

## Multispecies effects in the equilibrium and out-of-equilibrium thermostatics of overdamped motion

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Progress has been recently made, both theoretical and experimental, regarding the thermostatics of complex systems of interacting particles or agents (species) obeying a nonlinear Fokker-Planck dynamics. However, major advances along these lines have been restricted to systems consisting of only one type of species. The aim of the present contribution is to overcome that limitation, going beyond single-species scenarios. We investigate the dynamics of overdamped motion in interacting and confined many-body systems having two or more species that experience different intra- and interspecific forces in a regime where forces arising from standard thermal noise can be neglected. Even though these forces are neglected, the behavior of the system can be analyzed in terms of an appropriate thermostatical formalism. By recourse to a mean-field treatment, we derive a set of coupled nonlinear Fokker-Planck equations governing the behavior of these systems. We obtain an  $H$  theorem for this Fokker-Planck dynamics and discuss in detail an example admitting an exact, analytical stationary solution.

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### I. INTRODUCTION

Nonlinear Fokker-Planck equations [1–3] provide effective descriptions of various phenomena in complex systems [4–12]. In particular, they provide analytically (or semianalytically) tractable models illuminating the dynamics of some complex systems of interacting particles or agents. Among these systems we can mention vortices in type-II superconductors [13], granular media [14], and self-gravitating systems [15,16]. Variants or extensions of the nonlinear Fokker-Planck dynamics have also found applications. In this respect, we can mention the nonlinear Kramers equation [17] and the nonlinear Kramers-Klein equation [18], the first applied to the spreading behavior of the microorganism *Hydra-Boltzonaris*, and the second discussed in connection with thermodynamical systems described by nonadditive entropies.

The nonlinear Fokker-Planck equation governs the evolution of a density  $\mathcal{P}(\mathbf{x}, t)$ , where  $\mathbf{x} \in \text{Re}^N$  denotes a point in an  $N$ -dimensional configuration space. In many of the aforementioned applications,  $\mathcal{P}(\mathbf{x}, t)$  is a real physical density (as opposed to a statistical ensemble probability density). In these cases  $\mathcal{P}(\mathbf{x}, t)$  represents the time-dependent spatial distribution of a set of interacting particles [4,19–22]. There are two terms in the nonlinear Fokker-Planck equation: A nonlinear diffusion term [23,24] and a linear drift term. The nonlinear diffusion term provides an effective description of the interaction between the particles, and the drift term accounts for the effects due to external forces. The nonlinear

Fokker-Planck equation thus determines the behavior of the spatial density of a many-particle system, consisting of particles that perform overdamped motion under the effects of forces arising from the interactions between the particles and from an external potential. Evolution equations with nonlinear diffusion terms constitute also a useful phenomenological way of describing the interaction of particles or agents in other contexts, such as in the study of the dispersal of biological populations [25–27]. This kind of equations also govern the spread of energy in some many-body systems [28].

Virtually all previous efforts done to model complex systems of interacting particles with nonlinear Fokker-Planck equations have considered systems constituted only by one type of particle. This single-species assumption imposes a serious restriction on the types of systems or processes that can be described by the nonlinear Fokker-Planck dynamics. The aim of the present contribution is to overcome this limitation. We shall consider the nonlinear Fokker-Planck approach to the thermostatics of overdamped motion in many-particle systems consisting of different types (species) of particles.

In the next section we review briefly the nonlinear Fokker-Planck equation (NLFPE) associated with the nonextensive power-law entropy  $S_q$ , its stationary-state solution, and some of its applications. In Sec. III we derive NLFPEs describing a system constituted by two types of interacting particles in the overdamped motion regime, which result in a pair of coupled nonlinear equations. By introducing a free-energy-like functional, we prove an  $H$  theorem for this system. Moreover, we show that an optimization procedure on this functional yields equilibrium solutions that coincide with the stationary-state solutions of the pair of NLFPEs. In Sec. IV

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we analyze in detail a one-dimensional example of the pair of equations derived in Sec. III and obtain its stationary-state solutions in a semianalytical way. In Sec. V we investigate the more general case of multispecies systems with  $L$  different types of particles, described by a set of  $L$  coupled NLFPEs. We prove that these systems satisfy an  $H$  theorem in terms of a free-energy-like functional, that is, a linear combination of the confining external potential and of an appropriate, nonlinear functional of the  $L$  time-dependent densities respectively associated with each of the species constituting the system. We analyze the stationary states of these general systems. These results are also extended to systems involving nonlocal interactions between particles. Finally, in Sec. VI we present our main conclusions.

## II. THE NONLINEAR FOKKER-PLANCK EQUATION

One of the most intensively studied NLFPEs in recent years has the form

$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{D} \nabla^2 \left[ \mathcal{P} \left( \frac{\mathcal{P}}{\mathcal{P}_0} \right)^{1-q} \right] - \nabla \cdot [\mathcal{P} \mathbf{K}], \quad (1)$$

where  $\mathcal{P}(\mathbf{x}, t)$  is a time-dependent density with dimensions of inverse volume,  $\mathcal{P}_0$  is a constant with the same dimensions as  $\mathcal{P}(\mathbf{x}, t)$ ,  $\mathcal{D}$  is a diffusion constant,  $\mathbf{K}(\mathbf{x})$  is a drift force, and  $q$  is a real parameter characterizing the (power-law) nonlinearity in the Laplacian term. As already mentioned in the Introduction, the densities  $\mathcal{P}(\mathbf{x})$  that we are going to consider in this work are real physical densities, not ensemble probability densities. More precisely, the densities considered here are densities of particles. This means that  $\mathcal{P}(\mathbf{x}) d^N \mathbf{x}$  is the number of particles within the volume element  $d^N \mathbf{x}$  centered at the point  $\mathbf{x}$ . In most applications of the evolution Eq. (1), the drift field  $\mathbf{K}$  is assumed to arise from a potential function  $U(\mathbf{x})$ ,

$$\mathbf{K} = -\nabla U. \quad (2)$$

The stationary solutions of the NLFPE then satisfy

$$\nabla \cdot \left\{ \mathcal{D} \nabla \left[ \mathcal{P} \left( \frac{\mathcal{P}}{\mathcal{P}_0} \right)^{1-q} \right] + \mathcal{P}(\nabla U) \right\} = 0. \quad (3)$$

Let us consider the  $q$ -statistical ansatz [2,3,29]

$$\mathcal{P}_q = A \exp_q[-\beta U(\mathbf{x})] = A [1 - (1 - q)\beta U(\mathbf{x})]_+^{\frac{1}{1-q}}, \quad (4)$$

where  $A$  and  $\beta$  are constants to be determined, and the function  $\exp_q(z) = [1 + (1 - q)z]_+^{\frac{1}{1-q}}$ , usually referred to as the  $q$ -exponential function, is equal to  $[1 + (1 - q)z]_+^{\frac{1}{1-q}}$  when  $1 + (1 - q)z > 0$  and vanishes when  $1 + (1 - q)z \leq 0$ . One finds that the ansatz given by Eq. (4) complies with the relation

$$\mathcal{D} \nabla \left[ \mathcal{P} \left( \frac{\mathcal{P}}{\mathcal{P}_0} \right)^{1-q} \right] + \mathcal{P}(\nabla U) = 0, \quad (5)$$

if

$$(2 - q)\beta \mathcal{D} = \left( \frac{A}{\mathcal{P}_0} \right)^{q-1}. \quad (6)$$

It therefore satisfies also Eq. (3) and constitutes a stationary solution of the NLFPE. In summary, the  $q$ -exponential ansatz (4) is a stationary solution of the NLFPE, if the drift force  $\mathbf{K}$  is derived from a potential and  $A$  and  $\beta$  satisfy the relation (6). We shall assume that the stationary distribution  $\mathcal{P}_q$  has a finite norm, that is,  $\int \mathcal{P}_q d^N \mathbf{x} = I < \infty$ . The detailed conditions, such as the allowed range of values of the parameter  $q$ , required to have a stationary solution with finite norm depend on the particular shape of the potential function  $U(\mathbf{x})$  and, consequently, have to be established on a case-by-case fashion. Since in many applications the solution of the NLFPE represents a physical, spatial density of particles (instead of representing a probability density), we assume that the norm  $I$  is finite but not necessarily equal to 1.

The stationary density  $\mathcal{P}_q$  can be regarded as a  $q$ -maxent distribution, because it maximizes, for an appropriate value  $q^*$  of the entropic  $q$  parameter, the nonadditive  $q$ -entropic functional [29–33]

$$S_q[\mathcal{P}] = \frac{k}{q-1} \int \mathcal{P} \left[ 1 - \left( \frac{\mathcal{P}}{\mathcal{P}_0} \right)^{q-1} \right] d^N \mathbf{x}, \quad (7)$$

under the constraints corresponding to the norm and the mean value of the potential  $U$  [2,3]. More precisely, the density that maximizes  $S_{q^*}$ , with  $q^* = 2 - q$ , under the constraints of the mean value of  $U$  and normalization, has the form (4). In this work we use units such that  $k = 1$ .

In the limit  $q \rightarrow 1$ , the standard linear Fokker-Planck equation,

$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{D} \nabla^2 \mathcal{P} - \nabla \cdot [\mathcal{P} \mathbf{K}], \quad (8)$$

is recovered. In this limit, the  $q$ -maxent stationary density (4) reduces to the exponential, Boltzmann-Gibbs-like density,

$$\mathcal{P}_{\text{BG}} = \frac{1}{Z} \exp \left[ -\frac{1}{\mathcal{D}} U(\mathbf{x}) \right], \quad (9)$$

with the condition (6) becoming  $\beta \mathcal{D} = 1$ , independent of the normalization constant  $A$ . The density  $\mathcal{P}_{\text{BG}}$  is normalized to one provided that  $Z = \int \exp[-\frac{1}{\mathcal{D}} U(\mathbf{x})] d^N \mathbf{x}$ . The density  $\mathcal{P}_{\text{BG}}$  optimizes the Boltzmann-Gibbs entropy  $S_{\text{BG}} = -k \int \mathcal{P} \ln(\mathcal{P}/\mathcal{P}_0) d^N \mathbf{x}$ , under the constraints of normalization and the mean value  $\langle U \rangle$  of the potential  $U$  (consistently with the fact that in the limit  $q \rightarrow 1$  the entropic functional  $S_q$  reduces to the standard Boltzmann-Gibbs entropy).

In some important applications the nonlinear Fokker-Planck equations have exact analytical, time-dependent solutions of the  $q$ -Gaussian form. The  $q$ -Gaussian densities optimize the nonadditive, power-law entropies  $S_q$  under constraints that are quadratic on the relevant phase-space variables [29]. These densities are observed in the study of various phenomena in complex systems [34–37]. The connection between the power-law, nonlinear Fokker-Planck equations and the  $S_q$ -based thermostatics, first pointed out in Ref. [2], has been the focus of increasing attention in recent years, stimulating diverse developments concerning the application of those equations to the study of complex systems. A notable experimental achievement along these lines is given by the results concerning experiments on granular media reported in Ref. [14], verifying within a 2% error a scale relation theoretically obtained by Tsallis and Bukman [3] from the

exact  $q$ -Gaussian time-dependent solutions of the nonlinear Fokker-Planck equation with a quadratic potential.

### III. THERMOSTATISTICS OF OVERDAMPED MOTION IN TWO-SPECIES SYSTEMS OF INTERACTING PARTICLES

In this section we are going to derive a set of coupled nonlinear Fokker-Planck equations describing the thermostatics of a confined many-body system comprising different types of particles interacting via short-range forces and performing overdamped motion. First we are going to consider a system constituted by two types of particles, referred to as type-1 and type-2 particles, with masses respectively equal to  $m_1$  and  $m_2$ , that move in an  $N$ -dimensional space and interact through short-range, repulsive forces. The spatial distribution of these particles are described by the densities  $\mathcal{P}_{1,2}(\mathbf{x})$ , so that the number of particles of type  $i$  within the volume element  $d^N\mathbf{x}$  centered at the point  $\mathbf{x}$  is  $\mathcal{P}_i(\mathbf{x})d^N\mathbf{x}$ . These particles are also under the effects of external confining potentials  $W_1$  and  $W_2$  (the  $W_i$  potential is the one acting on the  $i$ -type particles) and of drag forces due to a resisting medium. Therefore, there are several contributions to the total force acting on each of these particles. The contributions to the force acting on particles of a given type  $i$  include the forces arising from the interaction with particles of the same type and of different types, the force derived from the potential  $W_i$ , and the drag force

$$\mathbf{F}_{\text{drag}}^{(i)} = -\alpha_i \dot{\mathbf{x}}, \quad (10)$$

characterized by the constants  $\alpha_i > 0$ . At a given time, each particle of the system interacts only with particles located within its immediate neighborhood, because the interaction between the particles is of a short-range nature [4]. Let  $\mathcal{F}_{ij}(|\mathbf{x} - \mathbf{x}'|) \geq 0$  be the strength of the force  $\mathbf{F}_{ij}$  felt by a particle of type  $i$  located at  $\mathbf{x}$  due to a particle of type  $j$  at  $\mathbf{x}'$ . The vector representation of this (repulsive) force is  $\mathbf{F}_{ij}(\mathbf{x}, \mathbf{x}') = \mathcal{F}_{ij}(|\mathbf{x} - \mathbf{x}'|)(\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|$ . The force on a particle of type  $i$ , located at  $\mathbf{x}$ , due to its interaction with the other particles of the system is

$$\mathbf{F}_{\text{int}}^{(i)}(\mathbf{x}) = \sum_j \int \mathbf{F}_{ij}(\mathbf{x}, \mathbf{x}') \mathcal{P}_j(\mathbf{x}') d^N\mathbf{x}', \quad (11)$$

where the sum on the index  $j$  appearing in the right-hand side of the above equation runs over all the types of particles in the system. It is here assumed that  $\mathcal{F}_{ij}$  is a smooth function of  $r = |\mathbf{x} - \mathbf{x}'|$ , decaying fast enough so that the integral  $\int_0^\infty r^N \mathcal{F}_{ij}(r) dr$  converges. In addition, due to the short-range nature of the interaction, one can assume that the typical length scales of the system are large compared with the range of  $r$  values within which  $\mathcal{F}_{ij}(r)$  is appreciably different from zero. In particular, over this range of  $r$  values, the spatial densities  $\mathcal{P}_j(\mathbf{x})$  of  $j$ -type particles can be approximated as  $\mathcal{P}_j(\mathbf{x}') = \mathcal{P}_j(\mathbf{x}) + (\mathbf{x}' - \mathbf{x}) \cdot (\nabla \mathcal{P}_j)$ . Under these conditions, this expansion for the densities  $\mathcal{P}_j(\mathbf{x}')$  can be substituted into the expression (11) for the forces  $\mathbf{F}_{\text{int}}^{(i)}(\mathbf{x})$  acting on an  $i$ -type particle located at  $\mathbf{x}$ , yielding

$$\mathbf{F}_{\text{int}}^{(i)} = - \sum_j \mathcal{G}_{ij} \nabla \mathcal{P}_j, \quad (12)$$

where

$$\mathcal{G}_{ij} = \frac{1}{N} \int r \mathcal{F}_{ij}(r) d^N\mathbf{x} = \frac{\sigma_{N-1}}{N} \int_0^\infty r^N \mathcal{F}_{ij}(r) dr, \quad (13)$$

and  $\sigma_{N-1}$  denotes the hypersolid angle associated with an  $(N-1)$ -dimensional sphere (in other words, the hypersurface of an  $N$ -dimensional ball of radius 1). One has, for instance,  $\sigma_0 = 2$ ,  $\sigma_1 = 2\pi$ , and  $\sigma_2 = 4\pi$ . In one dimension, the forces  $\mathbf{F}_{\text{int}}^{(i)}$  coincide with the ones arising from the interaction potentials  $\mathcal{V}_{ij}(x_1, x_2) = \mathcal{G}_{ij} \delta(x_2 - x_1)$ , where  $\delta(x)$  is Dirac's  $\delta$  function and  $x_{1,2}$  are the positions of the two particles.

Before continuing, some remarks are in order concerning the conditions under which the expression (12) for  $\mathbf{F}_{\text{int}}^{(i)}$  is valid. As already explained, the validity of (12) requires the convergence,

$$\int_0^\infty r^N \mathcal{F}_{ij}(r) dr < \infty, \quad (14)$$

of the integrals appearing in the right-hand side of (13). This requirement is our present criterium for characterizing the force law  $\mathcal{F}_{ij}(r)$  as a short-range law force. This criterium is not fulfilled if  $\mathcal{F}_{ij}(r)$  decreases with  $r$  more slowly than  $r^{-(1+N)}$ . Therefore, in the case of power-law forces, the criterium used for the force to be a short-range one [and, consequently, for the validity of (12)] is that it decreases as  $r^{-\gamma}$  with the exponent  $\gamma$  satisfying

$$\gamma > 1 + N. \quad (15)$$

For strict power-law forces, however, even if satisfying the criterium (15), the expression (12) for  $\mathbf{F}_{\text{int}}^{(i)}$  is not valid, because the integrals  $\int_0^\infty r^N \mathcal{F}_{ij}(r) dr$  diverge due to the singularity of  $\mathcal{F}_{ij}(r)$  at  $r = 0$ . Expression (12) holds for forces  $\mathcal{F}_{ij}(r)$  that are short-range (according to the above explained criterium) and that are also well behaved at  $r = 0$ . For instance, the expression (12) is valid for softened power-law forces given by

$$\mathcal{F}_{ij}(r) = \mathcal{F}_0 \left[ 1 + \left( \frac{r}{r_0} \right) \right]^{-\gamma}, \quad (16)$$

where  $\gamma$  complies with the condition (15) and  $\mathcal{F}_0$  and  $r_0$  are positive constants.

As we shall presently see, the overdamped dynamics of systems of particles for which  $\mathbf{F}_{\text{int}}^{(i)}$  can be described by (12) and (13) is associated to the  $S_q$  thermostatics with  $q = 0$ . Therefore, it follows from the previous considerations that, for the kind of systems studied here, the  $q = 0$  case is associated with interaction forces that comply with the condition (14). For the softened power-law forces (16), that condition reduces to the inequality (15).

In a system with two types of particles one has

$$\begin{aligned} \mathbf{F}_{\text{int}}^{(1)} &= -\mathcal{G}_{11} \nabla \mathcal{P}_1 - \mathcal{G}_{12} \nabla \mathcal{P}_2, \\ \mathbf{F}_{\text{int}}^{(2)} &= -\mathcal{G}_{21} \nabla \mathcal{P}_1 - \mathcal{G}_{22} \nabla \mathcal{P}_2, \end{aligned} \quad (17)$$

The aforementioned components of the total force acting on a test particle of type  $i$  (due to the interaction with other particles, to the external confining potential, and to drag) lead

to the equations of motion

$$\begin{aligned} m_1 \dot{\mathbf{x}} + \mathcal{G}_{11} \nabla \mathcal{P}_1 + \mathcal{G}_{12} \nabla \mathcal{P}_2 + \nabla W_1 + \alpha_1 \dot{\mathbf{x}} &= \mathbf{0}, \\ m_2 \dot{\mathbf{x}} + \mathcal{G}_{21} \nabla \mathcal{P}_1 + \mathcal{G}_{22} \nabla \mathcal{P}_2 + \nabla W_2 + \alpha_2 \dot{\mathbf{x}} &= \mathbf{0}, \end{aligned} \quad (18)$$

for type-1 and type-2 particles, respectively.

Following the treatments in Refs. [4,19–22] we are going to assume that the random forces associated with standard thermal noise are much weaker than the forces of interaction between the particles. In terms of our mean-field description of the system's dynamics, this amounts to assuming that the standard diffusion coefficients  $D_i^{(\text{diff.})}$  for each type of particle are much smaller than  $\lambda^{-1} D_{ij}$ , where the  $D_{ij}$ 's are the effective diffusion coefficients associated with the nonlinear terms of the Fokker-Planck equation [see Eq. (21)] and  $\lambda$  is a characteristic length of the system. That is, we are going to work in a regime characterized by the inequality  $D_i^{(\text{diff.})} \ll \lambda^{-1} D_{ij}$ ,  $i, j = 1, 2$ . By recourse to the Einstein-Smoluchowsky relation,  $k_B T = \alpha_i D_i^{(\text{diff.})}$ , where  $k_B$  is Boltzmann's constant and  $T$  is the absolute temperature of the resisting medium generating the drag forces, this regime can be interpreted as a low temperature one, corresponding to the inequality  $k_B T \ll \lambda^{-1} \mathcal{G}_{ij}$ . Here we use the relation  $D_{ij} = \frac{\mathcal{G}_{ij}}{\alpha_i}$  explained before Eq. (21). Note also that previous work dealt with only one species of particles, in which case the associated diffusion constant, say  $D_{11}$ , is to be identified with  $2\mathcal{D}/\mathcal{P}_0$ ,  $\lambda$  with  $1/\mathcal{P}_0$ , and the nonlinear evolution equation reduces to (1) with  $q = 0$ .

When the above-explained inequalities hold, the effects of the random forces coming from thermal noise can be neglected, leading to the nonlinear evolution equation (21). It was shown in Refs. [4,20] that this approximation provides a good description, for instance, of systems of interaction vortices in type-II superconductors. Indeed, in those systems  $k_B T$  is several orders of magnitude smaller than  $\lambda^{-1} \mathcal{G}_{11}$  [20]. Moreover, even though the systems investigated in Refs. [4,19–22] (and in the present contribution) are studied in a regime where thermal noise is negligible, these systems still exhibit a dynamics that can be interpreted in terms of a thermostistical formalism associated with the  $S_q$  nonadditive entropies.

When the terms  $m_{1,2} \ddot{\mathbf{x}}$ , associated with inertial effects, are much smaller than the other terms appearing in (18), one deals with the overdamped motion regime, described by the equations of motion,

$$\begin{aligned} \dot{\mathbf{x}} &= -\frac{1}{\alpha_1} [\mathcal{G}_{11} \nabla \mathcal{P}_1 + \mathcal{G}_{12} \nabla \mathcal{P}_2 + \nabla W_1], \\ \dot{\mathbf{x}} &= -\frac{1}{\alpha_2} [\mathcal{G}_{21} \nabla \mathcal{P}_1 + \mathcal{G}_{22} \nabla \mathcal{P}_2 + \nabla W_2]. \end{aligned} \quad (19)$$

The particle densities  $\mathcal{P}_{1,2}(\mathbf{x}, t)$ , associated with a system consisting of two types of interacting particles, whose motion are governed by Eqs. (19), satisfy the continuity equations  $\frac{\partial \mathcal{P}_{1,2}}{\partial t} + \nabla \cdot \mathbf{J}_{1,2} = 0$ , where the density currents  $\mathbf{J}_{1,2}$  are

$$\begin{aligned} \mathbf{J}_1 &= -\frac{\mathcal{P}_1}{\alpha_1} [\mathcal{G}_{11} \nabla \mathcal{P}_1 + \mathcal{G}_{12} \nabla \mathcal{P}_2 + \nabla W_1], \\ \mathbf{J}_2 &= -\frac{\mathcal{P}_2}{\alpha_2} [\mathcal{G}_{21} \nabla \mathcal{P}_1 + \mathcal{G}_{22} \nabla \mathcal{P}_2 + \nabla W_2]. \end{aligned} \quad (20)$$

Making the identifications  $D_{ij} \rightarrow \frac{\mathcal{G}_{ij}}{\alpha_i}$  and  $U_i \rightarrow \frac{W_i}{\alpha_i}$ , the above-mentioned continuity equations can be recast as

$$\begin{aligned} \frac{\partial \mathcal{P}_1}{\partial t} &= \nabla \cdot [\mathcal{P}_1 \nabla (D_{11} \mathcal{P}_1 + D_{12} \mathcal{P}_2)] + \nabla \cdot [\mathcal{P}_1 \nabla U_1], \\ \frac{\partial \mathcal{P}_2}{\partial t} &= \nabla \cdot [\mathcal{P}_2 \nabla (D_{21} \mathcal{P}_1 + D_{22} \mathcal{P}_2)] + \nabla \cdot [\mathcal{P}_2 \nabla U_2], \end{aligned} \quad (21)$$

which constitutes a pair of coupled nonlinear Fokker-Planck equations. In summary, Eqs. (21) are the evolution equations governing the time-dependent densities of two types of interacting particles, constituting a confined many-body system in the overdamped motion regime. Note that when  $D_{12} = D_{21} = 0$  the equations reduce to a pair of uncoupled nonlinear Fokker-Planck equations of the form (1), with  $q = 0$ , in which either  $D_{11}$  or  $D_{22}$  have to be identified with  $2\mathcal{D}/\mathcal{P}_0$ . Even when  $D_{12}$  and  $D_{21}$  do not vanish, Eqs. (21) admit special limit solutions, for which one of the two densities  $\mathcal{P}_{1,2}$  vanish (for all  $\mathbf{x}$  and all  $t$ ), while the other (nonvanishing) density evolves according to the power-law nonlinear Fokker-Planck Eq. (1), with  $q = 0$ . In these special limit situations, only one of the two types of particles is present.

If  $D_{12} = D_{21}$  (consistently with Newton's principle of action and reaction), then the coupled nonlinear Fokker-Planck equations (21) admit an  $H$  theorem. Let us define

$$\begin{aligned} \Omega(\mathcal{P}_1, \mathcal{P}_2) &= \frac{\mathcal{P}_0}{2} (D_{11} \mathcal{P}_1 + D_{22} \mathcal{P}_2) \\ &\quad - \frac{1}{2} [D_{11} \mathcal{P}_1^2 + 2D_{12} \mathcal{P}_1 \mathcal{P}_2 + D_{22} \mathcal{P}_2^2], \end{aligned} \quad (22)$$

where  $\mathcal{P}_0$  is a constant with the same dimensions as  $\mathcal{P}_{1,2}$ . We define the functional

$$F = \langle U_1 \rangle + \langle U_2 \rangle - \left[ \int \Omega d^N \mathbf{x} \right], \quad (23)$$

where  $\langle U_1 \rangle = \int \mathcal{P}_1 U_1 d^N \mathbf{x}$  and  $\langle U_2 \rangle = \int \mathcal{P}_2 U_2 d^N \mathbf{x}$ . We then have

$$\begin{aligned} \frac{dF}{dt} &= \int \left[ \left( U_1 - \frac{\partial \Omega}{\partial \mathcal{P}_1} \right) \frac{\partial \mathcal{P}_1}{\partial t} + \left( U_2 - \frac{\partial \Omega}{\partial \mathcal{P}_2} \right) \frac{\partial \mathcal{P}_2}{\partial t} \right] d^N \mathbf{x} \\ &= \int \left\{ \left( \frac{\partial \Omega}{\partial \mathcal{P}_1} - U_1 \right) \nabla \cdot \left[ \mathcal{P}_1 \nabla \left( \frac{\partial \Omega}{\partial \mathcal{P}_1} - U_1 \right) \right] \right. \\ &\quad \left. + \left( \frac{\partial \Omega}{\partial \mathcal{P}_2} - U_2 \right) \nabla \cdot \left[ \mathcal{P}_2 \nabla \left( \frac{\partial \Omega}{\partial \mathcal{P}_2} - U_2 \right) \right] \right\} d^N \mathbf{x} \\ &= - \int \left\{ \mathcal{P}_1 \left[ \nabla \left( \frac{\partial \Omega}{\partial \mathcal{P}_1} - U_1 \right) \right]^2 \right. \\ &\quad \left. + \mathcal{P}_2 \left[ \nabla \left( \frac{\partial \Omega}{\partial \mathcal{P}_2} - U_2 \right) \right]^2 \right\} d^N \mathbf{x} \leq 0. \end{aligned} \quad (24)$$

In the last step in (24) we performed an integration by parts and made the standard assumption that  $\mathcal{P}_{1,2} \rightarrow 0$  fast enough when  $|\mathbf{x}| \rightarrow \infty$ , for the concomitant surface terms to vanish. Thus, we see that the functional  $F$  is a nonincreasing function of time. The densities that minimize the functional  $F$ , under the constraints imposed by the normalization of  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , are stationary solutions of the evolution equations (21). Indeed, introducing the Lagrange multipliers  $\mathcal{L}_1$  and

$\mathcal{L}_2$ , the above-mentioned optimization scheme leads to the variational problem  $\delta[F + \mathcal{L}_1 \int \mathcal{P}_1 d^N \mathbf{x} + \mathcal{L}_2 \int \mathcal{P}_2 d^N \mathbf{x}] = 0$ , in turn yielding the relations

$$\begin{aligned} D_{11}\mathcal{P}_1 + D_{12}\mathcal{P}_2 + U_1 &= \frac{1}{2}\mathcal{P}_0 D_{11} + \mathcal{L}_1 = \text{const}, \\ D_{22}\mathcal{P}_2 + D_{12}\mathcal{P}_1 + U_2 &= \frac{1}{2}\mathcal{P}_0 D_{22} + \mathcal{L}_2 = \text{const}. \end{aligned} \quad (25)$$

It is plain that the densities  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , satisfying relations (25), are also stationary solutions of (21).

The functional  $F$  given in (23) is reminiscent of a free energy. The quantity  $\int \Omega d^N \mathbf{x}$  is not, however, itself an entropic functional in the sense discussed, for instance, in Refs. [31–33]. This quantity is, nevertheless, closely related to the power-law, nonadditive entropic measure  $S_2$ . The minimization of the functional  $F$ , with respect to each one of the two densities  $\mathcal{P}_{1,2}$  (say,  $\mathcal{P}_i$ ) keeping the other density (say,  $\mathcal{P}_j$ ) fixed, is tantamount to the optimization of the entropy  $S_{q^*}$  (with  $q^* = 2$ ) under the constraints of normalization and the mean value of an effective potential  $\tilde{U}_i = D_{ij}\mathcal{P}_j + U_i$ . The effective potential  $\tilde{U}_i$  takes into account both the external potential  $U_i$  as well as the potential  $D_{ij}\mathcal{P}_j$  that particles of type  $i$  feel due to their interaction with particles of type  $j$ . Consequently, the stationary solutions of (21) can be formally cast as maximum  $S_q$ -entropy densities

$$\begin{aligned} \mathcal{P}_1 &= A_1 \exp_q[-\beta_1 \tilde{U}_1] = A_1 \exp_q[-\beta_1(U_1 + D_{12}\mathcal{P}_2)], \\ \mathcal{P}_2 &= A_2 \exp_q[-\beta_2 \tilde{U}_2] = A_2 \exp_q[-\beta_2(U_2 + D_{12}\mathcal{P}_1)], \end{aligned} \quad (26)$$

with  $A_1\beta_1 D_{11} = A_2\beta_2 D_{22} = 1$  and  $q = 2 - q^* = 0$ .

The structure of the intertwined densities (26) has a curious resemblance with the structure of the probability distributions discussed by Harré *et al.* in Ref. [38] that result from a maximum entropy approach to economic games. As happens with (26), the probability distributions discussed in Ref. [38] are interlinked maximum entropy distributions, each one maximizing an entropy under a constraint that is a linear function of the other distribution. Possible connections between the dynamics leading to the densities (26) and the dynamics generating the maximum entropy distributions studied in Ref. [38] is a subject worth exploring.

It is worthwhile now to compare our present multispecies evolution equations with the Fokker-Planck equation for systems of two particles discussed, for instance, by Risken in Ref. [39]. The two-particle Fokker-Planck equation considered in Ref. [39] governs the time evolution of a joint probability density  $P(\mathbf{x}_1, \mathbf{x}_2, t)$ , where  $\mathbf{x}_1$  and  $\mathbf{x}_2$  denote the locations of the first and the second particles, respectively, and  $P(\mathbf{x}_1, \mathbf{x}_2, t)d^N \mathbf{x}_1 d^N \mathbf{x}_2$  is the joint probability of having the first particle within the volume element  $d^N \mathbf{x}_1$  (centered at  $\mathbf{x}_1$ ), and the second particle within the volume element  $d^N \mathbf{x}_2$  (centered at  $\mathbf{x}_2$ ). In contrast with that scenario, the system (21) of coupled, nonlinear Fokker-Planck equations governs the coevolution of two densities  $\mathcal{P}_1(\mathbf{x}, t)$  and  $\mathcal{P}_2(\mathbf{x}, t)$ , respectively, describing the evolving spatial distribution of particles of type 1 and type 2.

#### IV. A ONE-DIMENSIONAL EXAMPLE

We shall now consider a one-dimensional example admitting an exact analytical stationary solution. We are going to

consider a pair of evolution equations of the form

$$\begin{aligned} \frac{\partial \mathcal{P}_1}{\partial t} &= \frac{\partial}{\partial x} \left[ \mathcal{P}_1 \frac{\partial}{\partial x} (D_{11}\mathcal{P}_1 + D_{12}\mathcal{P}_2) \right] + \frac{\partial}{\partial x} \left[ \mathcal{P}_1 \frac{\partial U_1}{\partial x} \right], \\ \frac{\partial \mathcal{P}_2}{\partial t} &= \frac{\partial}{\partial x} \left[ \mathcal{P}_2 \frac{\partial}{\partial x} (D_{12}\mathcal{P}_1 + D_{22}\mathcal{P}_2) \right] + \frac{\partial}{\partial x} \left[ \mathcal{P}_2 \frac{\partial U_2}{\partial x} \right], \end{aligned} \quad (27)$$

which constitute a one-dimensional instance of (21). We assume harmonic external confining potentials,  $U_1 = \frac{1}{2}C_1 x^2$  and  $U_2 = \frac{1}{2}C_2 x^2$ , with  $C_{1,2} > 0$ . The evolution equations (27) are continuity equations for the two densities  $\mathcal{P}_1(x, t)$  and  $\mathcal{P}_2(x, t)$ . They preserve the norms  $N_1 = \int \mathcal{P}_1(x, t) dx$  and  $N_2 = \int \mathcal{P}_2(x, t) dx$  [remember that  $\mathcal{P}_{1,2}(x, t)$  represent the physical spatial densities of the particles of type 1 and 2 and, consequently, are not necessarily equal to 1].

The coupled, nonlinear Fokker-Planck equations (27) admit stationary solutions  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , both of which exhibit, within a restricted range of  $x$  values, a  $q$ -Gaussian form. We have

$$\begin{aligned} \mathcal{P}_1 &= \mathcal{A}_1 \exp_q[-b_1 x^2], \quad (|x| \leq x_c), \\ \mathcal{P}_2 &= \mathcal{A}_2 \exp_q[-b_2 x^2], \quad (|x| \leq x_c), \end{aligned} \quad (28)$$

where  $\mathcal{A}_{1,2}$  and  $b_{1,2}$  are positive constants and  $q = 0$ . The stationary solutions have the  $q$ -Gaussian shape within the range of  $x$  values corresponding to  $|x| \leq x_c$ , where  $x_c = \min(x_1, x_2)$ , with  $x_{1,2} = \sqrt{1/b_{1,2}}$ . Without loss of generality we shall assume that  $x_c = x_1$  (that is, we assume that  $x_1 \leq x_2$  and  $b_1 \geq b_2$ ). For  $|x| > x_c$  the solutions have the form

$$\begin{aligned} \mathcal{P}_1 &= \mathcal{A}_1 \exp_q[-b_1 x^2], \quad (|x| > x_c), \\ \mathcal{P}_2 &= \mathcal{A}'_2 \exp_q[-b'_2 x^2], \quad (|x| > x_c), \end{aligned} \quad (29)$$

with  $\mathcal{A}'_2$  and  $b'_2$  positive constants and  $q = 0$ . That is, the stationary density  $\mathcal{P}_1(x)$  is a  $q$  Gaussian for all values of  $x$ , while  $\mathcal{P}_2(x)$  is a piecewise continuous function defined in terms of two different  $q$  Gaussians. Note that both stationary densities have a compact support. Indeed,  $\mathcal{P}_1 = 0$ , for  $|x| > 1/\sqrt{b_1}$ , and  $\mathcal{P}_2 = 0$ , for  $|x| > 1/\sqrt{b'_2}$ .

The ansatz (28) and (29) is characterized by the six parameters  $\mathcal{A}_{1,2}$ ,  $\mathcal{A}'_2$ ,  $b_{1,2}$ , and  $b'_2$ . To determine completely the stationary solution of the Fokker-Planck equations (27) one has to find the values of these parameters. Inserting (28) and (29) into (27) one sees that the appropriate values of the parameters satisfy the equations

$$\begin{aligned} 2(D_{11}\mathcal{A}_1 b_1 + D_{12}\mathcal{A}_2 b_2) &= C_1, \\ 2(D_{12}\mathcal{A}_1 b_1 + D_{22}\mathcal{A}_2 b_2) &= C_2, \\ \mathcal{A}'_2 [1 - b'_2 x_c^2] &= \mathcal{A}_2 [1 - b_2 x_c^2], \\ 2D_{22}\mathcal{A}'_2 b'_2 &= C_2. \end{aligned} \quad (30)$$

It is worthwhile, for the sake of clarity, to make a few comments on the above equations and their relationship with the form of the ansatz (28) and (29). The first two equations guarantee that the densities  $\mathcal{P}_1$  and  $\mathcal{P}_2$  comply with the Fokker-Planck equations (27) within the spatial region  $|x| \leq x_c$ , where both densities are different from zero. The third equation is the matching condition arising from the continuity of the density  $\mathcal{P}_2$  at  $x_c$ . One has this condition because the

density  $\mathcal{P}_2$  is defined in a piecewise form, with different analytical expressions in the region  $|x| \leq x_c$  and in the region  $|x| > x_c$ . Finally, the fourth equation in (30) guarantees that  $\mathcal{P}_2$  satisfies (28) and (29) in the region  $|x| > x_c$ , where  $\mathcal{P}_1$  is equal to zero.

The solutions of Eqs. (30) are

$$\begin{aligned} \mathcal{A}_1 b_1 &= \frac{D_{22}C_1 - D_{12}C_2}{2(D_{11}D_{22} - D_{12}^2)}, \\ \mathcal{A}_2 b_2 &= \frac{D_{11}C_2 - D_{12}C_1}{2(D_{11}D_{22} - D_{12}^2)}, \\ \mathcal{A}'_2 b'_2 &= \frac{C_2}{2D_{22}}, \\ \mathcal{A}'_2 &= \mathcal{A}_2 \left(1 - \frac{b_2}{b_1}\right) + \frac{C_2}{2b_1 D_{22}}, \end{aligned} \quad (31)$$

provided that  $D_{11}D_{22} - D_{12}^2 \neq 0$ . This corresponds to a physical solution (that can be realized with positive values of the parameters  $\mathcal{A}_{1,2}$ ,  $\mathcal{A}'_2$ ,  $b_{1,2}$ , and  $b'_2$ , corresponding to positive densities  $\mathcal{P}_{1,2}$ ), if the quantities  $D_{11}D_{22} - D_{12}^2$ ,  $D_{22}C_1 - D_{12}C_2$ , and  $D_{11}C_2 - D_{12}C_1$  are all different from zero and have the same sign. The norms corresponding to the two types of particles are

$$\begin{aligned} N_1 &= \frac{4\mathcal{A}_1}{3\sqrt{b_1}}, \\ N_2 &= \frac{2\mathcal{A}_2}{\sqrt{b_1}} \left[1 - \frac{b_2}{3b_1}\right] + 2\mathcal{A}'_2 \left\{ \frac{2}{3\sqrt{b'_2}} - \frac{1}{\sqrt{b_1}} \left[1 - \frac{b'_2}{3b_1}\right] \right\}. \end{aligned} \quad (32)$$

The number of particles of type 2 within the overlap region where both particles coexist is

$$N_2^* = 2\mathcal{A}_2 \left[ \frac{1}{\sqrt{b_1}} - \frac{\mathcal{A}_2 b_2}{3b_1^{3/2}} \right]. \quad (33)$$

The four relations (31) determine a biparametric family of stationary solutions of (28), since we have only four equations for the six parameters  $\mathcal{A}_1$ ,  $\mathcal{A}_2$ ,  $\mathcal{A}'_2$ ,  $b_1$ ,  $b_2$ , and  $b'_2$ . The members of this family of solutions correspond to different normalizations for the densities  $\mathcal{P}_1$  and  $\mathcal{P}_2$ .

The stationary densities (28) and (29), and some of their main properties, are illustrated in Figs. 1–4. In the examples represented in these figures we choose  $C_1 = C_2 = C$ ,  $D_{12} = D_{21} = \frac{1}{2}D_{11}$ , and  $D_{22} = \delta D_{11}$ , where  $D_{11} > 0$  and  $\delta > \frac{1}{2}$  is a dimensionless parameter. The normalization  $N_1$  was set equal to 1 and we consider  $b_1 \geq b_2$ . The quantity  $(D_{11}/C)^{1/3}$  has dimensions of length. It provides a natural length scale for the stationary solutions of (27). Therefore, in Figs. 1–4 we used  $(D_{11}/C)^{1/3}$  as our unit of length. Expressed using this length unit, and under the above explained conditions, the values of the physical parameters determining the stationary solutions satisfy  $\mathcal{A}_1 = \left[\frac{9}{16} \left(\frac{2\delta-1}{4\delta-1}\right)\right]^{1/3}$ ,  $b_1 = \left[\frac{16}{9} \left(\frac{2\delta-1}{4\delta-1}\right)^2\right]^{1/3}$ , and  $\mathcal{A}_2 = \frac{1}{b_2(4\delta-1)}$ . The values of the remaining parameters,  $\mathcal{A}'_2$  and  $b'_2$ , can then be expressed in terms of the values of the previously found ones, using the last two relations in (31). This completely determines a subfamily of stationary

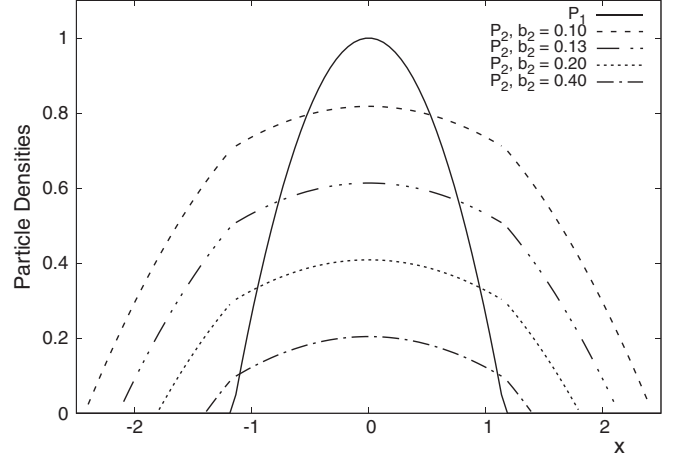


FIG. 1. Particle densities for particles of type 1 ( $\mathcal{P}_1$ ) and 2 ( $\mathcal{P}_2$ ), for different values of  $b_2$ . Length is measured in units of  $(D_{11}/C)^{1/3}$  and the particle densities in units of  $\mathcal{A}_1$ . We set  $N_1 = 1$  and  $\delta = 5$ . Other model parameters are obtained from these, as explained in the main text of Sec. IV. For these parameter values and units,  $x_c = 1/\sqrt{b_1} = 1.17$  marks the cutoff of the density of type-1 particles. It is also the point where  $\mathcal{P}_2$  ceases to be defined by Eq. (28) and starts to be described by Eq. (29).

solutions of (28), parameterized by  $b_2$ , whose properties are shown in Figs. 1–4.

In Fig. 1, the densities  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are plotted for different values of the parameter  $b_2$ . Note that the density  $\mathcal{P}_1$  is a  $q$  Gaussian, but the density  $\mathcal{P}_2$  is not. Figure 2 depicts the ratio of the number of type-2 particles within the overlap region in which both particles coexist,  $N_2^*$  [see Eq. (33)], to the total number of type-2 particles, as a function of the quotient  $N_2/N_1$  between the total number of particles of type 2 and the total number of particles of type 1. We see that the fraction of particles of type 2 in the coexistence region decreases monotonically with the ratio  $N_2/N_1$ . Figure 3 exhibits the behavior of the quotient of the total number of type-2 particles,  $N_2$  [see

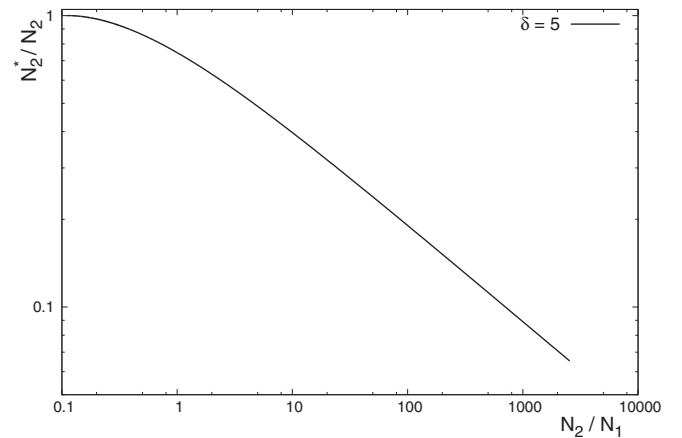


FIG. 2. Ratio of the number of type-2 particles within the overlap region in which both particles coexist,  $N_2^*$  [see Eq. (33)], to the total number of type-2 particles, as a function of  $N_2/N_1$ . All depicted quantities are dimensionless. Values of model parameters are the same as specified for Fig. 1.

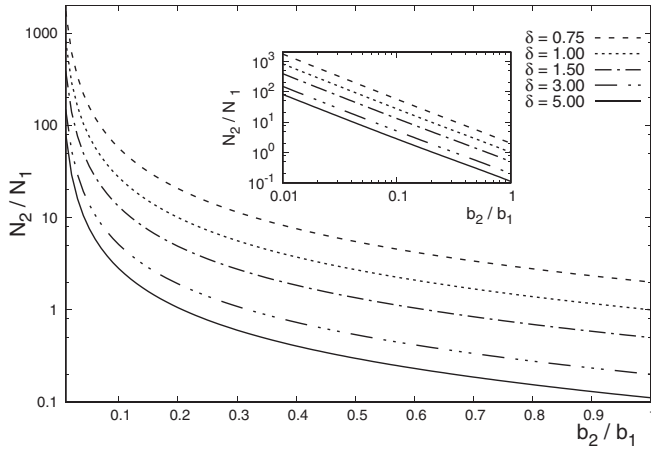


FIG. 3. Ratio of the number of type-2 particles,  $N_2$  [Eq. (32)], to the number of type-1 particles, as a function of  $b_2/b_1$ , for different values of  $\delta$ . All depicted quantities are dimensionless. The other model parameters are obtained as explained in Sec. IV. We note in the inset of this figure that this relation is described by power laws.

Eqs. (32)], to the total number of type-1 particles  $N_1$ , as a function of  $b_2/b_1$ . In Fig. 4, we plot the ratio of the number of type-2 particles in the overlap region,  $N_2^*$ , to the total number of type-2 particles, as a function of  $b_2/b_1$ .

It is worth mentioning that the analysis carried out in this section can be extended to an  $N$ -dimensional scenario. The  $N$ -dimensional counterpart of the system investigated in this section is described by the equations of motion,

$$\begin{aligned} \frac{\partial \mathcal{P}_1}{\partial t} &= \nabla \cdot [\mathcal{P}_1 \nabla (D_{11} \mathcal{P}_1 + D_{12} \mathcal{P}_2)] + \nabla \cdot [\mathcal{P}_1 \nabla U_1], \\ \frac{\partial \mathcal{P}_2}{\partial t} &= \nabla \cdot [\mathcal{P}_2 \nabla (D_{12} \mathcal{P}_1 + D_{22} \mathcal{P}_2)] + \nabla \cdot [\mathcal{P}_2 \nabla U_2]. \end{aligned} \quad (34)$$

It admits an exact stationary solution having essentially the same shape as the one described by Eqs. (28) and (29). The

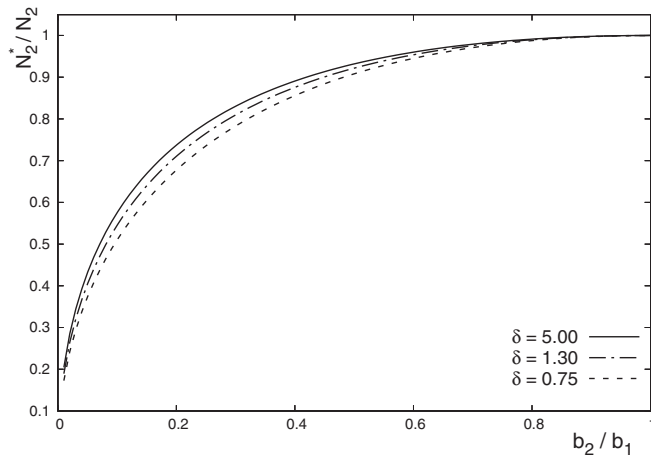


FIG. 4. Ratio of the number of type-2 particles in the overlap region,  $N_2^*$  [see Eq. (33)], to the total number of type-2 particles as a function of  $b_2/b_1$ . All depicted quantities are dimensionless. Values of model parameters are the same as specified for Fig. 3.

stationary solution of (34) is given by

$$\begin{aligned} \mathcal{P}_1 &= \mathcal{A}_1 \exp_q[-b_1 x^2], & (|x| \leq x_c), \\ \mathcal{P}_2 &= \mathcal{A}_2 \exp_q[-b_2 x^2], & (|x| \leq x_c), \end{aligned} \quad (35)$$

and

$$\begin{aligned} \mathcal{P}_1 &= \mathcal{A}'_1 \exp_q[-b_1 x^2], & (|x| > x_c), \\ \mathcal{P}_2 &= \mathcal{A}'_2 \exp_q[-b'_2 x^2], & (|x| > x_c), \end{aligned} \quad (36)$$

with  $x_c = 1/\sqrt{b_1}$  and the parameters  $\mathcal{A}'_2$  and  $b'_2$  determined by (30). Thus, we see that the basic structure of the solution (28) and (29) is not restricted to one-dimensional scenarios. We focused on the one-dimensional instance only for illustrative purposes.

## V. MULTISPECIES NONLINEAR FOKKER-PLANCK EQUATIONS

We shall now investigate a more general Fokker-Planck dynamics for a system consisting of  $L$  types of particles, interacting through short-range forces. The spatial distribution of the particles of type  $i$  is given by the density  $\mathcal{P}_i(\mathbf{x}, t)$ , ( $i = 1, \dots, L$ ). These densities evolve according to the set of coupled nonlinear Fokker-Planck equations

$$\begin{aligned} \frac{\partial \mathcal{P}_i}{\partial t} &= \nabla \cdot \left[ \mathcal{P}_i \nabla \left( \sum_{j=1}^L D_{ij} \mathcal{P}_j \right) \right] \\ &+ \nabla \cdot [\mathcal{P}_i \nabla U_i], \quad i = 1, \dots, L, \end{aligned} \quad (37)$$

where the coefficients  $D_{ij}$  are related to the interaction forces between particles of type  $i$  and particles of type  $j$ . We assume that  $D_{ij} = D_{ji}$ . The potential  $U_i(\mathbf{x})$  is the external confining potential acting on particles of type  $i$ . For  $L = 2$  the system (37) reduces to the one discussed in Sec. III.

The system of coupled Fokker-Planck equations (37) could be analyzed in a similar way as that used in Sec. III. It is more useful, however, to cast the evolution equations (37) in a more abstract way that clarifies some of its most fundamental properties. We first introduce the functions

$$\Omega(\mathcal{P}_1, \dots, \mathcal{P}_L) = \frac{\mathcal{P}_0}{2} \left( \sum_{i=1}^L D_{ii} \mathcal{P}_i \right) - \frac{1}{2} \left( \sum_{i,j=1}^L D_{ij} \mathcal{P}_i \mathcal{P}_j \right), \quad (38)$$

and

$$\Psi_i(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_L) = -\frac{\partial \Omega}{\partial \mathcal{P}_i}, \quad i = 1, \dots, L. \quad (39)$$

Note that  $\Omega = 0$  and  $\Psi_1 = \dots = \Psi_L = 0$  when  $\mathcal{P}_1 = \mathcal{P}_2 = \dots = \mathcal{P}_L = 0$ . For  $L = 2$  the function and  $\Omega$  defined in (38) coincides with the one discussed in Sec. III, given by (22).

With the definitions (38) and (39), the set of coupled evolution equations (37) can be cast as

$$\frac{\partial \mathcal{P}_i}{\partial t} = \nabla \cdot [\mathcal{P}_i (\nabla \Psi_i)] + \nabla \cdot [\mathcal{P}_i (\nabla U_i)], \quad i = 1, \dots, L. \quad (40)$$

The quantity  $\Psi_i(\mathbf{x})$  can be interpreted as the effective potential acting on the particles of type  $i$  due to their interaction with other particles, both of the same and of different types.

There are compelling reasons for rewriting the system of coupled nonlinear Fokker-Planck equations (37) in the more abstract form (40). First, some of the most fundamental features of the evolution equation (40), such as it admitting an  $H$  theorem, do not depend on the detailed, particular form (38) of the function  $\Omega$ . Those essential features follow directly from the general structure encapsulated in Eqs. (39) and (40). Consequently, the results obtained exploiting this general structure are not restricted to the evolution equations (37). They hold for a wide family of coupled nonlinear Fokker-Planck equations and are therefore applicable to a variety of scenarios, both within and without  $S_q$  thermostats. For example, within the general setting described by Eqs. (39) and (40), it may be possible to consider multispecies extensions of the systems considered in Ref. [12], consisting of particles interacting through a force obeying the power law

$$\mathcal{F}(r) = \mathcal{F}_c \left[ \frac{r}{r_0} \right]^{-\gamma}, \quad (41)$$

where  $\gamma$  is a positive dimensionless number and  $r_0$  and  $\mathcal{F}_c$  are positive constants with dimensions of length and force, respectively. It was shown in Ref. [12] that, under appropriate conditions, the overdamped motion of particles interacting through the force law (41) can be described by nonlinear Fokker-Planck equations involving nonlinear diffusion terms of the form  $\mathcal{D}\nabla[\mathcal{P}\nabla[(\frac{\mathcal{P}}{\mathcal{P}_0})^{1-q}]]$ , with the parameter  $q$  related to the exponent  $\gamma$  and to the spatial dimension  $N$ , through

$$q = 1 - \frac{\gamma - 1}{N} \quad (42)$$

(for an intuitive derivation of this relation see Ref. [40]). Therefore, the results reported in Ref. [12] suggest that the overdamped dynamics of systems comprising two or more species of particles interacting through the power-law forces (41) may be described by coupled evolution equations of the form (40), with the potentials  $\Psi$  derived from a function akin to (38), but involving a combination of powers of the densities with exponents  $1 - q$ . The investigation of this possibility is worth pursuing, although it is beyond the scope of the present

work. The general form (40) of coupled nonlinear Fokker-Planck equations provides also a framework for studying in multispecies scenarios the thermostats aspects of general nonlinear Fokker-Planck equations, extending the analysis performed in Ref. [8], where it was shown that a large family of nonlinear Fokker-Planck equations admit  $H$  theorems in terms of free-energy functionals related to generalized entropic measures.

The system of evolution equations (40) constitutes a set of coupled continuity equations,

$$\frac{\partial \mathcal{P}_i}{\partial t} + \nabla \cdot \mathcal{J}_i = 0, \quad i = 1, \dots, L, \quad (43)$$

where the flow  $\mathcal{J}_i$  corresponding to particles of type  $i$  is given by

$$\mathcal{J}_i = -\mathcal{P}_i \nabla(\Psi_i + U_i), \quad i = 1, \dots, L. \quad (44)$$

The evolution equations (40) admit stationary solutions  $\mathcal{P}_i^{(st)}(\mathbf{x})$  satisfying

$$\mathcal{P}_i^{(st)}(\mathbf{x}) \nabla(\Psi_i + U_i) = 0, \quad i = 1, \dots, L, \quad (45)$$

with all the fluxes  $\mathcal{J}_i$  vanishing. For these solutions, within any connected spatial region where the density of the  $i$ -type particles does not vanish, one has

$$\Psi_i[\mathcal{P}_1^{(st)}, \dots, \mathcal{P}_L^{(st)}] = \mathcal{L}_i - U_i(\mathbf{x}), \quad (46)$$

where  $\mathcal{L}_i$  is an integration constant.

#### A. $H$ theorem

We are now going to explore the possibility of formulating an  $H$  theorem for the more general form of the multispecies, nonlinear Fokker-Planck equations given by (40). To that effect, we introduce the free-energy-like functional

$$F = \int \left[ \left( \sum_{i=1}^L U_i \mathcal{P}_i \right) - \Omega(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_L) \right] d^N \mathbf{x}. \quad (47)$$

The time derivative of  $F$  is

$$\begin{aligned} \frac{dF}{dt} &= \int \left[ \sum_{i=1}^L \left( U_i - \frac{\partial \Omega}{\partial \mathcal{P}_i} \right) \frac{\partial \mathcal{P}_i}{\partial t} \right] d^N \mathbf{x} = \int \left[ \sum_{i=1}^L \left( -\frac{\partial \Omega}{\partial \mathcal{P}_i} + U_i \right) \{ \nabla \cdot [\mathcal{P}_i(\nabla \Psi_i)] + \nabla \cdot [\mathcal{P}_i(\nabla U_i)] \} \right] d^N \mathbf{x} \\ &= \int \left[ \sum_{i=1}^L (\Psi_i + U_i) \{ \nabla \cdot [\mathcal{P}_i(\nabla \Psi_i)] + \nabla \cdot [\mathcal{P}_i(\nabla U_i)] \} \right] d^N \mathbf{x} = - \int \left[ \sum_{i=1}^L \mathcal{P}_i (\nabla \Psi_i + \nabla U_i)^2 \right] d^N \mathbf{x} \leq 0, \end{aligned} \quad (48)$$

where, to obtain the last equality in the above equation, we performed an integration by parts, under the standard assumption that the densities  $\mathcal{P}_i$  decrease fast enough when  $|\mathbf{x}| \rightarrow \infty$ , so that the surface terms arising from the integration by parts vanish.

A set of densities  $\{\mathcal{P}_1(\mathbf{x}), \dots, \mathcal{P}_L(\mathbf{x})\}$  that makes the functional (47) stationary, under the constraints imposed by the normalization of each of the densities  $\mathcal{P}_i$ , constitutes a stationary solution of the evolution equations (40). Indeed, introducing appropriate Lagrange multipliers  $\mathcal{L}_i$  ( $i = 1, \dots, L$ ),

associated with the above-mentioned constraints, one sees that a set of densities for which the first variation of the functional  $(\sum_{i=1}^L U_i) - (\int \Omega d^N \mathbf{x}) - (\sum_{i=1}^L \mathcal{L}_i \int \mathcal{P}_i d^N \mathbf{x})$  vanishes,

$$\delta \int d^N \mathbf{x} \left\{ \left( \sum_{i=1}^L U_i \mathcal{P}_i \right) - \Omega(\mathcal{P}_1, \dots, \mathcal{P}_L) - \left( \sum_{i=1}^L \mathcal{L}_i \mathcal{P}_i \right) \right\} = 0, \quad (49)$$

comply with the relations

$$-\frac{\partial \Omega}{\partial \mathcal{P}_i} + U_i - \mathcal{L}_i = 0, \quad i = 1, \dots, L, \quad (50)$$



which are equivalent to the Eqs. (46) for the stationary solutions of (40).

## VI. CONCLUSIONS

We investigated the dynamics of confined, overdamped many-body systems consisting of more than one type of interacting particles or agents. We showed that a mean-field approach to this kind of systems leads to a set of coupled, multispecies nonlinear Fokker-Planck equations, governing the densities  $\mathcal{P}_i(\mathbf{x}, t)$  associated with each of the different types of particles involved. These evolution equations, in turn, allowed us to establish some features of the equilibrium and out of equilibrium thermostistical behavior of the aforementioned physical systems. In particular, we proved that the set of coupled, nonlinear Fokker-Planck equations satisfy an  $H$  theorem. That is, we obtained a free-energy-like functional  $F$  of the time-dependent densities associated with the different types of particles that always exhibits a nonpositive time derivative and, consequently, provide a quantitative manifestation of the “arrow of time” for the types of systems under consideration in the present work. The particle densities that make the functional  $F$  stationary, under the constraints of normalization, constitute stationary solutions of the aforementioned set of coupled evolution equations. We investigated in detail a particular system comprising two types of particles interacting via short-range forces that admits exact semianalytical stationary solutions.

The results reported here could be applied to the study of multispecies effects in the thermostistical properties of various systems recently addressed in the research literature, such as interacting vortices in type-II superconductors [13]

and overdamped many body systems with power-law interactions [12]. Another possible subject to which our results can be applied is to the recently reported connection between scaling laws in granular materials and the  $q$ -Gaussian solutions of power-law, nonlinear Fokker-Planck equations [14]. In the present work we emphasized applications to mechanical systems. However, other applications of the coupled, nonlinear Fokker-Planck equations discussed here immediately come to mind. Indeed, these equations provide a natural tool for extending to multispecies scenarios nonlinear models for the dispersal of biological populations [25–27]. Finally, an intriguing similarity between the form of the stationary solutions of the coupled, nonlinear Fokker-Planck equations addressed here, on the one hand, and the maximum entropy solutions of game-theoretical models advanced in Ref. [38], on the other hand, may also deserve to be further explored.

The present developments overcome a serious limitation of previous works that use the nonlinear Fokker-Planck equations to study interacting many-body complex systems. Virtually all those works were restricted to systems consisting of only one type of particles or agents. It is plain that this constitutes a severe restriction for the application of the nonlinear Fokker-Planck approach to the study of complex systems, most of which do contain particles or agents of diverse types, having different physical properties and, consequently, experiencing different intra- and interspecific forces.

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