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Lie structure on the Hochschild cohomology of a family of subalgebras of the Weyl algebra

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Abstract. For each nonzero $h \in \mathbb{F}[x]$, where $\mathbb{F}$ is a field, let A_h be the unital associative algebra generated by elements x, y, satisfying the relation $yx - xy = h$. This gives a parametric family of subalgebras of the Weyl algebra A_1, containing many well-known algebras which have previously been studied independently. In this paper, we give a full description of the Hochschild cohomology $\mathsf{HH}^\bullet(\mathsf{A}_h)$ over a field of an arbitrary characteristic. In case $\mathbb{F}$ has a positive characteristic, the center $\mathsf{Z}(\mathsf{A}_h)$ of A_h is nontrivial and we describe $\mathsf{HH}^\bullet(\mathsf{A}_h)$ as a module over $\mathsf{Z}(\mathsf{A}_h)$. The most interesting results occur when $\mathbb{F}$ has a characteristic 0. In this case, we describe $\mathsf{HH}^\bullet(\mathsf{A}_h)$ as a module over the Lie algebra $\mathsf{HH}^1(\mathsf{A}_h)$ and find that this action is closely related to the intermediate series modules over the Virasoro algebra. We also determine when $\mathsf{HH}^\bullet(\mathsf{A}_h)$ is a semisimple $\mathsf{HH}^1(\mathsf{A}_h)$-module.

1. Introduction

The Weyl algebra became an object of interest in the 1920s, together with the development of the quantum theories in physics. It has played an important role in $\mathcal{D}$-module theory. It is well known that the Weyl algebra is the algebra of differential operators over the one-dimensional affine space, where x acts by multiplication and y corresponds to the usual derivative $\frac{\partial}{\partial x}$. Of course, replacing this last action by $h \cdot \frac{\partial}{\partial x}$ for any fixed polynomial $h(x) \in \mathbb{F}[x]$ also corresponds to a derivation. If $h = 0$, the derivation would annihilate everywhere, so we will not consider this case. Precisely, the algebras we consider are Ore extensions of the polynomial algebra in one variable, whose only other possible Ore extensions – here we allow $h = 0$ – are a quantum plane or a quantum Weyl algebra.

Given a field $\mathbb{F}$ and a nonzero polynomial $h(x) \in \mathbb{F}[x]$, let A_h be the unital associative $\mathbb{F}$-algebra with two generators x and $\hat{y}$, subject to the relation $\hat{y}x - x\hat{y} = h$. There is an embedding of A_h in A_1 given by $x \mapsto x$, $\hat{y} \mapsto yh$, as in [2, Lem. 3.1]. We will thus henceforth take $\hat{y} = yh$ and consider A_h as the unital subalgebra of the Weyl algebra A_1 generated by x and $\hat{y} = yh$, where $[y, x] = 1$ and $[\hat{y}, x] = h$.

The family A_h parametrizes many well-known algebras, which we study simultaneously. As previously said, for $h = 1$, we retrieve the first Weyl algebra A_1. Other particular cases have attracted attention, such as A_x, which is the universal enveloping algebra of the two-dimensional non-abelian Lie algebra, and A_{x^2}, known as the *Jordan plane*, which is a

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Nichols algebra of non-diagonal type. More generally, taking $h = x^n$ with $n \geq 3$ and setting x in degree 1 and $\hat{y}$ in degree $n-1$, then, as observed by Stephenson [12], A_{x^n} is an Artin–Schelter regular of global dimension two, although it does not admit any regrading so that it becomes generated in degree one.

The aim of this article is to describe the structure – given by the Gerstenhaber bracket – of the Hochschild cohomology spaces $\mathsf{HH}^\bullet(\mathsf{A}_h)$ as Lie modules over $\mathsf{HH}^1(\mathsf{A}_h)$.

The Hochschild cohomology $\mathsf{HH}^\bullet(\mathsf{A}_h) = \bigoplus_{n\geq 0} \mathsf{HH}^n(\mathsf{A}_h)$ can be made into a Lie module for the Lie algebra $\mathsf{HH}^1(\mathsf{A}_h)$ of outer derivations of A_h, under the *Gerstenhaber bracket*. By the Hochschild–Kostant–Rosenberg theorem, under suitable assumptions, this bracket is the generalization to higher degrees of the Schouten–Nijenhuis bracket. In our setting, this is especially interesting in case $\mathsf{char}(\mathbb{F}) = 0$ and $\mathsf{gcd}(h, h') \neq 1$ as then the description of $\mathsf{HH}^1(\mathsf{A}_h)$ is related to the Witt algebra and, as we shall see, the $\mathsf{HH}^1(\mathsf{A}_h)$-Lie module structure of $\mathsf{HH}^2(\mathsf{A}_h)$ can be described in terms of the representation theory of the Witt algebra.

The paper is organized as follows. In Section 2, we prove a few technical lemmas about commutators, while in Section 3, we construct the minimal resolution of A_h as an A_h-bimodule. Since it has length 2, $\mathsf{HH}^i(\mathsf{A}_h)$ is zero for i greater than 2. In particular, this resolution allows us to give an explicit description of $\mathsf{HH}^2(\mathsf{A}_h)$ in a positive characteristic. The aim of Section 4 is to complete the construction of a contracting homotopy for the minimal resolution, and in Section 5, we recall the method developed by Suárez-Álvarez [13] to compute the brackets $[\mathsf{HH}^1(A), \mathsf{HH}^n(A)]$ for any associative unital algebra A. This allows us to obtain in Section 6 the main results of this article: the description, in a characteristic zero, of the Lie structure of $\mathsf{HH}^\bullet(\mathsf{A}_h)$ as an $\mathsf{HH}^1(\mathsf{A}_h)$-Lie module.

Below we summarize, in simplified form, the main results of the paper.

Theorem A (Theorem 3.24). *Assume that* $\mathsf{char}(\mathbb{F}) = p > 0$ *and let* $\mathsf{Z}(\mathsf{A}_h)$ *denote the center of* A_h. *Then,* $\mathsf{HH}^2(\mathsf{A}_h)$ *is a free* $\mathsf{Z}(\mathsf{A}_h)$*-module if and only if* $\mathsf{gcd}(h, h') = 1$. *In this case,* $\mathsf{HH}^2(\mathsf{A}_h)$ *has rank one over* $\mathsf{Z}(\mathsf{A}_h)$ *and, moreover,* $\mathsf{HH}^\bullet(\mathsf{A}_h)$ *is a free* $\mathsf{Z}(\mathsf{A}_h)$*-module.*

In a positive characteristic, an explicit description of $\mathsf{HH}^2(\mathsf{A}_h)$ is given in Theorem 3.21, although this is a bit involved. On the other hand, in a characteristic zero, $\mathsf{HH}^2(\mathsf{A}_h)$ can be presented as a space of polynomials.

Theorem B (cf. Corollary 3.11 and Remark 3.13). *Assume that* $\mathsf{char}(\mathbb{F}) = 0$. *There are isomorphisms*

$$\mathsf{HH}^2(\mathsf{A}_h) \cong \mathsf{A}_h / \mathsf{gcd}(h, h')\mathsf{A}_h \cong \mathsf{D}[\hat{y}],$$

where $\mathsf{D} = (\mathbb{F}[x]/\mathsf{gcd}(h, h')\mathbb{F}[x])$. *In particular,* $\mathsf{HH}^2(\mathsf{A}_h) = 0$ *if and only if* h *is a separable polynomial; otherwise,* $\mathsf{HH}^2(\mathsf{A}_h)$ *is infinite-dimensional.*

We also describe in detail $\mathsf{HH}^2(\mathsf{A}_h)$ as an $\mathsf{HH}^1(\mathsf{A}_h)$-Lie module. Please refer to Theorem 6.2 below, or [1, Thm. 5.1, Prop. 5.9], for a detailed description of the structure of $\mathsf{HH}^1(\mathsf{A}_h)$ as a Lie algebra.

Theorem C (cf. Theorem 6.19). *Assume that* $\mathsf{char}(\mathbb{F}) = 0$ *and* $\mathsf{gcd}(h, h') \neq 1$. *Let* $m_h + 1$ *be the largest exponent occurring in the decomposition of* h *in* $\mathbb{F}[x]$ *into irreducible factors. The structure of* $\mathsf{HH}^2(\mathsf{A}_h)$ *as a Lie module, under the Gerstenhaber bracket, for the Lie algebra* $\mathsf{HH}^1(\mathsf{A}_h)$ *is as follows:*

(a) *there is a filtration of length* m_h *by* $\mathsf{HH}^1(\mathsf{A}_h)$*-submodules,*

$$\mathsf{HH}^2(\mathsf{A}_h) = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_{m_h-1} \supsetneq P_{m_h} = 0,$$

such that each factor P_i/P_{i+1} *is semisimple;*

(b) *the irreducible summands of each* P_i/P_{i+1} *can be naturally seen as obtained from intermediate series modules for the Witt algebra, under a suitable finite field extension of* $\mathbb{F}$*;*

(c) $\mathsf{HH}^2(\mathsf{A}_h)$ *has a finite composition length, equal to the number of irreducible factors of* $\mathsf{gcd}(h, h')$*, counted with multiplicity;*

(d) $\mathsf{HH}^2(\mathsf{A}_h)$ *is a semisimple* $\mathsf{HH}^1(\mathsf{A}_h)$*-module if and only if* h *is not divisible by the cube of any non-constant polynomial.*

It is noteworthy that in case $\mathbb{F}$ is of a characteristic 0 and algebraically closed (so that each irreducible factor of h is linear and the corresponding factor algebra of $\mathbb{F}[x]$ is isomorphic to $\mathbb{F}$), then from Theorem C and the previous results obtained in [1] we can recover the number of irreducible factors appearing in h and the corresponding multiplicities. More specifically, let $\lambda(h)$ denote the partition encoding the multiplicities of the irreducible factors of h. We can conclude that if $\lambda(h)$ and $\lambda(g)$ are different partitions, then A_h is not derived equivalent to A_g.

We now fix some definitions and notation. Given an associative algebra A and elements $a, b \in A$, we use the commutator notation $[a, b] = ab - ba$. The center of A and the centralizer of an element $a \in A$ will be denoted by $\mathsf{Z}(A)$ and $\mathsf{C}_A(a)$, respectively. An element $c \in A$ is *normal* if $cA = Ac$ (an ideal of A). We remark that the set of normal elements of A forms a multiplicative monoid.

Given a two-sided ideal I of A, we will write $a \equiv b \pmod{I}$ to mean that $a - b \in I$. This yields an equivalence relation on A with the usual stability properties under addition and multiplication. If J is another ideal such that $J \subseteq I$, then obviously $a \equiv b \pmod{J}$ implies that $a \equiv b \pmod{I}$. In case $I = cA$ for some normal element $c \in A$, we also use the notation $a \equiv b \pmod{c}$.

Unadorned $\otimes$ will always mean $\otimes_{\mathbb{F}}$. For any set E, 1_E will denote the identity map on E. Given $f \in \mathbb{F}[x]$, $f^{(k)}$ stands for the k-th derivative of f with respect to x, which we also denote by f' and f'' in case $k = 1, 2$, respectively. If $f, g \in \mathbb{F}[x]$ are not both zero, then we tacitly assume that $\mathsf{gcd}(f, g)$ is monic.

An infinite-dimensional Lie algebra which plays an important role in the description of $\mathsf{HH}^1(\mathsf{A})$ is the *Witt algebra*. A confusion with terminology may arise here, since the term Witt algebra has been used in the literature to mean two different things: the complex Witt algebra is the Lie algebra of derivations of the ring $\mathbb{C}[z^{\pm 1}]$, with basis elements $w_n = z^{n+1}\frac{d}{dz}$, for $n \in \mathbb{Z}$; while over a field $\mathbb{K}$ of a characteristic $p > 0$, the Witt algebra is defined to be the Lie algebra of derivations of the ring $\mathbb{K}[z]/(z^p)$, spanned by w_n for $-1 \leq n \leq p-2$. Here, we are considering a subalgebra of the first one (defined over the field $\mathbb{F}$):

$$\mathsf{W} = \mathsf{span}_{\mathbb{F}}\{w_i \mid i \geq -1\}, \tag{1.1}$$

equipped with the Lie bracket $[w_m, w_n] = (n-m)w_{m+n}$, for $m, n \geq -1$. It is easy to check that if $\mathsf{char}(\mathbb{F}) = 0$, then W is a simple Lie algebra (cf. [1, Lem. 5.19]). For the sake of simplicity and in accordance with the usage in [1], we will abuse the terminology and refer to the algebra W defined above as the Witt algebra. To make the distinction clear, we will call the Lie algebra of derivations of $\mathbb{F}[z^{\pm 1}]$, with basis $\{w_i\}_{i\in\mathbb{Z}}$, the *full Witt algebra*.

A related Lie algebra of the utmost importance in theoretical physics is the *Virasoro algebra*, denoted by Vir. It has basis $\{w_i \mid i \in \mathbb{Z}\} \cup \{c\}$ over $\mathbb{F}$, with bracket

$$[c, \mathsf{Vir}] = 0 \quad \text{and} \quad [w_m, w_n] = (n-m)w_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c,$$

for all $m, n \in \mathbb{Z}$. We will see in (6.21) that the composition factors of $\mathsf{HH}^2(\mathsf{A}_h)$ can be naturally embedded into irreducible modules for the Virasoro algebra. These are the so-called intermediate series modules and it is a result of Mathieu [9] that a Harish-Chandra module for Vir is either a highest weight module, a lowest weight module or an intermediate series module.

2. Some technical results on commutators

In this short section, we gather some technical lemmas about commutators in A_h. We will need several additional results on centralizers and commutators in A_h from [2], which for convenience we combine below.

Proposition 2.1 (cf. [2, Lem. 3.4, 5.2, 6.1, 6.3; Prop. 5.5, 6.2; Thm. 5.3]). *Let $\delta : \mathbb{F}[x] \to \mathbb{F}[x]$ be the derivation defined by $\delta(f) = f'h$ for all $f \in \mathbb{F}[x]$.*

(a) *One has the following formula for computing in A_h:*

$$\hat{y}^n f = \sum_{j=0}^{n} \binom{n}{j} \delta^j(f)\hat{y}^{n-j}. \tag{2.2}$$

(b) *A_h is a free left $\mathbb{F}[x]$-module with basis $\{h^i y^i\}_{i\geq 0}$.*

(c) *If $\mathsf{char}(\mathbb{F}) = 0$, then $\mathsf{Z}(\mathsf{A}_h) = \mathbb{F}$; if $\mathsf{char}(\mathbb{F}) = p > 0$, then $\mathsf{Z}(\mathsf{A}_h)$ is the polynomial algebra in the variables x^p and $h^p y^p$.*

(d) *The centralizer $\mathsf{C}_{\mathsf{A}_h}(x)$ is generated by $\mathbb{F}[x]$ and $\mathsf{Z}(\mathsf{A}_h)$.*

(e) *A_h is free over $\mathsf{Z}(\mathsf{A}_h)$ and over $\mathsf{C}_{\mathsf{A}_h}(x)$. If $\mathsf{char}(\mathbb{F}) = p > 0$, then*

$$\mathsf{A}_h = \bigoplus_{i,j=0}^{p-1} \mathsf{Z}(\mathsf{A}_h)x^i h^j y^j = \bigoplus_{j=0}^{p-1} \mathsf{C}_{\mathsf{A}_h}(x)h^j y^j.$$

(f) *$[\mathsf{A}_h, \mathsf{A}_h] \subseteq h\mathsf{A}_h$. If $\mathsf{char}(\mathbb{F}) = 0$, then $[x, \mathsf{A}_h] = [\hat{y}, \mathsf{A}_h] = [\mathsf{A}_h, \mathsf{A}_h] = h\mathsf{A}_h$.*

Lemma 2.3. *For any* $0 \neq h \in \mathbb{F}[x]$, $[\mathbb{F}[x], \mathsf{A}_h] = [x, \mathsf{A}_h]$.

Proof. If $\mathsf{char}(\mathbb{F}) = 0$, then the claim follows from $[x, \mathsf{A}_h] = [\mathsf{A}_h, \mathsf{A}_h]$, by Proposition 2.1.

So assume that $\mathsf{char}(\mathbb{F}) = p > 0$. By [2, Lem. 6.3] and Proposition 2.1, we know that

$$[x, \mathsf{A}_h] = \bigoplus_{j=0}^{p-2} h\mathsf{C}_{\mathsf{A}_h}(x)h^j y^j \quad \text{and} \quad \mathsf{A}_h = \bigoplus_{j=0}^{p-1} \mathsf{C}_{\mathsf{A}_h}(x)h^j y^j.$$

Given that $f \in \mathbb{F}[x]$, $c \in \mathsf{C}_{\mathsf{A}_h}(x)$, and $0 \leq j \leq p-1$, we have, using (2.2),

$$[ch^j y^j, f] = ch^j [y^j, f] = ch^j \sum_{k=1}^{j} \binom{j}{k} f^{(k)} y^{j-k} = h \sum_{k=1}^{j} \binom{j}{k} ch^{k-1} f^{(k)} h^{j-k} y^{j-k}.$$

So, $[ch^j y^j, f] \in \bigoplus_{j=0}^{p-2} h\mathsf{C}_{\mathsf{A}_h}(x)h^j y^j = [x, \mathsf{A}_h]$. ■

Now, we can characterize the subspace $[x, \mathsf{A}_h] + [\hat{y}, \mathsf{A}_h]$ in case $\mathsf{char}(\mathbb{F}) = p > 0$.

Lemma 2.4. *Assume that* $\mathsf{char}(\mathbb{F}) = p > 0$. *The following hold:*

(a) *for all* $z \in \mathsf{Z}(\mathsf{A}_h)$, $f \in \mathbb{F}[x]$, *and* $0 \leq j \leq p-2$, *one has*

$$[\hat{y}, zfh^j y^j] \in [x, \mathsf{A}_h] \quad \text{and} \quad [\hat{y}, zfh^{p-1} y^{p-1}] = zhf'h^{p-1} y^{p-1};$$

(b) $[x, \mathsf{A}_h] + [\hat{y}, \mathsf{A}_h] = \bigoplus_{\substack{i,j=0 \\ (i,j)\neq(p-1,p-1)}}^{p-1} \mathsf{Z}(\mathsf{A}_h)hx^i h^j y^j$;

(c) $h\mathsf{A}_h = ([x, \mathsf{A}_h] + [\hat{y}, \mathsf{A}_h]) \oplus h\mathsf{Z}(\mathsf{A}_h)x^{p-1}h^{p-1}y^{p-1}$.

Proof. For the first part of (a), it suffices to show that $[\hat{y}, fh^j y^j] \in [x, \mathsf{A}_h]$ for all $0 \leq j \leq p-2$, as the latter is clearly a $\mathsf{Z}(\mathsf{A}_h)$-module. Since $\hat{y} - hy = h' \in \mathbb{F}[x]$ and $[\mathbb{F}[x], \mathsf{A}_h] = [x, \mathsf{A}_h]$, we need to prove that $[hy, fh^j y^j] \in [x, \mathsf{A}_h]$. Moreover,

$$[hy, fh^j y^j] = [hy, f]h^j y^j + f[hy, h^j y^j] = hf'h^j y^j + f[hy, h^j y^j]$$

and $hf'h^j y^j \in [x, \mathsf{A}_h]$, so we are left with showing that $[hy, h^j y^j] \in [x, \mathsf{A}_h]$. This is clear for $j = 0, 1$. For $2 \leq j \leq p-2$, we have, using (2.2),

$$[hy, h^j y^j] = -\sum_{\ell=1}^{j-1} \binom{j}{\ell-1} h^{j-\ell-1} h^{(j-\ell+1)} h^{\ell+1} y^\ell.$$

This proves that $[\hat{y}, zfh^j y^j] \in [x, \mathsf{A}_h]$ for all $z \in \mathsf{Z}(\mathsf{A}_h)$, $f \in \mathbb{F}[x]$, and $0 \leq j \leq p-2$.

Now, notice that, since $h^p, y^p \in \mathsf{Z}(\mathsf{A}_1)$, then

$$h^{p-1} y^{p-1} \hat{y} = h^{p-1} y^p h = h^p y^p = yh^p y^{p-1} = \hat{y} h^{p-1} y^{p-1}, \tag{2.5}$$

so $[\hat{y}, h^{p-1} y^{p-1}] = 0$. Thus, for $z \in \mathsf{Z}(\mathsf{A}_h)$ and $f \in \mathbb{F}[x]$, we have

$$[\hat{y}, zfh^{p-1} y^{p-1}] = z[\hat{y}, f]h^{p-1} y^{p-1} = zhf'h^{p-1} y^{p-1},$$

which finishes the proof of (a).

Since $\mathsf{Z}(\mathsf{A}_h)h \cdot \mathsf{im}(\frac{d}{dx})h^{p-1} y^{p-1} = \bigoplus_{i=0}^{p-2} \mathsf{Z}(\mathsf{A}_h)hx^i h^{p-1} y^{p-1}$, (b) is also established and (c) follows from (b), by Proposition 2.1. ■

3. Minimal free bimodule resolution of A_h

For simplicity, throughout the remainder of this paper, we denote A_h simply by A, reserving the notation A_h for situations in which we want to emphasize h or make particular choices for h, e.g., when referring to the Weyl algebra A_1.

In this section, we construct a free resolution of A as an A-bimodule or, equivalently, as a left A^e-module, where $\mathsf{A}^e = \mathsf{A} \otimes \mathsf{A}^{op}$ is the enveloping algebra of A and A^{op} is the opposite algebra of A.

We will follow the approach in [4]. Let $\mathsf{V} = \mathbb{F}x \oplus \mathbb{F}\hat{y}$ be the vector subspace of A spanned by x and $\hat{y}$ and let $\mathsf{R} = \mathbb{F}\mathsf{r}$ be a vector space of dimension 1. Consider the following sequence of right A-module maps:

$$0 \longrightarrow \mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A} \underset{\mathsf{s}_1}{\overset{\mathsf{d}_1}{\rightleftarrows}} \mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A} \underset{\mathsf{s}_0}{\overset{\mathsf{d}_0}{\rightleftarrows}} \mathsf{A} \otimes \mathsf{A} \underset{\mathsf{s}_{-1}}{\overset{\mu}{\rightleftarrows}} \mathsf{A} \longrightarrow 0. \tag{3.1}$$

The maps μ, d_0, and d_1 are in fact A-bimodule maps, whereas the maps s_{-1}, s_0, and s_1 are just right A-module maps. We describe them all below, except for s_1, which we discuss only in Section 4:

- μ is the multiplication map;
- $\mathsf{d}_0(1 \otimes v \otimes 1) = v \otimes 1 - 1 \otimes v$ for all $v \in \mathsf{V}$;
- $\mathsf{s}_{-1}(1) = 1 \otimes 1$;
- $\mathsf{s}_0(x^k \hat{y}^\ell \otimes 1) = \sum_{i=0}^{k-1} x^i \otimes x \otimes x^{k-1-i} \hat{y}^\ell + \sum_{j=0}^{\ell-1} x^k \hat{y}^j \otimes \hat{y} \otimes \hat{y}^{\ell-1-j}$, with the usual convention that an empty summation is null; in particular, $\mathsf{s}_0(1 \otimes 1) = 0$;
- $\mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1) = 1 \otimes \hat{y} \otimes x + \hat{y} \otimes x \otimes 1 - 1 \otimes x \otimes \hat{y} - x \otimes \hat{y} \otimes 1 - \mathsf{s}_0(h \otimes 1)$.

It is easy to check that

$$\mu \circ \mathsf{d}_0 = 0 = \mathsf{d}_0 \circ \mathsf{d}_1, \tag{3.2}$$

so (3.1) is a complex of A-bimodules. In fact, we already know that (3.1) is exact, and hence a free resolution of A, since its associated graded complex is exact (see [4]), but it will be useful for further computations to have an explicit contracting homotopy.

We claim that the right A-module maps s_{-1}, s_0, and s_1 form the desired contracting homotopy for (3.1), i.e., that the following hold:

$$\begin{aligned} \mu \circ \mathsf{s}_{-1} &= 1_{\mathsf{A}}, \\ \mathsf{s}_{-1} \circ \mu + \mathsf{d}_0 \circ \mathsf{s}_0 &= 1_{\mathsf{A}\otimes\mathsf{A}}, \\ \mathsf{s}_0 \circ \mathsf{d}_0 + \mathsf{d}_1 \circ \mathsf{s}_1 &= 1_{\mathsf{A}\otimes\mathsf{V}\otimes\mathsf{A}}, \\ \mathsf{s}_1 \circ \mathsf{d}_1 &= 1_{\mathsf{A}\otimes\mathsf{R}\otimes\mathsf{A}}. \end{aligned} \tag{3.3}$$

The first two equalities are easy to prove and are left as an exercise. So as not to stray from the main ideas of this section, we will defer the construction of the map s_1 and the proof of the last two relations in (3.3) until Section 4 (see Theorem 4.8).

Applying the functor $\mathsf{Hom}_{\mathsf{A}^e}(-, \mathsf{A})$ to the resolution associated with (3.1), we get the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathsf{Hom}_{\mathsf{A}^e}(\mathsf{A}\otimes\mathsf{A},\mathsf{A}) & \xrightarrow{\mathsf{d}_0^*} & \mathsf{Hom}_{\mathsf{A}^e}(\mathsf{A}\otimes\mathsf{V}\otimes\mathsf{A},\mathsf{A}) & \xrightarrow{\mathsf{d}_1^*} & \mathsf{Hom}_{\mathsf{A}^e}(\mathsf{A}\otimes\mathsf{R}\otimes\mathsf{A},\mathsf{A}) & \longrightarrow 0 \\ & \downarrow \rho_0 & & \downarrow \rho_1 & & \downarrow \rho_2 & \\ 0 \longrightarrow & \mathsf{A} & \xrightarrow{\phi_1} & \mathsf{A}\oplus\mathsf{A} & \xrightarrow{\phi_2} & \mathsf{A} & \longrightarrow 0, \end{array}$$

where d_i^* is right composition with d_i, for $i = 0, 1$, and the vector space isomorphisms ρ_j are defined as usual by

$$\rho_0(f) = f(1\otimes 1), \quad \rho_1(f) = \big(f(1\otimes x\otimes 1), f(1\otimes \hat{y}\otimes 1)\big), \quad \rho_2(f) = f(1\otimes \mathsf{r}\otimes 1).$$

The maps ϕ_1 and ϕ_2 are given by

$$\phi_1(\alpha) = \big([x,\alpha], [\hat{y},\alpha]\big) \tag{3.4}$$

and

$$\phi_2(\alpha,\beta) = [\beta, x] + [\hat{y},\alpha] - F_\alpha(h), \tag{3.5}$$

for all $\alpha, \beta \in \mathsf{A}$, where $F_\alpha : \mathbb{F}[x] \to \mathsf{A}$ is the linear map defined by

$$F_\alpha(x^s) = \sum_{\ell=0}^{s-1} x^\ell \alpha x^{s-\ell-1}, \quad \text{for } s \geq 0, \tag{3.6}$$

with the convention that $F_\alpha(1) = 0$.

Since $F_{z\alpha} = zF_\alpha$, for $z \in \mathsf{Z}(\mathsf{A})$, the maps ρ_i and ϕ_j are actually $\mathsf{Z}(\mathsf{A})$-module maps. It follows that, as a $\mathsf{Z}(\mathsf{A})$-module, the Hochschild cohomology of A can be determined from the maps ϕ_i:

- $\mathsf{HH}^0(\mathsf{A}) = \mathsf{Z}(\mathsf{A}) = \ker\phi_1$;
- $\mathsf{HH}^1(\mathsf{A}) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A})/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}) \cong \ker\phi_2/\operatorname{im}\phi_1$;
- $\mathsf{HH}^2(\mathsf{A}) \cong \mathsf{A}/\operatorname{im}\phi_2$ is the space of equivalence classes of infinitesimal deformations of A (see [6]);
- $\mathsf{HH}^i(\mathsf{A}) = 0$ for all $i \geq 3$.

The degree zero cohomology $\mathsf{HH}^0(\mathsf{A})$ has been computed in [2, Sec. 5], while the derivations and the Lie algebra structure of $\mathsf{HH}^1(\mathsf{A})$ were determined in [1], both over arbitrary fields.

Examples 3.7. Assume that $\mathsf{char}(\mathbb{F}) = 0$.

- If $h = 1$, then A_1 is the Weyl algebra and it is well known (see [11]) that $\mathsf{HH}^0(\mathsf{A}_1) = \mathbb{F}$ and $\mathsf{HH}^i(\mathsf{A}_1) = 0$ for all $i > 0$. In this case, A_1 is graded, setting $\deg(x) = 1$ and $\deg(y) = -1$.

- If $h = x$, then A_x is the universal enveloping algebra of the two-dimensional non-abelian Lie algebra. In this case, $\mathsf{HH}^0(\mathsf{A}_x) = \mathbb{F} = \mathsf{HH}^1(\mathsf{A}_x)$, by [1, Thm. 5.29]. We will see shortly that $\mathsf{HH}^2(\mathsf{A}_x) = 0$.
- If $h = x^2$, then A_{x^2} is the Jordan plane. In this case, A_{x^2} is graded, setting $\deg(x) = \deg(\hat{y}) = 1$. Note that $\mathsf{HH}^0(\mathsf{A}_{x^2}) = \mathbb{F}$ and by [1, Thm. 5.29], as a Lie algebra, $\mathsf{HH}^1(\mathsf{A}_{x^2}) = \mathbb{F}c \oplus \mathsf{W}$, where c is central and W is the Witt algebra given in (1.1). We will see that $\mathsf{HH}^2(\mathsf{A}_{x^2}) \cong \mathbb{F}[\hat{y}]$ is naturally a simple module for W and that this module can be embedded into a simple module for the Virasoro algebra.

Our main goal in this section will be to determine the image of ϕ_2 and the quotient $\mathsf{Z}(\mathsf{A})$-module $\mathsf{A}/\operatorname{im}\phi_2$. Later, we will determine the Lie action of $\mathsf{HH}^1(\mathsf{A})$ on $\mathsf{HH}^2(\mathsf{A})$ given by the Gerstenhaber bracket. Towards that goal, we start out by studying the map F_α given in (3.6). It will be convenient to introduce a mild generalization, so that F_α can be defined for all α in the Weyl algebra $\mathsf{A}_1 \supseteq \mathsf{A}$. With this extension, the range of F_α becomes A_1, but we will still use F_α to denote this map.

Lemma 3.8. *For $\alpha \in \mathsf{A}_1$, let $F_\alpha : \mathbb{F}[x] \to \mathsf{A}_1$ be the linear map defined by (3.6). The following hold for all $f, g \in \mathbb{F}[x]$:*

(a) *$F_\alpha(fg) = fF_\alpha(g) + F_\alpha(f)g$, i.e., F_α is a derivation;*

(b) *if $\alpha \in \mathsf{C}_{\mathsf{A}_1}(x)$, then $F_\alpha(f) = \alpha f'$;*

(c) *moreover, if $\alpha \in \mathsf{A}$, then $F_\alpha(f) \in f'\alpha + [x, \mathsf{A}]$.*

Proof. To show (a), it suffices to consider $f = x^k$ and $g = x^s$, with $k, s \geq 0$. Then,

$$\begin{aligned} F_\alpha(fg) &= F_\alpha(x^{k+s}) = \sum_{\ell=0}^{k+s-1} x^{k+s-\ell-1}\alpha x^\ell \\ &= x^k \sum_{\ell=0}^{s-1} x^{s-\ell-1}\alpha x^\ell + \left(\sum_{\ell=0}^{k-1} x^{k-\ell-1}\alpha x^\ell\right) x^s = fF_\alpha(g) + F_\alpha(f)g. \end{aligned}$$

This proves (a); (b) is clear and we proceed to prove (c). Again, we need only consider $\alpha \in \mathsf{A}$ and $f = x^k$, as above. We have

$$\begin{aligned} F_\alpha(x^k) &= \sum_{\ell=0}^{k-1} x^{k-\ell-1}\alpha x^\ell = \sum_{\ell=0}^{k-1} x^{k-1}\alpha + \sum_{\ell=0}^{k-1} x^{k-\ell-1}[\alpha, x^\ell] \\ &= kx^{k-1}\alpha + \sum_{\ell=0}^{k-1}[x^{k-\ell-1}\alpha, x^\ell] \in kx^{k-1}\alpha + [\mathbb{F}[x], \mathsf{A}] = f'\alpha + [x, \mathsf{A}]. \end{aligned}$$ ∎

In case $\operatorname{char}(\mathbb{F}) = 0$, the following result completely describes the image of the map ϕ_2.

Proposition 3.9. *The following hold:*

(a) *$\operatorname{im}\phi_2 \subseteq \gcd(h, h')\mathsf{A}$;*

(b) *if $\operatorname{char}(\mathbb{F}) = 0$, then $\operatorname{im}\phi_2 = \gcd(h, h')\mathsf{A}$.*

Proof. It is convenient to write $\phi_2 = \phi_2^1 \oplus \phi_2^2$, where

$$\begin{aligned} \phi_2^1\colon \mathsf{A} &\to \mathsf{A} & \phi_2^2\colon \mathsf{A} &\to \mathsf{A} \\ \alpha &\mapsto [\hat{y}, \alpha] - F_\alpha(h), & \beta &\mapsto [\beta, x]. \end{aligned} \tag{3.10}$$

Since, by Lemma 3.8 (c),

$$\phi_2^1(-\alpha) \in h'\alpha + [x, \mathsf{A}] + [\hat{y}, \mathsf{A}] \subseteq h'\mathsf{A} + h\mathsf{A} = \mathsf{gcd}(h, h')\mathsf{A},$$

for all $\alpha \in \mathsf{A}$, it follows that

$$\mathsf{im}\,\phi_2 = \mathsf{im}\,\phi_2^1 + \mathsf{im}\,\phi_2^2 \subseteq \mathsf{gcd}(h, h')\mathsf{A} + [x, \mathsf{A}] \subseteq \mathsf{gcd}(h, h')\mathsf{A} + h\mathsf{A} = \mathsf{gcd}(h, h')\mathsf{A}.$$

Now, assume that $\mathsf{char}(\mathbb{F}) = 0$. By Proposition 2.1, we know that $[x, \mathsf{A}] = [\hat{y}, \mathsf{A}] = h\mathsf{A}$ and thus $\mathsf{im}\,\phi_2^2 = [x, \mathsf{A}] = h\mathsf{A}$, which implies that $h\mathsf{A} \subseteq \mathsf{im}\,\phi_2$. Hence, we proceed to show that also $h'\mathsf{A} \subseteq \mathsf{im}\,\phi_2$. For $\alpha \in \mathsf{A}$, we have seen that

$$\phi_2^1(-\alpha) - h'\alpha \in [\alpha, \hat{y}] + [x, \mathsf{A}] \subseteq h\mathsf{A} \subseteq \mathsf{im}\,\phi_2.$$

Also, $\phi_2^1(-\alpha) \in \mathsf{im}\,\phi_2$, so it follows that $h'\alpha \in \mathsf{im}\,\phi_2$. Hence, $\mathsf{gcd}(h, h')\mathsf{A} = h'\mathsf{A} + h\mathsf{A} \subseteq \mathsf{im}\,\phi_2$ and the equality holds, by (a). ■

Corollary 3.11. *Assume that* $\mathsf{char}(\mathbb{F}) = 0$. *There are isomorphisms*

$$\mathsf{HH}^2(\mathsf{A}) \cong \mathsf{A}/\,\mathsf{gcd}(h, h')\mathsf{A} \cong \mathsf{D}[\hat{y}], \tag{3.12}$$

where $\mathsf{D} = (\mathbb{F}[x]/\,\mathsf{gcd}(h, h')\mathbb{F}[x])$. *In particular,* $\mathsf{HH}^2(\mathsf{A}) = 0$ *if and only if* $\mathsf{gcd}(h, h') = 1$, *i.e., if and only if h is a separable polynomial; otherwise,* $\mathsf{HH}^2(\mathsf{A})$ *is infinite-dimensional.*

Remark 3.13. In case $\mathsf{A}/\,\mathsf{gcd}(h, h')\mathsf{A}$ is graded, (3.12) is an isomorphism of graded vector spaces.

Let us now consider the case $\mathsf{char}(\mathbb{F}) = p > 0$. Suppose first that $h \in \mathbb{F}[x^p]$, a central polynomial. This is a particularly interesting case, not only because it includes the Weyl algebra A_1 but also since A_h is Calabi–Yau if and only if h is central. Indeed, more generally, A_h is twisted Calabi–Yau with Nakayama automorphism satisfying $x \mapsto x$, $\hat{y} \mapsto \hat{y} + h'$, a fact which can be derived from [8, Rem. 3.4, (2.10)].

Although we can retrieve the following result from Theorem 3.21 below, we think this particular case helps set the stage for our general result and offers a more concrete example.

Proposition 3.14. *Assume that* $\mathsf{char}(\mathbb{F}) = p > 0$ *and* $0 \neq h \in \mathbb{F}[x^p]$. *Then,* $\mathsf{im}\,\phi_2 = [x, \mathsf{A}] + [\hat{y}, \mathsf{A}]$. *Thus,*

$$\mathsf{HH}^2(\mathsf{A}) \cong \bigoplus_{\substack{i,j=0 \\ (i,j)\neq(p-1,p-1)}}^{p-1} \big(\mathsf{Z}(\mathsf{A}_h)/\,h\mathsf{Z}(\mathsf{A}_h)\big)x^i h^j y^j \oplus \mathsf{Z}(\mathsf{A})x^{p-1}h^{p-1}y^{p-1},$$

as $\mathsf{Z}(\mathsf{A})$*-modules.*

In particular, in case $h = 1$, we obtain $\mathsf{HH}^2(\mathsf{A}_1) \cong \mathsf{Z}(\mathsf{A}_1)x^{p-1}y^{p-1}$, a rank-one module over $\mathsf{Z}(\mathsf{A}_1) = \mathbb{F}[x^p, y^p]$.

Proof. We continue to use the maps ϕ_2^1 and ϕ_2^2 defined in (3.10). For $\alpha \in \mathsf{A}$, we have

$$\phi_2^1(\alpha) = [\hat{y}, \alpha] - F_\alpha(h) = [\hat{y}, \alpha] - h'\alpha - \Theta_\alpha = [\hat{y}, \alpha] - \Theta_\alpha, \tag{3.15}$$

for some $\Theta_\alpha \in [x, \mathsf{A}] = \operatorname{im}\phi_2^2$. Thus, $\operatorname{im}\phi_2^1 \subseteq [x, \mathsf{A}] + [\hat{y}, \mathsf{A}]$ and there are inclusions $[x, \mathsf{A}] \subseteq \operatorname{im}\phi_2 = \operatorname{im}\phi_2^1 + \operatorname{im}\phi_2^2 \subseteq [x, \mathsf{A}] + [\hat{y}, \mathsf{A}]$. Conversely, by (3.15) we also have that $[\hat{y}, \alpha] = \phi_2^1(\alpha) + \Theta_\alpha \in \operatorname{im}\phi_2^1 + \operatorname{im}\phi_2^2 = \operatorname{im}\phi_2$, so $[\hat{y}, \mathsf{A}] \subseteq \operatorname{im}\phi_2$, yielding the equality $\operatorname{im}\phi_2 = [x, \mathsf{A}] + [\hat{y}, \mathsf{A}]$.

The expression for $\mathsf{A}/\operatorname{im}\phi_2$ then comes from Lemma 2.4 (b) and Proposition 2.1. ∎

We now tackle the general case for $0 \neq h \in \mathbb{F}[x]$, which is a bit more intricate than the particular case studied above. Consider the decomposition $\mathsf{A} = \mathcal{I} \oplus \mathcal{J}$, where

$$\mathcal{I} = \mathsf{C}_\mathsf{A}(x)h^{p-1}y^{p-1} \quad \text{and} \quad \mathcal{J} = \bigoplus_{j=0}^{p-2} \mathsf{C}_\mathsf{A}(x)h^j y^j. \tag{3.16}$$

Thus, $\operatorname{im}\phi_2^1 = \operatorname{im}\phi_2^1|_\mathcal{I} + \operatorname{im}\phi_2^1|_\mathcal{J}$. Also, by [2, Lem. 6.3 (b)], $\operatorname{im}\phi_2^2 = [x, \mathsf{A}] = h\mathcal{J}$.

We wish to show that

$$\operatorname{im}\phi_2^1|_\mathcal{J} + \operatorname{im}\phi_2^2 = h\mathcal{J} + h'\mathcal{J} = \gcd(h, h')\mathcal{J}. \tag{3.17}$$

Let $\alpha \in \mathcal{J}$. Then, $[\hat{y}, \alpha] \in [x, \mathsf{A}] = h\mathcal{J}$, by Lemma 2.4 (a). As in (3.15), $\phi_2^1(\alpha) = [\hat{y}, \alpha] - h'\alpha - \Theta_\alpha$ for some $\Theta_\alpha \in [x, \mathsf{A}] = h\mathcal{J}$. Thus, $\operatorname{im}\phi_2^1|_\mathcal{J} \subseteq h\mathcal{J} + h'\mathcal{J}$; moreover, $h'\alpha = -\phi_2^1(\alpha) + [\hat{y}, \alpha] - \Theta_\alpha \in \operatorname{im}\phi_2^1|_\mathcal{J} + \operatorname{im}\phi_2^2$, and (3.17) is established.

So it remains to determine the image of $\phi_2^1|_\mathcal{I}$. Let $\alpha \in \mathcal{I}$. Without loss of generality, we can assume that $\alpha = zfh^{p-1}y^{p-1}$ with $z \in \mathsf{Z}(\mathsf{A})$ and $f \in \mathbb{F}[x]$. Then, using Lemma 2.4 (a), we have

$$\begin{aligned}\phi_2^1(\alpha) &= [\hat{y}, zfh^{p-1}y^{p-1}] - F_\alpha(h)\\ &= zf'hh^{p-1}y^{p-1} - zh'fh^{p-1}y^{p-1} - \Theta_\alpha\\ &= z(f'h - h'f)h^{p-1}y^{p-1} - \Theta_\alpha,\end{aligned} \tag{3.18}$$

with $\Theta_\alpha \in [x, \mathsf{A}] = h\mathcal{J}$.

Define the map

$$\varkappa = \varkappa_h : \mathbb{F}[x] \to \mathbb{F}[x], \quad \varkappa(g) = g'h - h'g. \tag{3.19}$$

By [1, Lem. 4.28 (d)], we know that $\ker\varkappa = \mathbb{F}[x^p](h/\varrho_h)$, where ϱ_h is the unique monic polynomial in $\mathbb{F}[x^p]$ of maximal degree dividing h (see [1, Def. 2.14] for a detailed description of ϱ_h). Since $\varkappa$ is clearly $\mathbb{F}[x^p]$-linear and $\mathbb{F}[x]$ is free of rank p over the hereditary algebra $\mathbb{F}[x^p]$, we conclude that $\mathcal{K} := \operatorname{im}\varkappa$ is a free $\mathbb{F}[x^p]$-submodule of $\mathbb{F}[x]$ of rank $p - 1$.

From the above and (3.18), we can conclude that

$$\operatorname{im}\phi_2^1|_{\mathcal{I}} + \operatorname{im}\phi_2^2 = h\mathcal{J} \oplus \mathsf{Z}(\mathsf{A})\mathcal{K}h^{p-1}y^{p-1}$$

and finally that

$$\operatorname{im}\phi_2 = \operatorname{gcd}(h, h')\mathcal{J} \oplus \mathsf{Z}(\mathsf{A})\mathcal{K}h^{p-1}y^{p-1}. \tag{3.20}$$

Thence, we obtain a description of $\mathsf{HH}^2(\mathsf{A})$ in a positive characteristic.

Theorem 3.21. *Assume that* $\operatorname{char}(\mathbb{F}) = p > 0$. *Then, the image of the map* ϕ_2 *defined in* (3.5) *is* $\operatorname{im}\phi_2 = \operatorname{gcd}(h, h')\mathcal{J} \oplus \mathsf{Z}(\mathsf{A})\mathcal{K}h^{p-1}y^{p-1}$, *where* $\mathcal{J}$ *and* $\varkappa$ *are given in* (3.16) *and* (3.19)*, respectively, and* $\mathcal{K}$ *is the image of* $\varkappa$*. Thus,*

$$\mathsf{HH}^2(\mathsf{A}) \cong \mathcal{J}/\operatorname{gcd}(h, h')\mathcal{J} \oplus \big(\mathsf{C}_{\mathsf{A}}(x)/\mathsf{Z}(\mathsf{A})\mathcal{K}\big)h^{p-1}y^{p-1},$$

as $\mathsf{Z}(\mathsf{A})$*-modules. In particular,* $\mathsf{HH}^2(\mathsf{A})$ *is nonzero for all* $0 \neq h \in \mathbb{F}[x]$.

Remark 3.22. Suppose that in Theorem 3.21, we take $0 \neq h \in \mathbb{F}[x^p]$. Then, $\operatorname{gcd}(h, h') = h$ and $\mathcal{K} = h \operatorname{im}\frac{d}{dx} = \bigoplus_{i=0}^{p-2}\mathbb{F}[x^p]hx^i$, so that

$$\operatorname{im}\phi_2 = h\mathcal{J} \oplus \bigoplus_{i=0}^{p-2}\mathsf{Z}(\mathsf{A})hx^i h^{p-1}y^{p-1} = [x, \mathsf{A}] + [\hat{y}, \mathsf{A}],$$

by Lemma 2.4 (b), in agreement with the statements in Proposition 3.14.

Examples 3.23. Let $\operatorname{char}(\mathbb{F}) = p > 0$.

(a) In case $h = 1$, then A_1 is the Weyl algebra and, as observed in Proposition 3.14, $\mathsf{HH}^2(\mathsf{A}_1) \cong \mathsf{Z}(\mathsf{A}_1)x^{p-1}y^{p-1}$ is a rank-one free module over $\mathsf{Z}(\mathsf{A}_1) = \mathbb{F}[x^p, y^p]$. It was shown in [1, Thm. 3.8] that $\mathsf{HH}^1(\mathsf{A}_1)$ is a rank-two free module over $\mathsf{Z}(\mathsf{A}_1)$.

(b) In case $h = x$, then A_x is the universal enveloping algebra of the two-dimensional non-abelian Lie algebra. We have $\operatorname{gcd}(h, h') = 1$ so that $\mathcal{J}/\operatorname{gcd}(h, h')\mathcal{J} = 0$. By computing the image under $\varkappa$ of the $\mathbb{F}[x^p]$-basis $\{x^i \mid 0 \leq i \leq p-1\}$ of $\mathbb{F}[x]$, we easily see that

$$\mathsf{Z}(\mathsf{A}_x)\mathcal{K} = \mathsf{Z}(\mathsf{A}_x) \oplus \bigoplus_{i=2}^{p-1}\mathsf{Z}(\mathsf{A}_x)x^i.$$

Hence, Theorem 3.21 yields

$$\mathsf{HH}^2(\mathsf{A}_x) \cong \mathsf{Z}(\mathsf{A}_x)x^p y^{p-1},$$

again a free rank-one module over $\mathsf{Z}(\mathsf{A}_x) = \mathbb{F}[x^p, x^p y^p]$.

(c) Assume that $h = x^2$. Then, A_{x^2} is the Jordan plane. We distinguish between two cases:

- *Case 1:* $p = 2$.
  In this case, x^2 is central and we use Proposition 3.14 to obtain the isomorphism

$$\mathsf{HH}^2(\mathsf{A}_{x^2}) \cong \mathsf{D} \oplus \mathsf{D}x \oplus \mathsf{D}x^2 y \oplus \mathsf{Z}(\mathsf{A}_{x^2})x^3 y,$$

  where $\mathsf{Z}(\mathsf{A}_{x^2}) = \mathbb{F}[x^2, x^4 y^2]$ and $\mathsf{D} = \mathsf{Z}(\mathsf{A}_{x^2})/x^2\mathsf{Z}(\mathsf{A}_{x^2})$.

- *Case 2: $p > 2$.*
  In this case, x^2 is not central and we use Theorem 3.21. Since $\mathsf{gcd}(h, h') = x$ and $\mathsf{C}_{\mathsf{A}_{x^2}}(x)/x\mathsf{C}_{\mathsf{A}_{x^2}}(x) \cong \mathsf{Z}(\mathsf{A}_{x^2})/x^p\mathsf{Z}(\mathsf{A}_{x^2})$, we can conclude that

$$\mathcal{J}/\,\mathsf{gcd}(h, h')\mathcal{J} \cong \bigoplus_{j=0}^{p-2} \big(\mathsf{Z}(\mathsf{A}_{x^2})/x^p\mathsf{Z}(\mathsf{A}_{x^2})\big)h^j y^j.$$

  Finally, as in the case $h = x$, it is easy to see that

$$\mathsf{Z}(\mathsf{A}_{x^2})\mathcal{K} = \bigoplus_{i=1}^{2} \mathsf{Z}(\mathsf{A}_{x^2})x^i \oplus \bigoplus_{i=4}^{p} \mathsf{Z}(\mathsf{A}_{x^2})x^i,$$

  where the last summand is zero in case $p = 3$. Hence, Theorem 3.21 gives

$$\mathsf{HH}^2(\mathsf{A}_{x^2}) \cong \mathsf{D} \oplus \mathsf{D}x^2y \oplus \mathsf{Z}(\mathsf{A}_{x^2})x^4y^2,$$

  in case $p = 3$, and

$$\begin{aligned}\mathsf{HH}^2(\mathsf{A}_{x^2}) &\cong \bigoplus_{j=0}^{p-2} \mathsf{D}x^{2j}y^j \oplus \mathsf{D}x^{2(p-1)}y^{p-1} \oplus \mathsf{Z}(\mathsf{A}_{x^2})x^{2p+1}y^{p-1}\\ &= \bigoplus_{j=0}^{p-1} \mathsf{D}x^{2j}y^j \oplus \mathsf{Z}(\mathsf{A}_{x^2})x^{2p+1}y^{p-1},\end{aligned}$$

  for all primes $p > 3$, where $\mathsf{Z}(\mathsf{A}_{x^2}) = \mathbb{F}[x^p, x^{2p}y^p]$ and $\mathsf{D} = \mathsf{Z}(\mathsf{A}_{x^2})/x^p\mathsf{Z}(\mathsf{A}_{x^2})$.

  Notice that in all cases, $\mathsf{HH}^2(\mathsf{A}_{x^2})$ is not a free module over $\mathsf{Z}(\mathsf{A}_{x^2})$, although it is composed of a torsion summand and a free summand of rank one.

We have seen in the examples that, in general, $\mathsf{HH}^2(\mathsf{A})$ is not a free module over $\mathsf{Z}(\mathsf{A})$. The next theorem provides a necessary and sufficient condition for $\mathsf{HH}^2(\mathsf{A})$ to be free.

Theorem 3.24. *Assume that $\mathsf{char}(\mathbb{F}) = p > 0$. Then, $\mathsf{HH}^2(\mathsf{A})$ is a free $\mathsf{Z}(\mathsf{A})$-module if and only if $\mathsf{gcd}(h, h') = 1$. In this case, $\mathsf{HH}^2(\mathsf{A})$ has rank one over $\mathsf{Z}(\mathsf{A})$ and, moreover, $\mathsf{HH}^\bullet(\mathsf{A})$ is a free $\mathsf{Z}(\mathsf{A})$-module.*

Proof. The last statement follows from the first by [1, Thm. 6.29], so we need only focus on $\mathsf{HH}^2(\mathsf{A})$.

The condition $\mathsf{gcd}(h, h') = 1$ is necessary, as otherwise, $\mathcal{J}/\,\mathsf{gcd}(h, h')\mathcal{J}$ would be nonzero and annihilated by the central element $(\mathsf{gcd}(h, h'))^p$. Next, we prove that it is sufficient.

Suppose that $\mathsf{gcd}(h, h') = 1$. Then, $\mathsf{HH}^2(\mathsf{A}) \cong (\mathsf{C}_\mathsf{A}(x)/\mathsf{Z}(\mathsf{A})\mathcal{K})h^{p-1}y^{p-1}$ and, since $\mathsf{C}_\mathsf{A}(x) = \mathsf{Z}(\mathsf{A})\mathbb{F}[x]$, it suffices to prove that $\mathcal{K}$ is a direct summand of $\mathbb{F}[x]$, as $\mathbb{F}[x^p]$-modules. The latter is equivalent to showing that $\mathbb{F}[x]/\mathcal{K}$ is torsion free, for then the canonical epimorphism $\mathbb{F}[x] \to \mathbb{F}[x]/\mathcal{K}$ will yield the decomposition $\mathbb{F}[x] = \mathcal{K} \oplus \mathbb{F}[x^p]\xi$, for some rank-one free $\mathbb{F}[x^p]$-submodule $\mathbb{F}[x^p]\xi$. It will follow that $\mathsf{HH}^2(\mathsf{A}) \cong \mathsf{Z}(\mathsf{A})\xi h^{p-1}y^{p-1}$, a free $\mathsf{Z}(\mathsf{A})$-module of rank one.

Claim. *The* $\mathbb{F}[x^p]$*-module* $\mathbb{F}[x]/\mathcal{K}$ *is torsion free.*

Proof of the claim. Recall that $\varkappa$ is defined in (3.19) and $\mathcal{K}$ is the image of $\varkappa$. Let $0 \neq \omega \in \mathbb{F}[x^p]$ and $f \in \mathbb{F}[x]$ be such that $\omega f \in \mathcal{K}$, say $\omega f = \varkappa(g)$. It needs to be shown that $f \in \mathcal{K}$. For such, it is enough to show that there exist $q \in \mathbb{F}[x]$ and $r \in \mathbb{F}[x^p]$ so that $g = \omega q + rh$. Indeed, if this is the case, then $\omega f = \omega\varkappa(q) + r\varkappa(h) = \omega\varkappa(q)$ and it follows that $f = \varkappa(q) \in \mathcal{K}$.

Subclaim 1. $g \in \omega\mathbb{F}[x] + h\mathbb{F}[x]$.

Proof of Subclaim 1. Let $t = \mathsf{gcd}(\omega, h)$. Then, $\omega\mathbb{F}[x] + h\mathbb{F}[x] = t\mathbb{F}[x]$ and the equality $\omega f = g'h - h'g$ implies that $h'g \in t\mathbb{F}[x]$. But t is a divisor of h and $\mathsf{gcd}(h, h') = 1$ so it follows that $g \in t\mathbb{F}[x]$, as required. ∎

Take $q, r \in \mathbb{F}[x]$ with $g = \omega q + rh$. Applying $\varkappa$ to this equality, we obtain $\varkappa(g) = \omega\varkappa(q) + \varkappa(rh)$ and thus ω divides $\varkappa(rh)$. So it suffices to prove that if ω divides $\varkappa(rh)$, then $rh \in \omega\mathbb{F}[x] + h\mathbb{F}[x^p]$. In other words, we may assume without loss of generality that $g = rh$.

Write $r = r_0 + r_1$, with $r_0 \in \mathbb{F}[x^p]$ and $r_1 \in \bigoplus_{i=1}^{p-1} \mathbb{F}[x^p]x^i$. As $\varkappa(rh) = \varkappa(r_1 h)$, we may assume that $r_0 = 0$. So, without loss of generality, we assume that $r \in \bigoplus_{i=1}^{p-1} \mathbb{F}[x^p]x^i$.

Subclaim 2. *ω divides rh.*

Proof of Subclaim 2. Note that $\varkappa(rh) = r'h^2$, so we need to show that if ω divides $r'h^2$, then ω divides rh. From this point on, our proof follows that of [1, Lem. 6.28 (iv)], although the details are a bit more intricate and some modifications are needed. Thus, we suspend the proof of the subclaim here and refer the interested reader to the proof of [1, Lem. 6.28 (iv)]. ∎

By the above arguments, the claim is also established, thus proving the theorem. ∎

4. The contracting homotopies s_{-1}, s_0, and s_1

Recall the definition of the right A-module maps s_{-1} and s_0, given at the beginning of Section 3. In this section, we prove the two final relations in (3.3), together with a few other useful identities. For the sake of brevity, we leave most of the details to the reader.

Lemma 4.1. *Let $f \in \mathbb{F}[x]$, $a, b \in \mathsf{A}$, and $\alpha \in \mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A}$. The following hold:*

(a) $\mathsf{s}_0(fa \otimes b) = f\mathsf{s}_0(a \otimes b) + \mathsf{s}_0(f \otimes ab)$;

(b) $\mathsf{s}_0(f\mathsf{d}_0(\alpha)) = f\mathsf{s}_0(\mathsf{d}_0(\alpha))$.

Recall that we have fixed r as the basis element of the one-dimensional vector space R. Consider the linear map $G : \mathbb{F}[x] \to \mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A}$ defined by

$$G(x^k) = \sum_{i=0}^{k-1} x^i \otimes \mathsf{r} \otimes x^{k-1-i}, \quad \text{for all } k \geq 0, \tag{4.2}$$

with $G(1) = 0$. Also, recall that δ denotes the derivation of $\mathbb{F}[x]$ defined by $\delta(f) = f'h$, so that $[\hat{y}, f] = \delta(f)$, for all $f \in \mathbb{F}[x]$.

Lemma 4.3. *The map G is a derivation and, for any $f \in \mathbb{F}[x]$,*

$$\mathsf{d}_1 \circ G(f) = 1 \otimes \hat{y} \otimes f - f \otimes \hat{y} \otimes 1 - \mathsf{s}_0(f \otimes \hat{y}) - \mathsf{s}_0(\delta(f) \otimes 1) + \hat{y}\mathsf{s}_0(f \otimes 1).$$

Proof. The first statement follows from Lemma 4.1 (a) and the second one can be verified through a computation, using the properties of s_0. ∎

We are finally ready to define the homotopy $\mathsf{s}_1 : \mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A} \to \mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A}$. This is the right A-module map defined inductively as follows, for $f \in \mathbb{F}[x]$, $a, b \in \mathsf{A}$, and $\ell \geq 0$:

- $\mathsf{s}_1(a \otimes \hat{y} \otimes b) = 0$;
- $\mathsf{s}_1(f\hat{y}^\ell \otimes x \otimes a) = f\mathsf{s}_1(\hat{y}^\ell \otimes x \otimes 1)a$;
- $\mathsf{s}_1(1 \otimes x \otimes 1) = 0$;
- $\mathsf{s}_1(\hat{y}^{\ell+1} \otimes x \otimes 1) = \hat{y}\mathsf{s}_1(\hat{y}^\ell \otimes x \otimes 1) + \sum_{j=0}^{\ell} \binom{\ell}{j}(G \circ \delta^j(x))\hat{y}^{\ell-j}$, where $\delta(f) = f'h$ and G is the linear map given by (4.2).

Lemma 4.4. *The map s_1 satisfies $\mathsf{s}_0 \circ \mathsf{d}_0 + \mathsf{d}_1 \circ \mathsf{s}_1 = 1_{\mathsf{A}\otimes\mathsf{V}\otimes\mathsf{A}}$.*

Now, we aim to prove the last relation in (3.3), namely, $\mathsf{s}_1 \circ \mathsf{d}_1 = 1_{\mathsf{A}\otimes\mathsf{R}\otimes\mathsf{A}}$. We start with a technical identity which just depends on the fact that G and δ are derivations.

Lemma 4.5. *Given $k \geq 1$ and $r \geq 0$,*

$$\sum_{i=0}^{k-1} \sum_{j=0}^{r} \sum_{t=0}^{r-j} \binom{r}{j}\binom{r-j}{t} \delta^j(x^i) G\big(\delta^t(x)\big)\delta^{r-j-t}(x^{k-i-1}) = G\big(\delta^r(x^k)\big).$$

Our next results concern the computation of s_1.

Proposition 4.6. *For all $\ell \geq 0$ and all $f \in \mathbb{F}[x]$, the following identity holds:*

$$\mathsf{s}_1\big(\hat{y}^{\ell+1}\mathsf{s}_0(f \otimes 1)\big) = \hat{y}\mathsf{s}_1\big(\hat{y}^\ell\mathsf{s}_0(f \otimes 1)\big) + \sum_{j=0}^{\ell}\binom{\ell}{j} G\big(\delta^j(f)\big)\hat{y}^{\ell-j}.$$

We are now able to determine the closed formulas for $\mathsf{s}_1(\hat{y}^{\ell+1}\mathsf{s}_0(f \otimes 1))$ and $\mathsf{s}_1(\hat{y}^{\ell+1} \otimes x \otimes 1)$.

Proposition 4.7. *For all $\ell \geq 0$ and $f \in \mathbb{F}[x]$, one has*

$$\mathsf{s}_1\big(\hat{y}^{\ell+1}\mathsf{s}_0(f \otimes 1)\big) = \sum_{j=0}^{\ell}\sum_{k=0}^{\ell-j}\binom{\ell-k}{j}\hat{y}^k G\big(\delta^j(f)\big)\hat{y}^{\ell-j-k}.$$

In particular, taking $f = x$, one obtains the following explicit formula for s_1:

$$\mathsf{s}_1(\hat{y}^{\ell+1} \otimes x \otimes 1) = \sum_{j=0}^{\ell}\sum_{k=0}^{\ell-j}\binom{\ell-k}{j}\hat{y}^k G\big(\delta^j(x)\big)\hat{y}^{\ell-j-k}.$$

Finally, we can prove the main result of this section.

Theorem 4.8. *The right* A*-module maps* s_{-1}, s_0, *and* s_1 *form a contracting homotopy for* (3.1).

Proof. It remains to prove the identity $\mathsf{s}_1 \circ \mathsf{d}_1 = 1_{\mathsf{A}\otimes\mathsf{R}\otimes\mathsf{A}}$ from (3.3), and it clearly suffices to check this identity on elements of the form $\hat{y}^\ell \otimes \mathsf{r} \otimes 1$, as s_1 is also a left $\mathbb{F}[x]$-module homomorphism. The case $\ell = 0$ is straightforward, so assume that $\ell \geq 1$. Then,

$$\begin{aligned}\mathsf{s}_1\big(\mathsf{d}_1(\hat{y}^\ell \otimes \mathsf{r} \otimes 1)\big) &= \mathsf{s}_1\big(\hat{y}^\ell \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1)\big)\\ &= \mathsf{s}_1(\hat{y}^{\ell+1} \otimes x \otimes 1) - \mathsf{s}_1(\hat{y}^\ell \otimes x \otimes 1)\hat{y} - \mathsf{s}_1\big(\hat{y}^\ell \mathsf{s}_0(\delta(x) \otimes 1)\big),\end{aligned}$$

and by Proposition 4.7, we have

$$\mathsf{s}_1(\hat{y}^{\ell+1} \otimes x \otimes 1) = \sum_{j=0}^{\ell} \sum_{k=0}^{\ell-j} \binom{\ell-k}{j} \hat{y}^k G\big(\delta^j(x)\big)\hat{y}^{\ell-j-k}.$$

Using adequate combinatorial identities, we obtain

$$\begin{aligned}\mathsf{s}_1(\hat{y}^{\ell+1} \otimes x \otimes 1) &= \sum_{j=1}^{\ell-1} \sum_{k=0}^{\ell-j-1} \binom{\ell-k-1}{j} \hat{y}^k G\big(\delta^j(x)\big)\hat{y}^{\ell-j-k}\\ &\quad + \sum_{j=1}^{\ell} \sum_{k=0}^{\ell-j} \binom{\ell-k-1}{j-1} \hat{y}^k G\big(\delta^j(x)\big)\hat{y}^{\ell-j-k}\\ &\quad + \sum_{k=0}^{\ell-1} \binom{\ell-k-1}{0} \hat{y}^k G\big(\delta^0(x)\big)\hat{y}^{\ell-k} + \hat{y}^\ell G(x)\\ &= \sum_{j=0}^{\ell-1} \sum_{k=0}^{\ell-j-1} \binom{\ell-k-1}{j} \hat{y}^k G\big(\delta^j(x)\big)\hat{y}^{\ell-j-k}\\ &\quad + \sum_{j=0}^{\ell-1} \sum_{k=0}^{\ell-j-1} \binom{\ell-k-1}{j} \hat{y}^k G\big(\delta^{j+1}(x)\big)\hat{y}^{\ell-j-k-1} + \hat{y}^\ell \otimes \mathsf{r} \otimes 1\\ &= \mathsf{s}_1(\hat{y}^\ell \otimes x \otimes 1)\hat{y} + \mathsf{s}_1\big(\hat{y}^\ell \mathsf{s}_0(\delta(x) \otimes 1)\big) + \hat{y}^\ell \otimes \mathsf{r} \otimes 1,\end{aligned}$$

which proves the desired identity. ∎

5. The Gerstenhaber bracket: general remarks

The Hochschild cohomology $\mathsf{HH}^\bullet(\mathsf{A}) = \bigoplus_{n\geq 0} \mathsf{HH}^n(\mathsf{A})$ has a rich structure, including an associative, graded-commutative product (relative to homological degree), given by the cup product, and also a graded Lie bracket $[\,,]$ of (homological) degree -1; these are related by the graded Poisson identity. In particular, the graded anti-symmetric property of $[\,,]$ means

$$[\alpha, \beta] = -(-1)^{(m-1)(n-1)}[\beta, \alpha], \quad \text{for all } \alpha \in \mathsf{HH}^m(\mathsf{A}) \text{ and } \beta \in \mathsf{HH}^n(\mathsf{A}),$$

and there is a corresponding graded version of the Jacobi identity (see [5]). Under this construction, $\mathsf{HH}^\bullet(\mathsf{A})$ becomes a *Gerstenhaber algebra*. In particular, the Jacobi identity implies that $\mathsf{HH}^\bullet(\mathsf{A})$ is a Lie module for the Lie algebra $\mathsf{HH}^1(\mathsf{A})$, extending the usual Lie bracket of derivations on $\mathsf{HH}^1(\mathsf{A})$. In case A is a smooth finitely generated $\mathbb{F}$-algebra and $\mathbb{F}$ is perfect, the Hochschild–Kostant–Rosenberg theorem gives an isomorphism of Gerstenhaber algebras between the Hochschild cohomology of A and the exterior algebra over A of the k-linear derivations of A, as R. Hermann proved in [7], telling that, in this situation, the Gerstenhaber bracket is the generalization to higher degrees of the Schouten–Nijenhuis bracket.

The Gerstenhaber structure of Hochschild cohomology is particularly interesting for us since in case $\mathsf{char}(\mathbb{F}) = 0$ and $\mathsf{gcd}(h, h') \neq 1$, the description of $\mathsf{HH}^1(\mathsf{A})$ involves the Witt algebra W. In a prime characteristic, most of the computations of the Gerstenhaber structure in Hochschild cohomology concern group algebras and tame blocks; see, for example, [3, 10].

Although the Gerstenhaber bracket does not depend on the chosen bimodule projective resolution of A, it is, in general, difficult to compute it on an arbitrary resolution other than the bar resolution. In spite of this, we always have $[\overline{D}, z] = \overline{D}(z)$ and $[\overline{D}, \overline{D'}] = \overline{[D, D']}$ for $D, D' \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A})$ and $z \in \mathsf{Z}(\mathsf{A})$, so it remains to compute $[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^2(\mathsf{A})]$, which is what we undertake in this section. Notice that, in our case, we already have the contracting homotopy of the minimal resolution, from which the comparison maps can be obtained. Nevertheless, we will use an easier method that, for the family of algebras we consider, also needs the contracting homotopy.

To avoid cumbersome notation, we identify $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A})$ with its canonical image $\overline{D} \in \mathsf{HH}^1(\mathsf{A})$. We will often refer to the map $[D, -] : \mathsf{HH}^i(\mathsf{A}) \to \mathsf{HH}^i(\mathsf{A})$ as the (Lie) action of $D \in \mathsf{HH}^1(\mathsf{A})$ on $\mathsf{HH}^i(\mathsf{A})$.

5.1. The method of Suárez-Álvarez for computing $[\mathsf{HH}^1(\mathsf{A}), -]$

In this subsection, we will describe a method devised by Suárez-Álvarez in [13] to compute the Gerstenhaber bracket $[\mathsf{HH}^1(\mathsf{A}), -]$ in terms of an arbitrary projective resolution of A as a bimodule. The reader is advised to consult [13] for further details and all the proofs.

Fix an $\mathbb{F}$-algebra B and a derivation $\psi : \mathsf{B} \to \mathsf{B}$. Given a left B-module M, we say that a linear map $f : M \to M$ satisfying $f(bm) = bf(m) + \psi(b)m$ for all $b \in \mathsf{B}$ and $m \in M$ is a *ψ-operator* on M. Given a projective resolution

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

of M, a *ψ-lifting* of the ψ-operator f to $P_\bullet$ is a sequence $f_\bullet = (f_i)_{i \geq 0}$ of ψ-operators $f_i : P_i \to P_i$ such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0.
\end{array}$$

It was shown in [13, Lem. 1.4] that every ψ-operator f admits a unique (up to B-module homotopy) ψ-lifting.

Given a ψ-operator f and a ψ-lifting $f_\bullet$ of f to $P_\bullet$, define a sequence $f_\bullet^\sharp = (f_i^\sharp)_{i\geq 0}$ of linear maps $f_i^\sharp : \mathsf{Hom}_{\mathsf{B}}(P_i, M) \to \mathsf{Hom}_{\mathsf{B}}(P_i, M)$ by

$$f_i^\sharp(\phi)(p) = f\big(\phi(p)\big) - \phi\big(f_i(p)\big),$$

for $\phi \in \mathsf{Hom}_{\mathsf{B}}(P_i, M)$ and $p \in P_i$. In fact, $f_\bullet^\sharp$ is an endomorphism of the complex of vector spaces $\mathsf{Hom}_{\mathsf{B}}(P_\bullet, M)$ and the induced map on cohomology

$$\nabla^\bullet_{f,P_\bullet} : \mathsf{H}\big(\mathsf{Hom}_{\mathsf{B}}(P_\bullet, M)\big) \to \mathsf{H}\big(\mathsf{Hom}_{\mathsf{B}}(P_\bullet, M)\big)$$

depends only on f and not on the choice of ψ-lifting $f_\bullet$. What is more, noticing that $\mathsf{H}(\mathsf{Hom}_{\mathsf{B}}(P_\bullet, M))$ is canonically isomorphic to $\mathsf{Ext}^\bullet_{\mathsf{B}}(M, M)$, we obtain a canonical morphism of graded vector spaces

$$\nabla^\bullet_f : \mathsf{Ext}^\bullet_{\mathsf{B}}(M, M) \to \mathsf{Ext}^\bullet_{\mathsf{B}}(M, M)$$

which depends only on f and not on the chosen projective resolution of M (see [13, Thm. A]).

Returning to the problem at hand, which is the computation of the bracket $[\mathsf{HH}^1(\mathsf{A}), -]$ in terms of a chosen bimodule projective resolution $\mu : P_\bullet \twoheadrightarrow \mathsf{A}$ of A, set $\mathsf{B} = \mathsf{A}^e$ and $M = \mathsf{A}$, so that $\mu : P_\bullet \twoheadrightarrow \mathsf{A}$ can be identified with a projective resolution of A as a left B-module. Given a derivation D of A, construct a new derivation $D^e = D \otimes 1_{\mathsf{A}} + 1_{\mathsf{A}} \otimes D$ of B. It can be readily seen that D is a D^e-operator on A. Since $\mathsf{Ext}^\bullet_{\mathsf{B}}(\mathsf{A}, \mathsf{A})$ is naturally identified with the Hochschild cohomology $\mathsf{HH}^\bullet(\mathsf{A})$, the above construction yields a map $\nabla^\bullet_D : \mathsf{HH}^\bullet(\mathsf{A}) \to \mathsf{HH}^\bullet(\mathsf{A})$, which by [13, Sec. 2.2] turns out to be $[D, -]$ and which can be computed using any bimodule projective resolution of A, provided that a D^e-lifting $D_\bullet$ of D to the given resolution is found.

Going back to the case under study, with $\mathsf{A} = \mathsf{A}_h$, $\epsilon = \mu$ (the multiplication map), $P_0 = \mathsf{A} \otimes \mathsf{A}$, $P_1 = \mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A}$, and $P_2 = \mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A}$, it can be checked that $D \circ \mu = \mu \circ D^e$ and D^e is trivially a D^e-operator on $\mathsf{A} \otimes \mathsf{A}$, so we can choose $D_0 = D^e$. Taking $i = 2$ and using the map ρ_2 from Section 3 to identify $\mathsf{HH}^2(\mathsf{A})$ with a homomorphic image of A, we obtain the formula describing the Lie action of $\mathsf{HH}^1(\mathsf{A})$ on $\mathsf{HH}^2(\mathsf{A})$:

$$[D, a] = D(a) - \chi_a\big(D_2(1 \otimes \mathsf{r} \otimes 1)\big), \tag{5.1}$$

for $a \in \mathsf{A}$ and $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A})$, where $\chi_a \in \mathsf{Hom}_{\mathsf{A}^e}(\mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A}, \mathsf{A})$ is defined by $\chi_a(1 \otimes \mathsf{r} \otimes 1) = a$.

5.2. The D^e-lifting of D to (3.1)

In order to make use of (5.1), it remains to determine the D^e-lifting D_2 of D, which we do in this subsection. We begin with a few general observations aimed at simplifying computations; then, we determine the D^e-liftings D_1 and D_2.

The proof of the lemma that follows is standard and is thus omitted.

Lemma 5.2. *Let* B *be an algebra,* $\psi : \mathsf{B} \to \mathsf{B}$ *a derivation,* M *and* N *left* B*-modules,* $X \subseteq M$ *a generating set for* M *as a* B*-module, and* $Y \subseteq \mathsf{B}$ *a generating set for* B *as a vector space.*

(a) *If* X *is a free* B*-basis for* M*, then for any function* $f' : X \to M$ *there is a unique* ψ*-operator* $f : M \to M$ *such that* $f|_X = f'$*.*

(b) *Let* $\phi : M \to N$ *be a morphism of* B*-modules and let* $f : M \to M$ *and* $g : N \to N$ *be* ψ*-operators. If* $g \circ \phi|_X = \phi \circ f|_X$*, then the following square commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{\phi} & N. \end{array}$$

(c) *If* $f : M \to M$ *is a linear map such that* $f(bm) = bf(m) + \psi(b)m$ *for all* $b \in Y \subseteq \mathsf{B}$ *and all* $m \in X \subseteq M$*, then* f *is a* ψ*-operator.*

Throughout the rest of this subsection, fix $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A})$ and let $D_0 = D^e : \mathsf{A}^e \to \mathsf{A}^e$. Next, we define a D_0-lifting $D_1 : \mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A} \to \mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A}$ in terms of the homotopy s_0.

Lemma 5.3. *Let* $D_1(a \otimes v \otimes b) = a\mathsf{s}_0(D(v) \otimes b) + D(a) \otimes v \otimes b + a \otimes v \otimes D(b)$*, for all* $a, b \in \mathsf{A}$ *and all* $v \in \mathsf{V} = \mathbb{F}x \oplus \mathbb{F}\hat{y}$*. Then, extending linearly to* $\mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A}$*, this rule defines a* D_0*-operator such that* $D_0 \circ \mathsf{d}_0 = \mathsf{d}_0 \circ D_1$*.*

Proof. Define first $D_1(1 \otimes v \otimes 1) = \mathsf{s}_0(D(v) \otimes 1)$ for $v \in \{x, \hat{y}\}$. Since $\{1 \otimes x \otimes 1, 1 \otimes \hat{y} \otimes 1\}$ is a free basis for $\mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A}$ as an A^e-module, Lemma 5.2 (a) guarantees the existence of a unique D_0-operator, which we still denote by D_1, defined on $\mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A}$ and extending the above rule.

First, notice that by linearity of D and s_0, one has $D_1(1 \otimes v \otimes 1) = \mathsf{s}_0(D(v) \otimes 1)$ for all $v \in \mathsf{V}$. Given $a, b \in \mathsf{A}$, the definition of a D_0-operator implies that

$$\begin{aligned} D_1(a \otimes v \otimes b) &= D_1\big((a \otimes b)(1 \otimes v \otimes 1)\big) \\ &= (a \otimes b)D_1(1 \otimes v \otimes 1) + D_0(a \otimes b)(1 \otimes v \otimes 1) \\ &= a\mathsf{s}_0(D(v) \otimes 1)b + D(a) \otimes v \otimes b + a \otimes v \otimes D(b). \end{aligned}$$

As s_0 is a right A-module map, this expression matches the one in the statement.

Now, by Lemma 5.2 (b), it suffices to check the equality $D_0 \circ \mathsf{d}_0 = \mathsf{d}_0 \circ D_1$ on elements of the form $1 \otimes v \otimes 1$. Thus, using the second identity in (3.3), we establish the final claim:

$$\begin{aligned} \mathsf{d}_0 \circ D_1(1 \otimes v \otimes 1) &= \mathsf{d}_0\big(\mathsf{s}_0(D(v) \otimes 1)\big) = D(v) \otimes 1 - \mathsf{s}_{-1} \circ \mu\big(D(v) \otimes 1\big) \\ &= D(v) \otimes 1 - \mathsf{s}_{-1}\big(D(v)\big) = D(v) \otimes 1 - 1 \otimes D(v) \\ &= D_0(v \otimes 1 - 1 \otimes v) = D_0 \circ \mathsf{d}_0(1 \otimes v \otimes 1). \end{aligned}$$ ∎

Before we proceed to define the D_0-lifting D_2, we prove some auxiliary relations which will simplify several expressions, including one for $D_2(1 \otimes \mathsf{r} \otimes 1)$.

Lemma 5.4. *Let $g \in \mathbb{F}[x]$, $\alpha \in \mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A}$, $b \in \mathsf{A}$, and $k, \ell \geq 0$. The following hold:*

(a) $\mathsf{s}_1(g\alpha) = g\mathsf{s}_1(\alpha)$;

(b) $\mathsf{s}_1 \circ \mathsf{s}_0 = 0$;

(c) $\mathsf{s}_1(\hat{y}\mathsf{s}_0(g\hat{y}^\ell \otimes b)) = G(g)\hat{y}^\ell b$, *where G is given in* (4.2);

(d) $\mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(x^k \otimes 1) = \sum_{i=1}^{k-1} \mathsf{s}_1(D(x^i) \otimes x \otimes x^{k-i-1})$, *where this sum is understood to be* 0 *in case $k \in \{0, 1\}$.*

Proof. Both (a) and (b) follow trivially from the definitions, so we proceed to prove (c). As before, we can assume that $b = 1$. Furthermore, using (a), (b), Lemma 4.1 (a), the definition of s_1, and Proposition 4.6, we get

$$\begin{aligned}
\mathsf{s}_1\big(\hat{y}\mathsf{s}_0(g\hat{y}^\ell \otimes 1)\big) &= \mathsf{s}_1\big(\hat{y}\big(g\mathsf{s}_0(\hat{y}^\ell \otimes 1) + \mathsf{s}_0(g \otimes \hat{y}^\ell)\big)\big) \\
&= \mathsf{s}_1\big(g\hat{y}\mathsf{s}_0(\hat{y}^\ell \otimes 1)\big) + g'h\mathsf{s}_1\big(\mathsf{s}_0(\hat{y}^\ell \otimes 1)\big) + \mathsf{s}_1\big(\hat{y}\mathsf{s}_0(g \otimes 1)\big)\hat{y}^\ell \\
&= g\mathsf{s}_1\big(\hat{y}\mathsf{s}_0(\hat{y}^\ell \otimes 1)\big) + \mathsf{s}_1\big(\hat{y}\mathsf{s}_0(g \otimes 1)\big)\hat{y}^\ell \\
&= G(g)\hat{y}^\ell.
\end{aligned}$$

Finally, for the proof of (d), we have, using the definition of D_1, parts (a) and (b), and the definition of s_1:

$$\begin{aligned}
\mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(x^k \otimes 1) &= \sum_{i=0}^{k-1} \mathsf{s}_1 \circ D_1(x^i \otimes x \otimes x^{k-i-1}) \\
&= \sum_{i=0}^{k-1} \mathsf{s}_1\big(D(x^i) \otimes x \otimes x^{k-i-1}\big) \\
&= \sum_{i=1}^{k-1} \mathsf{s}_1\big(D(x^i) \otimes x \otimes x^{k-i-1}\big).
\end{aligned}$$ ■

Motivated by Lemma 5.4 (c), we extend the map G linearly to A, by setting

$$G(f\hat{y}^\ell) = G(f)\hat{y}^\ell, \quad \text{for all } f \in \mathbb{F}[x] \text{ and all } \ell \geq 0. \tag{5.5}$$

Thus, we can rewrite Lemma 5.4 (c) as

$$\mathsf{s}_1\big(\hat{y}\mathsf{s}_0(a \otimes b)\big) = G(a)b, \quad \text{for all } a, b \in \mathsf{A}. \tag{5.6}$$

We are now ready to define the D_0-operator D_2 in terms of D_1 and the homotopy s_1.

Lemma 5.7. *There is a unique D_0-operator $D_2 : \mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A} \to \mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A}$ such that $D_2(1 \otimes \mathsf{r} \otimes 1) = \mathsf{s}_1 \circ D_1 \circ \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1)$. Then, $D_1 \circ \mathsf{d}_1 = \mathsf{d}_1 \circ D_2$ and*

$$D_2(1 \otimes \mathsf{r} \otimes 1) = G\big(D(x)\big) + \mathsf{s}_1\big(D(\hat{y}) \otimes x \otimes 1\big) - \mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(h \otimes 1). \tag{5.8}$$

Proof. By Lemma 5.2 (a), there exists a unique D_0-operator D_2 defined on $\mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A}$ and such that $D_2(1 \otimes \mathsf{r} \otimes 1) = \mathsf{s}_1 \circ D_1 \circ \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1)$. The exact expression for $D_2(a \otimes \mathsf{r} \otimes b)$ can be computed as in the proof of Lemma 5.3.

Now, using Lemma 5.4 and (5.6), we have

$$\begin{aligned} D_2(1 \otimes \mathsf{r} \otimes 1) &= \mathsf{s}_1\big(D_1(1 \otimes \hat{y} \otimes x)\big) + \mathsf{s}_1\big(D_1(\hat{y} \otimes x \otimes 1)\big) - \mathsf{s}_1\big(D_1(1 \otimes x \otimes \hat{y})\big) \\ &\quad - \mathsf{s}_1\big(D_1(x \otimes \hat{y} \otimes 1)\big) - \mathsf{s}_1\big(D_1\big(\mathsf{s}_0(h \otimes 1)\big)\big) \\ &= \mathsf{s}_1\big(\mathsf{s}_0\big(D(\hat{y}) \otimes x\big)\big) + \mathsf{s}_1\big(1 \otimes \hat{y} \otimes D(x)\big) + \mathsf{s}_1\big(\hat{y}\mathsf{s}_0\big(D(x) \otimes 1\big)\big) \\ &\quad + \mathsf{s}_1\big(D(\hat{y}) \otimes x \otimes 1\big) - \mathsf{s}_1\big(\mathsf{s}_0\big(D(x) \otimes \hat{y}\big)\big) - \mathsf{s}_1\big(1 \otimes x \otimes D(\hat{y})\big) \\ &\quad - \mathsf{s}_1\big(x\mathsf{s}_0\big(D(\hat{y}) \otimes 1\big)\big) - \mathsf{s}_1\big(D(x) \otimes \hat{y} \otimes 1\big) - \mathsf{s}_1\big(D_1\big(\mathsf{s}_0(h \otimes 1)\big)\big) \\ &= \mathsf{s}_1\big(\hat{y}\mathsf{s}_0\big(D(x) \otimes 1\big)\big) + \mathsf{s}_1\big(D(\hat{y}) \otimes x \otimes 1\big) - \mathsf{s}_1\big(D_1\big(\mathsf{s}_0(h \otimes 1)\big)\big) \\ &= G\big(D(x)\big) + \mathsf{s}_1\big(D(\hat{y}) \otimes x \otimes 1\big) - \mathsf{s}_1\big(D_1\big(\mathsf{s}_0(h \otimes 1)\big)\big). \end{aligned}$$

Finally, by Lemma 5.2 (b), it is enough to show that $D_1 \circ \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1) = \mathsf{d}_1 \circ D_2(1 \otimes \mathsf{r} \otimes 1)$, so we compute, using Lemma 4.4 and Lemma 5.3,

$$\begin{aligned} \mathsf{d}_1 \circ D_2(1 \otimes \mathsf{r} \otimes 1) &= \mathsf{d}_1 \circ \mathsf{s}_1 \circ D_1 \circ \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1) \\ &= D_1 \circ \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1) - \mathsf{s}_0 \circ \mathsf{d}_0 \circ D_1 \circ \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1) \\ &= D_1 \circ \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1) - \mathsf{s}_0 \circ D_0 \circ \mathsf{d}_0 \circ \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1) \\ &= D_1 \circ \mathsf{d}_1(1 \otimes \mathsf{r} \otimes 1), \end{aligned}$$

as $\mathsf{d}_0 \circ \mathsf{d}_1 = 0$. ∎

5.3. Technical lemmas

We need to prove yet some more technical results which will allow us to simplify the computation of the Gerstenhaber bracket given in (5.1). Although these will be particularly useful in case $\mathrm{char}(\mathbb{F}) = 0$, most statements hold over an arbitrary field, so we include them here.

Following [1, Lem. 2.13], it will be useful to define, for $0 \neq f \in \mathbb{F}[x]$, the element π_f such that

(1) $\pi_f \in \mathbb{F}[x]$ is monic;

(2) $\pi_f = \frac{f}{\mathrm{gcd}(f, f')}$, up to a nonzero scalar.

In particular, if $f' = 0$, then $\pi_f = 1$.

In this subsection, we will mostly work over some homomorphic image of A and we will extensively use the notations $a \equiv b \pmod{I}$ and $a \equiv b \pmod{c}$, defined in the introduction to mean that $a - b \in I$ and $a - b \in c\mathsf{A} = \mathsf{A}c$, for a two-sided ideal I and a normal element c, respectively. We remark that the monoid of normal elements of A was described in [2, Thm. 7.2] and, in particular, any product of factors of h is normal in A.

Lemma 5.9. *Let $D \in \mathrm{Der}_{\mathbb{F}}(\mathsf{A})$, $a \in \mathsf{A}$, and $k \geq 0$. The following hold:*

(a) $D(h) \in h\mathsf{A}$ *and* $D(x) \in \pi_h\mathsf{A}$;

(b) $D(a^k) \equiv ka^{k-1}D(a) \pmod{h}$;

(c) $D(\mathsf{gcd}(h, h')) \in \mathsf{gcd}(h, h')\mathsf{A}$.

Proof. The defining relation for A implies that

$$D(h) = -\big[D(x), \hat{y}\big] - \big[x, D(\hat{y})\big] \in [\mathsf{A}, \mathsf{A}] \subseteq h\mathsf{A}.$$

So $D(h\mathsf{A}) \subseteq h\mathsf{A}$ and D induces a derivation $\overline{D} : \mathsf{A}/h\mathsf{A} \to \mathsf{A}/h\mathsf{A}$ with $\overline{D}(a + h\mathsf{A}) = D(a) + h\mathsf{A}$. Since $\mathsf{A}/h\mathsf{A}$ is commutative, we have

$$D(a^k) + h\mathsf{A} = \overline{D}\big((a + h\mathsf{A})^k\big) = ka^{k-1}D(a) + h\mathsf{A},$$

which proves (b).

In particular, $0 \equiv D(h) \equiv h'D(x) \pmod{h}$, and it follows that $h'D(x) \in h\mathsf{A}$. Since for any $f \in \mathbb{F}[x]$ we have that h divides $h'f$ if and only if π_h divides f, we conclude that $D(x) \in \pi_h\mathsf{A}$, finishing the proof of (a).

Let $g = \mathsf{gcd}(h, h')$. Up to a nonzero scalar, $h = \pi_h g$. Write $D(x) = \pi_h b$ for some $b \in \mathsf{A}$. By (b),

$$D(g) \in g'\pi_h b + h\mathsf{A} \subseteq g'\pi_h\mathsf{A} + h\mathsf{A}.$$

As $h' = \pi_h g' + \pi_h' g$ and g divides h', we deduce that g divides $\pi_h g'$, so $D(g) \in g\mathsf{A} + h\mathsf{A} = g\mathsf{A}$. ∎

Lemma 5.10. *Let ν be a divisor of h, $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A})$, $\chi \in \mathsf{Hom}_{\mathsf{A}^e}(\mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A}, \mathsf{A})$, and $f \in \mathbb{F}[x]$. The following hold:*

(a) $\mathsf{s}_1(\nu\mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A} + \mathsf{A} \otimes \mathsf{V} \otimes \nu\mathsf{A}) \subseteq \nu\mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A} + \mathsf{A} \otimes \mathsf{R} \otimes \nu\mathsf{A}$;

(b) $\chi(\nu\mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A} + \mathsf{A} \otimes \mathsf{R} \otimes \nu\mathsf{A}) \subseteq \nu\mathsf{A}$;

(c) $\chi \circ G(f) \equiv f'\chi(1 \otimes \mathsf{r} \otimes 1) \pmod{h}$; *in particular,* $\chi \circ G(h\mathsf{A}) \subseteq \mathsf{gcd}(h, h')\mathsf{A}$;

(d) *if* $\mathsf{char}(\mathbb{F}) \neq 2$, *then* $\chi \circ \mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(f \otimes 1) \in \pi_h f''\mathsf{A} + h\mathsf{A}$; *in particular,* $\chi \circ \mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(h \otimes 1) \in \mathsf{gcd}(h, h')\mathsf{A}$;

(e) $\chi \circ \mathsf{s}_1(\hat{y}^\ell \otimes x \otimes 1) \equiv \ell\chi(1 \otimes \mathsf{r} \otimes 1)\hat{y}^{\ell-1} \pmod{\mathsf{gcd}(h, h')}$, *for all* $\ell \geq 0$.

Proof. The claim in (a) is clear because ν is normal, $\mathsf{s}_1(\nu\mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A}) = \nu\mathsf{s}_1(\mathsf{A} \otimes \mathsf{V} \otimes \mathsf{A}) \subseteq \nu\mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A}$, by Lemma 5.4, and s_1 is a right A-module map. Claim (b) is proved similarly.

Take $f = x^k$, with $k \geq 0$. Then,

$$\begin{aligned} \chi \circ G(x^k) = \sum_{i=0}^{k-1} x^i \chi(1 \otimes \mathsf{r} \otimes 1)x^{k-i-1} &\equiv \sum_{i=0}^{k-1} x^{k-1}\chi(1 \otimes \mathsf{r} \otimes 1) \\ &\equiv kx^{k-1}\chi(1 \otimes \mathsf{r} \otimes 1) \pmod{h}, \end{aligned}$$

establishing the first claim in (c). Thus, for all $\ell \geq 0$,

$$\chi \circ G(hf\hat{y}^\ell) = \chi\big(G(hf)\big)\hat{y}^\ell \in (h'f + hf')\chi(1 \otimes \mathsf{r} \otimes 1)\hat{y}^\ell + h\mathsf{A} \subseteq \mathsf{gcd}(h, h')\mathsf{A},$$

proving that $\chi \circ G(h\mathsf{A}) \subseteq \mathsf{gcd}(h, h')\mathsf{A}$.

For (d), consider $f = x^k$, with $k \geq 0$. By Lemma 5.9, there is $a \in \mathsf{A}$ such that $D(x) = \pi_h a$ and $D(x^i) - ix^{i-1}D(x) \in h\mathsf{A}$, for all $i \geq 0$. Set $\theta_i = D(x^i) - ix^{i-1}D(x)$. By Lemma 5.4, we have

$$\begin{aligned}\chi \circ \mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(x^k \otimes 1) &= \sum_{i=1}^{k-1} \chi \circ \mathsf{s}_1\big(D(x^i) \otimes x \otimes x^{k-i-1}\big)\\ &= \sum_{i=1}^{k-1} \chi \circ \mathsf{s}_1\big(\big(ix^{i-1}D(x) + \theta_i\big) \otimes x \otimes x^{k-i-1}\big).\end{aligned}$$

By (a) and (b), $\sum_{i=1}^{k-1} \chi \circ \mathsf{s}_1(\theta_i \otimes x \otimes x^{k-i-1}) \in h\mathsf{A}$. Thus, working modulo $h\mathsf{A}$ and using the commutativity of $\mathsf{A}/h\mathsf{A}$ and the hypothesis that $\mathsf{char}(\mathbb{F}) \neq 2$, we obtain

$$\begin{aligned}\chi \circ \mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(x^k \otimes 1) &\equiv \sum_{i=1}^{k-1} \chi \circ \mathsf{s}_1(ix^{i-1}\pi_h a \otimes x \otimes x^{k-i-1})\\ &\equiv \sum_{i=1}^{k-1} ix^{i-1}\pi_h \chi\big(\mathsf{s}_1(a \otimes x \otimes 1)\big)x^{k-i-1}\\ &\equiv \binom{k}{2} x^{k-2}\pi_h \chi\big(\mathsf{s}_1(a \otimes x \otimes 1)\big)\\ &\equiv (x^k)''\pi_h \frac{1}{2}\chi\big(\mathsf{s}_1(a \otimes x \otimes 1)\big) \pmod{h},\end{aligned}$$

so indeed $\chi \circ \mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(f \otimes 1) \in f''\pi_h\mathsf{A} + h\mathsf{A}$. In particular,

$$\chi \circ \mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(h \otimes 1) \in h''\pi_h\mathsf{A} + h\mathsf{A} \subseteq \mathsf{gcd}(h, h')\mathsf{A}$$

because $\mathsf{gcd}(h, h')$ divides $h''\pi_h$.

Lastly, we prove (e) by induction on $\ell \geq 0$. As $\chi \circ \mathsf{s}_1(1 \otimes x \otimes 1) = 0$, the base step is established and we assume that

$$\chi \circ \mathsf{s}_1(\hat{y}^\ell \otimes x \otimes 1) \equiv \ell\chi(1 \otimes \mathsf{r} \otimes 1)\hat{y}^{\ell-1} \big(\mathsf{mod}\ \mathsf{gcd}(h, h')\big)$$

holds for some $\ell \geq 0$. Then, by the definition of s_1, the commutativity of $\mathsf{A}/\mathsf{gcd}(h, h')\mathsf{A}$, and part (c) above, as $\delta^j(x) \in h\mathsf{A}$ for all positive j,

$$\begin{aligned}\chi \circ \mathsf{s}_1(\hat{y}^{\ell+1} \otimes x \otimes 1) &= \hat{y}\chi\big(\mathsf{s}_1(\hat{y}^\ell \otimes x \otimes 1)\big) + \sum_{j=0}^{\ell} \binom{\ell}{j} \chi \circ G\big(\delta^j(x)\big)\hat{y}^{\ell-j}\\ &\equiv \ell\chi(1 \otimes \mathsf{r} \otimes 1)\hat{y}^\ell + \chi \circ G(x)\hat{y}^\ell\\ &\equiv \ell\chi(1 \otimes \mathsf{r} \otimes 1)\hat{y}^\ell + \chi(1 \otimes \mathsf{r} \otimes 1)\hat{y}^\ell \big(\mathsf{mod}\ \mathsf{gcd}(h, h')\big).\end{aligned}$$

■

Lemma 5.11. *Let $\chi \in \mathsf{Hom}_{\mathsf{A}^e}(\mathsf{A} \otimes \mathsf{R} \otimes \mathsf{A}, \mathsf{A})$, $f \in \mathbb{F}[x]$, and $k \geq 0$. Then, the following hold.*

(a) $\pi_h h^{k-1}[y^{k+1}, h] \equiv (k+1)\pi_h h' h^{k-1} y^k + \binom{k+1}{2}\pi_h h'' h^{k-1} y^{k-1} \pmod{h}$. *(Notice that in case $k = 0$, the above expression still makes sense, as $\frac{\pi_h h'}{h} = \frac{h'}{\mathsf{gcd}(h,h')} \in \mathbb{F}[x]$.)*

(b) $\hat{y}^k \equiv h^k y^k \pmod{\mathsf{gcd}(h,h')}$.

(c) $\chi \circ G(fh^k y^k) \equiv f'\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^k - \binom{k+1}{2} fh''\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^{k-1} \pmod{\mathsf{gcd}(h,h')}$.

Proof. Working modulo $h\mathsf{A}$, we deduce (a):

$$\pi_h h^{k-1}[y^{k+1},h] = \sum_{j=1}^{k+1}\binom{k+1}{j}\pi_h h^{(j)} h^{k-1} y^{k+1-j}$$
$$\equiv (k+1)\pi_h h' h^{k-1} y^k + \binom{k+1}{2}\pi_h h'' h^{k-1} y^{k-1} \pmod{h}.$$

In particular, multiplying both sides of (a) by $\mathsf{gcd}(h,h') = h/\pi_h$, we obtain

$$h^k[y^{k+1},h] \equiv (k+1)h'h^k y^k + \binom{k+1}{2} h'' h^k y^{k-1} \equiv (k+1)h'h^k y^k \pmod{h}, \quad (5.12)$$

and it follows that $h^k[y^{k+1},h] \in \mathsf{gcd}(h,h')\mathsf{A}$.

We are now ready to prove (b) by induction on $k \geq 0$, the base case being trivial. Supposing that (b) holds for a certain $k \geq 0$, we get

$$\hat{y}^{k+1} \equiv h^k y^{k+1} h = h^{k+1} y^{k+1} + h^k[y^{k+1},h] \equiv h^{k+1} y^{k+1} \big(\mathsf{mod}\ \mathsf{gcd}(h,h')\big).$$

We also prove (c) by induction on $k \geq 0$. The case $k = 0$ is immediate from Lemma 5.10 (c). For the inductive step, assume that the congruence holds for $k \geq 0$. By (5.12), we have

$$h^{k+1} y^{k+1} = h^k y^{k+1} h - h^k[y^{k+1},h] \equiv h^k y^k \hat{y} - (k+1)h'h^k y^k \pmod{h}.$$

By Lemma 5.10 (c),

$$\begin{aligned}
&\chi \circ G(fh^{k+1}y^{k+1})\\
&\equiv \chi \circ G(fh^k y^k)\hat{y} - (k+1)\chi \circ G(fh'h^k y^k)\\
&\equiv f'\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^{k+1} - \binom{k+1}{2} fh''\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^k\\
&\quad - (k+1)(fh')'\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^k + (k+1)\binom{k+1}{2} fh'h''\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^{k-1}\\
&\equiv f'\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^{k+1} - \binom{k+1}{2} fh''\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^k - (k+1)fh''\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^k\\
&\equiv f'\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^{k+1} - \binom{k+2}{2} fh''\chi(1\otimes \mathsf{r}\otimes 1)\hat{y}^k \big(\mathsf{mod}\ \mathsf{gcd}(h,h')\big).
\end{aligned}$$

■

6. The Gerstenhaber bracket

In this section, we determine the structure of $\mathsf{HH}^2(\mathsf{A})$ as a module over the Lie algebra $\mathsf{HH}^1(\mathsf{A})$ under the Gerstenhaber bracket, always under the assumption that $\mathsf{char}(\mathbb{F}) = 0$. We will prove some of the main results of this article. In the first subsection, we will describe two different subspaces of the space of linear derivations of our algebra that will act on $\mathsf{HH}^2(\mathsf{A})$ in a very different way. Next, we will describe the action of the classes of these derivations on $\mathsf{HH}^2(\mathsf{A})$. Then, we achieve our goal of giving an explicit description of $\mathsf{HH}^2(\mathsf{A})$ as an $\mathsf{HH}^1(\mathsf{A})$-Lie module. We finish the section by relating this action of $\mathsf{HH}^1(\mathsf{A})$ on $\mathsf{HH}^2(\mathsf{A})$ with the representation theory of the Virasoro algebra, and then by discussing several special cases. More explicitly, we will describe the composition series of $\mathsf{HH}^2(\mathsf{A})$ as an $\mathsf{HH}^1(\mathsf{A})$-Lie module, whose length equals the maximum of the multiplicities of the irreducible factors of h minus 1. The successive quotients associated to this composition series turn out to be completely reducible. Moreover, they decompose as direct sums of intermediate series modules over a Witt algebra. The intermediate series modules are naturally graded and the dimensions of the homogeneous components are uniformly bounded.

6.1. The Lie algebra structure of $\mathsf{HH}^1(\mathsf{A})$

The Lie algebra structure of $\mathsf{HH}^1(\mathsf{A})$ in case $\mathsf{char}(\mathbb{F}) = 0$ is described explicitly in [1, Sec. 5] and we briefly collect the results we need below.

There are two types of derivations of A, which together describe $\mathsf{Der}_{\mathbb{F}}(\mathsf{A})$ and $\mathsf{HH}^1(\mathsf{A})$.

- For any $g \in \mathbb{F}[x]$, let D_g be the derivation of A such that $D_g(x) = 0$ and $D_g(\hat{y}) = g$. Then, $\{D_g \mid g \in \mathbb{F}[x]\}$ is an abelian Lie subalgebra of $\mathsf{Der}_{\mathbb{F}}(\mathsf{A})$ and $D_g \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A})$ if and only if $g \in h\mathbb{F}[x]$.
- Viewing, as usual, $\mathsf{A} = \mathsf{A}_h \subseteq \mathsf{A}_1$ with $\hat{y} = yh$, define the elements $a_n = \pi_h h^{n-1} y^n \in \{u \in \mathsf{A}_1 \mid [u, \mathsf{A}] \subseteq \mathsf{A}\}$ (the normalizer of A in A_1), for all $n \geq 1$. It will also be convenient to consider the element $a_0 = \pi_h / h = \frac{1}{\mathsf{gcd}(h,h')}$ in the localization of A_1 at the Ore set formed by the powers of h. Then, $\mathsf{ad}_{ga_n} \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A})$ for all $n \geq 0$ and $g \in \mathbb{F}[x]$. Moreover, $\mathsf{ad}_{ga_n} \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A})$ if and only if $g \in \mathsf{gcd}(h, h')\mathbb{F}[x]$.

Next, we recall the definition in [1, Sec. 4.3] of the linear endomorphism $\delta_0 : \mathbb{F}[x] \to \mathbb{F}[x]$ given by

$$\delta_0(g) = \delta(ga_0) = (g\pi_h h^{-1})'h = (g\pi_h)' - g\frac{\pi_h h'}{h}, \tag{6.1}$$

where $\delta(f) = f'h$. By [1, Lem. 4.14], $\mathsf{ad}_{ga_0} = -D_{\delta_0(g)}$.

For notational simplicity, by [2, Thm. 8.2], we can assume that h is monic, say $h = \mathsf{u}_1^{\alpha_1} \cdots \mathsf{u}_t^{\alpha_t}$, where $\mathsf{u}_1, \ldots, \mathsf{u}_t$ are the distinct monic prime factors of h, with multiplicities $\alpha_1, \ldots, \alpha_t$. Up to changing the order of the factors, we can further assume that there is $0 \leq k \leq t$ such that $\alpha_1, \ldots, \alpha_k \geq 2$ and $\alpha_{k+1} = \cdots = \alpha_t = 1$. Moreover, if $k = 0$, then $\mathsf{gcd}(h, h') = 1$ and in this case $\mathsf{HH}^2(\mathsf{A}) = 0$, so there is nothing to prove.

We have the following result (see also [1, Thm. 5.1, Prop. 5.9]).

Theorem 6.2. *Assume that* $\mathsf{char}(\mathbb{F}) = 0$. *Then, there is a decomposition* $\mathsf{HH}^1(\mathsf{A}) = \mathsf{Z}(\mathsf{HH}^1(\mathsf{A})) \oplus [\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})]$. *Moreover, using the above notations, there are isomorphisms of Lie algebras:*

(a) $\mathcal{N} = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ga_n} \mid g \in \mathsf{u}_1 \cdots \mathsf{u}_k\mathbb{F}[x],\ n \geq 0\}$ *is the unique maximal nilpotent ideal of* $[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})]$*;*

(b) $\mathsf{Z}(\mathsf{HH}^1(\mathsf{A})) \cong \{D_g \mid g \in \mathsf{gcd}(h,h')\mathbb{F}[x],\ \deg g < \deg h\}$;

(c) $[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})] = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ga_n} \mid g \in \mathbb{F}[x],\ \deg g < \deg \mathsf{gcd}(h,h'),\ n \geq 0\}$;

(d) $[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})]/\mathcal{N} \cong \mathsf{W}_1 \oplus \cdots \oplus \mathsf{W}_k$, *where* $\mathsf{W}_i = (\mathbb{F}[x]/\mathsf{u}_i\mathbb{F}[x]) \otimes \mathsf{W}$ *is a field extension of the Witt algebra.*

6.2. Formulas for the Gerstenhaber bracket $[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^2(\mathsf{A})]$

Recall that by Corollary 3.11, $\mathsf{HH}^2(\mathsf{A}) \cong \mathsf{A}/\mathsf{gcd}(h,h')\mathsf{A}$ can be identified with the polynomial ring $\mathsf{D}[\hat{y}]$, where $\mathsf{D} = (\mathbb{F}[x]/\mathsf{gcd}(h,h')\mathbb{F}[x])$. We will use (5.1) and also the identification introduced there between $\mathsf{A}/\mathsf{gcd}(h,h')\mathsf{A}$ and $\mathsf{Hom}_{\mathsf{A}^e}(\mathsf{A}\otimes\mathsf{R}\otimes\mathsf{A},\mathsf{A})/\operatorname{im}\mathsf{d}_1^*$, which associates the element $a \in \mathsf{A}$ with the map $\chi_a \in \mathsf{Hom}_{\mathsf{A}^e}(\mathsf{A}\otimes\mathsf{R}\otimes\mathsf{A},\mathsf{A})$ defined by $\chi_a(1\otimes\mathsf{r}\otimes 1) = a$, and similarly for the corresponding homomorphic images.

Fix $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A})$ and let D_1 be the lifting as in Lemma 5.3. Now, Lemma 5.10 (d) implies that for all $a \in \mathsf{A}$, the image of $\chi_a \circ \mathsf{s}_1 \circ D_1 \circ \mathsf{s}_0(h \otimes 1)$ in $\mathsf{HH}^2(\mathsf{A})$ is zero. Thus, we have, using Lemma 5.7,

$$[D,a] = D(a) - \chi_a\big(G\big(D(x)\big)\big) - \chi_a\big(\mathsf{s}_1\big(D(\hat{y}) \otimes x \otimes 1\big)\big), \tag{6.3}$$

for all $a \in \mathsf{A}$ and $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A})$. Moreover, by Lemma 5.9 (c), the image of $D(a)$ in $\mathsf{HH}^2(\mathsf{A})$ depends only on the class $a + \mathsf{gcd}(h,h')\mathsf{A}$ and similarly, $\chi_a(G(D(x)))$ and $\chi_a(\mathsf{s}_1(D(\hat{y}) \otimes x \otimes 1))$ depend only on the classes $D(x) + h\mathsf{A}$ and $D(\hat{y}) + \mathsf{gcd}(h,h')\mathsf{A}$, respectively, by Lemma 5.10.

We will first consider the derivations of the form D_g, for $g \in \mathbb{F}[x]$. Fix g and let $D = D_g$. Take $a = p(x)\hat{y}^k$ for some $p(x) \in \mathbb{F}[x]$ and $k \geq 0$. Then, $D(x) = 0 = \mathsf{s}_1(D(\hat{y})\otimes x \otimes 1)$ and by Lemma 5.9, $D(p(x)\hat{y}^k) = p(x)D(\hat{y}^k) \equiv kp(x)\hat{y}^{k-1}g \equiv kgp(x)\hat{y}^{k-1} \pmod{h}$. Thus, $[D_g, p(x)\hat{y}^k] \equiv kgp(x)\hat{y}^{k-1} \pmod{\mathsf{gcd}(h,h')}$. So,

$$[D_g, -] = g\frac{d}{d\hat{y}} \quad \text{on } \mathsf{D}[\hat{y}]. \tag{6.4}$$

In particular, $[\mathsf{Z}(\mathsf{HH}^1(\mathsf{A})), \mathsf{HH}^2(\mathsf{A})] = 0$, by Theorem 6.2 (b).

Now, we can turn our attention to the derivations of the form ad_{ga_n}, with $g \in \mathbb{F}[x]$ and $n \geq 0$.

Lemma 6.5. *Let* $D = \mathsf{ad}_{ga_n}$ *and* $a = p(x)\hat{y}^k \in \mathsf{A}$*, as above. Then,*

(a) $D(x) = n\pi_h gh^{n-1}y^{n-1} \equiv n\pi_h g\hat{y}^{n-1} \pmod{\mathsf{gcd}(h,h')}$;

(b) $D(\hat{y}) \equiv -\delta_0(g)\hat{y}^n \pmod{\mathsf{gcd}(h,h')}$;

(c) $D(a) \equiv (n\pi_h gp'(x) - kp(x)\delta_0(g))\hat{y}^{n+k-1} \pmod{\mathsf{gcd}(h,h')}$.

Proof. We have

$$D(x) = [\pi_h gh^{n-1}y^n, x] = n\pi_h gh^{n-1}y^{n-1} \equiv n\pi_h g\hat{y}^{n-1} \pmod{\mathsf{gcd}(h,h')},$$

where the last congruence comes from Lemma 5.11 (b). Also,

$$\begin{aligned} D(\hat{y}) &= [\pi_h gh^{n-1}y^n, \hat{y}] = \pi_h gh^{n-1}y^{n+1}h - y\pi_h gh^n y^n \\ &= \pi_h gh^n y^{n+1} + \pi_h gh^{n-1}[y^{n+1}, h] - \pi_h gh^n y^{n+1} - [y, \pi_h gh^n]y^n \end{aligned}$$

$$
\begin{aligned}
&\equiv (n+1)\pi_h h' g h^{n-1} y^n + \binom{n+1}{2}\pi_h g h'' h^{n-1} y^{n-1} - (\pi_h g h^n)' y^n \pmod{h} \\
&\equiv (n+1)\pi_h h' g h^{n-1} y^n - (\pi_h g h^n)' y^n \big(\bmod \operatorname{gcd}(h,h')\big) \\
&\equiv (n+1)\pi_h h' g h^{n-1} y^n - n\pi_h g h' h^{n-1} y^n - (\pi_h g)' h^n y^n \big(\bmod \operatorname{gcd}(h,h')\big) \\
&\equiv \pi_h h' g h^{n-1} y^n - (\pi_h g)' h^n y^n \big(\bmod \operatorname{gcd}(h,h')\big) \\
&\equiv \left(\frac{\pi_h h' g}{h} - (\pi_h g)'\right) h^n y^n \big(\bmod \operatorname{gcd}(h,h')\big) \\
&\equiv -\delta_0(g)\hat{y}^n \big(\bmod \operatorname{gcd}(h,h')\big),
\end{aligned}
$$

using Lemma 5.11 (a) and (b), the fact that $\operatorname{gcd}(h,h')$ divides $h''\pi_h$, and (6.1).

Finally, using Lemma 5.9 (b),

$$
\begin{aligned}
D(a) &\equiv D\big(p(x)\big)\hat{y}^k + p(x)D(\hat{y}^k) \\
&\equiv p'(x)D(x)\hat{y}^k + kp(x)D(\hat{y})\hat{y}^{k-1} \\
&\equiv \big(n\pi_h g p'(x) - kp(x)\delta_0(g)\big)\hat{y}^{n+k-1} \big(\bmod \operatorname{gcd}(h,h')\big). \qquad \blacksquare
\end{aligned}
$$

Hence, for $D = \mathsf{ad}_{ga_n}$ and $a = p(x)\hat{y}^k \in \mathsf{A}$, we can now compute $[D,a]$ as an element of $\mathsf{D}[\hat{y}]$, using (6.3), Lemma 5.10 (e), and Lemma 5.11 (c) and recalling that $\operatorname{gcd}(h,h')$ divides $h''\pi_h$:

$$
\begin{aligned}
D(a) &\equiv \big(n\pi_h g p'(x) - kp(x)\delta_0(g)\big)\hat{y}^{n+k-1} \big(\bmod \operatorname{gcd}(h,h')\big), \\
\chi_a\Big(G\big(D(x)\big)\Big) &= \chi_a\big(G(n\pi_h g h^{n-1} y^{n-1})\big) \\
&\equiv n(\pi_h g)' p(x)\hat{y}^{n+k-1} - n\binom{n}{2}\pi_h g h'' p(x)\hat{y}^{n+k-2} \\
&\equiv n(\pi_h g)' p(x)\hat{y}^{n+k-1} \big(\bmod \operatorname{gcd}(h,h')\big), \\
\chi_a\big(\mathsf{s}_1(D(\hat{y})\otimes x\otimes 1)\big) &\equiv -\delta_0(g)\chi_a\big(\mathsf{s}_1(\hat{y}^n\otimes x\otimes 1)\big) \\
&\equiv -n\delta_0(g)p(x)\hat{y}^{n+k-1} \big(\bmod \operatorname{gcd}(h,h')\big).
\end{aligned}
$$

It thus follows that, working in $\mathsf{HH}^2(\mathsf{A}) = \mathsf{A}/\operatorname{gcd}(h,h')\mathsf{A}$ and recalling (6.1),

$$
\begin{aligned}
[D,a] &\equiv n\big(\pi_h g p'(x) - (\pi_h g)' p(x)\big)\hat{y}^{n+k-1} + (n-k)p(x)\delta_0(g)\hat{y}^{n+k-1} \\
&\equiv \left(n\pi_h g p'(x) - ng\frac{\pi_h h'}{h}p(x) - k\delta_0(g)p(x)\right)\hat{y}^{n+k-1} \big(\bmod \operatorname{gcd}(h,h')\big).
\end{aligned}
$$

Therefore, we have proved the main result of this subsection.

Theorem 6.6. *Assume that* $\operatorname{char}(\mathbb{F}) = 0$. *The Lie action of* $\mathsf{HH}^1(\mathsf{A})$ *on* $\mathsf{HH}^2(\mathsf{A})$ *under the Gerstenhaber bracket is given by the following formulas:*

$$
\big[\mathsf{Z}\big(\mathsf{HH}^1(\mathsf{A})\big), \mathsf{HH}^2(\mathsf{A})\big] = 0, \tag{6.7}
$$

$$
[\mathsf{ad}_{ga_n}, -] = n\pi_h g\hat{y}^{n-1}\frac{d}{dx} - \delta_0(g)\hat{y}^n\frac{d}{d\hat{y}} - ng\frac{\pi_h h'}{h}\hat{y}^{n-1}1_{\mathsf{D}[\hat{y}]}, \tag{6.8}
$$

for all $g \in \mathbb{F}[x]$ *and* $n \geq 0$, *where* $a_n = \pi_h h^{n-1} y^n$.

6.3. The structure of $\mathsf{HH}^2(\mathsf{A})$ as a Lie module over $\mathsf{HH}^1(\mathsf{A})$

Recall that $h = \mathsf{u}_1^{\alpha_1} \cdots \mathsf{u}_t^{\alpha_t}$, where $\mathsf{u}_1, \dots, \mathsf{u}_t$ are the prime factors of h, ordered so that $\alpha_1, \dots, \alpha_k \geq 2$ and $\alpha_{k+1} = \cdots = \alpha_t = 1$ for $0 \leq k \leq t$, as in Theorem 6.2. If $k = 0$, then $\mathsf{gcd}(h, h') = 1$ and in this case, $\mathsf{HH}^2(\mathsf{A}) = 0$. Thus, we suppose throughout this subsection that $k \geq 1$. Then,

$$\pi_h = \mathsf{u}_1 \cdots \mathsf{u}_t, \quad \mathsf{gcd}(h, h') = h/\pi_h = \mathsf{u}_1^{\alpha_1 - 1} \cdots \mathsf{u}_k^{\alpha_k - 1}, \quad \pi_{(h/\pi_h)} = \mathsf{u}_1 \cdots \mathsf{u}_k.$$

Let us fix $m_h = \max\{\alpha_j - 1 \mid 1 \leq j \leq k\} \geq 1$.

We make the identification $\mathsf{HH}^2(\mathsf{A}) = \mathsf{D}[\hat{y}]$, where $\mathsf{D} = \mathbb{F}[x]/\,\mathsf{gcd}(h, h')\mathbb{F}[x]$. Since $\mathsf{u}_i^{\alpha_i - 1}$, $1 \leq i \leq k$, are pairwise coprime,

$$\mathsf{D} \cong \mathbb{F}[x]/\mathsf{u}_1^{\alpha_1 - 1}\mathbb{F}[x] \oplus \cdots \oplus \mathbb{F}[x]/\mathsf{u}_k^{\alpha_k - 1}\mathbb{F}[x],$$

and there exist nonzero pairwise orthogonal idempotents $e_1, \dots, e_k \in \mathsf{D}$ with $e_1 + \cdots + e_k = 1$, $\mathsf{D} = \mathsf{D}e_1 \oplus \cdots \oplus \mathsf{D}e_k$, and $\mathsf{D}e_i \cong \mathbb{F}[x]/\mathsf{u}_i^{\alpha_i - 1}\mathbb{F}[x]$ (these isomorphisms are both as algebras and as left $\mathbb{F}[x]$-modules). Define $\mathsf{D}_i = \mathsf{D}e_i$. Then, $\mathsf{HH}^2(\mathsf{A}) = \mathsf{D}_1[\hat{y}] \oplus \cdots \oplus \mathsf{D}_k[\hat{y}]$.

Let $\overline{\mathsf{D}} = \mathbb{F}[x]/\mathsf{u}_1 \cdots \mathsf{u}_k\mathbb{F}[x] \cong \mathbb{F}[x]/\mathsf{u}_1\mathbb{F}[x] \oplus \cdots \oplus \mathbb{F}[x]/\mathsf{u}_k\mathbb{F}[x]$. Then, by Theorem 6.2 (d), we have

$$\left[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})\right]\big/\mathcal{N} \cong \overline{\mathsf{D}} \otimes \mathsf{W} \cong \mathsf{W}_1 \oplus \cdots \oplus \mathsf{W}_k,$$

with $\mathsf{W}_i = (\mathbb{F}[x]/\mathsf{u}_i\mathbb{F}[x]) \otimes \mathsf{W}$. As the notation suggests, the algebra $\overline{\mathsf{D}}$ is a quotient of D by the ideal $\mathsf{u}_1 \cdots \mathsf{u}_k\mathsf{D}$. Let $\overline{e_1}, \dots, \overline{e_k} \in \overline{\mathsf{D}}$ be the images of the idempotents $e_1, \dots, e_k \in \mathsf{D}$ under the canonical epimorphism. It is straightforward to see that these are still nonzero pairwise orthogonal idempotents in $\overline{\mathsf{D}}$ with $\overline{e_1} + \cdots + \overline{e_k} = 1$, $\overline{\mathsf{D}} = \overline{\mathsf{D}}\overline{e_1} \oplus \cdots \oplus \overline{\mathsf{D}}\overline{e_k}$, and $\overline{\mathsf{D}}\overline{e_i} \cong \mathbb{F}[x]/\mathsf{u}_i\mathbb{F}[x]$. Denote this field $\overline{\mathsf{D}}\overline{e_i} = \overline{\mathsf{D}_i}$ by $\overline{\mathsf{D}}_i$. Then,

$$\left[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})\right]\big/\mathcal{N} \cong (\overline{\mathsf{D}}_1 \otimes \mathsf{W}) \oplus \cdots \oplus (\overline{\mathsf{D}}_k \otimes \mathsf{W}). \tag{6.9}$$

For $i \geq 0$, set

$$\Theta_i = \prod_{j=1}^{k} \mathsf{u}_j^{\min\{\alpha_j - 1, i\}}.$$

Thus, $\Theta_0 = 1$, $\Theta_1 = \mathsf{u}_1 \cdots \mathsf{u}_k = \pi_{(h/\pi_h)}$ and for any $i \geq m_h$, $\Theta_i = \mathsf{gcd}(h, h')$. Finally, define

$$P_i = \Theta_i\mathsf{D}[\hat{y}] \subseteq \mathsf{HH}^2(\mathsf{A}).$$

We record a few useful facts below.

Lemma 6.10. *For $i \geq 0$, one has*

(a) $\Theta_{i+1} = \Theta_i(\prod_{\alpha_j \geq i+2} \mathsf{u}_j)$;

(b) $\pi_h\Theta_i' \equiv i\,\Theta_i\pi_h' \pmod{\Theta_{i+1}\mathbb{F}[x]}$;

(c) $P_i = \Theta_i \mathsf{D}[\hat{y}]$ *is a Lie* $\mathsf{HH}^1(\mathsf{A})$*-submodule of* $\mathsf{HH}^2(\mathsf{A})$ *and there is a strictly decreasing filtration*

$$\mathsf{HH}^2(\mathsf{A}) = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_{m_h-1} \supsetneq P_{m_h} = 0. \tag{6.11}$$

Proof. (a) is clear from the definition. The identity in (b) holds trivially for $i = 0$ and we prove it by induction on $i \geq 0$. So assume that $\pi_h \Theta_i' = i\Theta_i \pi_h' + \Theta_{i+1} f$, for some $f \in \mathbb{F}[x]$. As $\Theta_{i+1}(\prod_{\alpha_j \geq i+2} \mathsf{u}_j) \in \Theta_{i+2}\mathbb{F}[x]$, by (a), we have

$$\begin{aligned}
\pi_h \Theta_{i+1}' &= \pi_h \Big(\Theta_i \prod_{\alpha_j \geq i+2} \mathsf{u}_j\Big)' = \pi_h \Theta_i' \Big(\prod_{\alpha_j \geq i+2} \mathsf{u}_j\Big) + \pi_h \Theta_i \Big(\prod_{\alpha_j \geq i+2} \mathsf{u}_j\Big)' \\
&= (i\Theta_i \pi_h' + \Theta_{i+1} f)\Big(\prod_{\alpha_j \geq i+2} \mathsf{u}_j\Big) + \pi_h \Theta_i \Big(\prod_{\alpha_j \geq i+2} \mathsf{u}_j\Big)' \\
&\equiv i\Theta_i \pi_h' \Big(\prod_{\alpha_j \geq i+2} \mathsf{u}_j\Big) + \pi_h \Theta_i \Big(\prod_{\alpha_j \geq i+2} \mathsf{u}_j\Big)' \ \big(\mathsf{mod}\ \Theta_{i+2}\mathbb{F}[x]\big) \\
&= i\Theta_{i+1}\pi_h' + \Theta_{i+1}\Big(\prod_{1 \leq \alpha_j \leq i+1} \mathsf{u}_j\Big)\Big(\prod_{\alpha_j \geq i+2} \mathsf{u}_j\Big)' \\
&= i\Theta_{i+1}\pi_h' + \Theta_{i+1}\Big(\pi_h' - \Big(\prod_{1 \leq \alpha_j \leq i+1} \mathsf{u}_j\Big)'\Big(\prod_{\alpha_j \geq i+2} \mathsf{u}_j\Big)\Big) \\
&\equiv i\Theta_{i+1}\pi_h' + \Theta_{i+1}\pi_h' \ \big(\mathsf{mod}\ \Theta_{i+2}\mathbb{F}[x]\big).
\end{aligned}$$

The fact that (6.11) is a decreasing filtration of vector spaces is clear because Θ_i divides Θ_{i+1}. Since the quotient $\prod_{\alpha_j \geq i+2} \mathsf{u}_j$ of these polynomials is not a unit, for $0 \leq i \leq m_h - 1$, by the definition of m_h, the filtration is strict. Thus, it remains to show that $[\mathsf{ad}_{ga_n}, P_i] \subseteq P_i$, for all $g \in \mathbb{F}[x]$ and $n, i \geq 0$. By (6.8), given $f \in \mathbb{F}[x]$ and $\ell \geq 0$,

$$\begin{aligned}
[\mathsf{ad}_{ga_n}, \Theta_i f \hat{y}^\ell] = n\pi_h g \Theta_i f' \hat{y}^{n+\ell-1} + n\pi_h g \Theta_i' f \hat{y}^{n+\ell-1} \\
- \ell\delta_0(g)\Theta_i f \hat{y}^{n+\ell-1} - ng\frac{\pi_h h'}{h}\Theta_i f \hat{y}^{n+\ell-1},
\end{aligned}$$

which is in P_i because $\pi_h \Theta_i' \in \Theta_i \mathbb{F}[x]$. ■

Set $S_i = P_i/P_{i+1}$, for $0 \leq i \leq m_h - 1$. We have seen that S_i is a nonzero $\mathsf{HH}^1(\mathsf{A})$-module under the action induced from the Gerstenhaber bracket. Noting that $\delta_0(g) = g\delta_0(1) + g'\pi_h$ (see [1, Lem. 4.14]) and $\pi_h \Theta_i \in \Theta_{i+1}\mathbb{F}[x]$, we see that this action is completely described by the following computation in S_i:

$$\begin{aligned}
[\mathsf{ad}_{ga_n}, \Theta_i f \hat{y}^\ell] &\equiv fg\Big(n\pi_h\Theta_i' - \ell\delta_0(1)\Theta_i - n\frac{\pi_h h'}{h}\Theta_i\Big)\hat{y}^{n+\ell-1}, \\
&\equiv fg\Theta_i\Big(in\pi_h' - \ell\delta_0(1) - n\frac{\pi_h h'}{h}\Big)\hat{y}^{n+\ell-1} \ \ (\mathsf{mod}\ P_{i+1}).
\end{aligned} \tag{6.12}$$

In particular, $[\mathrm{ad}_{ga_n}, S_i] = 0$ if $g \in \mathsf{u}_1 \cdots \mathsf{u}_k \mathbb{F}[x] = \Theta_1 \mathbb{F}[x]$ because $\Theta_1 \Theta_i \in \Theta_{i+1} \mathbb{F}[x]$. So, $[\mathcal{N}, S_i] = 0$ for all $i \geq 0$, where $\mathcal{N}$ is the unique maximal nilpotent ideal of $[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})]$, as in Theorem 6.2. It follows that S_i is naturally an $[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})]/\mathcal{N}$-module.

Note that $S_i \cong (\Theta_i \mathsf{D}/\Theta_{i+1}\mathsf{D})[\hat{y}]$. Then, the definitions of D, Θ_i, and $m_h - 1$, along with Lemma 6.10 (a), imply that there is a natural isomorphism of vector spaces induced by the natural map $\mathsf{D} \twoheadrightarrow \Theta_i \mathsf{D}/\Theta_{i+1}\mathsf{D}$:

$$S_i \cong \frac{\mathsf{D}}{(\prod_{\alpha_j \geq i+2} \mathsf{u}_j)\mathsf{D}}[\hat{y}] \cong \bigoplus_{\alpha_j \geq i+2} \overline{\mathsf{D}}_j[\hat{y}], \quad \text{for all } 0 \leq i \leq m_h - 1. \tag{6.13}$$

By the above isomorphisms, the element $\Theta_i f \hat{y}^\ell + \Theta_{i+1}\mathsf{D}[\hat{y}] \in S_i$ is identified with the element $\sum_{\alpha_j \geq i+2} f\overline{e_j}\hat{y}^\ell \in \bigoplus_{\alpha_j \geq i+2} \overline{\mathsf{D}}_j[\hat{y}]$.

Our next step is to describe the Lie algebra isomorphism (6.9). We will need the following.

Lemma 6.14. *There is an element $\nu \in \mathbb{F}[x]$, determining a unique class modulo $\Theta_1\mathbb{F}[x]$, such that $\nu\delta_0(1) \equiv 1 \pmod{\Theta_1\mathbb{F}[x]}$. For such an element, the following hold:*

(a) $\nu\pi_h' - 1 \equiv \nu\frac{\pi_h h'}{h} \pmod{\Theta_1\mathbb{F}[x]}$;

(b) $\nu\pi_h' \equiv \frac{1}{1-\alpha_j} \pmod{\mathsf{u}_j\mathbb{F}[x]}$, *for all $1 \leq j \leq k$.*

Proof. We have that $\pi_h' = \sum_{i=1}^t \mathsf{u}_1 \cdots \widehat{\mathsf{u}_i} \cdots \mathsf{u}_t \mathsf{u}_i'$ and $\frac{\pi_h h'}{h} = \sum_{i=1}^t \alpha_i \mathsf{u}_1 \cdots \widehat{\mathsf{u}_i} \cdots \mathsf{u}_t \mathsf{u}_i'$, so, in particular,

$$\delta_0(1) = \pi_h' - \frac{\pi_h h'}{h} = \mathsf{u}_{k+1} \cdots \mathsf{u}_t \sum_{i=1}^k (1-\alpha_i)\mathsf{u}_1 \cdots \widehat{\mathsf{u}_i} \cdots \mathsf{u}_k \mathsf{u}_i'$$

and $\gcd(\delta_0(1), \Theta_1) = 1$. This shows the existence of ν with $\nu\delta_0(1) \equiv 1 \pmod{\Theta_1\mathbb{F}[x]}$ and also proves (a).

Fix $1 \leq j \leq k$. Then,

$$\pi_h' \equiv \mathsf{u}_1 \cdots \widehat{\mathsf{u}_j} \cdots \mathsf{u}_t \mathsf{u}_j' \pmod{\mathsf{u}_j\mathbb{F}[x]},$$
$$\frac{\pi_h h'}{h} \equiv \alpha_j \mathsf{u}_1 \cdots \widehat{\mathsf{u}_j} \cdots \mathsf{u}_t \mathsf{u}_j' \equiv \alpha_j \pi_h' \pmod{\mathsf{u}_j\mathbb{F}[x]}.$$

But, by (a), we also have $\nu\pi_h' - \nu\frac{\pi_h h'}{h} \equiv 1 \pmod{\mathsf{u}_j\mathbb{F}[x]}$, so $(1-\alpha_j)\nu\pi_h' \equiv 1 \pmod{\mathsf{u}_j\mathbb{F}[x]}$ and (b) follows since $\alpha_j \geq 2$. ∎

Based on the proof of [1, Lem. 5.19] and the definition of $\overline{\mathsf{D}}_q$, we can deduce that under the isomorphism (6.9), the element $g\overline{e_q} \otimes w_m \in \overline{\mathsf{D}}_q \otimes \mathsf{W}$ is mapped to $-\mathrm{ad}_{ge_q\nu a_{m+1}} + \mathcal{N} \in [\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})]/\mathcal{N}$, for $1 \leq q \leq k$, $g \in \mathbb{F}[x]$, and $m \geq -1$, where ν is as in Lemma 6.14. Using these identifications and those in (6.13), we have

$$(g\overline{e_q} \otimes w_m) \cdot \left(\sum_{\alpha_j \geq i+2} f\overline{e_j}\hat{y}^\ell \right)$$
$$= -[\mathrm{ad}_{ge_q\nu a_{m+1}}, \Theta_i f \hat{y}^\ell]$$

$$= \Theta_i fg\overline{e_q}\Big(-i(m+1)\nu\pi_h' + \ell\nu\delta_0(1) + (m+1)\nu\frac{\pi_h h'}{h}\Big)\hat{y}^{m+\ell} \pmod{P_{i+1}}$$
$$= \Theta_i fg\overline{e_q}\big((1-i)(m+1)\nu\pi_h' + \ell - (m+1)\big)\hat{y}^{m+\ell} \pmod{P_{i+1}}$$
$$= \sum_{\alpha_j \geq i+2} fg\overline{e_j}\overline{e_q}\big(\ell - (m+1)(1-(1-i)\nu\pi_h')\big)\hat{y}^{m+\ell}$$
$$= \begin{cases} fg\overline{e_q}\big(\ell - (m+1)(1-(1-i)\nu\pi_h')\big)\hat{y}^{m+\ell} & \text{if } \alpha_q \geq i+2, \\ 0 & \text{if } \alpha_q \leq i+1, \end{cases}$$

by (6.12) and Lemma 6.14, as Θ_{i+1} divides $\Theta_1\Theta_i$. Moreover, we can use Lemma 6.14 (b) since $\mathsf{u}_q\overline{e_q} = 0$ in $\overline{D}_q$, yielding

$$(g\overline{e_q} \otimes w_m)\cdot\Big(\sum_{\alpha_j\geq i+2} f\overline{e_j}\hat{y}^\ell\Big) = \begin{cases} fg\overline{e_q}\big(\ell - (m+1)\frac{\alpha_q - i}{\alpha_q - 1}\big)\hat{y}^{m+\ell} & \text{if } \alpha_q \geq i+2, \\ 0 & \text{if } \alpha_q \leq i+1. \end{cases}$$

The above shows that $\overline{\mathsf{D}}_q \otimes \mathsf{W}$ acts trivially on $\overline{\mathsf{D}}_j[\hat{y}] \subseteq S_i$ except if $j = q$ and $\alpha_q \geq i+2$. In the latter case, the action of $\overline{\mathsf{D}}_q \otimes \mathsf{W}$ on $\overline{\mathsf{D}}_q[\hat{y}]$ is given by

$$(g\overline{e_q} \otimes w_m)\cdot(f\overline{e_q}\hat{y}^\ell) = fg\overline{e_q}\Big(\ell - (m+1)\frac{\alpha_q - i}{\alpha_q - 1}\Big)\hat{y}^{m+\ell}. \tag{6.15}$$

In particular, each $\overline{\mathsf{D}}_j[\hat{y}] \subseteq S_i$ in the decomposition (6.13) is an $\mathsf{HH}^1(\mathsf{A})$-submodule of S_i.

Notice that in (6.15), the elements $f\overline{e_q}$ and $g\overline{e_q}$ are scalars in the field extension $\overline{\mathsf{D}}_q \cong \mathbb{F}[x]/\mathsf{u}_q\mathbb{F}[x]$ of $\mathbb{F}$ and the action (6.15) is $\overline{\mathsf{D}}_q$-linear. This motivates the following definition. Fix a scalar $\mu \in \mathbb{F}$ and let $V_\mu = \mathbb{F}[\hat{y}]$. Define an action of the Witt algebra W on V_μ by

$$w_m \cdot \hat{y}^\ell = \big(\ell - (m+1)\mu\big)\hat{y}^{m+\ell}, \quad \text{for all } m \geq -1 \text{ and } \ell \geq 0. \tag{6.16}$$

It can be verified that this indeed defines an action of W on V_μ, for any $\mu \in \mathbb{F}$ (for μ of the form $\frac{\alpha - i}{\alpha - 1}$ with $\alpha \geq i+2$, this statement is implied by (6.15)).

The module V_μ is related to the intermediate series modules for the Witt and Virasoro algebras (compare (6.21), ahead). Next, we record irreducibility and isomorphism criteria for these modules.

Lemma 6.17. *For $\mathbb{F}$ an arbitrary field of a characteristic 0 and $\mu \in \mathbb{F}$, let V_μ be the W-module defined in (6.16). Then,*

(a) *V_μ is irreducible if and only if $\mu \neq 0$;*

(b) *$V_\mu \cong V_{\mu'}$ if and only if $\mu = \mu'$.*

Proof. The proof is straightforward, so we just sketch it. First, if $\mu = 0$, then $\mathbb{F}\hat{y}^0$ is a submodule of V_0, so V_0 is reducible. Suppose now that $\mu \neq 0$. Let X be a nonzero submodule of V_μ. Since $w_{-1}^\ell \cdot \hat{y}^\ell = \ell!\hat{y}^0$, it follows by the usual argument that $\hat{y}^0 \in X$. Taking into account that $w_m \cdot \hat{y}^0 = -(m+1)\mu\hat{y}^m \in X$ for all $m \geq 0$ and $\mu \neq 0$, we deduce that $X = V_\mu$. Thus, V_μ is irreducible and (a) is proved.

The action of w_0 on V_μ is diagonalizable with eigenvalues $\{\ell - \mu\}_{\ell \geq 0}$, with $-\mu$ being the unique eigenvalue such that $-\mu - 1$ is no longer an eigenvalue. Thus, the action of W on V_μ determines μ, which proves (b). ■

It follows from the above that for all $0 \leq i \leq m_h - 1$ and all j such that $\alpha_j \geq i + 2$, the $\overline{\mathsf{D}}_j \otimes \mathsf{W}$-module $\overline{\mathsf{D}}_j[\hat{y}] \subseteq S_i$ is irreducible and it is isomorphic to $\overline{\mathsf{D}}_j \otimes V_{\mu_{ij}}$, where $\mu_{ij} = \frac{\alpha_j - i}{\alpha_j - 1} \neq 0$. As the action depends on i, it is convenient to introduce i into the notation for this module. Thus, we henceforth denote this module by $\overline{V}_{ij}$:

$$\overline{V}_{ij} = \overline{\mathsf{D}}_j[\hat{y}] \subseteq S_i \quad \text{and} \quad \overline{V}_{ij} \cong \overline{\mathsf{D}}_j \otimes V_{\mu_{ij}},$$

for all $0 \leq i \leq m_h - 1$ and j such that $\alpha_j \geq i + 2$. Moreover, $\overline{\mathsf{D}}_q \otimes \mathsf{W}$ acts trivially on $\overline{V}_{ij}$ for $q \neq j$, so it follows by Theorem 6.2 and (6.9) that $\overline{V}_{ij}$ is an irreducible $\mathsf{HH}^1(\mathsf{A})$-submodule of S_i on which both $\mathsf{Z}(\mathsf{HH}^1(\mathsf{A}))$ and the nilpotent radical $\mathcal{N}$ of $[\mathsf{HH}^1(\mathsf{A}), \mathsf{HH}^1(\mathsf{A})]$ act trivially. As a result of this analysis, we conclude that S_i is a completely reducible $\mathsf{HH}^1(\mathsf{A})$-module with semisimple decomposition (cf. (6.13)):

$$S_i = \bigoplus_{\alpha_j \geq i+2} \overline{V}_{ij}. \tag{6.18}$$

We summarize these results in the following, which constitutes the main result of this paper.

Theorem 6.19. *Assume that* $\mathsf{char}(\mathbb{F}) = 0$ *and* $\mathsf{A} = \mathsf{A}_h$ *for* $0 \neq h \in \mathbb{F}[x]$. *Let* $h = \mathsf{u}_1^{\alpha_1} \cdots \mathsf{u}_t^{\alpha_t}$ *be the decomposition of h into irreducible factors with* $0 \leq k \leq t$ *such that* $\alpha_1, \dots, \alpha_k \geq 2$ *and* $\alpha_{k+1} = \cdots = \alpha_t = 1$. *Since* $\mathsf{HH}^2(\mathsf{A}) \neq 0$ *if and only if* $k \geq 1$, *assume that* $k \geq 1$ *and set* $m_h = \max\{\alpha_j - 1 \mid 1 \leq j \leq k\}$.

The structure of $\mathsf{HH}^2(\mathsf{A})$ *as Lie module over the Lie algebra* $\mathsf{HH}^1(\mathsf{A})$ *under the Gerstenhaber bracket is as follows.*

(a) *There is a filtration of length* m_h *by* $\mathsf{HH}^1(\mathsf{A})$*-submodules*

$$\mathsf{HH}^2(\mathsf{A}) = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_{m_h - 1} \supsetneq P_{m_h} = 0.$$

(b) *For each* $0 \leq i \leq m_h - 1$, *the factor module* $S_i = P_i / P_{i+1}$ *is completely reducible with semisimple decomposition* $S_i = \bigoplus_{\alpha_j \geq i+2} \overline{V}_{ij}$, *where*

  (i) *the nilpotent radical* $\mathsf{Z}(\mathsf{HH}^1(\mathsf{A})) \oplus \mathcal{N}$ *of* $\mathsf{HH}^1(\mathsf{A})$ *acts trivially on* S_i, *so* S_i *becomes a* $(\overline{\mathsf{D}}_1 \otimes \mathsf{W}) \oplus \cdots \oplus (\overline{\mathsf{D}}_k \otimes \mathsf{W})$*-module, where* $\overline{D}_j \cong \mathbb{F}[x]/\mathsf{u}_j\mathbb{F}[x]$ *and* $\mathsf{W} = \mathsf{span}_{\mathbb{F}}\{w_i \mid i \geq -1\}$ *is the Witt algebra;*

  (ii) $\overline{V}_{ij} \cong \overline{\mathsf{D}}_j \otimes V_{\mu_{ij}}$, *where* $\mu_{ij} = \frac{\alpha_j - i}{\alpha_j - 1}$ *and the irreducible* W*-module* V_μ *is described in* (6.16);

  (iii) $\overline{\mathsf{D}}_q \otimes \mathsf{W}$ *acts trivially on* $\overline{V}_{ij}$ *for* $q \neq j$ *and* $\overline{\mathsf{D}}_j \otimes \mathsf{W}$ *acts on* $\overline{V}_{ij}$ *via* (6.16), *under scalar extension;*

  (iv) $\overline{V}_{ij} \cong \overline{V}_{i'j'}$ *as* $\mathsf{HH}^1(\mathsf{A})$*-modules if and only if* $(i, j) = (i', j')$.

(c) $\mathsf{HH}^2(\mathsf{A})$ *has finite composition length equal to* $\sum_{j=1}^{k}(\alpha_j - 1)$*, the number of irreducible factors of* $\mathsf{gcd}(h, h')$ *counted with multiplicity; the composition factors are* $\{\overline{V}_{ij} \mid 0 \leq i \leq m_h - 1, \alpha_j \geq i + 2\}$*, representing distinct isomorphism classes.*

(d) $\mathsf{HH}^2(\mathsf{A})$ *is a semisimple* $\mathsf{HH}^1(\mathsf{A})$*-module if and only if* $m_h \leq 1$*, i.e., if and only if* h *is not divisible by the cube of any non-constant polynomial.*

Remark 6.20. It turns out that under the same conditions that ensure that $\mathsf{HH}^2(\mathsf{A})$ is semisimple, both $\mathsf{HH}^0(\mathsf{A})$ and $\mathsf{HH}^1(\mathsf{A})$ are also semisimple $\mathsf{HH}^1(\mathsf{A})$-modules: since $\mathsf{char}(\mathbb{F}) = 0$, $\mathsf{HH}^0(\mathsf{A}) = \mathbb{F}$ is always simple and by [1, Cor. 5.22 (ii)], $\mathsf{HH}^1(\mathsf{A}_h)$ is a direct sum of its center – a sum of trivial modules – and simple Lie ideals.

Proof. All of the above statements have been proved, except for (iv) and (d). We start with (iv). If $\overline{V}_{ij} \cong \overline{V}_{i'j'}$, then $\overline{\mathsf{D}}_j \otimes \mathsf{W}$ acts non-trivially on $\overline{V}_{i'j'}$, so $j = j'$, by (iii). Thus, by Lemma 6.17 (b), $\mu_{ij} = \mu_{i'j}$, which in turn implies that $i = i'$.

For the proof of (d), if h is not divisible by the cube of any non-constant polynomial, then $m_h = 1$ and $\mathsf{HH}^2(\mathsf{A}) = S_0$, which we have seen in (b) is semisimple. Conversely, if $m_h \geq 2$, then there is some i such that $\alpha_i \geq 3$, say $i = 1$. By (6.8),

$$[\mathsf{ad}_{\mathsf{u}_1 \cdots \mathsf{u}_k a_1}, \hat{y}^0] = -\mathsf{u}_1 \cdots \mathsf{u}_k \sum_{i=1}^{t} \alpha_i \mathsf{u}_1 \cdots \widehat{\mathsf{u}_i} \cdots \mathsf{u}_t \mathsf{u}_i' \notin \mathsf{gcd}(h, h')\mathbb{F}[x]$$

because u_1^2 divides $\mathsf{gcd}(h, h')$ but it does not divide $[\mathsf{ad}_{\mathsf{u}_1 \cdots \mathsf{u}_k a_1}, \hat{y}^0]$. But $\mathsf{ad}_{\mathsf{u}_1 \cdots \mathsf{u}_k a_1} \in \mathcal{N}$ and $\mathcal{N}$ annihilates all the composition factors of $\mathsf{HH}^2(\mathsf{A})$, by (i), so $\mathsf{HH}^2(\mathsf{A})$ cannot be semisimple in this case. ■

Before we proceed to illustrate our result with some special cases, we first want to establish a connection between the representations $\overline{V}_{ij}$ and the Virasoro algebra. Recall that the Virasoro algebra is the unique (up to isomorphism) central extension of the full Witt algebra of derivations of $\mathbb{F}[z^{\pm 1}]$. This Lie algebra is defined as $\mathsf{Vir} = \bigoplus_{i \in \mathbb{Z}} \mathbb{F} \cdot w_i \oplus \mathbb{F} \cdot c$, where

$$[c, \mathsf{Vir}] = 0 \quad \text{and} \quad [w_m, w_n] = (n - m)w_{m+n} + \delta_{m+n,0}\frac{m^3 - m}{12}c \quad \forall m, n \in \mathbb{Z}.$$

Define, for $\mu \in \mathbb{F}$, the Vir-module $U_\mu = \mathbb{F}[\hat{y}^{\pm 1}]$ with action

$$w_m \cdot \hat{y}^\ell = (\ell - (m+1)\mu)\hat{y}^{m+\ell} \quad \text{and} \quad c \cdot \hat{y}^\ell = 0, \quad \forall \ell, m \in \mathbb{Z}. \tag{6.21}$$

The module U_μ is an intermediate series module (see [9] for details).

The following can be readily checked by the reader:

(a) W is a Lie subalgebra of Vir;

(b) formula (6.21) gives a well-defined action of Vir on U_μ;

(c) $V_\mu \subseteq U_\mu$ as W-modules;

(d) U_μ is irreducible as a Vir-module if and only if $\mu \neq 0$ and $\mu \neq 1$;

(e) $U_\mu \cong U_{\mu'}$ as Vir-modules if and only if $\mu = \mu'$.

6.4. Special cases

We end this section with a discussion of some examples of special interest. To avoid trivial cases, in all examples, the polynomial h is assumed to be divisible by the square of some non-constant polynomial. We continue to assume that $\mathsf{char}(\mathbb{F}) = 0$.

Example 6.22 ($h = x^n$). Let's consider the case where h has a unique irreducible factor. For the sake of simplicity, we will assume that this factor is x, that is, $h = x^n$ with $n \geq 2$; the more general case of an irreducible factor of higher degree is entirely analogous. In this case,

$$\begin{gathered}\mathsf{Z}\big(\mathsf{HH}^1(\mathsf{A}_{x^n})\big) = \mathbb{F} D_{x^{n-1}}, \quad \text{where } D_{x^{n-1}}(x) = 0 \text{ and } D_{x^{n-1}}(\hat{y}) = x^{n-1},\\ \mathcal{N} = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{x^i a_m} \mid 1 \leq i \leq n-2,\ m \geq 0\},\\ \big[\mathsf{HH}^1(\mathsf{A}_{x^n}), \mathsf{HH}^1(\mathsf{A}_{x^n})\big]/\mathcal{N} \cong \mathsf{W} \quad \text{(the Witt algebra)},\\ \mathsf{HH}^2(\mathsf{A}_{x^n}) = \mathsf{D}[\hat{y}], \quad \text{where } \mathsf{D} = \big(\mathbb{F}[x]/x^{n-1}\mathbb{F}[x]\big).\end{gathered}$$

For $0 \leq i \leq n-1$, let $P_i = x^i \mathsf{D}[\hat{y}]$, so that we get the following filtration of $\mathsf{HH}^1(\mathsf{A}_{x^n})$-submodules of $\mathsf{HH}^2(\mathsf{A}_{x^n})$:

$$\mathsf{HH}^2(\mathsf{A}_{x^n}) = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_{n-2} \supsetneq P_{n-1} = 0.$$

Set $S_i = P_i/P_{i+1} \cong \mathbb{F}[\hat{y}]$, for $i \leq n-2$. Then, $D_{x^{n-1}}.\mathsf{HH}^2(\mathsf{A}_{x^n}) = 0$ and $\mathcal{N}.P_i \subseteq P_{i+1}$, so S_i is naturally a module for the Witt algebra W, with action

$$w_m \cdot \hat{y}^\ell = \left(\ell - (m+1)\frac{n-i}{n-1}\right)\hat{y}^{m+\ell}, \quad \text{for all } m \geq -1 \text{ and } \ell \geq 0.$$

Thus, $S_i \cong V_{\frac{n-i}{n-1}}$ is simple and the composition factors $\{S_i\}_{0\leq i\leq n-2}$ of $\mathsf{HH}^2(\mathsf{A}_{x^n})$ are pairwise non-isomorphic. In particular, $\mathsf{HH}^2(\mathsf{A}_{x^n})$ has length $n-1$ as an $\mathsf{HH}^1(\mathsf{A}_{x^n})$-module, with distinct composition factors.

The next example, a particular case of the previous one, focuses on the Jordan plane.

Example 6.23 (The Jordan plane). Taking $h = x^2$, we obtain the algebra A_{x^2}, known as the Jordan plane, with homogeneous defining relation $\hat{y}x = x\hat{y} + x^2$. The description here is

$$\mathsf{HH}^1(\mathsf{A}_{x^2}) = \mathbb{F} D_x \oplus \mathsf{W} \quad \text{and} \quad \mathsf{HH}^2(\mathsf{A}_{x^2}) = \mathbb{F}[\hat{y}],$$

where $D_x(x) = 0$, $D_x(\hat{y}) = x$, and W is the Witt algebra.

It follows that $\mathsf{HH}^2(\mathsf{A}_{x^2})$ is a simple $\mathsf{HH}^1(\mathsf{A}_{x^2})$-module annihilated by D_x and such that, as a W-module, $\mathsf{HH}^2(\mathsf{A}_{x^2}) \cong V_2$.

The Lie subalgebra $\mathbb{F}w_{-1} \oplus \mathbb{F}w_0 \oplus \mathbb{F}w_1 \subseteq \mathsf{W}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$, under the identification $e = w_{-1}$, $h = -2w_0$, and $f = -w_1$, where $e = E_{12}$, $f = E_{21}$, and $h = [e, f]$ are the canonical generators of $\mathfrak{sl}_2(\mathbb{F})$. The restriction of the $\mathsf{HH}^1(\mathsf{A}_{x^2})$-module structure of $\mathsf{HH}^2(\mathsf{A}_{x^2}) = \mathbb{F}[\hat{y}]$ to $\mathfrak{sl}_2(\mathbb{F})$ is determined by the relations

$$e \cdot \hat{y}^\ell = \ell \hat{y}^{\ell-1}, \quad h \cdot \hat{y}^\ell = (4 - 2\ell)\hat{y}^\ell, \quad f \cdot \hat{y}^\ell = (4-\ell)\hat{y}^{\ell+1}, \quad \forall \ell \geq 0.$$

Whence, it is easy to see that $L(4) := \mathbb{F}\hat{y}^0 \oplus \mathbb{F}\hat{y}^1 \oplus \mathbb{F}\hat{y}^2 \oplus \mathbb{F}\hat{y}^3 \oplus \mathbb{F}\hat{y}^4$ is a simple $\mathfrak{sl}_2(\mathbb{F})$-submodule of $\mathsf{HH}^2(\mathsf{A}_{x^2})$. In fact, $L(4)$ is the simple $\mathfrak{sl}_2(\mathbb{F})$-module of highest weight 4 and the quotient module $\mathsf{HH}^2(\mathsf{A}_{x^2})/L(4) \cong M(-6)$ is the irreducible Verma module of highest weight -6.

Our last example deals with the case where $\mathsf{HH}^2(\mathsf{A})$ is a semisimple Lie module.

Example 6.24 (h is cube free). By Theorems 6.2 and 6.19 (d), the following conditions are equivalent:

- $\mathsf{HH}^2(\mathsf{A})$ is a semisimple $\mathsf{HH}^1(\mathsf{A})$-module;
- $\mathcal{N} = 0$;
- $\mathsf{HH}^1(\mathsf{A})$ is a reductive Lie algebra;
- h is cube free.

Here, we study the case in which these conditions hold, so the decomposition of h into irreducible factors is of the form $h = \mathsf{u}_1^2 \cdots \mathsf{u}_k^2 \mathsf{u}_{k+1} \cdots \mathsf{u}_t$, for some $1 \le k \le t$. We have

$$\begin{aligned} \dim_{\mathbb{F}} \mathsf{Z}\big(\mathsf{HH}^1(\mathsf{A})\big) &= \deg \mathsf{u}_1 \cdots \mathsf{u}_t, \\ \mathsf{HH}^1(\mathsf{A}) &= \mathsf{Z}\big(\mathsf{HH}^1(\mathsf{A})\big) \oplus (\overline{D}_1 \otimes \mathsf{W}) \oplus \cdots \oplus (\overline{D}_k \otimes \mathsf{W}), \\ \mathsf{HH}^2(\mathsf{A}) &= \overline{D}_1[\hat{y}] \oplus \cdots \oplus \overline{D}_k[\hat{y}], \end{aligned}$$

where $\overline{D}_j \cong \mathbb{F}[x]/\mathsf{u}_j\mathbb{F}[x]$ and W is the Witt algebra.

Then, $\mathsf{Z}(\mathsf{HH}^1(\mathsf{A}))$ acts trivially on $\mathsf{HH}^2(\mathsf{A})$ and $\overline{D}_i \otimes \mathsf{W}$ acts trivially on $\overline{D}_j[\hat{y}]$, if $i \neq j$. As a $\overline{D}_j \otimes \mathsf{W}$-module, $\overline{D}_j[\hat{y}] \cong \overline{D}_j \otimes V_2$. Thus, the irreducible summands of $\mathsf{HH}^2(\mathsf{A})$ are $\{\overline{D}_j[\hat{y}]\}_{1 \le j \le k}$, they are pairwise non-isomorphic as $\mathsf{HH}^1(\mathsf{A})$-modules, and the composition length of $\mathsf{HH}^2(\mathsf{A})$ is k.

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