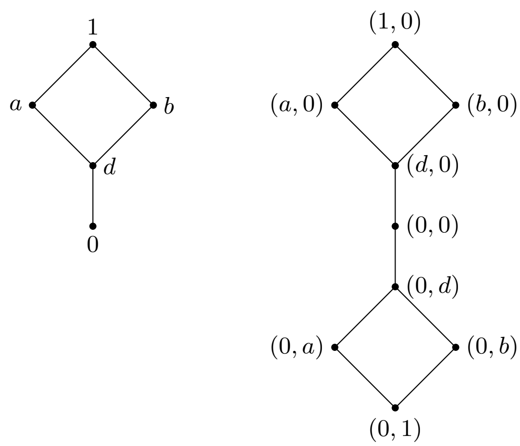

Figure 2: L and $\mathbb{K}(L)$ from Example 4.9.

x	$\nabla_1 x$	$\nabla_2 x$	$\nabla_3 x$	$\nabla_4 x$	$\nabla_5 x$
0	0				
1	1				
a	a	a	1	1	1
b	b	1	b	1	1
d	d	a	b	d	1

Table 3:

x	$\Omega_{\nabla_1} x$	$\Omega_{\nabla_2} x$	$\Omega_{\nabla_3} x$	$\Omega_{\nabla_4} x$	$\Omega_{\nabla_5} x$
$(0,1)$	$(0,1)$				
$(0,a)$	$(0,1)$				
$(0,b)$	$(0,1)$				
$(0,d)$	$(0,1)$				
$(0,0)$	$(0,1)$				
$(1,0)$	$(1,0)$				
$(a,0)$	$(a,0)$	$(a,0)$	$(1,0)$	$(1,0)$	$(1,0)$
$(b,0)$	$(b,0)$	$(1,0)$	$(b,0)$	$(1,0)$	$(1,0)$
$(d,0)$	$(d,0)$	$(a,0)$	$(b,0)$	$(d,0)$	$(1,0)$

Table 4:

5. Some additional results

In this section we study some consequences of the categorical equivalences and related topics. We show that in general it is not true that a KAN-algebra is the reduct of a Nelson algebra. We also introduce and study a particular subvariety of KAN. Then we give an alternative (algebraic) proof of Theorem 2.17.

5.1 A KAN-algebra that is not isomorphic to the reduct of a Nelson algebra.

Although it is part of the folklore that there are pseudomplemented distributive lattices which are not the reduct of any Heyting algebra, to make this paper self-contained we illustrate this fact with an example, which will be used to show that not every KAN-algebra is the reduct of a Nelson algebra.

Consider the lattice $L = \mathbb{N}^2 \cup \{\bot, \top\}$, consisting in the lattice $\mathbb{N}^2$ (defined coordinatewise) with added top $\top$ and bottom $\bot$ (here $\mathbb{N}$ is the set of natural numbers). This can be easily seen as a pseudocomplemented distributive lattice by defining $\neg\top = \bot$, $\neg\bot = \top$ and $\neg(m, n) = \bot$ for any $m, n \in \mathbb{N}$. However, a Heyting implication cannot be defined. For example, $(2, 1) \to (1, 2)$ cannot exist. Indeed, observe that

$$(2, 1) \wedge (m, n) = (2 \wedge m, 1) \leq (1, 2) \text{ if and only if } m = 1, n \in \mathbb{N}.$$

Since the set $\{(1, n) : n \in \mathbb{N}\}$ does not have a maximum, $(2, 1) \to (1, 2)$ does not exist.

Consider now the KAN-algebra $T \coloneqq \mathbb{K}(L, L)$. Assume that there is a Nelson algebra $(N, \wedge, \vee, \sim, \to, 0, 1)$ such that after defining $\neg x \coloneqq x \to 0$ we have that $(N, \wedge, \vee, \sim, \neg, 0, 1)$ is the algebra T. From Theorem 3.8 we know that T can be extended to a centered KAN-algebra T_c. It is easy to verify that N_c is a centered Nelson algebra. We also know that the restriction of the functor C to the subcategory of centered Nelson algebras into the category of Heyting algebras yields a categorical equivalence (details in [5]). Therefore, $C(N_c) = C(T_c)$ is a Heyting algebra. From Theorem 3.7 we have that $C(T_c) \cong T/\theta \cong L$ but this contradicts the fact that L was not a Heyting algebra.

Remark 5.1. However, if $T \in \mathsf{KAN}$ is finite, then it is isomorphic to the reduct of a Nelson algebra. T/θ is a finite pseudocomplemented distributive lattice and

$$a \to b = \max\{c : a \wedge c \leq b\}$$

is a well-defined implication which is compatible with the negation $\neg$. Hence $(T/\theta, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra and $\mathbb{K}(T/\theta, T^+/\theta)$ can be equipped with the implication

$$(a, b) \to (d, e) \coloneqq (a \to d, a \wedge e)$$

so that it is a Nelson algebra.

5.2 Strengthening the negation

Following [11], a negation t defined on a bounded poset T (that is, a unary operator satisfying $t(0) = 1$, $t(1) = 0$ and $t(y) \leq t(x)$ whenever $x \leq y$) is a *weak negation* if it also satisfies

$$x \leq t(t(x)) \tag{7}$$

and a strong negation if

$$t(t(x)) = x.$$

The algebras in the variety KAN have therefore a strong negation $\sim$ and a negation $\neg$, which we call intuitionistic. But unlike the intuitionistic negation in Heyting algebras, which is weak in the preceding sense, the operator $\neg$ is not necessarily a weak negation. In this section we shall investigate the class of KAN-algebras for which $\neg$ is a weak negation.

Definition 5.2. KAN_w is the variety of KAN-algebras satisfying $x \leq \neg\neg x$.

Recalling Example 2.3, if $B = (B, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and we set $\sim x = \neg x$, then $\tilde{B} := (B, \wedge, \vee, \sim, \neg, 0, 1)$ is in KAN_w, as $\neg\neg x = x$. It turns out that all algebras in KAN_w will be of this form. Thus, we cannot *strengthen* (in the sense of requiring it to satisfy $\neg\neg x \geq x$) the intuitionistic negation in KAN-algebras without collapsing it into the involution.

Lemma 5.3. *If $T \in \mathsf{KAN}_w$, then $T^+ = \{1\}$ and $T^- = \{0\}$.*

Proof. If $x \in T^+$, then $\neg \sim x = 1$, so $\neg\neg(\sim x) = 0$. But as equation (7) holds, $\sim x = 0$ and $x = 1$. If $x \in T^-$, then $\sim x \in T^+$ and $\sim x = 1$, so the lemma holds. □

Theorem 5.4. *If $T \in \mathsf{KAN}_w$, then T/θ is a Boolean algebra.*

Proof. As $T/\theta \in \mathsf{PDL}$, we only need to show that

$$(x/\theta) \vee \neg(x/\theta) = (1/\theta).$$

This is immediate, as $x \vee \neg x \geq x \vee \sim x \in T^+$ and T^+ is a filter, therefore $x \vee \neg x \in T^+$ and $x \vee \neg x = 1$. □

If B is Boolean, then 1 is the only element satisfying $\neg x = 0$, so the filter $\{1\}$ is a Boolean filter.

Lemma 5.5. *If B is a Boolean algebra, then $\mathbb{K}(B,\{1\})$ is in KAN_w and satisfies $\sim(a,b) = \neg(a,b)$. Moreover, $\tilde{B} \cong \mathbb{K}(B,\{1\})$.*

Proof. From the previous observation, $\mathbb{K}(B,\{1\})$ is a KAN-algebra. Observe that $(a,b) \in \mathrm{K}(B,\{1\})$ if and only if $a \wedge b = 0$ and $a \vee b = 1$, if and only if $b = \neg a$. Therefore it is immediate that

$$\sim(a,b) = \sim(a,\neg a) = (\neg a, a) = \neg(a,\neg a) = \neg(a,b)$$

and also that (7) holds. The assignment $b \in B \mapsto (b, \neg b)$ is clearly an isomorphism between $\tilde{B}$ and $\mathbb{K}(B,\{1\})$. □

Observe that for $T \in \mathsf{KAN}_w$ the congruence θ is just the diagonal $\Delta = \{(x,x) : x \in T\}$.

From Theorem 2.15 and the previous results, we obtain the following.

Theorem 5.6. *Let $T \in \mathsf{KAN}_w$. Then $\sim$ and $\neg$ coincide, and $(T, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra.*

5.3 Comparison with the variety $\mathcal{K}_\omega$

The rest of our work is devoted to providing an alternative (direct) proof of Theorem 2.17. We have divided it into a sequence of lemmas, starting with those from which the inclusion $\mathsf{KAN} \subseteq \mathcal{K}_\omega$ can be deduced.

Lemma 5.7. *If $T \in \mathsf{KAN}$ and $x, y \in T$, then $x \wedge y \leq \sim x$ if and only if $y \leq \neg x$.*

Proof. Let $x, y \in T$. If $y \leq \neg x$, an application of (N3) yields

$$x \wedge y \leq \neg x \wedge x = \sim x \wedge x \leq \sim x,$$

so $x \wedge y \leq \sim x$. We now will see that if $x \wedge y \leq \sim x$, then

$$\sim\neg(y \wedge \neg x) = \sim\neg y \tag{8}$$

$$\neg\sim(y \wedge \neg x) = \neg\sim y. \tag{9}$$

Indeed, since $x \wedge y \leq \sim x$, $x \wedge y = x \wedge \sim x \wedge y$. Recalling that the operator $\neg$ reverses the order, by (N6) we get

$$1 = \neg(x \wedge \sim x) \leq \neg(x \wedge \sim x \wedge y);$$

that is to say that $\neg(x\wedge\sim x\wedge y)=1$. Hence

$$\neg y=\neg(y\wedge 1)=\neg(y\wedge\neg(x\wedge\sim x\wedge y))=\neg(y\wedge\neg(x\wedge y))=\neg(y\wedge\neg x),$$

where the last equality is due to (N1). Then $\sim\neg(y\wedge\neg x)=\sim\neg y$ and equality (8) is proved. In order to prove equality (9), note that since $x\leq\sim x\vee\sim y$,

$$x=x\wedge(\sim x\vee\sim y)=(x\wedge\sim x)\vee(x\wedge\sim y).$$

Then

$$\begin{aligned}\neg x&=\neg((x\wedge\sim x)\vee(x\wedge\sim y))=\neg(x\wedge\sim x)\wedge\neg(x\wedge\sim y)=1\wedge\neg(x\wedge\sim y)\\&=\neg(x\wedge\sim y)\end{aligned}$$

by (N2) and (N6). Hence, by (N2) and (N9),

$$\begin{aligned}\neg\sim(y\wedge\neg x)&=\neg(\sim y\vee\sim\neg x)=\neg\sim y\wedge\neg\sim\neg x=\neg\sim y\wedge\neg x\\&=\neg\sim y\wedge\neg(x\wedge\sim y).\end{aligned}$$

But since $x\wedge\sim y\leq\sim y$, $\neg(x\wedge\sim y)\geq\neg\sim y$, and therefore $\neg\sim(y\wedge\neg x)=\neg\sim y$.

From (8), (9) and Lemma 2.6 we obtain that $y\wedge\neg x=y$, or its equivalent $y\leq\neg x$, which was our aim. □

Lemma 5.8. *If $T\in\mathsf{KAN}$ and $x,y\in T$, then $\neg(x\wedge y)=1$ if and only if $\neg\neg x\leq\neg y$.*

Proof. Assume that $\neg(x\wedge y)=1$ holds. Then, arguing as in Lemma 2.8,

$$\begin{aligned}\neg\neg x\wedge\neg y&=\neg\neg x\wedge\neg(y\wedge\neg(x\wedge y))=\neg\neg x\wedge\neg(y\wedge\neg x)\\&=\neg(\neg x\vee(y\wedge\neg x))=\neg\neg x.\end{aligned}$$

Hence $\neg\neg x\leq\neg y$. In case $\neg\neg x\leq\neg y$, $\neg\neg x\wedge y\leq\neg y\wedge y$, so that $\neg(\neg\neg x\wedge y)\geq\neg(\neg y\wedge y)=1$ by (N6). Then

$$\neg(\neg\neg x\wedge y)=1. \tag{10}$$

On the other hand, since $\sim\neg x\leq\neg\neg x$ by (N4), $(\sim\neg x)\wedge y\leq\neg\neg x\wedge y$ an consequently

$$\neg(x\wedge y)=\neg((\sim\neg x)\wedge y)\geq\neg(\neg\neg x\wedge y)=1$$

by (N5) and equation (10) above. Thus $\neg(x\wedge y)=1$, as desired. □

Corollary 5.9. $\mathsf{KAN} \subseteq \mathcal{K}_\omega$.

For the proof of the opposite inclusion we will use (i), (iii), (iv), (vi) and (vii) of [25, Lemma 2.1] as well as the known fact that the binary relation $\theta(T)$ defined on a Kleene algebra T as

$$(x, y) \in \theta(T) \text{ if and only if there exists } z \leq\sim z \text{ such that } x \vee z = y \vee z$$

is a congruence of lattices.

Every member of $\mathcal{K}_\omega$ fulfils (N2), (N3) and (N4), as they are trivial from (iv), (i) and (iii), respectively. Every member of $\mathcal{K}_\omega$ also satisfies (N5). Indeed, let $T \in \mathcal{K}_\omega$. Since $(x, \sim \neg x) \in \theta(T)$ by (vi) and $\theta(T)$ being a congruence of lattices, $(x \wedge y, (\sim \neg x) \wedge y) \in \theta(T)$. Then

$$\neg(x \wedge y) = \neg((\sim \neg x) \wedge y)$$

by (vii), and condition (N5) is proved.

The following lemma will be used to show that every member of $\mathcal{K}_\omega$ satisfies (N1).

Lemma 5.10. *Let $T \in \mathcal{K}_\omega$ and $x, y \in T$. If $\neg(x \wedge y) = 1$, then $\neg(x \wedge \neg y) = \neg x$.*

Proof. Let $x, y \in T$ such that $\neg(x \wedge y) = 1$. Then $\neg\neg y \leq \neg x$ because $T \in \mathcal{K}_\omega$, so that $\neg\neg y = \neg(\neg y \vee x)$ by (iv) and hence $(\neg y, \neg y \vee x) \in \theta(T)$ by (vii). Since $\theta(T)$ is a congruence of lattices, $(x \wedge \neg y, x \wedge (\neg y \vee x)) = (x \wedge \neg y, x) \in \theta(T)$, so again by (vii) we get $\neg(x \wedge \neg y) = \neg x$, which proves the lemma. □

Let $T \in \mathcal{K}_\omega$. From $\neg\neg x \leq \neg\neg x$ it is easily seen that $\neg(x \wedge \neg x) = 1$ for every $x \in T$. Let $x, y \in T$. Then

$$\neg((x \wedge y) \wedge \neg(x \wedge y)) = \neg((x \wedge \neg(x \wedge y)) \wedge y) = 1.$$

Hence, by Lemma 5.10,

$$\neg(x \wedge \neg(x \wedge y)) = \neg((x \wedge \neg(x \wedge y)) \wedge \neg y) = \neg(x \wedge (\neg(x \wedge y) \wedge \neg y)).$$

But since $y = (x \wedge y) \vee y$, $\neg y = \neg((x \wedge y) \vee y) = \neg(x \wedge y) \wedge \neg y$ by (iv). Thus we get

$$\neg(x \wedge \neg(x \wedge y)) = \neg(x \wedge \neg y),$$

which is condition (N1).

Corollary 5.11. $\mathcal{K}_\omega \subseteq \mathsf{KAN}$.

Therefore, it follows from corollaries 5.9 and 5.11 that $\mathsf{KAN} = \mathcal{K}_\omega$. In particular, $\mathcal{K}_\omega^{\mathrm{c}} = \mathsf{KAN}_{\mathrm{c}}$.

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