

# *Number Theory Revealed: An Introduction* by Andrew Granville

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# Number Theory Revealed: An Introduction

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AMERICAN MATHEMATICAL SOCIETY, 2019, XXIV+264 PP., \$69.00, ISBN 978-1-4704-4157-9

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# Number Theory Revealed: A Masterclass

by Andrew Granville

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AMERICAN MATHEMATICAL SOCIETY, 2019, XXVIII+587 PP., \$99.00, ISBN 978-1-4704-4158-6

REVIEWED BY MARCO ABATE

What? Yet more introductory books on number theory? Aren't there enough of them already? And why are we discussing them here? This is the *Mathematical Intelligencer*, not *Mathematical Reviews*! Indeed, the simple fact that this review is appearing here is a first hint that we have got something special.

For starters, we are not presenting just one book; there are two, or three, or four, or even five of them, depending on how one counts (and counting is very important in number theory, as you know). Indeed, the pair of books under review here is intended to be followed by two (or three) other books—*The Distribution of Primes: Analytic Number Theory Revealed* and *Rational Points on Curves: Arithmetic Geometry Revealed*—that will present in detail deeper results in modern number theory (and they will be followed by *Gauss's Disquisitiones Arithmeticae Revealed*, a modern reworking of Gauss's classic book, one of the most influential texts in number theory).

*Number Theory Revealed* is a series of books intended to constitute a first introduction to number theory, giving a survey of the subject starting from the very beginning and proceeding up to some glimpses of contemporary research. The *Introduction* is a condensed version of the *Masterclass*, containing just what is needed for a first course in number theory; but if you have even the slightest interest in number theory, I strongly suggest that you go for the *Masterclass*, which in its almost 600 pages contains a wonderful wealth of material.

But the interest of this book series goes beyond the included material and depends also on a number of particularly effective stylistic and structural choices made by the author. Usually, mathematical textbooks proceed in a linear way. They start from the basic definitions and

progress in an orderly fashion to more advanced material via theorems, examples, more definitions, and sometimes exercises. The material is supposed to be read in the order presented in the book, with no digressions; possibly at the end one might have a couple of chapters independent of the others. Graphically, a standard textbook can be represented by a line, with maybe just a few branches at the end.

The topological structure of this book is instead much more complex: every chapter ramifies into many appendices, variously interconnected with each other and with future (or past) chapters. Moreover, some chapters (not necessarily the last ones) are very interesting but somewhat optional, and can be skipped on a first reading. The overall structure is very rich but complicated; this is one of the main reasons for the existence of two versions of the book. As mentioned above, the *Introduction* contains only the essential material for a first course in number theory (but every chapter still has an appendix); the full richness of the chosen structure appears in the *Masterclass*, with up to nine appendices for each chapter and five chapters more than the *Introduction*.

Full disclosure: In my textbooks I use similar tools—appendices, optional chapters, nonlinear reading order—and so I cannot help appreciating how well Granville is able to balance the many ingredients he puts in play, making the whole reading experience truly gratifying. It is a difficult task, and Granville manages it with deceptive ease, like a professional conjurer.

One important outcome of such a structure is that it represents well—and allows the student/reader to appreciate from the very beginning—the organic nature of mathematical research. The progress of knowledge in mathematics is not a straight line going from point A to point B; it is full of digressions, unexpected shortcuts connecting apparently unrelated subjects, cul-de-sacs that at the very last minute (or after thirty thousand different trials) reveal a concealed door leading to a hidden garden of delights—but there are also cul-de-sacs that remain as such, no matter how hard we try to find a back door. Doing mathematical research is like wandering in a dark forest and suddenly discovering a vast clearing where many paths join and from where many new paths start toward new destinations—or even toward old destinations reached via a different route.

The author does a very good job of representing this aspect of mathematical research, often returning to the same topic with a different point of view, as allowed by some new material introduced meanwhile; or, conversely, when he points out how new ideas (sometimes presented in later chapters, or even to be presented in forthcoming volumes in the series) will shed a different light on the problem at hand.

The historical development of the theory is also well presented, from Fermat, Euler, and Gauss to contemporary research (there are references as recent as 2019); Gauss's contribution in particular receives the attention it deserves. Furthermore, Granville makes very good use of a peculiar characteristic of number theory, the existence of many intriguing open problems, easy to state and devilishly difficult to solve, introducing them from the very beginning

(there is an open problem mentioned already in Chapter 1). The final outcome is the reader's strong feeling that number theory is a subject very much alive, and this is a wonderful feat for an introductory textbook.

Learning mathematics is not a passive endeavor; to learn mathematics, one has to do mathematics. Granville fully embraces this approach, and thus the book is filled with hundreds of exercises (indeed, perhaps more than one thousand), ranging from basic verifications to challenging problems, with hints for solutions at the end of the book. Often, some part of the proof of a result, even a main result, is contained in one or more exercises, to be solved by the reader (while reading, not later) to better appreciate the ideas supporting the proof.

This leads us to another peculiar but very effective stylistic choice: the author often presents several proofs, not just one, of the main results. For instance, the book contains at least five different proofs of Fermat's little theorem, and at least as many proofs that there are infinitely many primes. Different proofs highlight different aspects of the result, or indicate different connections with other parts of the theory. Some proofs may be generalized to different contexts, or may have different consequences—not to mention the classical distinction between constructive and nonconstructive proofs. Exposing the student as soon as possible to a large variety of proofs is pedagogically very important (and not so often done in textbooks), since it underlines that in mathematics, the arguments used are as relevant as the results obtained. A good mathematician knows not only many theorems, but also—and perhaps mainly—many arguments that can be used to prove theorems.

Granville is fully aware of what he is doing; in the first chapter there is even a footnote discussing questions like, "Which type of proof is preferable?" and "Which type of proof has the greatest clarity?" This is yet another aspect of the book to be highlighted: it contains many metamathematical comments explaining why mathematics is written or presented in a certain way; what outcomes one would like to achieve with a particular way of writing; which effects a different way of presenting an argument might have; and how a given symbol or convention has developed historically. Bringing attention to such topics helps the reader achieve a fuller understanding not only of the matter at hand but also of the ways it can be expressed and the ways language or stylistic choices can be used to facilitate (or impede) reaching certain goals—and such awareness may turn out to be quite useful outside mathematics as well.

A few words on the contents of the book are in order. As already mentioned, the basic topics in number theory are all treated: the Euclidean algorithm, congruences, prime

numbers, Diophantine problems, power and quadratic residues, etc. The *Introduction* ends with more advanced topics, such as rational approximation of real numbers, quadratic forms, and relations between factorization algorithms and encryption algorithms. The *Masterclass* then goes on with quite interesting chapters on the anatomy of integers (readers of Granville's graphic novel *Prime Suspects* will love it), on rational points on curves, and on combinatorial number theory, introducing the reader to some of the more exciting aspects of contemporary number theory.

Summing up, I strongly recommend the reading of *Number Theory Revealed* (the *Masterclass* in particular) not only to all mathematicians but also to anybody scientifically inclined and curious about what mathematics is and how it is done. Not only are the topics well chosen and well presented, but this book is a real page-turner. How often can you say that about a mathematical textbook? *Chapeau!*

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