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Analysis of fully discrete, quasi non-conforming approximations of evolution equations and applications

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Abstract In this paper we consider fully discrete approximations of abstract evolution equations, by means of a quasi non-conforming spatial approximation and finite differences in time (Rothe–Galerkin method). The main result is the convergence of the discrete solutions to a weak solution of the continuous problem. Hence, the result can be interpreted either as a justification of the numerical method, or as an alternative way of constructing weak solutions.

We set the problem in the very general and abstract setting of pseudo-monotone operators, which allows for a unified treatment of several evolution problems. The examples –which fit into our setting and which motivated our research– are problems describing the motion of incompressible fluids, since the quasi non-conforming approximation allows to handle problems with prescribed divergence.

Our abstract results for pseudo-monotone operators allow to show convergence just by verifying a few natural assumptions on the operator time-by-time and on the discretization spaces. Hence, applications and extensions to several other evolution problems can be easily performed. The results of some numerical experiments are reported in the final section.

Keywords Fully discrete · Pseudo-monotone operator · Evolution equation

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1 Introduction

We consider the numerical approximation of an abstract evolution equation[1]

$$\begin{aligned} \frac{d\boldsymbol{u}}{dt}(t) + A(t)(\boldsymbol{u}(t)) &= \boldsymbol{f}(t) \qquad \text{in } V^*, \\ \boldsymbol{u}(0) &= u_0 \qquad \text{in } H, \end{aligned} \tag{1.1}$$

by means of a quasi non-conforming Rothe–Galerkin scheme. Here, $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ is a given evolution triple, $I := (0,T)$ a finite time horizon, $u_0 \in H$ an initial value, $\boldsymbol{f} \in L^{p'}(I, V^*)$, $p \in (1,\infty)$, a right-hand side and $A(t) : V \to V^*$, $t \in I$, a family of operators.

In order to make (1.1) accessible to non-conforming approximation methods, we will additionally require that there exists a further evolution triple $X \hookrightarrow Y \cong Y^* \hookrightarrow X^*$, such that $V \subseteq X$ with $\|\cdot\|_V = \|\cdot\|_X$ on V and $H \subseteq Y$ with $(\cdot,\cdot)_H = (\cdot,\cdot)_Y$ in H, and extensions $\hat{A}(t) : X \to X^*$, $t \in I$, and $\hat{\boldsymbol{f}} \in L^{p'}(I, X^*)$ of $\{A(t)\}_{t\in I}$ and $\boldsymbol{f}$, resp., i.e., $\langle \hat{A}(t)v, w\rangle_X = \langle A(t)v, w\rangle_V$ and $\langle \hat{\boldsymbol{f}}(t), v\rangle_X = \langle \boldsymbol{f}(t), v\rangle_V$ for all $v, w \in V$ and almost every $t \in I$. For sake of readability we set $A(t) := \hat{A}(t)$ and $\boldsymbol{f}(t) := \hat{\boldsymbol{f}}(t)$ for almost every $t \in I$.

Recently, the existence theory for abstract evolution problems with Bochner pseudo-monotone operators in [35], based on the convergence of a Galerkin approximation, was extended in [7] to the convergence proof of a fully discrete Rothe–Galerkin approximation. However, the result in [7] is not applicable to the treatment of problems describing the flow of incompressible fluids. The main aim of this paper is develop an abstract framework which shows that a fully discrete Rothe–Galerkin approximation converges also for such problems (cf. Theorem 6.16). Even though the framework is rather abstract it is easily applicable to many problems, since we show that it is enough to check a few easily verifiable conditions (cf. conditions (A.1)–(A.4) in Proposition 4.6 and conditions (QNC.1)–(QNC.2) in Definition 3.1).

A prototypical example for the flow of incompressible non-Newtonian fluids are the following equations describing the unsteady motion of incompressible shear-dependent fluids

$$\begin{aligned} \partial_t \boldsymbol{u} - \operatorname{div}\big((\kappa + |\mathcal{D}\boldsymbol{u}|)^{p-2}\mathcal{D}\boldsymbol{u}\big) + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla\boldsymbol{q} &= \boldsymbol{f} \qquad \text{in } I\times\Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 \qquad \text{in } I\times\Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \qquad \text{on } I\times\partial\Omega, \\ \boldsymbol{u}(0) &= u_0 \qquad \text{in } \Omega. \end{aligned} \tag{1.2}$$

Here, $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a domain, $I = (0,T)$ a time interval, and $\kappa \geq 0$ and $p \in (1,\infty)$ are material parameters of the shear-dependent fluid. Moreover, $\boldsymbol{u} : I\times\Omega \to \mathbb{R}^d$ denotes the velocity, $\boldsymbol{f} : I\times\Omega \to \mathbb{R}^d$ is a given external force, $u_0 : \Omega \to \mathbb{R}^d$ is an initial condition, $\boldsymbol{q} : I\times\Omega \to \mathbb{R}$ is the pressure and $\mathcal{D}\boldsymbol{u} := \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)$ denotes the symmetric gradient.

We define for $p > \frac{3d+2}{d+2}$ the function spaces $X := W_0^{1,p}(\Omega)^d$, $Y := L^2(\Omega)^d$, $V := W_{0,\operatorname{div}}^{1,p}(\Omega)$ as the closure of $\mathcal{V} := \{v \in C_0^\infty(\Omega)^d \mid \operatorname{div} v \equiv 0\}$ in X, $H := L^2_{\operatorname{div}}(\Omega)$ as the closure of $\mathcal{V}$ in Y, and the operators $S, B : X \to X^*$ for all $u, v \in X$ via

$$\langle Su, v\rangle_X := \int_\Omega (\kappa + |\mathcal{D}u|)^{p-2}\mathcal{D}u : \mathcal{D}v\,dx \quad \text{and} \quad \langle Bu, v\rangle_X := -\int_\Omega u\otimes u : \nabla v\,dx. \tag{1.3}$$

[1] If not specified differently, we will denote in boldface elements of Bochner spaces (as the solution $\boldsymbol{u}(t)$ and the external source $\boldsymbol{f}(t)$), to highlight the difference with elements belonging to standard Banach spaces, denoted by usual symbols.

Then, (1.2) for $u_0 \in H$ and $\boldsymbol{f} \in L^{p'}(I, X^*)$ can be re-written as the abstract evolution equation

$$\begin{aligned} \frac{d\boldsymbol{u}}{dt}(t) + S(\boldsymbol{u}(t)) + B(\boldsymbol{u}(t)) &= \boldsymbol{f}(t) \quad \text{in } V^*, \\ \boldsymbol{u}(0) &= u_0 \quad \text{in } H. \end{aligned} \tag{1.4}$$

Typical fully discrete approximations of (1.4) are often based on appropriate finite element spaces $(V_n)_{n\in\mathbb{N}}$. As the construction of finite element spaces which meet the divergence constraint exactly, i.e., satisfy $V_n \subseteq V$ for all $n \in \mathbb{N}$, highly restricts the flexibility of the approximation, one usually works with finite element spaces satisfying a discrete divergence constraint only. Thus, we have $V_n \nsubseteq V$, which is the reason that the theory in [7] is not applicable. However, the spaces V_n often satisfy $V_n \subseteq X$, which allows us to develop a convergence theory for such a setting. Note that this problem is treated from a different point of view, namely the theory of maximal monotone graphs, in the recent contributions [54] and [51].

1.1 The numerical scheme

A quasi non-conforming Rothe–Galerkin approximation of the initial value problem (1.1) usually consists of two parts:

The first part is a spatial discretization, often called Galerkin approximation, which consists in the approximation of V by a sequence of closed subspaces $(V_n)_{n\in\mathbb{N}}$ of X. We emphasize that we do not require $(V_n)_{n\in\mathbb{N}}$ to be a sequence of subspaces of V, which motivates the prefix *non-conforming*. Hence, we do *not* have $V_n \subseteq V$ and $V_n \nearrow V$ (approximation from below). The prefix *quasi*[2] indicates that the subspaces V_n are for all $n \in \mathbb{N}$ equipped with the norm of the space X. In this case we have, under appropriate assumptions, that $V_n \searrow V$, i.e., we have an approximation of the space V from above. The assumption that the subspaces V_n are equipped with the norm of the space X (and not with a norm depending on n), together with the assumption that $\|\cdot\|_X = \|\cdot\|_V$ on V reflects the fact that the spaces V_n and X have the same „regularity". This excludes spaces V_n resulting from spatial discontinuous Galerkin approximations (cf. [18]). For a spatial discontinuous Galerkin approximations in the above example one would choose $X = L^2(\Omega)^d$ and $V_n \subseteq X$ as a subspace[3] of $\mathcal{P}_k(\mathcal{T}_{h_n})^d$ satisfying a discrete divergence constraint and being equipped with the norm of the *broken* Sobolev space $W^{1,p}_{0,\mathrm{div},\mathrm{DG}}(\mathcal{T}_{h_n})^d$, where $\mathcal{T}_{h_n}$ is an appropriate triangulation of Ω. These choices violate both our assumptions and are thus not included in our treatment.

The second part is a temporal discretization, also called Rothe scheme, which consists in the approximation of the unsteady problem (1.1) by a sequence of piece-wise constant, steady problems. This is achieved by replacing the time derivative $\frac{d}{dt}$ by so-called backwards difference quotients. These are for a given step-size $\tau := \frac{T}{K} > 0$, where $K \in \mathbb{N}$, and a given finite sequence $(u^k)_{k=0,\dots,K} \subseteq X$ defined via

$$d_\tau u^k := \frac{1}{\tau}(u^k - u^{k-1}) \quad \text{in } X \quad \text{for all } k = 1, \dots, K.$$

Moreover, the operator family $A(t) : X \to X^*$, $t \in I$, and the right-hand side $\boldsymbol{f} \in L^{p'}(I, X^*)$ are discretized by means of the Clemeńt 0-order quasi interpolant. This means that for a given

[2] Observe that the subspaces $(V_n)_{n\in\mathbb{N}}$ are spatially conforming in X.

[3] $\mathcal{P}_k(\mathcal{T}_h)$, $k \in \mathbb{N}_0$, denotes the space of possibly discontinuous scalar functions, which are polynomials of degree at most m on each simplex $K \in \mathcal{T}_h$.

step-size $\tau = \frac{T}{K} > 0$, where $K \in \mathbb{N}$, we replace them piece-wise by their local temporal means, i.e., by $[A]_k^\tau : X \to X^*$, $k = 1, \dots, K$, and $([\boldsymbol{f}]_k^\tau)_{k=1,\dots,K} \subseteq X^*$, resp., for every $k = 1, \dots, K$ and $u \in X$ given via

$$[A]_k^\tau u := \fint_{\tau(k-1)}^{\tau k} A(t)u\,dt \qquad \text{and} \qquad [\boldsymbol{f}]_k^\tau := \fint_{\tau(k-1)}^{\tau k} \boldsymbol{f}(t)\,dt \qquad \text{in } X^*.$$

Altogether, using these two levels of approximation, we formulate the following fully discrete or Rothe–Galerkin scheme of the evolution problem (1.1):

Algorithm 1.5 (quasi non-conforming Rothe–Galerkin scheme) For given $K, n \in \mathbb{N}$ and $u_n^0 \in V_n$ the sequence of iterates $(u_n^k)_{k=0,\dots,K} \subseteq V_n$ is given solving the implicit scheme for $\tau = \frac{T}{K}$ and $k = 1, \dots, K$

$$(d_\tau u_n^k, v_n)_Y + \langle [A]_k^\tau u_n^k, v_n \rangle_X = \langle [\boldsymbol{f}]_k^\tau, v_n \rangle_X \qquad \text{for all } v_n \in V_n. \tag{1.6}$$

Traditionally, the verification of the convergence of a Rothe–Galerkin scheme like (1.6) to a weak solution of the evolution equation (1.1) causes a certain effort. In the case that quasi non-conforming approximations are used in (1.6), to the best of the authors' knowledge, there are no abstract results guaranteeing the weak convergence of such a scheme. Therefore, the purposes of this article are (i) to give general and easily verifiable assumptions on both the operator family $A(t) : X \to X^*$, $t \in I$, and the sequence of approximative spaces $(V_n)_{n\in\mathbb{N}}$ which provide both the existence of iterates $(u_n^k)_{k=0,\dots,K} \subseteq V_n$, solving (1.6), for a sufficiently small step-size $\tau = \frac{T}{K} \in (0, \tau_0)$, where $\tau_0 > 0$, and $K, n \in \mathbb{N}$; and (ii) to prove the stability of the scheme, i.e., the boundedness of the piece-wise constant interpolants $\overline{\boldsymbol{u}}_n^\tau \in L^\infty(I, V_n)$, $K, n \in \mathbb{N}$ with $\tau = \frac{T}{K}$, in $L^p(I, X) \cap L^\infty(I, Y)$; and finally (iii) to show the weak convergence of a diagonal subsequence $(\overline{\boldsymbol{u}}_{m_n}^{\tau_n})_{n\in\mathbb{N}} \subseteq L^\infty(I, X)$, where $\tau_n = \frac{T}{K_n}$ and $K_n, m_n \to \infty$ $(n \to \infty)$, towards a weak solution of problem (1.1). All these results are formulated exactly and proved in Section 6 (cf. Proposition 6.6, Proposition 6.8 and the main Theorem 6.16).

Surprisingly, there are only few contributions with a rigorous convergence analysis of fully discrete Rothe–Galerkin schemes towards weak solutions. Most authors consider only semi-discrete schemes, i.e., either a pure Rothe scheme (cf. [44]) or a pure Galerkin scheme (cf. [28], [56], [50], [35]). Much more results are concerned with explicit convergence rates for more regular data and more regular solutions (cf. [3], [45], [39], [26], [19], [42], [15], [6], [25], [5], [12]). Even in the conforming case, that is if the sequence $(V_n)_{n\in\mathbb{N}}$ satisfies the following two natural conditions:[4]

(C.1) $(V_n)_{n\in\mathbb{N}}$ is an increasing sequence of closed subspaces of V, i.e., $V_n \subseteq V_{n+1} \subseteq V$ for all $n \in \mathbb{N}$.
(C.2) $\bigcup_{n\in\mathbb{N}} V_n$ is dense in V.

there are very few results concerning the convergence analysis of a fully discrete Rothe–Galerkin scheme. We are only aware of the early contribution [1] which treats the porous media equation and [7] dealing with a setting similar to the one proposed in the present paper.

Let us shortly explain the strategy used in [35] and [7], since it will be extended in the present paper to handle also a quasi non-conforming setting. Using the properties of the operator family $A(t) : V \to V^*$, $t \in I$ (cf. [35, condition (C.1)–(C.4)]) and the properties of the Rothe–Galerkin scheme (1.6) one can show that the iterates $(u_n^k)_{k=0,\dots,K} \subseteq V_n$, $K, n \in \mathbb{N}$, solving (1.6), generate for sufficiently small $\tau = \frac{T}{K}$ a family of piece-wise constant interpolants $\overline{\boldsymbol{u}}_n^\tau$ for which the following holds:

[4] If (C.1) and (C.2) are satisfied one can choose $X = V$ in the above setting.

There exists a sequence $(\overline{\boldsymbol{u}}_n)_{n\in\mathbb{N}} := (\overline{\boldsymbol{u}}_{m_n}^{\tau_n})_{n\in\mathbb{N}}$ and an element $\overline{\boldsymbol{u}} \in L^p(I,V)\cap L^\infty(I,H)$, such that

$$\begin{aligned}
\overline{\boldsymbol{u}}_n &\rightharpoonup \overline{\boldsymbol{u}} && \text{in } L^p(I,V) && (n\to\infty),\\
\overline{\boldsymbol{u}}_n &\overset{*}{\rightharpoonup} \overline{\boldsymbol{u}} && \text{in } L^\infty(I,H) && (n\to\infty),\\
\overline{\boldsymbol{u}}_n(t) &\rightharpoonup \overline{\boldsymbol{u}}(t) && \text{in } H \quad \text{for a.e. } t\in I && (n\to\infty),\\
\limsup_{n\to\infty}\, &\langle \boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}_n, \overline{\boldsymbol{u}}_n - \overline{\boldsymbol{u}}\rangle_{L^p(I,V)} \le 0,
\end{aligned}$$

where $\boldsymbol{\mathcal{A}} : L^p(I,V)\cap L^\infty(I,H) \to (L^p(I,V))^*$ denotes the induced operator, which is for every $\boldsymbol{u} \in L^p(I,V)\cap L^\infty(I,H)$ and $\boldsymbol{v}\in L^p(I,V)$ given via $\langle \boldsymbol{\mathcal{A}}\boldsymbol{u}, \boldsymbol{v}\rangle_{L^p(I,V)} := \int_I \langle A(t)(\boldsymbol{u}(t)), \boldsymbol{v}(t)\rangle_V\, dt$. Using these properties and the fact that the induced operator $\boldsymbol{\mathcal{A}}$ is *Bochner pseudo-monotone* one can conclude that $\boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}_n \rightharpoonup \boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}$ in $(L^p(I,V))^*$ $(n\to\infty)$, and therefore the weak convergence of the scheme (1.6). In this argumentation one has used on several places the fact that the sequence $(V_n)_{n\in\mathbb{N}}$ satisfies the conditions (C.1) and (C.2).

Without the conditions (C.1) and (C.2), i.e., $V \neq X$ and $V_n \not\subseteq V$ for all $n\in\mathbb{N}$ one could hope to prove the above properties with V and H replaced by X and Y, respectively. Even if this works it is not clear whether the weak limit lies in the right function space, i.e., whether $\overline{\boldsymbol{u}} \in L^p(I,V)\cap L^\infty(I,H)$. To guarantee that this procedure works in an appropriate sense, we will assume that $(V_n)_{n\in\mathbb{N}}$ satisfies the assumptions (QNC.1) and (QNC.2) from Definition 3.1. Moreover, we have to adapt the notion of Bochner pseudo-monotone operators to the quasi non-conforming setting.

1.2 The example of the p-Navier-Stokes equations

Let us indicate that the prototypical example (1.2) fits into the abstract setting of the previous section. Full details will be given in Section 7 where two different problems from the field of incompressible fluid flows are treated. In fact, one of these examples contains problem (1.2) as a special case.

Let $Z := L^{p'}(\Omega)$. For a given family of shape regular triangulations (cf. [13]) $(\mathcal{T}_h)_{h>0}$ of a polygonal Lipschitz domain Ω and for given $m,\ell\in\mathbb{N}_0$, we denote by $X_h \subset \mathcal{P}_m(\mathcal{T}_h)^d \cap X$, equipped with the X-norm, and $Z_h \subset \mathcal{P}_\ell(\mathcal{T}_h)\cap Z$, equipped with the Z-norm, appropriate finite element spaces. In addition, we define for $h>0$ the *discretely divergence free finite element spaces*

$$V_h := \{v_h \in X_h \,\big|\, \langle \operatorname{div} v_h, \eta_h\rangle_Z = 0 \text{ for all } \eta_h \in Z_h\}.$$

For a null sequence[5] $(h_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ and $V_n := V_{h_n}$, $n\in\mathbb{N}$, we formulate the following algorithm of a space-time discrete approximation of (1.2):

Algorithm 1.7 For given $K,n\in\mathbb{N}$ and $u_n^0 \in V_n$ the sequence of iterates $(u_n^k)_{k=0,\dots,K} \subseteq V_n$ is given solving the implicit Rothe–Galerkin scheme for $\tau = \frac{T}{K}$ and $k = 1,\dots,K$

$$(d_\tau u_n^k, v_n)_Y + \langle [S]u_n^k, v_n\rangle_X + \langle \hat{B}u_n^k, v_n\rangle_X = \langle [\boldsymbol{f}]_k^\tau, v_n\rangle_X \quad \text{for all } v_n \in V_n, \tag{1.8}$$

where $\hat{B} : X \to X^*$ is given via $\langle \hat{B}u, v\rangle_X := \frac{1}{2}\int_\Omega v\otimes u : \nabla u\, dx - \frac{1}{2}\int_\Omega u\otimes u : \nabla v\, dx$ for all $u,v\in X$.

[5] A null sequence is a sequence converging to zero.

The operator $\hat{B}$ can be viewed as a symmetrized extension of B, as $\langle \hat{B}u, v\rangle_X = \langle Bu, v\rangle_X$ for all $u, v \in V$, which in contrast to B fulfils $\langle \hat{B}u, u\rangle_X = 0$ for all $u \in X$ (and not only for all $u \in V$), and therefore guarantees the stability of the scheme (1.8).

The sequence $(V_n)_{n\in\mathbb{N}}$ violates the conditions (C.1) and (C.2). However, the assumptions (QNC.1) and (QNC.2) on the discrete spaces $(V_n)_{n\in\mathbb{N}}$ are often fulfilled under mild assumptions, e.g., if one assumes that $\mathcal{P}_1(\mathcal{T}_h)^d \subset X_h$, $\mathbb{R} \subset Z_h$, and that there exist linear interpolation operators $\Pi_h^{\mathrm{div}} : X \to X_h$ and $\Pi_h^Z : Z \to Z_h$ which are locally $W^{1,1}$-stable and locally L^1-stable, resp., and that Π_h^{div} preserves the divergence in Z_h^* (cf. Section 7 or [9], [54] for more details).

Plan of the paper: In Section 2 we recall some basic definitions and results concerning the theory of pseudo-monotone operators and evolution equations. In Section 3 we introduce the concept of quasi non-conforming approximations. In Section 4 we introduce quasi non-conforming Bochner pseudo-monotonicity, and give sufficient and easily verifiable conditions on families of operators such that the corresponding induced operator satisfies this concept. In Section 5 we recall some basic facts about the Rothe scheme. In Section 6 we formulate the scheme of a fully discrete, quasi non-conforming approximation of an evolution equation, prove that this scheme is well-defined, i.e., the existence of iterates, that the corresponding family of piece-wise constant interpolants satisfies certain a-priori estimates. Moreover, we formulate and prove the main result of this paper, Theorem 6.16, which shows the existence of a diagonal subsequence which weakly converges to a weak solution of the corresponding evolution equation. In Section 7 we apply this approximation scheme to two problems describing incompressible non-Newtonian fluid flow. In Section 8 we present some numerical experiments for one of the problems.

2 Preliminaries

2.1 Operators

For a Banach space X with norm $\|\cdot\|_X$ we denote by X^* its dual space equipped with the norm $\|\cdot\|_{X^*}$. The duality pairing is denoted by $\langle\cdot,\cdot\rangle_X$. All occurring Banach spaces are assumed to be real.

Definition 2.1 Let X and Y be Banach spaces. The operator $A : X \to Y$ is said to be

(i) **bounded**, if for all bounded subsets $M \subseteq X$ the image $A(M) \subseteq Y$ is bounded.

(ii) **pseudo-monotone**, if $Y = X^*$, and for $(u_n)_{n\in\mathbb{N}} \subseteq X$ from $u_n \rightharpoonup u$ in X $(n \to \infty)$ and $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X \le 0$, it follows $\langle Au, u-v\rangle_X \le \liminf_{n\to\infty} \langle Au_n, u_n - v\rangle_X$ for every $v \in X$.

(iii) **coercive**, if $Y = X^*$ and $\lim_{\|u\|_X\to\infty} \frac{\langle Au,u\rangle_X}{\|u\|_X} = \infty$.

Proposition 2.2 If X is a reflexive Banach space and $A : X \to X^*$ a bounded, pseudo-monotone, and coercive operator, then $R(A) = X^*$.

Proof See [56, Corollary 32.26]. □

Lemma 2.3 If X is a reflexive Banach space and $A : X \to X^*$ a locally bounded and pseudo-monotone operator, then A is demi-continuous.

Proof See [56, Proposition 27.7]. □

2.2 Evolution equations

We call (V, H, j) an **evolution triple**, if V is a reflexive Banach space, H is a Hilbert space and $j : V \to H$ is a dense embedding, i.e., j is a linear, injective and bounded operator with $\overline{j(V)}^{\|\cdot\|_H} = H$. Let $R : H \to H^*$ be the Riesz isomorphism with respect to $(\cdot,\cdot)_H$. As j is a dense embedding the adjoint $j^* : H^* \to V^*$ and therefore $e := j^*Rj : V \to V^*$ are embeddings as well. We call e the **canonical embedding** of (V, H, j). Note that

$$\langle ev, w\rangle_V = (jv, jw)_H \quad \text{ for all } v, w \in V.$$

For an evolution triple (V, H, j), $I := (0, T)$, $T < \infty$, and $1 \le p \le q \le \infty$ we define operators $\boldsymbol{j} : L^p(I, V) \to L^p(I, H)$: $\boldsymbol{u} \to \boldsymbol{j}\boldsymbol{u}$ and $\boldsymbol{j}^* : L^{q'}(I, H^*) \to L^{q'}(I, V^*)$: $\boldsymbol{v} \to \boldsymbol{j}^*\boldsymbol{v}$, where $\boldsymbol{j}\boldsymbol{u}$ and $\boldsymbol{j}^*\boldsymbol{v}$ are for every $\boldsymbol{u} \in L^p(I, V)$ and $\boldsymbol{v} \in L^{q'}(I, H^*)$ given via

$$\begin{aligned}(\boldsymbol{j}\boldsymbol{u})(t) &:= j(\boldsymbol{u}(t)) \quad &&\text{in } H \quad &&\text{for a.e. } t \in I,\\ (\boldsymbol{j}^*\boldsymbol{v})(t) &:= j^*(\boldsymbol{v}(t)) \quad &&\text{in } V^* \quad &&\text{for a.e. } t \in I.\end{aligned}$$

It is shown in [35, Proposition 2.19] that both $\boldsymbol{j}$ and $\boldsymbol{j}^*$ are embeddings, which we call **induced embeddings**. Moreover, we define the intersection space

$$L^p(I, V) \cap_{\boldsymbol{j}} L^q(I, H) := \{\boldsymbol{u} \in L^p(I, V) \,|\, \boldsymbol{j}\boldsymbol{u} \in L^q(I, H)\},$$

which forms a Banach space equipped with the canonical sum norm

$$\|\cdot\|_{L^p(I,V)\cap_{\boldsymbol{j}} L^q(I,H)} := \|\cdot\|_{L^p(I,V)} + \|\boldsymbol{j}(\cdot)\|_{L^q(I,H)}.$$

If $1 < p \le q < \infty$, then $L^p(I, V) \cap_{\boldsymbol{j}} L^q(I, H)$ is additionally reflexive. Furthermore, for each $\boldsymbol{u}^* \in (L^p(I, V) \cap_{\boldsymbol{j}} L^q(I, H))^*$ there exist functions $\boldsymbol{g} \in L^{p'}(I, V^*)$ and $\boldsymbol{h} \in L^{q'}(I, H^*)$, such that for every $\boldsymbol{u} \in L^p(I, V) \cap_{\boldsymbol{j}} L^q(I, H)$ it holds

$$\langle \boldsymbol{u}^*, \boldsymbol{u}\rangle_{L^p(I,V)\cap_{\boldsymbol{j}} L^q(I,H)} = \int_I \langle \boldsymbol{g}(t) + (\boldsymbol{j}^*\boldsymbol{h})(t), \boldsymbol{u}(t)\rangle_V \, dt, \tag{2.4}$$

and $\|\boldsymbol{u}^*\|_{(L^p(I,V)\cap_{\boldsymbol{j}} L^q(I,H))^*} := \|\boldsymbol{g}\|_{L^{p'}(I,V^*)} + \|\boldsymbol{h}\|_{L^{q'}(I,H^*)}$, i.e., $(L^p(I, V) \cap_{\boldsymbol{j}} L^q(I, H))^*$ is isometrically isomorphic to the sum $L^{p'}(I, V^*) + \boldsymbol{j}^*(L^{q'}(I, H^*))$ (cf. [28, Kapitel I, Bemerkung 5.13 & Satz 5.13]), which is a Banach space equipped with the norm

$$\|\boldsymbol{f}\|_{L^{p'}(I,V^*)+\boldsymbol{j}^*(L^{q'}(I,H^*))} := \min_{\substack{\boldsymbol{g}\in L^{p'}(I,V^*)\\ \boldsymbol{h}\in L^{q'}(I,H^*)\\ \boldsymbol{f}=\boldsymbol{g}+\boldsymbol{j}^*\boldsymbol{h}}} \|\boldsymbol{g}\|_{L^{p'}(I,V^*)} + \|\boldsymbol{h}\|_{L^{q'}(I,H^*)}.$$

Definition 2.5 (Generalized time derivative) Let (V, H, j) be an evolution triple, $I := (0, T)$, $T < \infty$, and $1 < p \le q < \infty$. A function $\boldsymbol{u} \in L^p(I, V) \cap_{\boldsymbol{j}} L^q(I, H)$ possesses a **generalized time derivative with respect to the canonical embedding** e **of** (V, H, j) if there exists a function $\boldsymbol{u}^* \in L^{p'}(I, V^*) + \boldsymbol{j}^*(L^{q'}(I, H^*))$ such that for all $v \in V$ and $\varphi \in C_0^\infty(I)$

$$-\int_I (j(\boldsymbol{u}(s)), jv)_H \varphi'(s)\, ds = \int_I \langle \boldsymbol{u}^*(s), v\rangle_V \varphi(s)\, ds.$$

As this function $\boldsymbol{u}^* \in L^{p'}(I,V^*) + \boldsymbol{j}^*(L^{q'}(I,H^*))$ is unique (cf. [55, Proposition 23.18]), $\frac{d_e\boldsymbol{u}}{dt} := \boldsymbol{u}^*$ is well-defined. By

$$\boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H) := \left\{ \boldsymbol{u} \in L^p(I,V) \cap_{\boldsymbol{j}} L^q(I,H) \;\middle|\; \exists\, \frac{d_e\boldsymbol{u}}{dt} \in L^{p'}(I,V^*) + \boldsymbol{j}^*(L^{q'}(I,H^*)) \right\}$$

we denote the **Bochner-Sobolev space with respect to** e.

Proposition 2.6 (Formula of integration by parts) Let (V,H,j) be an evolution triple, $I := (0,T)$, $T < \infty$, and $1 < p \le q < \infty$. Then, it holds:

(i) The space $\boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$ forms a Banach space equipped with the norm

$$\|\cdot\|_{\boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)} := \|\cdot\|_{L^p(I,V)\cap_{\boldsymbol{j}} L^q(I,H)} + \left\| \frac{d_e\,\cdot}{dt} \right\|_{L^{p'}(I,V^*)+\boldsymbol{j}^*(L^{q'}(I,H^*))}.$$

(ii) Given $\boldsymbol{u} \in \boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$ the function $\boldsymbol{j}\boldsymbol{u} \in L^q(I,H)$ possesses a unique representation $\boldsymbol{j}_c\boldsymbol{u} \in C^0(\overline{I},H)$, and the resulting mapping $\boldsymbol{j}_c : \boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H) \to C^0(\overline{I},H)$ is an embedding.

(iii) **Generalized integration by parts formula:** It holds

$$\int_{t'}^{t} \left\langle \frac{d_e\boldsymbol{u}}{dt}(s), \boldsymbol{v}(s) \right\rangle_V ds = \left[((\boldsymbol{j}_c\boldsymbol{u})(s), (\boldsymbol{j}_c\boldsymbol{v})(s))_H\right]_{s=t'}^{s=t} - \int_{t'}^{t} \left\langle \frac{d_e\boldsymbol{v}}{dt}(s), \boldsymbol{u}(s) \right\rangle_V ds,$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$ and $t, t' \in \overline{I}$ with $t' \le t$.

Proof See [28, Kapitel IV, Satz 1.16 & Satz 1.17]. □

For an evolution triple (V,H,j), $I := (0,T)$, $T < \infty$, and $1 < p \le q < \infty$ we call an operator $\boldsymbol{\mathcal{A}} : L^p(I,V) \cap_{\boldsymbol{j}} L^q(I,H) \to (L^p(I,V) \cap_{\boldsymbol{j}} L^q(I,H))^*$ **induced** by a family of operators $A(t) : V \to V^*$, $t \in I$, if for every $\boldsymbol{u}, \boldsymbol{v} \in L^p(I,V) \cap_{\boldsymbol{j}} L^q(I,H)$ it holds

$$\langle \boldsymbol{\mathcal{A}}\boldsymbol{u}, \boldsymbol{v} \rangle_{L^p(I,V)\cap_{\boldsymbol{j}} L^q(I,H)} = \int_I \langle A(t)(\boldsymbol{u}(t)), \boldsymbol{v}(t) \rangle_V \, dt. \tag{2.7}$$

Remark 2.8 (Need for $L^p(I,V) \cap_{\boldsymbol{j}} L^q(I,H)$) Note that an operator family $A(t) : V \to V^*$, $t \in I$ can define an induced operator in different spaces. In [35], [34] the induced operator $\boldsymbol{\mathcal{A}}$ is considered as an operator from $L^p(I,V) \cap_{\boldsymbol{j}} L^\infty(I,H)$ into $(L^p(I,V))^*$. Here, we consider the induced operator $\boldsymbol{\mathcal{A}}$ as an operator from $L^p(I,V) \cap_{\boldsymbol{j}} L^q(I,H)$ into $(L^p(I,V) \cap_{\boldsymbol{j}} L^q(I,H))^*$, which is more general and enables us to consider operator families with significantly worse growth behavior. Here, the so-called *Temam modification* $\hat{B} : X \to X^*$, tracing back to [52], [53], of the convective term $B : X \to X^*$ defined in (1.3), defined for $p > \frac{3d+2}{d+2}$ and all $u, v \in X$ via

$$\langle \hat{B}u, v \rangle_X = \frac{1}{2} \int_\Omega v \otimes u : \nabla u \, dx - \frac{1}{2} \int_\Omega u \otimes u : \nabla v \, dx,$$

serves as a prototypical example. In fact, following [35, Example 5.1], one can prove that $B : X \to X^*$ satisfies for $d = 3$ and $p \ge \frac{11}{5}$ the estimate

$$\|Bu\|_{X^*} \le c(1 + \|u\|_Y)(1 + \|u\|_X^{p-1}), \tag{2.9}$$

for all $u \in X$ and that corresponding induced operator $\boldsymbol{\mathcal{B}}$ is well-defined and bounded as an operator from $L^p(I,X) \cap L^\infty(I,Y)$ to $(L^p(I,X))^*$. Regrettably, for the remaining term in Temam's modification, i.e., for the operator $\tilde{B} := \hat{B} - \frac{1}{2}B : X \to X^*$, we can prove (2.9) for $d = 3$ only for $p > \frac{13}{5}$. In order to reach $p > \frac{11}{5}$ for $d = 3$, one is forced to use a larger target space, i.e., we view the induced operator of $\tilde{B}$ as an operator from $L^p(I,X) \cap L^q(I,Y)$ to $(L^p(I,X) \cap L^q(I,Y))^*$, where $q \in [p,\infty)$ is specified in the proof Proposition 7.4.

Definition 2.10 (Weak solution) Let (V,H,j) be an evolution triple, $I := (0,T)$, $T < \infty$, and $1 < p \le q < \infty$. Moreover, let $u_0 \in H$ be an initial value, $\boldsymbol{f} \in L^{p'}(I,V^*)$ a right-hand side, and $\boldsymbol{\mathcal{A}} : L^p(I,V) \cap_{\boldsymbol{j}} L^q(I,H) \to (L^p(I,V) \cap_{\boldsymbol{j}} L^q(I,H))^*$ induced by a family of operators $A(t) : V \to V^*$, $t \in I$. A function $\boldsymbol{u} \in \boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$ is called **weak solution** of the initial value problem (1.1) if $(\boldsymbol{j}_c\boldsymbol{u})(0) = u_0$ in H and for all $\boldsymbol{\phi} \in C_0^1(I,V)$ there holds

$$\int_I \left\langle \frac{d_e\boldsymbol{u}}{dt}(t), \boldsymbol{\phi}(t) \right\rangle_V dt + \int_I \langle A(t)(\boldsymbol{u}(t)), \boldsymbol{\phi}(t) \rangle_V \, dt = \int_I \langle \boldsymbol{f}(t), \boldsymbol{\phi}(t) \rangle_V \, dt.$$

Here, the initial condition is well-defined since due to Proposition 2.6 (ii) there exists the unique continuous representation $\boldsymbol{j}_c\boldsymbol{u} \in C^0(\overline{I},H)$ of $\boldsymbol{u} \in \boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$.

3 Quasi non-conforming approximation

In this section we introduce the concept of quasi non-conforming approximations.

Definition 3.1 (Quasi non-conforming approximation) Let (V,H,j) and (X,Y,j) be evolution triples such that $V \subseteq X$ with $\|\cdot\|_V = \|\cdot\|_X$ in V and $H \subseteq Y$ with $(\cdot,\cdot)_H = (\cdot,\cdot)_Y$ in $H \times H$. Moreover, let $I := (0,T)$, $T < \infty$, and let $1 < p < \infty$. We call a sequence of closed subspaces $(V_n)_{n\in\mathbb{N}}$ of X a **quasi non-conforming approximation of** V **in** X, if the following properties are satisfied:

(QNC.1) There exists a dense subset $C \subseteq V$, such that for each $v \in C$ there exist elements $v_n \in V_n$, $n \in \mathbb{N}$, such that $v_n \to v$ in X $(n \to \infty)$.

(QNC.2) For each sequence $\boldsymbol{u}_n \in L^p(I,V_{m_n})$, $n \in \mathbb{N}$, where $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $m_n \to \infty$ $(n \to \infty)$, from $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in $L^p(I,X)$ $(n \to \infty)$, it follows that $\boldsymbol{u} \in L^p(I,V)$.

The next proposition shows that the notion of a quasi non-conforming approximation is indeed a generalization of the usual notion of a conforming approximation. In Section 7 we will show that our motivating example, namely the approximation of divergence-free Sobolev functions through discretely divergence-free finite element spaces, fits into the framework of quasi non-conforming approximations.

Proposition 3.2 Let (X,Y,j) and (V,H,j) be as in Definition 3.1. Then, it holds:

(i) The constant approximation $V_n = V$, $n \in \mathbb{N}$, is a quasi non-conforming approximation of V in X.

(ii) If $(V_n)_{n\in\mathbb{N}}$ is a conforming approximation of V, i.e., $(V_n)_{n\in\mathbb{N}}$ satisfy (C.1) and (C.2), then $(V_n)_{n\in\mathbb{N}}$ is a quasi non-conforming approximation of V in X.

Proof **ad (i)** Follows right from the definition.

ad (ii) We set $C := \bigcup_{n\in\mathbb{N}} V_n$. Then, for each $v \in C$ there exists an integer $n_0 \in \mathbb{N}$ such that $v \in V_n$ for every $n \ge n_0$. Therefore, the sequence $v_n \in V_n$, $n \in \mathbb{N}$, given via $v_n := 0$ if $n < n_0$ and $v_n := v$ if $n \ge n_0$, satisfies $v_n \to v$ in V $(n \to \infty)$, i.e., $(V_n)_{n\in\mathbb{N}}$ satisfies (QNC.1). Apart from that, $(V_n)_{n\in\mathbb{N}}$ obviously fulfills (QNC.2). □

The following proposition will be crucial in verifying that the induced operator $\boldsymbol{\mathcal{A}}$ of a family of operators $(A(t))_{t\in I}$ is quasi non-conforming Bochner pseudo-monotone (cf. Definition 4.1).

Proposition 3.3 Let (V,H,j) and (X,Y,j) be as in Definition 3.1 and let $(V_n)_{n\in\mathbb{N}}$ be a quasi non-conforming approximation of V in X. Then, the following statements hold true:

(i) For a sequence $v_n \in V_{m_n}$, $n\in\mathbb{N}$, where $(m_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}$ with $m_n\to\infty$ $(n\to\infty)$, from $v_n \rightharpoonup v$ in X $(n\to\infty)$, it follows that $v\in V$.

(ii) For a sequence $v_n \in V_{m_n}$, $n\in\mathbb{N}$, where $(m_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}$ with $m_n\to\infty$ $(n\to\infty)$, with $\sup_{n\in\mathbb{N}}\|v_n\|_X<\infty$, and $v\in V$ the following statements are equivalent:
  (a) $v_n \rightharpoonup v$ in X $(n\to\infty)$.
  (b) $P_H j v_n \rightharpoonup jv$ in H $(n\to\infty)$, where $P_H : Y\to H$ is the orthogonal projection of Y into H.

(iii) For each $\eta\in H$ there exists a sequence $v_n\in V_{m_n}$, $n\in\mathbb{N}$, where $(m_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}$ with $m_n\to\infty$ $(n\to\infty)$, such that $jv_n\to\eta$ in Y $(n\to\infty)$.

Proof **ad (i)** Immediate consequence of (QNC.2).

ad (ii) (a) ⇒ (b) Follows from the weak continuity of $j:X\to Y$ and $P_H:Y\to H$.

(b) ⇒ (a) From the reflexivity of X, we obtain a subsequence $(v_n)_{n\in\Lambda}$, with $\Lambda\subseteq\mathbb{N}$, and an element $\tilde{v}\in X$, such that $v_n\rightharpoonup\tilde{v}$ in X $(\Lambda\ni n\to\infty)$. Due to **(i)** we infer $\tilde{v}\in V$. From the weak continuity of $j:X\to Y$ and $P_H:Y\to H$ we conclude $P_H j v_n \rightharpoonup P_H j\tilde{v} = j\tilde{v}$ in H $(\Lambda\ni n\to\infty)$. In consequence, we have $j\tilde{v}=jv$ in H, which in virtue of the injectivity of $j:V\to H$ implies that $\tilde{v}=v$ in V, and therefore

$$v_n \rightharpoonup v \quad \text{in } X \quad (\Lambda\ni n\to\infty). \tag{3.4}$$

Since this argumentation remains valid for each subsequence of $(v_n)_{n\in\mathbb{N}}\subseteq X$, $v\in V$ is weak accumulation point of each subsequence of $(v_n)_{n\in\mathbb{N}}\subseteq X$. Therefore, the standard convergence principle (cf. [28, Kap. I, Lemma 5.4]) guarantees that (3.4) remains true even if $\Lambda=\mathbb{N}$.

ad (iii) Since (V,H,j) is an evolution triple, $j(V)$ is dense in H. As a result, for fixed $\eta\in H$ there exists a sequence $(v_n)_{n\in\mathbb{N}}\subseteq V$, such that $\|\eta - jv_n\|_H\le 2^{-n}$ for all $n\in\mathbb{N}$. Due to (QNC.1) there exist a sequence $(w_n)_{n\in\mathbb{N}}\subseteq C$, such that $\|v_n-w_n\|_V\le 2^{-n-1}$ for all $n\in\mathbb{N}$ and a double sequence $(v_k^n)_{n,k\in\mathbb{N}}\subseteq X$, with $v_k^n\in V_k$ for all $k,n\in\mathbb{N}$, such that $v_k^n\to w_n$ in X $(k\to\infty)$ for all $n\in\mathbb{N}$. Thus, for each $n\in\mathbb{N}$ there exists $m_n\in\mathbb{N}$, such that $\|w_n-v_k^n\|_X\le 2^{-n-1}$ for all $k\ge m_n$. Then, we have $v_{m_n}^n\in V_{m_n}$ for all $n\in\mathbb{N}$ and $\|\eta - jv_{m_n}^n\|_Y\le(1+c)2^{-n}$ for all $n\in\mathbb{N}$, where $c>0$ is the embedding constant of j. □

4 Quasi non-conforming Bochner pseudo-monotonicity

In this section we introduce an extended notion of Bochner pseudo-monotonicity (cf. [35], [34]), which incorporates a given quasi non-conforming approximation $(V_n)_{n\in\mathbb{N}}$.

Definition 4.1 Let (X,Y,j) and (V,H,j) be as in Definition 3.1 and let $(V_n)_{n\in\mathbb{N}}$ be a quasi non-conforming approximation of V in X, $I:=(0,T)$, with $0<T<\infty$, and $1<p\le q<\infty$. An operator $\boldsymbol{\mathcal{A}}: L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)\to(L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*$ is said to be **quasi non-conforming Bochner pseudo-monotone with respect to** $(V_n)_{n\in\mathbb{N}}$ if for a sequence $\boldsymbol{u}_n\in L^\infty(I,V_{m_n})$, $n\in\mathbb{N}$, where $(m_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}$ with $m_n\to\infty$ $(n\to\infty)$, from

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \quad \text{in } L^p(I,X) \quad (n\to\infty), \tag{4.2}$$

$$\boldsymbol{j}\boldsymbol{u}_n \overset{*}{\rightharpoondown} \boldsymbol{j}\boldsymbol{u} \quad \text{in } L^\infty(I,Y) \quad (n\to\infty), \tag{4.3}$$

$$P_H(\boldsymbol{j}\boldsymbol{u}_n)(t) \rightharpoonup (\boldsymbol{j}\boldsymbol{u})(t) \quad \text{in } H \quad (n\to\infty) \quad \text{for a.e. } t\in I, \tag{4.4}$$

and

$$\limsup_{n\to\infty}\langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{u}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)}\le 0, \tag{4.5}$$

it follows for all $\boldsymbol{v}\in L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)$ that

$$\langle \boldsymbol{\mathcal{A}}\boldsymbol{u},\boldsymbol{u}-\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)}\le \liminf_{n\to\infty}\langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)}.$$

Note that (4.2) and (4.3) guarantee that $\boldsymbol{u}\in L^p(I,V)\cap_{\boldsymbol{j}}L^\infty(I,H)$ due to Definition 3.1.

The basic idea of quasi non-conforming Bochner pseudo-monotonicity, in comparison to the original notion of Bochner pseudo-monotonicity tracing back to [35], consists in incorporating the finite dimensional approximation $(V_n)_{n\in\mathbb{N}}$ into the definition. We will see in the proof of Theorem 6.16 that (4.2)–(4.5) are natural properties of a sequence $\boldsymbol{u}_n\in L^p(I,V_{m_n})$, $n\in\mathbb{N}$, coming from (1.6) (which is a quasi non-conforming Rothe–Galerkin approximation of (1.1)), if $\boldsymbol{\mathcal{A}}$ satisfies appropriate additional assumptions. In fact, (4.2) usually is a consequence of the coercivity of $\boldsymbol{\mathcal{A}}$, (4.3) stems from the time derivative, while (4.4) and (4.5) follow directly from the approximative scheme.

Proposition 4.6 Let (X,Y,j) and (V,H,j) be as in Definition 3.1 and let $(V_n)_{n\in\mathbb{N}}$ be a quasi non-conforming approximation of V in X, $I:=(0,T)$, $T<\infty$, and $1<p\le q<\infty$. Moreover, let $A(t):X\to X^*$, $t\in I$, be a family of operators with the following properties:

(A.1) $A(t):X\to X^*$ is pseudo-monotone for almost every $t\in I$.
(A.2) $A(\cdot)u:I\to X^*$ is Bochner measurable for every $u\in X$.
(A.3) For some constants $c_0>0$ and $c_1,c_2\ge 0$ holds for almost every $t\in I$ and every $u\in X$

$$\langle A(t)u,u\rangle_X\ge c_0\|u\|_X^p-c_1\|ju\|_Y^2-c_2.$$

(A.4) For constants $\gamma\ge 0$ and $\lambda\in(0,c_0)$ holds for almost every $t\in I$ and every $u,v\in X$

$$|\langle A(t)u,v\rangle_X|\le\lambda\|u\|_X^p+\gamma\big(1+\|ju\|_Y^q+\|jv\|_Y^q+\|v\|_X^p\big).$$

Then, the induced operator $\boldsymbol{\mathcal{A}}:L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)\to(L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y))^*$, given via (2.7), is well-defined, bounded and quasi non-conforming Bochner pseudo-monotone with respect to the subspaces $(V_n)_{n\in\mathbb{N}}$.

Proof **1. Well-definiteness:** For $\boldsymbol{u}_1,\boldsymbol{u}_2\in L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)$ there exists sequences of simple functions $(\boldsymbol{s}_n^m)_{n\in\mathbb{N}}\subseteq L^\infty(I,X)$, $m=1,2$, i.e., $\boldsymbol{s}_n^m(t)=\sum_{i=1}^{k_n^m}s_{n,i}^m\chi_{E_{n,i}^m}(t)$ for $t\in I$ and $m=1,2$, where $s_{n,i}^m\in X$, $k_n^m\in\mathbb{N}$ and $E_{n,i}^m\in\mathcal{L}^1(I)$ with $\bigcup_{i=1}^{k_n^m}E_{n,i}^m=I$ and $E_{n,i}^m\cap E_{n,j}^m=\emptyset$ for $i\neq j$, such that $\boldsymbol{s}_n^m(t)\to\boldsymbol{u}_m(t)$ in X for almost every $t\in I$ and $m=1,2$. Moreover, it follows from Lemma 2.3

that $A(t):X\to X^*$ is for almost every $t\in I$ demi-continuous, since it is for almost every $t\in I$ pseudo-monotone (cf. (A.1)) and bounded (cf. (A.4)). This yields for almost every $t\in I$

$$\langle A(t)(\boldsymbol{s}_n^1(t)),\boldsymbol{s}_n^2(t)\rangle_X=\sum_{i=1}^{k_n^1}\sum_{j=1}^{k_n^2}\langle A(t)s_{n,i}^1,s_{n,j}^2\rangle_X\chi_{E_{n,i}^1\cap E_{n,j}^2}(t)\overset{n\to\infty}{\to}\langle A(t)(\boldsymbol{u}_1(t)),\boldsymbol{u}_2(t)\rangle_X. \tag{4.7}$$

Thus, since the functions $(t\mapsto\langle A(t)s_{n,i}^1,s_{n,j}^2\rangle_X:I\to\mathbb{R}$, $i=0,\dots,k_n^1$, $j=1,\dots,k_n^2$, $n\in\mathbb{N}$, are Lebesgue measurable due to (A.2), we conclude from (4.7) that $(t\mapsto\langle A(t)(\boldsymbol{u}_1(t)),\boldsymbol{u}_2(t)\rangle_X):I\to\mathbb{R}$ is Lebesgue measurable. In addition, using (A.4), we obtain

$$\begin{aligned}\int_I\langle A(t)(\boldsymbol{u}_1(t)),\boldsymbol{u}_2(t)\rangle_X\,dt&\le\lambda\|\boldsymbol{u}_1\|_{L^p(I,X)}^p\\&\quad+\gamma[T+\|\boldsymbol{j}\boldsymbol{u}_1\|_{L^q(I,Y)}^q+\|\boldsymbol{j}\boldsymbol{u}_2\|_{L^q(I,Y)}^q+\|\boldsymbol{u}_2\|_{L^p(I,X)}^p],\end{aligned}\tag{4.8}$$

i.e., $\boldsymbol{\mathcal{A}}:L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)\to(L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y))^*$ is well-defined.

2. Boundedness: As $\|\boldsymbol{v}\|_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)}\le 1$ implies that $\|\boldsymbol{v}\|_{L^p(I,X)}^p+\|\boldsymbol{j}\boldsymbol{v}\|_{L^q(I,Y)}^q\le 2$ for every $\boldsymbol{v}\in L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)$, we infer from (4.8) for every $\boldsymbol{u}\in L^p(I,X)\cap_{\boldsymbol{j}}L^\infty(I,Y)$ that

$$\begin{aligned}\|\boldsymbol{\mathcal{A}}\boldsymbol{u}\|_{(L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y))^*}&=\sup_{\|\boldsymbol{v}\|_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)}\le 1}\langle\boldsymbol{\mathcal{A}}\boldsymbol{u},\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)}\\&\le\lambda\|\boldsymbol{u}\|_{L^p(I,X)}^p+\gamma\|\boldsymbol{j}\boldsymbol{u}\|_{L^q(I,Y)}^q+\gamma[T+2],\end{aligned}$$

i.e., $\boldsymbol{\mathcal{A}}:L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)\to(L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y))^*$ is bounded.

3. Quasi non-conforming Bochner pseudo-monotonicity with respect to $(V_n)_{n\in\mathbb{N}}$: In principle, we proceed analogously to [35, Proposition 3.13]. However, as we have solely almost everywhere weak convergence of the orthogonal projections available, i.e., (4.4), in the definition of quasi-nonconforming Bochner pseudo-monotonicity (cf. Definition 4.1), the arguments in [35] ask for some slight modifications. In fact, in this context the properties of the quasi non-conforming approximation $(V_n)_{n\in\mathbb{N}}$ come into play. Especially the role of Proposition 3.3 will be crucial. We split the proof of the quasi non-conforming Bochner pseudo-monotonicity into four steps:

3.1. Collecting information: Let $\boldsymbol{u}_n\in L^\infty(I,V_{m_n})$, $n\in\mathbb{N}$, where $(m_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}$ with $m_n\to\infty$ $(n\to\infty)$, be a sequence satisfying (4.2)–(4.5). We fix an arbitrary $\boldsymbol{v}\in L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)$, and choose a subsequence $(\boldsymbol{u}_n)_{n\in\Lambda}$, with $\Lambda\subseteq\mathbb{N}$, such that

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}}\langle\boldsymbol{\mathcal{A}}\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)}=\liminf_{n\to\infty}\langle\boldsymbol{\mathcal{A}}\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)}. \tag{4.9}$$

Due to (4.4) there exists a subset $E\subseteq I$, with $I\setminus E$ a null set[6], such that for all $t\in E$

$$P_H(\boldsymbol{j}\boldsymbol{u}_n)(t)\ \rightharpoonup\ (\boldsymbol{j}\boldsymbol{u})(t)\quad\text{in }H\quad(n\to\infty). \tag{4.10}$$

Using (A.3) and (A.4), we get for every $\boldsymbol{z}\in L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)$ and almost every $t\in I$

$$\begin{aligned}&\langle A(t)(\boldsymbol{u}_n(t)),\boldsymbol{u}_n(t)-\boldsymbol{z}(t)\rangle_X\\&\quad\ge c_0\|\boldsymbol{u}_n(t)\|_X^p-c_1\|j(\boldsymbol{u}_n(t))\|_Y^2-c_2-\langle A(t)(\boldsymbol{u}_n(t)),\boldsymbol{z}(t)\rangle_X\\&\quad\ge(c_0-\lambda)\|\boldsymbol{u}_n(t)\|_X^p-c_1K^2-c_2-\gamma\big[1+K^q+\|(\boldsymbol{j}\boldsymbol{z})(t)\|_Y^q+\|\boldsymbol{z}(t)\|_X^p\big],\end{aligned}\tag{4.11}$$

[6] A null set is a set of zero Lebesgue measure.

where $K := \sup_{n\in\mathbb{N}} \|\boldsymbol{j}\boldsymbol{u}_n\|_{L^\infty(I,Y)} < \infty$ (cf. (4.3)). If we set $\mu_{\boldsymbol{z}}(t) := -c_1K^2 - c_2 - \gamma\big[1 + K^q + \|(\boldsymbol{j}\boldsymbol{z})(t)\|_Y^q + \|\boldsymbol{z}(t)\|_X^p\big] \in L^1(I)$, then (4.11) reads

$$\langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{z}(t)\rangle_X \geq (c_0 - \lambda)\|\boldsymbol{u}_n(t)\|_X^p - \mu_{\boldsymbol{z}}(t), \qquad (*)_{\boldsymbol{z},n,t}$$

for almost every $t \in I$ and all $n \in \Lambda$. Next, we define

$$E_1 := \big\{t \in E \,\big|\, A(t) : X \to X^* \text{ is pseudo-monotone}, |\mu_{\boldsymbol{u}}(t)| < \infty \text{ and } (*)_{\boldsymbol{u},n,t} \text{ holds for all } n \in \Lambda\big\}.$$

From the defining properties of E_1 it follows directly that $I \setminus E_1$ is a null set.

3.2. Intermediate objective: Our next objective is to verify that for all $t \in E_1$ there holds

$$\liminf_{\substack{n\to\infty \\ n\in\Lambda}} \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X \geq 0. \qquad (**)_t$$

To this end, let us fix an arbitrary $t \in E_1$ and define

$$\Lambda_t := \{n \in \Lambda \,\big|\, \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X < 0\}.$$

We assume without loss of generality that Λ_t is not finite. Otherwise, $(**)_t$ would already hold true for this specific $t \in E_1$ and nothing would be left to do. But if Λ_t is not finite, then

$$\limsup_{\substack{n\to\infty \\ n\in\Lambda_t}} \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X \leq 0. \tag{4.12}$$

The definition of Λ_t and $(*)_{\boldsymbol{u},n,t}$ imply for all $n \in \Lambda_t$

$$(c_0 - \lambda)\|\boldsymbol{u}_n(t)\|_X^p \leq \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X + |\mu_{\boldsymbol{u}}(t)| < |\mu_{\boldsymbol{u}}(t)| < \infty. \tag{4.13}$$

This and $\lambda < c_0$ yield that the sequence $(\boldsymbol{u}_n(t))_{n\in\Lambda_t}$ is bounded in X. In view of (4.10), Proposition 3.3 (ii) implies that

$$\boldsymbol{u}_n(t) \quad\rightharpoonup\quad \boldsymbol{u}(t) \quad \text{ in } X \quad (\Lambda_t \ni n \to \infty). \tag{4.14}$$

Since $A(t) : X \to X^*$ is pseudo-monotone, we get from (4.14) and (4.12) that

$$\liminf_{\substack{n\to\infty \\ n\in\Lambda_t}} \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X \geq 0.$$

Due to $\langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X \geq 0$ for all $n \in \Lambda \setminus \Lambda_t$, $(**)_t$ holds for all $t \in E_1$.

3.3. Switching to the image space level: In this passage we verify the existence of a subsequence $(\boldsymbol{u}_n)_{n\in\Lambda_0} \subseteq L^p(I,X) \cap_{\boldsymbol{j}} L^\infty(I,Y)$, with $\Lambda_0 \subseteq \Lambda$, such that for almost every $t \in I$

$$\begin{aligned} &\boldsymbol{u}_n(t) \quad\rightharpoonup\quad \boldsymbol{u}(t) \quad \text{ in } X \quad (\Lambda_0 \ni n \to \infty), \\ &\limsup_{\substack{n\to\infty \\ n\in\Lambda_0}} \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X \leq 0. \end{aligned} \tag{4.15}$$

As a consequence, we are in a position to exploit the almost everywhere pseudo-monotonicity of the operator family. Thanks to $\langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X \geq -\mu_{\boldsymbol{u}}(t)$ for all $t \in E_1$ and $n \in \Lambda$ (cf. $(*)_{\boldsymbol{u},n,t}$), Fatou's lemma (cf. [44, Theorem 1.18]) is applicable. It yields, also using (4.5)

$$\begin{aligned} 0 &\leq \int_I \liminf_{\substack{n\to\infty \\ n\in\Lambda}} \langle A(s)(\boldsymbol{u}_n(s)), \boldsymbol{u}_n(s) - \boldsymbol{u}(s)\rangle_X \, ds \\ &\leq \liminf_{\substack{n\to\infty \\ n\in\Lambda}} \int_I \langle A(s)(\boldsymbol{u}_n(s)), \boldsymbol{u}_n(s) - \boldsymbol{u}(s)\rangle_X \, ds \\ &\leq \limsup_{n\to\infty} \langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \\ &\leq 0. \end{aligned} \tag{4.16}$$

Let us define $g_n(t) := \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X$. Then, $(**)_t$ and (4.16) read:

$$\liminf_{\substack{n\to\infty \\ n\in\Lambda}} g_n(t) \geq 0 \quad \text{ for all } t \in E_1. \tag{4.17}$$

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} \int_I g_n(s)\, ds = 0. \tag{4.18}$$

As $s \mapsto s^- := \min\{0, s\}$ is continuous and non-decreasing we deduce from (4.17) that

$$0 \geq \limsup_{\substack{n\to\infty \\ n\in\Lambda}} g_n(t)^- \geq \liminf_{\substack{n\to\infty \\ n\in\Lambda}} g_n(t)^- \geq \min\Big\{0, \liminf_{\substack{n\to\infty \\ n\in\Lambda}} g_n(t)\Big\} = 0,$$

i.e., $g_n(t)^- \to 0$ $(\Lambda \ni n \to \infty)$ for all $t \in E_1$. Since $0 \geq g_n(t)^- \geq -\mu_{\boldsymbol{u}}(t)$ for all $t \in E_1$ and $n \in \Lambda$, Vitali's theorem yields $g_n^- \to 0$ in $L^1(I)$ $(\Lambda \ni n \to \infty)$. From the latter, $|g_n| = g_n - 2g_n^-$ and (4.18), we conclude that $g_n \to 0$ in $L^1(I)$ $(\Lambda \ni n \to \infty)$. This provides a further subsequence $(\boldsymbol{u}_n)_{n\in\Lambda_0}$ with $\Lambda_0 \subseteq \Lambda$ and a subset $F \subseteq I$ such that $I \setminus F$ is a null set and for all $t \in F$

$$\lim_{\substack{n\to\infty \\ n\in\Lambda_0}} \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X = 0. \tag{4.19}$$

This and (4.13) implies for all $t \in E_1 \cap F$ that

$$\limsup_{\substack{n\to\infty \\ n\in\Lambda_0}} (c_0 - \lambda)\|\boldsymbol{u}_n(t)\|_X^p \leq \limsup_{\substack{n\to\infty \\ n\in\Lambda_0}} \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle_X + |\mu_{\boldsymbol{u}}(t)| = |\mu_{\boldsymbol{u}}(t)| < \infty,$$

i.e., $(\boldsymbol{u}_n(t))_{n\in\Lambda_0}$ is bounded in X for all $t \in E_1 \cap F$. Thus, (4.10) and Proposition 3.3 (ii) yield for all $t \in E_1 \cap F$

$$\boldsymbol{u}_n(t) \quad\rightharpoonup\quad \boldsymbol{u}(t) \quad \text{ in } X \quad (\Lambda_0 \ni n \to \infty). \tag{4.20}$$

The relations (4.19) and (4.20) are just (4.15).

3.4. Switching to the Bochner-Lebesgue level: From the pseudo-monotonicity of the operators $A(t) : X \to X^*$ for all $t \in E_1 \cap F$ we obtain almost every $t \in I$

$$\langle A(t)(\boldsymbol{u}(t)), \boldsymbol{u}(t) - \boldsymbol{v}(t)\rangle_X \leq \liminf_{\substack{n\to\infty \\ n\in\Lambda_0}} \langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t) - \boldsymbol{v}(t)\rangle_X.$$

Due to $(*)_{\boldsymbol{v},n,t}$, we have $\langle A(t)(\boldsymbol{u}_n(t)), \boldsymbol{u}_n(t)-\boldsymbol{v}(t)\rangle_X \geq -\mu_{\boldsymbol{v}}(t)$ for almost every $t \in I$ and all $n \in \Lambda_0$. Thus, using the definition of the induced operator (2.7), Fatou's lemma and (4.9) we deduce

$$\begin{aligned}\langle \boldsymbol{\mathcal{A}u}, \boldsymbol{u}-\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} &\leq \int_I \liminf_{\substack{n\to\infty\\ n\in\Lambda_0}} \langle A(s)(\boldsymbol{u}_n(s)), \boldsymbol{u}_n(s)-\boldsymbol{v}(s)\rangle_X\, ds \\ &\leq \liminf_{\substack{n\to\infty\\ n\in\Lambda_0}} \int_I \langle A(s)(\boldsymbol{u}_n(s)), \boldsymbol{u}_n(s)-\boldsymbol{v}(s)\rangle_X\, ds \\ &= \lim_{\substack{n\to\infty\\ n\in\Lambda}} \langle \boldsymbol{\mathcal{A}u}_n, \boldsymbol{u}_n-\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \\ &= \liminf_{n\to\infty} \langle \boldsymbol{\mathcal{A}u}_n, \boldsymbol{u}_n-\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)}.\end{aligned}$$

As $\boldsymbol{v} \in L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)$ was chosen arbitrary, this completes the proof of Proposition 4.6. $\square$

5 Rothe scheme

Let X be a Banach space, and $I := (0,T)$, $T<\infty$, be a finite time interval. For $K \in \mathbb{N}$ we set $\tau := \frac{T}{K}$, $I_k^\tau := ((k-1)\tau, k\tau]$, $k=1,\dots,K$, and $\mathcal{I}_\tau := \{I_k^\tau\}_{k=1,\dots,K}$. Moreover, we denote by

$$\boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X) := \{\boldsymbol{u} : I \to X \,|\, \boldsymbol{u}(s) = \boldsymbol{u}(t) \text{ in } X \text{ for all } t,s \in I_k^\tau, k=1,\dots,K\} \subset L^\infty(I,X)$$

the **space of piece-wise constant functions with respect to** $\mathcal{I}_\tau$. For a given finite sequence $(u^k)_{k=0,\dots,K} \subseteq X$ the **backward difference quotient** operator is defined via

$$d_\tau u^k := \frac{1}{\tau}(u^k - u^{k-1}) \quad \text{in } X, \quad k=1,\dots,K.$$

Furthermore, we denote for a given finite sequence $(u^k)_{k=0,\dots,K} \subseteq X$ by $\overline{\boldsymbol{u}}^\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$ the **piece-wise constant interpolant**, and by $\hat{\boldsymbol{u}}^\tau \in W^{1,\infty}(I,X)$ the **piece-wise affine interpolant**, for every $t \in I_k^\tau$ and $k=1,\dots,K$ given via

$$\overline{\boldsymbol{u}}^\tau(t) := u^k, \qquad \hat{\boldsymbol{u}}^\tau(t) := \Big(\frac{t}{\tau} - (k-1)\Big)u^k + \Big(k - \frac{t}{\tau}\Big)u^{k-1} \quad \text{in } X. \tag{5.1}$$

In addition, if (X,Y,j) is an evolution triple and $(u^k)_{k=0,\dots,K} \subseteq X$ a finite sequence, then it holds for $k,l = 0,\dots,K$ the **discrete integration by parts formula**

$$\int_{k\tau}^{l\tau} \Big\langle \frac{d_e \hat{\boldsymbol{u}}^\tau}{dt}(t), \overline{\boldsymbol{u}}^\tau(t)\Big\rangle_X dt \geq \frac{1}{2}\|ju^l\|_Y^2 - \frac{1}{2}\|ju^k\|_Y^2, \tag{5.2}$$

which is an immediate consequence of the identity $\langle d_\tau e u^k, u^k\rangle_X = \frac{1}{2} d_\tau \|ju^k\|_Y^2 + \frac{\tau}{2}\|d_\tau j u^k\|_Y^2$ for every $k=1,\dots,K$.

For the discretization of the right-hand side in (1.1) we use the following construction. Let X be a Banach space, $I=(0,T)$, $T<\infty$, $K\in\mathbb{N}$, $\tau := \frac{T}{K} > 0$ and $1<p<\infty$. The **Clemént 0-order quasi-interpolation operator** $\mathscr{J}_\tau : L^p(I,X) \to \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$ is defined for every $\boldsymbol{u} \in L^p(I,X)$ via

$$\mathscr{J}_\tau[\boldsymbol{u}] := \sum_{k=1}^K [\boldsymbol{u}]_k^\tau \chi_{I_k^\tau} \quad \text{in } \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X), \qquad [\boldsymbol{u}]_k^\tau := \fint_{I_k^\tau} \boldsymbol{u}(s)\, ds \in X.$$

Proposition 5.3 For every $\boldsymbol{u} \in L^p(I,X)$ it holds:

(i) $\mathscr{J}_\tau[\boldsymbol{u}] \to \boldsymbol{u}$ in $L^p(I,X)$ $(\tau \to 0)$, i.e., $\bigcup_{\tau>0} \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$ is dense in $L^p(I,X)$.
(ii) $\sup_{\tau>0} \|\mathscr{J}_\tau[\boldsymbol{u}]\|_{L^p(I,X)} \le \|\boldsymbol{u}\|_{L^p(I,X)}$.

Proof See [44, Remark 8.15]. □

Since we treat non-autonomous evolution equations we also need to discretize the time dependent family of operators in (1.1). This will also be obtained by means of the Clemént 0-order quasi-interpolant. Let (X,Y,j) be an evolution triple, $I := (0,T)$, $T < \infty$, $K \in \mathbb{N}$, $\tau = \frac{T}{K} > 0$ and $1 < p \le q < \infty$. Let $A(t) : X \to X^*$, $t \in I$, be a family of operators satisfying the conditions (A.1)–(A.4), and denote by $\boldsymbol{\mathcal{A}} : L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y) \to (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$ the induced operator (cf. (2.7)). The **k-th temporal mean** $[A]^\tau_k : X \to X^*$, $k = 1,\dots,K$, of $A(t) : X \to X^*$, $t \in I$, is defined for every $u \in X$ via

$$[A]^\tau_k u := \fint_{I^\tau_k} A(s)u\,ds \quad \text{ in } X^*.$$

The **Clement 0-order quasi-interpolant** $\mathscr{J}_\tau[A](t) : X \to X^*$, $t \in I$, of $A(t) : X \to X^*$, $t \in I$, is defined for almost every $t \in I$ and $u \in X$ via

$$\mathscr{J}_\tau[A](t)u := \sum_{k=1}^{K} \chi_{I^\tau_k}(t)[A]^\tau_k u \quad \text{ in } X^*.$$

The **Clement 0-order quasi-interpolant** $\mathscr{J}_\tau[\boldsymbol{\mathcal{A}}] : L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y) \to (L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*$, of $\boldsymbol{\mathcal{A}} : L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y) \to (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$ is for all $\boldsymbol{u},\boldsymbol{v} \in L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)$ defined via

$$\langle \mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]\boldsymbol{u},\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} := \int_I \langle \mathscr{J}_\tau[A](t)(\boldsymbol{u}(t)),\boldsymbol{v}(t)\rangle_X\,dt.$$

Note that $\mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]$ is the induced operator of the family of operators $\mathscr{J}_\tau[A](t) : X \to X^*$, $t \in I$.

Proposition 5.4 (Clement 0-order quasi-interpolant for induced operators) Let $A(t) : X \to X^*$, $t \in I$, be a family of operators satisfying the conditions (A.1)–(A.4), and denote by $\boldsymbol{\mathcal{A}} : L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y) \to (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$ the induced operator (cf. (2.7)). Then, there holds:

(i) $[A]^\tau_k : X \to X^*$ is well-defined, bounded, pseudo-monotone, and satisfies:
  (i.a) $\langle [A]^\tau_k u, v\rangle_X \le \lambda\|u\|_X^p + \gamma[1 + \|ju\|_Y^q + \|jv\|_Y^q + \|v\|_X^p]$ for all $u,v \in X$.
  (i.b) $\langle [A]^\tau_k u, u\rangle_X \ge c_0\|u\|_X^p - c_1\|ju\|_Y^2 - c_2$ for all $u \in X$.
(ii) $\mathscr{J}_\tau[A](t) : X \to X^*$, $t \in I$, satisfies the conditions (A.1)–(A.4).
(iii) $\mathscr{J}_\tau[\boldsymbol{\mathcal{A}}] : L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y) \to (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$ is well-defined, bounded and satisfies:
  (iii.a) For all $\boldsymbol{u}_\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$, $\boldsymbol{v} \in L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y)$ holds

$$\langle \mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]\boldsymbol{u}_\tau,\boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} = \langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_\tau, \mathscr{J}_\tau[\boldsymbol{v}]\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)}.$$

  (iii.b) If the functions $\boldsymbol{u}_\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$, $\tau > 0$, are bounded in $L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y)$, then

$$\boldsymbol{\mathcal{A}}\boldsymbol{u}_\tau - \mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]\boldsymbol{u}_\tau \;\rightharpoonup\; \boldsymbol{0} \quad \text{ in } (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^* \quad (\tau \to 0).$$

(iii.c)

$$\|\mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]\boldsymbol{u}_\tau\|_{(L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*} \leq \|\boldsymbol{\mathcal{A}}\boldsymbol{u}_\tau\|_{(L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*}.$$

Proof **ad (i)** Let $u \in X$. Due to (A.2) the function $A(\cdot)u : I \to X^*$ is Bochner measurable. (A.4) guarantees that $\|A(\cdot)u\|_{X^*} \in L^1(I)$, and thus the Bochner integrability of $A(\cdot)u : I \to X^*$. As a result, the Bochner integral $[A]_k^\tau u = \fint_{I_k^\tau} A(s)u\,ds \in X^*$ exists, i.e., $[A]_k^\tau : X \to X^*$ is well-defined. The inequalities **(i.a)** and **(i.b)** are obvious. In particular, we gain from inequality **(i.a)** the boundedness of $[A]_k^\tau : X \to X^*$. So, it is left to show the pseudo-monotonicity. Therefore, let $(u_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence such that

$$u_n \rightharpoonup u \quad \text{in } X \quad (n \to \infty), \tag{5.5}$$

$$\limsup_{n\to\infty} \langle [A]_k^\tau u_n, u_n - u\rangle_X \leq 0. \tag{5.6}$$

If we set $\boldsymbol{u}_n := u_n \chi_{I_k^\tau} \in L^\infty(I, X)$, $n \in \mathbb{N}$, and $\boldsymbol{u} := u\chi_{I_k^\tau} \in L^\infty(I, X)$, then (5.5), the Lebesgue theorem on dominated convergence and the properties of the induced embedding $\boldsymbol{j}$ imply

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \qquad \text{in } L^p(I,X) \quad (n \to \infty), \tag{5.7}$$

$$\boldsymbol{j}\boldsymbol{u}_n \overset{*}{\rightharpoonup} \boldsymbol{j}\boldsymbol{u} \qquad \text{in } L^\infty(I,Y) \quad (n \to \infty), \tag{5.8}$$

$$(\boldsymbol{j}\boldsymbol{u}_n)(t) \overset{n\to\infty}{\rightharpoonup} (\boldsymbol{j}\boldsymbol{u})(t) \qquad \text{in } Y \quad \text{for a.e. } t \in I. \tag{5.9}$$

In addition, from (5.6) we infer

$$\limsup_{n\to\infty} \langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} = \tau \limsup_{n\to\infty} \langle [A]_k^\tau u_n, u_n - u\rangle_X \leq 0. \tag{5.10}$$

Note that the constant approximation $V_n = X$, $n \in \mathbb{N}$, is trivially a quasi non-conforming approximation of X in X (cf. Remark 6.5 (i)). Thus, Proposition 4.6 yields that the induced operator $\boldsymbol{\mathcal{A}} : L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y) \to (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$ is quasi non-conforming Bochner pseudo-monotone with respect to $V_n = X$, $n \in \mathbb{N}$. Consequently, we obtain from (5.7)–(5.10) that for all $\boldsymbol{v} \in L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y)$ there holds

$$\langle \boldsymbol{\mathcal{A}}\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \leq \liminf_{n\to\infty} \langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)}. \tag{5.11}$$

If we choose in (5.11) $\boldsymbol{v} := v\chi_{I_k^\tau} \in L^\infty(I, X)$ with $v \in X$ and divide by $\tau > 0$, we conclude

$$\langle [A]_k^\tau u, u - v\rangle_X \leq \liminf_{n\to\infty} \langle [A]_k^\tau u_n, u_n - v\rangle_X.$$

In other words, $[A]_k^\tau : X \to X^*$ is pseudo-monotone.

ad (ii) The assertion follows immediately from **(i)** and the definition of $\mathscr{J}_\tau[A](t)$, $t \in I$.

ad (iii) Since $\mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]$ is the induced operator of the family of operators $\mathscr{J}_\tau[A](t)$, $t \in I$, the well-definiteness and boundedness of $\mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]$ results from **(ii)** in conjunction with Proposition 4.6 applied again in the trivial setting of the constant approximation $V_n = X$, $n \in \mathbb{N}$.

ad (iii.a) Let $\boldsymbol{u}_\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$ and $\boldsymbol{v} \in L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y)$. Then, using for every $t, s \in I_k^\tau$, $k = 1,\ldots,K$, that $\langle A(s)(\boldsymbol{u}_\tau(t)), \boldsymbol{v}(t)\rangle_X = \langle A(s)(\boldsymbol{u}_\tau(s)), \boldsymbol{v}(t)\rangle_X$ and Fubini's theorem, we infer

$$\langle \mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]\boldsymbol{u}_\tau, \boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} = \int_I \langle \mathscr{J}_\tau[A](t)(\boldsymbol{u}(t)), \boldsymbol{v}(t)\rangle_X \,dt$$

$$
\begin{aligned}
&= \sum_{k=1}^{K} \int_{I_k^\tau} \Big\langle \fint_{I_k^\tau} A(s)(\boldsymbol{u}_\tau(t))\,ds, \boldsymbol{v}(t)\Big\rangle_X dt \\
&= \sum_{k=1}^{K} \int_{I_k^\tau} \Big\langle A(s)(\boldsymbol{u}_\tau(s)), \fint_{I_k^\tau} \boldsymbol{v}(t)\,dt\Big\rangle_X ds \\
&= \int_I \langle A(s)(\boldsymbol{u}(s)), \mathscr{J}_\tau[\boldsymbol{v}](s)\rangle_X\, ds \\
&= \langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_\tau, \mathscr{J}_\tau[\boldsymbol{v}]\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)}.
\end{aligned}
$$

ad (iii.b) Let the family $\boldsymbol{u}_\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$, $\tau > 0$, be bounded in $L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y)$. Then, by the boundedness of $\boldsymbol{\mathcal{A}} : L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y) \to (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$ (cf. Proposition 4.6), the family $(\boldsymbol{\mathcal{A}}\boldsymbol{u}_\tau)_{\tau>0} \subseteq (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$ is bounded as well. Therefore, also using **(iii.a)**, we conclude for every $\boldsymbol{v} \in L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y)$ that

$$
\langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_\tau - \mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]\boldsymbol{u}_\tau, \boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} = \langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_\tau, \boldsymbol{v} - \mathscr{J}_\tau[\boldsymbol{v}]\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \quad\to\quad 0 \quad (\tau \to 0),
$$

where we also used Proposition 5.3 (i).

ad (iii.c) Using **(iii.a)** and Proposition 5.3 (ii), we deduce

$$
\begin{aligned}
\|\mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]\boldsymbol{u}_\tau\|_{(L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*} &= \sup_{\|\boldsymbol{v}\|_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \le 1} \langle \mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]\boldsymbol{u}_\tau, \boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \\
&= \sup_{\|\boldsymbol{v}\|_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \le 1} \langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_\tau, \mathscr{J}_\tau[\boldsymbol{v}]\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \\
&\le \|\boldsymbol{\mathcal{A}}\boldsymbol{u}_\tau\|_{(L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*}.
\end{aligned}
$$

□

6 Fully discrete, quasi non-conforming approximation

In this section we formulate the exact framework of a quasi non-conforming Rothe–Galerkin approximation, prove its well-posedness, i.e., the existence of iterates, and its stability, i.e., the boundedness of the corresponding double sequence of piece-wise constant interpolants. Moreover, we prove the main result of this paper, Theorem 6.16, which shows the weak convergence of a diagonal subsequence towards a weak solution of problem (1.1).

Assumption 6.1 Let $I := (0,T)$, $T < \infty$ and $1 < p \le q < \infty$. We make the following assumptions:

(i) **Spaces:** (V,H,j) and (X,Y,j) are as in Definition 3.1 and $(V_n)_{n\in\mathbb{N}}$ is a quasi non-conforming approximation of V in X.
(ii) **Initial data:** $u_0 \in H$ and there is a sequence $u_n^0 \in V_n$, $n \in \mathbb{N}$, such that $u_n^0 \to u_0$ in Y $(n \to \infty)$ and $\sup_{n\in\mathbb{N}} \|ju_n^0\|_Y \le \|u_0\|_H$.[7]
(iii) **Right-hand side**: $\boldsymbol{f} \in L^{p'}(I,X^*)$.
(iv) **Operators**: $A(t) : X \to X^*$, $t \in I$, is a family of operators satisfying (A.1)–(A.4) and $\boldsymbol{\mathcal{A}} : L^p(I,X) \cap_{\boldsymbol{j}} L^\infty(I,Y) \to (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$ the corresponding induced operator.

[7] For a quasi non-conforming approximation Proposition 3.3 guarantees the existence of such a sequence.

Furthermore, we set $H_n := j(V_n) \subseteq Y$ equipped with $(\cdot,\cdot)_Y$, denote by $j_n : V_n \to H_n$ the restriction of j to V_n and by $R_n : H_n \to H_n^*$ the corresponding Riesz isomorphism with respect to $(\cdot,\cdot)_Y$. As j_n is an isomorphism, the triple (V_n, H_n, j_n) is an evolution triple with canonical embedding $e_n := j_n^* R_n j_n : V_n \to V_n^*$, which satisfies

$$\langle e_n v_n, w_n\rangle_{V_n} = (j_n v_n, j_n w_n)_Y \quad \text{for all } v_n, w_n \in V_n. \tag{6.2}$$

Putting all together leads us to the following algorithm:

Algorithm 6.3 (Quasi non-conforming Rothe–Galerkin scheme) Let Assumption (6.1) be satisfied. For given $K, n \in \mathbb{N}$ the sequence of iterates $(u_n^k)_{k=0,\dots,K} \subseteq V_n$ is given solving the implicit Rothe–Galerkin scheme for $\tau = \frac{T}{K}$ and $k = 1,\dots,K$

$$(d_\tau j u_n^k, j v_n)_Y + \langle [A]_k^\tau u_n^k, v_n\rangle_X = \langle [\boldsymbol{f}]_k^\tau, v_n\rangle_X \quad \text{for all } v_n \in V_n. \tag{6.4}$$

Remark 6.5 Note that the Rothe–Galerkin scheme (6.4) also covers pure Rothe schemes, i.e., without spatial approximation, and fully discrete conforming approximations:

(i) If $X = V$, $Y = H$, and $V_n = X$, $n \in \mathbb{N}$, then (6.4) is a pure Rothe scheme.
(ii) If $X = V$, $Y = H$, and the closed subspaces $(V_n)_{n\in\mathbb{N}}$ satisfy (C.1)–(C.2), then (6.4) is a conforming Rothe–Galerkin scheme.

Proposition 6.6 (Well-posedness of (6.4)) Let Assumption (6.1) be satisfied and set $\tau_0 := \frac{1}{4c_1}$. Then, for all $K, n \in \mathbb{N}$ with $\tau = \frac{T}{K} < \tau_0$ there exist iterates $(u_n^k)_{k=1,\dots,K} \subseteq V_n$, solving (6.4).

Proof Using (6.2) and the identity mapping $\mathrm{id}_{V_n} : V_n \to X$, we see that (6.4) is equivalent to

$$(\mathrm{id}_{V_n})^*\big([\boldsymbol{f}]_k^\tau\big) + \frac{1}{\tau} e_n u_n^{k-1} \in R\Big(\frac{1}{\tau} e_n + (\mathrm{id}_{V_n})^* \circ [A]_k^\tau \circ \mathrm{id}_{V_n}\Big), \quad \text{for all } k = 1,\dots,K. \tag{6.7}$$

We fix an arbitrary $k = 1,\dots,K$. Apparently, $\frac{1}{\tau} e_n : V_n \to V_n^*$ is linear and continuous. Using (6.2), we infer that $\langle \frac{1}{\tau} e_n u, u\rangle_{V_n} = \frac{1}{\tau}\|j_n u\|_Y^2 \geq 0$ for all $u \in V_n$, i.e., $\frac{1}{\tau} e_n : V_n \to V_n^*$ is positive definite, and thus monotone. In consequence, $\frac{1}{\tau} e_n : V_n \to V_n^*$ is pseudo-monotone. Since the conditions (A.1)–(A.4) are inherited from $A : X \to X^*$ to $(\mathrm{id}_{V_n})^* \circ A \circ \mathrm{id}_{V_n} : V_n \to V_n^*$ and since $(\mathrm{id}_{V_n})^* \circ [A]_k^\tau \circ \mathrm{id}_{V_n} = [(\mathrm{id}_{V_n})^* \circ A \circ \mathrm{id}_{V_n}]_k^\tau$, Proposition 5.4 (i) guarantees that the operator $(\mathrm{id}_{V_n})^* \circ [A]_\tau^k \circ \mathrm{id}_{V_n} : V_n \to V_n^*$ is bounded and pseudo-monotone. Altogether, we conclude that the sum $\frac{1}{\tau} e_n + (\mathrm{id}_{V_n})^* \circ [A]_\tau^k \circ \mathrm{id}_{V_n} : V_n \to V_n^*$ is bounded and pseudo-monotone. In addition, as $\tau < \frac{1}{2c_1}$, combining (6.2) and Proposition 5.4 (i.b), provides for all $u \in V_n$

$$\Big\langle \Big(\frac{1}{\tau} e_n + (\mathrm{id}_{V_n})^* \circ [A]_k^\tau \circ \mathrm{id}_{V_n}\Big) u, u\Big\rangle_{V_n} \geq 3c_1\|j_n u\|_Y^2 + c_0\|u\|_X^p - c_2,$$

i.e., $\frac{1}{\tau} e_n + (\mathrm{id}_{V_n})^* \circ [A]_k^\tau \circ \mathrm{id}_{V_n} : V_n \to V_n^*$ is coercive. Hence, Proposition 2.2 proves (6.7). □

Proposition 6.8 (Stability of (6.4)) Let Assumption (6.1) be satisfied and set $\tau_0 := \frac{1}{4c_1}$. Then, there exists a constant $M > 0$ (not depending on $K, n \in \mathbb{N}$), such that the piece-wise constant interpolants $\overline{\boldsymbol{u}}_n^\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$, $K, n \in \mathbb{N}$ with $\tau = \frac{T}{K} \in (0, \tau_0)$, and piece-wise affine interpolants

$\hat{\boldsymbol{u}}_n^\tau \in W^{1,\infty}(I,X)$, $n \in \mathbb{N}$, $\tau \in (0,\tau_0)$ (cf. (5.1)) generated by iterates $(u_n^k)_{k=0,\dots,K} \subseteq V_n$, $K, n \in \mathbb{N}$ with $\tau = \frac{T}{K} \in (0,\tau_0)$, solving (6.4), satisfy the following estimates:

$$\|\overline{\boldsymbol{u}}_n^\tau\|_{L^p(I,X)\cap_{\boldsymbol{j}} L^\infty(I,Y)} \le M, \tag{6.9}$$

$$\|\boldsymbol{j}\hat{\boldsymbol{u}}_n^\tau\|_{L^\infty(I,Y)} \le M, \tag{6.10}$$

$$\|\boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}_n^\tau\|_{(L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*} \le M, \tag{6.11}$$

$$\|e_n(\hat{\boldsymbol{u}}_n^\tau - \overline{\boldsymbol{u}}_n^\tau)\|_{L^{q'}(I,V_n^*)} \le \tau\big(\|\boldsymbol{f}\|_{L^{p'}(I,X^*)} + M\big). \tag{6.12}$$

Proof We use $v_n = u_n^k \in V_n$, $k = 1,\dots,l$, for arbitrary $l = 1,\dots,K$ in (6.4), multiply by $\tau \in (0,\tau_0)$, sum with respect to $k = 1,\dots,l$, use (5.2) and $\sup_{n\in\mathbb{N}} \|ju_n^0\|_Y \le \|u_0\|_H$, to obtain for every $l = 1,\dots,K$

$$\frac{1}{2}\|ju_n^l\|_Y^2 + \sum_{k=1}^{l} \tau\langle [A]_\tau^k u_n^k, u_n^k\rangle_X \le \frac{1}{2}\|u_0\|_H^2 + \sum_{k=1}^{l} \tau\langle [\boldsymbol{f}]_\tau^k, u_n^k\rangle_X. \tag{6.13}$$

Applying the weighted ε-Young inequality with constant $c(\varepsilon) := (p\varepsilon)^{1-p'}/p'$ for all $\varepsilon > 0$, using $\|\mathscr{J}_\tau[\boldsymbol{f}]\|_{L^{p'}(I,X^*)} \le \|\boldsymbol{f}\|_{L^{p'}(I,X^*)}$ (cf. Proposition 5.3 (ii)), we deduce for every $l = 1,\dots,K$

$$\sum_{k=1}^{l} \tau\langle [\boldsymbol{f}]_\tau^k, u_n^k\rangle_X = \langle \mathscr{J}_\tau[\boldsymbol{f}], \overline{\boldsymbol{u}}_n^\tau \chi_{[0,l\tau]}\rangle_{L^p(I,X)} \le c(\varepsilon)\|\boldsymbol{f}\|_{L^{p'}(I,X^*)}^{p'} + \varepsilon\int_0^{l\tau} \|\overline{\boldsymbol{u}}_n^\tau(s)\|_X^p\, ds.$$

In addition, using Proposition 5.4 (i.b), we obtain for every $l = 1,\dots,K$

$$\sum_{k=1}^{l} \tau\langle [A]_\tau^k u_n^k, u_n^k\rangle_X \ge c_0\int_0^{l\tau} \|\overline{\boldsymbol{u}}_n^\tau(s)\|_X^p\, ds - \tau c_1\|ju_n^l\|_Y^2 - \sum_{k=1}^{l-1} \tau c_1\|ju_n^k\|_Y^2 - c_2 T. \tag{6.14}$$

We set $\varepsilon := \frac{c_0}{2}$, $\alpha := \frac{1}{2}\|u_0\|_H^2 + c(\varepsilon)\|\boldsymbol{f}\|_{L^{p'}(I,X^*)}^{p'} + c_2 T$, $\beta := 4\tau c_1 < 1$ and $y_n^k := \frac{1}{4}\|ju_n^k\|_Y^2$ for $k = 1,\dots,K$. Thus, we infer for every $l = 1,\dots,K$ from (6.13), (6.14) that

$$y_n^l + \frac{c_0}{2}\int_0^{l\tau} \|\overline{\boldsymbol{u}}_n^\tau(s)\|_X^p\, ds \le \alpha + \beta\sum_{k=1}^{l-1} y_n^k. \tag{6.15}$$

The discrete Gronwall inequality applied on (6.15) yields

$$\frac{1}{4}\|\boldsymbol{j}\overline{\boldsymbol{u}}_n^\tau\|_{L^\infty(I,Y)}^2 + \frac{c_0}{2}\|\overline{\boldsymbol{u}}_n^\tau\|_{L^p(I,X)}^p \le \alpha\exp(K\beta) = \alpha\exp(4Tc_1) =: C_0,$$

which proves (6.9). From the boundedness of $\boldsymbol{\mathcal{A}} : L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y) \to (L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*$ (cf. Proposition 4.6) and (6.9) we infer $\|\boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}_n^\tau\|_{(L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*} \le C_1$ for some $C_1 > 0$, i.e., (6.11). In addition, it holds $\|\boldsymbol{j}\hat{\boldsymbol{u}}_n^\tau\|_{L^\infty(I,Y)}^2 \le \|\boldsymbol{j}\overline{\boldsymbol{u}}_n^\tau\|_{L^\infty(I,Y)}^2 \le 4C_0$ for every $n \in \mathbb{N}$ and $\tau \in (0,\tau_0)$, i.e., (6.10). Moreover, since $e_n\big(\hat{\boldsymbol{u}}_n^\tau(t) - \overline{\boldsymbol{u}}_n^\tau(t)\big) = (t - k\tau)d_\tau e_n\hat{\boldsymbol{u}}_n^\tau(t) = (t - k\tau)\frac{d_{e_n}\hat{\boldsymbol{u}}_n^\tau}{dt}(t)$ in V_n^* and $|t - k\tau| \le \tau$ for every $t \in I_k^\tau$, $k = 1,\dots,K$, $K, n \in \mathbb{N}$, there holds for every $n \in \mathbb{N}$ and $\tau \in (0,\tau_0)$

$$\begin{aligned}\big\|e_n\big(\hat{\boldsymbol{u}}_n^\tau - \overline{\boldsymbol{u}}_n^\tau\big)\big\|_{L^{q'}(I,V_n^*)} &\le \tau\left\|\frac{d_{e_n}\hat{\boldsymbol{u}}_n^\tau}{dt}\right\|_{L^{q'}(I,V_n^*)}\\ &= \tau\big\|(\mathrm{id}_{L^q(I,V_n)})^*\big(\mathscr{J}_\tau[\boldsymbol{f}] - \mathscr{J}_\tau[\boldsymbol{\mathcal{A}}]\overline{\boldsymbol{u}}_n^\tau\big)\big\|_{L^{q'}(I,V_n^*)} \le \tau\big(\|\boldsymbol{f}\|_{L^{p'}(I,X^*)} + C_1\big),\end{aligned}$$

i.e., the estimate (6.12), where we used Proposition 5.3 (ii) and Proposition 5.4 (iii.c). $\square$

We can now prove the abstract convergence result, which is the main result of this paper.

Theorem 6.16 Let Assumption (6.1) be satisfied and set $\tau_0 := \frac{1}{4c_1}$. If $(\overline{\boldsymbol{u}}_n)_{n\in\mathbb{N}} := (\overline{\boldsymbol{u}}^{\tau_n}_{m_n})_{n\in\mathbb{N}} \subseteq L^\infty(I,X)$, where $\tau_n = \frac{T}{K_n}$ and $K_n, m_n \to \infty$ $(n\to\infty)$, is an arbitrary diagonal sequence of piece-wise constant interpolants $\overline{\boldsymbol{u}}^\tau_n \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$, $K, n \in \mathbb{N}$ with $\tau = \frac{T}{K} \in (0,\tau_0)$, from Proposition 6.8. Then, there exists a not relabelled subsequence and a weak limit $\overline{\boldsymbol{u}} \in L^p(I,V) \cap_{\boldsymbol{j}} L^\infty(I,H)$ such that

$$\begin{aligned} \overline{\boldsymbol{u}}_n &\rightharpoonup \overline{\boldsymbol{u}} \quad \text{in } L^p(I,X), \\ \overline{\boldsymbol{u}}_n &\overset{*}{\rightharpoondown} \overline{\boldsymbol{u}} \quad \text{in } L^\infty(I,Y), \end{aligned} \qquad (n\to\infty).$$

Furthermore, it follows that $\overline{\boldsymbol{u}} \in \boldsymbol{\mathcal{W}}^{1,p,q}_e(I,V,H)$ is a weak solution of the initial value problem (1.1).

Proof We split the proof into four steps:

1. Convergences: From the estimates (6.9)–(6.12), the reflexivity of $L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)$, also using Proposition 5.4 (iii.b), we obtain not relabelled subsequences $(\overline{\boldsymbol{u}}_n)_{n\in\mathbb{N}}, (\hat{\boldsymbol{u}}_n)_{n\in\mathbb{N}} \subseteq L^p(I,X)\cap_{\boldsymbol{j}} L^\infty(I,Y)$, where $\hat{\boldsymbol{u}}_n := \hat{\boldsymbol{u}}^{\tau_n}_{m_n}$ for all $n\in\mathbb{N}$, as well as $\overline{\boldsymbol{u}} \in L^p(I,X)\cap_{\boldsymbol{j}} L^\infty(I,Y)$, $\boldsymbol{j}\hat{\boldsymbol{u}} \in L^\infty(I,Y)$ and $\boldsymbol{\chi} \in (L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^*$ such that

$$\begin{aligned} \overline{\boldsymbol{u}}_n &\rightharpoonup \overline{\boldsymbol{u}} && \text{in } L^p(I,X) \quad (n\to\infty), \\ \boldsymbol{j}\overline{\boldsymbol{u}}_n &\overset{*}{\rightharpoondown} \boldsymbol{j}\overline{\boldsymbol{u}} && \text{in } L^\infty(I,Y) \quad (n\to\infty), \\ \boldsymbol{j}\hat{\boldsymbol{u}}_n &\overset{*}{\rightharpoondown} \boldsymbol{j}\hat{\boldsymbol{u}} && \text{in } L^\infty(I,Y) \quad (n\to\infty), \\ \mathscr{J}_{\tau_n}[\boldsymbol{\mathcal{A}}]\overline{\boldsymbol{u}}_n &\rightharpoonup \boldsymbol{\chi} && \text{in } (L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^* \quad (n\to\infty), \\ \boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}_n &\rightharpoonup \boldsymbol{\chi} && \text{in } (L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y))^* \quad (n\to\infty). \end{aligned} \tag{6.17}$$

From (QNC.2) we immediately obtain that $\overline{\boldsymbol{u}} \in L^p(I,V)\cap_{\boldsymbol{j}} L^\infty(I,H)$. In particular, there exists $\boldsymbol{g} \in L^{p'}(I,V^*) + \boldsymbol{j}^*(L^{q'}(I,H^*))$ (cf. (2.4)), such that for every $\boldsymbol{v} \in L^p(I,V)\cap_{\boldsymbol{j}} L^q(I,H)$

$$\langle \boldsymbol{\chi}, \boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} = \int_I \langle \boldsymbol{g}(t), \boldsymbol{v}(t)\rangle_V \, dt. \tag{6.18}$$

Due to (6.12) there exists a subset $E \subset I$, with $I \setminus E$ a null set, such that for every $t \in E$

$$\|e_{m_n}(\hat{\boldsymbol{u}}_n(t) - \overline{\boldsymbol{u}}_n(t))\|_{V^*_{m_n}} \;\to\; 0 \quad (n\to\infty). \tag{6.19}$$

Owing to (QNC.1) we can choose for every element v of the dense subset $C \subseteq V$ a sequence $v_n \in V_{m_n}$, $n\in\mathbb{N}$, such that $v_n \to v$ in X $(n\to\infty)$. Then, using the definition of P_H, (6.2), (6.9), (6.10) and (6.19), we infer for every $t\in E$ that

$$\begin{aligned} |(P_H[(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t) - (\boldsymbol{j}\overline{\boldsymbol{u}}_n)(t)], jv)_H| &= |((\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t) - (\boldsymbol{j}\overline{\boldsymbol{u}}_n)(t), jv)_Y| \\ &\le |\langle e_{m_n}[(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t) - (\boldsymbol{j}\overline{\boldsymbol{u}}_n)(t)], v_n\rangle_{V_{m_n}}| + |((\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t) - (\boldsymbol{j}\overline{\boldsymbol{u}}_n)(t), jv - jv_n)_Y| \\ &\le \|e_{m_n}[\hat{\boldsymbol{u}}_n(t) - \overline{\boldsymbol{u}}_n(t)]\|_{V^*_{m_n}} \|v_n\|_X + \|(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t) - (\boldsymbol{j}\overline{\boldsymbol{u}}_n)(t)\|_Y \|jv - jv_n\|_Y \\ &\le \|e_{m_n}[\hat{\boldsymbol{u}}_n(t) - \overline{\boldsymbol{u}}_n(t)]\|_{V^*_{m_n}} \|v_n\|_X + 2M\|jv - jv_n\|_Y \;\to\; 0 \quad (n\to\infty). \end{aligned} \tag{6.20}$$

Since C is dense in V and $j(V)$ is dense in H, we conclude from (6.20) for every $t\in E$ that

$$P_H[(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t) - (\boldsymbol{j}\overline{\boldsymbol{u}}_n)(t)] \;\rightharpoonup\; 0 \quad \text{in } H \quad (n\to\infty). \tag{6.21}$$

Since the sequences $(P_H\boldsymbol{j}\overline{\boldsymbol{u}}_n)_{n\in\mathbb{N}}, (P_H\boldsymbol{j}\hat{\boldsymbol{u}})_{n\in\mathbb{N}} \subseteq L^\infty(I,H)$ are bounded (cf. (6.9) and (6.10)), [35, Proposition 2.15] yields, due to (6.21), that $P_H(\boldsymbol{j}\hat{\boldsymbol{u}}_n - \boldsymbol{j}\overline{\boldsymbol{u}}_n) \rightharpoonup \mathbf{0}$ in $L^q(I,H)$ $(n\to\infty)$. From $(6.17)_{2,3}$ we easily deduce that $P_H(\boldsymbol{j}\hat{\boldsymbol{u}}_n - \boldsymbol{j}\overline{\boldsymbol{u}}_n) \rightharpoonup P_H(\boldsymbol{j}\hat{\boldsymbol{u}} - \boldsymbol{j}\overline{\boldsymbol{u}})$ in $L^q(I,H)$ $(n\to\infty)$. Thus, $P_H(\boldsymbol{j}\hat{\boldsymbol{u}}) = P_H(\boldsymbol{j}\overline{\boldsymbol{u}}) = \boldsymbol{j}\overline{\boldsymbol{u}}$ in $L^\infty(I,H)$, where we used that $\overline{\boldsymbol{u}} \in L^p(I,V)\cap_{\boldsymbol{j}} L^\infty(I,H)$.

2. Regularity and trace of the weak limit: Let $v\in C$ and $v_n \in V_{m_n}$, $n\in\mathbb{N}$, be a sequence such that $v_n \to v$ in X $(n\to\infty)$. Testing (6.4) for $n\in\mathbb{N}$ by $v_n\in V_{m_n}$, multiplication by $\varphi\in C^\infty(\overline{I})$ with $\varphi(T)=0$, integration over I, and integration by parts yields for every $n\in\mathbb{N}$

$$\begin{aligned}\langle \mathscr{I}_{\tau_n}[\boldsymbol{\mathcal{A}}]\overline{\boldsymbol{u}}_n, v_n\varphi\rangle_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)} &- \int_I \langle \mathscr{I}_{\tau_n}[\boldsymbol{f}](s), v_n\rangle_X \varphi(s)\,ds \\ &= \int_I ((\boldsymbol{j}\hat{\boldsymbol{u}}_n)(s), jv_n)_Y \varphi'(s)\,ds + (u^0_{m_n}, jv_n)_Y\varphi(0).\end{aligned} \tag{6.22}$$

By passing in (6.22) for $n\to\infty$, using (6.17), (6.18), Proposition 5.3 (i), $P_H(\boldsymbol{j}\hat{\boldsymbol{u}}) = \boldsymbol{j}\overline{\boldsymbol{u}}$ in $L^\infty(I,H)$, $u^0_{m_n}\to u_0$ in Y $(n\to\infty)$ and the density of C in V, we obtain that for all $v\in V$ and $\varphi\in C^\infty(\overline{I})$ with $\varphi(T)=0$ there holds

$$\begin{aligned}\int_I \langle \boldsymbol{g}(s) - \boldsymbol{f}(s), v\rangle_V \varphi(s)\,ds &= \int_I ((\boldsymbol{j}\hat{\boldsymbol{u}})(s), jv)_Y\varphi'(s)\,ds + (u_0, jv)_Y\varphi(0) \\ &= \int_I ((\boldsymbol{j}\overline{\boldsymbol{u}})(s), jv)_H\varphi'(s)\,ds + (u_0, jv)_H\varphi(0).\end{aligned} \tag{6.23}$$

In the case $\varphi\in C_0^\infty(I)$ in (6.23), recalling Definition 2.5, we conclude that $\overline{\boldsymbol{u}}\in\boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$ with continuous representation $\boldsymbol{j}_c\overline{\boldsymbol{u}}\in C^0(\overline{I},H)$ and

$$\frac{d_e\overline{\boldsymbol{u}}}{dt} = \boldsymbol{f} - \boldsymbol{g} \quad \text{in } L^{p'}(I,V^*) + \boldsymbol{j}^*(L^{q'}(I,H^*)). \tag{6.24}$$

Thus, we are able to apply the generalized integration by parts formula in $\boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$ (cf. Proposition 2.6) in (6.23) in the case $\varphi\in C^\infty(\overline{I})$ with $\varphi(T)=0$ and $\varphi(0)=1$, which yields for all $v\in V$

$$((\boldsymbol{j}_c\overline{\boldsymbol{u}})(0) - u_0, jv)_H = 0.$$

As $j(V)$ is dense in H and $(\boldsymbol{j}_c\overline{\boldsymbol{u}})(0)\in H$, we deduce from (6.24) that $(\boldsymbol{j}_c\overline{\boldsymbol{u}})(0) = u_0$ in H.

3. Pointwise weak convergence: Next, we show that $P_H(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t) \rightharpoonup (\boldsymbol{j}_c\overline{\boldsymbol{u}})(t)$ in H $(n\to\infty)$ for all $t\in\overline{I}$, which due to (6.21) in turn yields that $P_H(\boldsymbol{j}\overline{\boldsymbol{u}}_n)(t)\rightharpoonup(\boldsymbol{j}\overline{\boldsymbol{u}})(t)$ in H $(n\to\infty)$ for almost every $t\in\overline{I}$. To this end, let us fix an arbitrary $t\in I$. From the a-priori estimate $\|(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t)\|_Y\le M$ for all $t\in\overline{I}$ and $n\in\mathbb{N}$ (cf. (6.10)) we obtain a subsequence $((\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t))_{n\in\Lambda_t}\subseteq Y$ with $\Lambda_t\subseteq\mathbb{N}$, initially depending on this fixed t, and an element $\hat{\boldsymbol{u}}_{\Lambda_t}\in Y$ such that

$$(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t) \quad\rightharpoonup\quad \hat{\boldsymbol{u}}_{\Lambda_t} \quad \text{in } Y \quad (\Lambda_t\ni n\to\infty). \tag{6.25}$$

Let $v\in C$ and $v_n\in V_{m_n}$, $n\in\mathbb{N}$, be such that $v_n\to v$ in X $(n\to\infty)$. Then, we test (6.4) for $n\in\Lambda_t$ by $v_n\in V_{m_n}$, multiply by $\varphi\in C^\infty(\overline{I})$ with $\varphi(0)=0$ and $\varphi(t)=1$, integrate over $[0,t]$ and integrate by parts, to obtain for all $n\in\Lambda_t$

$$\begin{aligned}&\big\langle \mathscr{I}_{\tau_n}[\boldsymbol{\mathcal{A}}]\overline{\boldsymbol{u}}_n, v_n\varphi\chi_{[0,t]}\big\rangle_{L^p(I,X)\cap_{\boldsymbol{j}}L^q(I,Y)} - \int_0^t \big\langle \mathscr{I}_{\tau_n}[\boldsymbol{f}](s), v_n\big\rangle_X\varphi(s)\,ds \\ &= \int_0^t ((\boldsymbol{j}\hat{\boldsymbol{u}}_n)(s), jv_n)_Y\varphi'(s)\,ds - ((\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t), jv_n)_Y.\end{aligned} \tag{6.26}$$

By passing in (6.26) for $n \in \Lambda_t$ to infinity, using (6.17), (6.18), Proposition 5.3 (i), (6.25) and the density of C in V, we obtain for all $v \in V$

$$\int_0^t \langle \boldsymbol{g}(s) - \boldsymbol{f}(s), v\rangle_V \varphi(s)\, ds = \int_0^t ((\boldsymbol{j}\overline{\boldsymbol{u}})(s), jv)_H \varphi'(s)\, ds - (\hat{\boldsymbol{u}}_{\Lambda_t}, jv)_Y. \tag{6.27}$$

From (6.24), (6.27), the integration by parts formula in $\boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$ and the properties of P_H we obtain

$$0 = ((\boldsymbol{j}_c\overline{\boldsymbol{u}})(t) - \hat{\boldsymbol{u}}_{\Lambda_t}, jv)_Y = ((\boldsymbol{j}_c\overline{\boldsymbol{u}})(t) - P_H\hat{\boldsymbol{u}}_{\Lambda_t}, jv)_H, \tag{6.28}$$

for all $v \in V$. Thanks to the density of $j(V)$ in H, (6.28) yields $(\boldsymbol{j}_c\overline{\boldsymbol{u}})(t) = P_H\hat{\boldsymbol{u}}_{\Lambda_t}$ in H, i.e.,

$$P_H(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t) \quad \rightharpoonup \quad (\boldsymbol{j}_c\overline{\boldsymbol{u}})(t) \quad \text{ in } H \quad (\Lambda_t \ni n \to \infty). \tag{6.29}$$

As this argumentation remains valid for each subsequence of $(P_H(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t))_{n\in\mathbb{N}} \subseteq H$, the element $(\boldsymbol{j}_c\overline{\boldsymbol{u}})(t) \in H$ is a weak accumulation point of each subsequence of $(P_H(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(t))_{n\in\mathbb{N}} \subseteq H$. The standard convergence principle (cf. [28, Kap. I, Lemma 5.4]) yields $\Lambda_t = \mathbb{N}$ in (6.29). Therefore, using (6.21) and that $(\boldsymbol{j}_c\overline{\boldsymbol{u}})(t) = (\boldsymbol{j}\boldsymbol{u})(t)$ in H for almost every $t \in I$, there holds for almost every $t \in I$

$$P_H(\boldsymbol{j}\overline{\boldsymbol{u}}_n)(t) \quad \rightharpoonup \quad (\boldsymbol{j}\overline{\boldsymbol{u}})(t) \quad \text{ in } H \quad (n \to \infty). \tag{6.30}$$

4. Identification of $\boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}$ and $\boldsymbol{\chi}$: Inequality (6.13) in the case $\tau = \tau_n$, $n = m_n$ and $l = K_n$, using Proposition 5.4 (iii.a), $(\boldsymbol{j}_c\overline{\boldsymbol{u}})(0) = u_0$ in H, $\|P_H(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(T)\|_H \le \|(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(T)\|_Y = \|(\boldsymbol{j}\overline{\boldsymbol{u}}_n)(T)\|_Y$ and $\langle \mathscr{I}_{\tau_n}[\boldsymbol{f}], \overline{\boldsymbol{u}}_n\rangle_{L^p(I,X)} = \langle \boldsymbol{f}, \overline{\boldsymbol{u}}_n\rangle_{L^p(I,X)}$ for all $n \in \mathbb{N}$, yields for all $n \in \mathbb{N}$

$$\langle \boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}_n, \overline{\boldsymbol{u}}_n\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \le -\frac{1}{2}\|P_H(\boldsymbol{j}\hat{\boldsymbol{u}}_n)(T)\|_H^2 + \frac{1}{2}\|(\boldsymbol{j}_c\overline{\boldsymbol{u}})(0)\|_H^2 + \langle \boldsymbol{f}, \overline{\boldsymbol{u}}_n\rangle_{L^p(I,X)}. \tag{6.31}$$

Thus, the limit superior with respect to $n \in \mathbb{N}$ on both sides in (6.31), (6.17), (6.18), (6.29) with $\Lambda_t = \mathbb{N}$ in the case $t = T$, the weak lower semi-continuity of $\|\cdot\|_H$, the integration by parts formula in $\boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$ and (6.24) yield

$$\begin{aligned}\limsup_{n\to\infty} \langle \boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}_n, \overline{\boldsymbol{u}}_n - \overline{\boldsymbol{u}}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} &\le -\frac{1}{2}\|(\boldsymbol{j}_c\overline{\boldsymbol{u}})(T)\|_H^2 + \frac{1}{2}\|(\boldsymbol{j}_c\overline{\boldsymbol{u}})(0)\|_H^2 \\ &\quad + \int_I \langle \boldsymbol{f}(t) - \boldsymbol{g}(t), \overline{\boldsymbol{u}}(t)\rangle_V\, dt \\ &= -\int_I \Big\langle \frac{d_e\overline{\boldsymbol{u}}}{dt}(t) + \boldsymbol{f}(s) - \boldsymbol{g}(s), \overline{\boldsymbol{u}}(t)\Big\rangle_V\, dt = 0.\end{aligned} \tag{6.32}$$

As a result of (6.17), (6.30), (6.32) and the quasi non-conforming Bochner pseudo-monotonicity of $\boldsymbol{\mathcal{A}} : L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y) \to (L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$ (cf. Proposition 4.6), there holds

$$\begin{aligned}\langle \boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}} - \boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} &\le \liminf_{n\to\infty} \langle \boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}}_n, \overline{\boldsymbol{u}}_n - \boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)} \\ &\le \langle \boldsymbol{\chi}, \overline{\boldsymbol{u}} - \boldsymbol{v}\rangle_{L^p(I,X)\cap_{\boldsymbol{j}} L^q(I,Y)},\end{aligned}$$

for any $\boldsymbol{v} \in L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y)$, which in turn implies that $\boldsymbol{\mathcal{A}}\overline{\boldsymbol{u}} = \boldsymbol{\chi}$ in $(L^p(I,X) \cap_{\boldsymbol{j}} L^q(I,Y))^*$. This completes the proof of Theorem 6.16. □

7 Applications

In this section, we apply the abstract theory developed in the previous sections to two problems stemming from incompressible non-Newtonian fluid flows. In particular, we treat the motion of micropolar electrorheological fluids and a variant of the Smagorinsky model in turbulence. In both cases we show that solutions of a fully discrete implicit Rothe–Galerkin scheme converge to a weak solution of the corresponding problem. These results are new and, to the best of the authors' knowledge, can not be found in the literature. We restrict our treatment to the three-dimensional case. All results have an analogue in the two-dimensional setting (for a formulation of (7.1) in two dimensions see (8.1) in Section 8).

In this section we always assume that we have given a family of shape regular triangulations (cf. [13]) $(\mathcal{T}_h)_{h>0}$ of a bounded polygonal Lipschitz domain $\Omega \subset \mathbb{R}^3$, and that $I = (0, T)$ is a finite time interval.

In order to formulate the problems we need some additional notation. We denote by $\boldsymbol{\varepsilon}$ the anti-symmetric Levi–Civita symbol. For a tensor P and a vector ω, resp., we denote by $\boldsymbol{\varepsilon} : P$ the vector having the components $\varepsilon_{ijk} P_{jk}$, $i = 1, 2, 3$ and by $\boldsymbol{\varepsilon} \cdot \omega$ the tensor with components $\varepsilon_{ijk}\omega_k$, $i, j = 1, 2, 3$, respectively. In both cases the summation convention over repeated indices is used. For a tensor P and a vector ω the symbol $P\omega \in \mathbb{R}^3$ denotes the matrix-vector product, i.e., $(P\omega)_i = P_{ij}\omega_j$ for $i = 1, 2, 3$. $\mathbb{M}^{3\times3}_{\text{sym}}$ is the vector space of all symmetric 3×3 tensors P and $\mathbb{M}^{3\times3}_{\text{skew}}$ is the vector space of all skew-symmetric 3×3 tensors P. We equip the vector space $\mathbb{M}^{3\times3}$ of all 3×3 tensors P with the scalar product $P : Q := P_{ij}Q_{ij}$ and the norm $|P| := (P : P)^{\frac{1}{2}}$. Moreover, we denote the symmetric and the skew-symmetric part, resp., of a tensor $P \in \mathbb{M}^{3\times3}$ by $P^{\text{sym}} := \frac{1}{2}(P + P^\top)$ and $P^{\text{skew}} := \frac{1}{2}(P - P^\top)$, respectively. The particular case of the skew-symmetric part of the velocity gradient is denoted by $Wu := \frac{1}{2}(\nabla u - \nabla u^\top)$.

7.1 Micropolar electrorheological fluids

We consider the following system describing the motion of micropolar electrorheological fluids

$$
\begin{aligned}
\partial_t \boldsymbol{u} - \operatorname{div} \mathrm{S} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \boldsymbol{q} &= \boldsymbol{f} && \text{in } I \times \Omega,\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } I \times \Omega,\\
\partial_t \boldsymbol{\omega} - \operatorname{div} \mathrm{N} + \operatorname{div}(\boldsymbol{\omega} \otimes \boldsymbol{u}) &= \boldsymbol{\ell} - \boldsymbol{\varepsilon} : \mathrm{S} && \text{in } I \times \Omega,\\
\boldsymbol{u} = \boldsymbol{0}, \qquad \boldsymbol{\omega} &= \boldsymbol{0} && \text{on } I \times \partial\Omega,\\
\boldsymbol{u}(0) = u_0, \qquad \boldsymbol{\omega}(0) &= \omega_0 && \text{in } \Omega.
\end{aligned}
\tag{7.1}
$$

In these equations $\boldsymbol{u}$ denotes the velocity, $\boldsymbol{\omega}$ the micro-rotation, $\boldsymbol{q}$ the pressure, S the mechanical extra stress tensor, N the couple stress tensor, $\boldsymbol{\ell}$ the electromagnetic couple force, $\boldsymbol{f} = \hat{\boldsymbol{f}} + \chi^E \operatorname{div}(\boldsymbol{E} \otimes \boldsymbol{E})$ the body force, where $\hat{\boldsymbol{f}}$ is the mechanical body force, χ^E the dielectric susceptibility and $\boldsymbol{E}$ the electric field. The electric field solves the quasi-static Maxwell's equations

$$
\begin{aligned}
\operatorname{div} \boldsymbol{E} = 0 \qquad \operatorname{curl} \boldsymbol{E} = \boldsymbol{0} \qquad &\text{in } I \times \Omega\,,\\
\boldsymbol{E} \cdot \boldsymbol{n} = \boldsymbol{E}_0 \cdot \boldsymbol{n} \qquad &\text{on } I \times \partial\Omega,
\end{aligned}
\tag{7.2}
$$

where $\boldsymbol{n}$ is the outer unit normal vector of the boundary $\partial\Omega$ and $\boldsymbol{E}_0$ is a given time-dependent electric field.

This model was developed in [24] to obtain a more realistic description for the motion of electrorheological fluids. A representative example for a constitutive relation for the stress tensors in (7.1) reads (cf. [24], [46])

$$\begin{aligned} \mathrm{S} &= (\alpha_{31} + \alpha_{33}|E|^2)(1+|D|)^{p-2}D + \alpha_{51}(1+|D|)^{p-2}\big(DE\otimes E + E\otimes DE\big) \\ &\quad + \alpha_{71}|E|^2(1+|R|)^{p-2}R + \alpha_{91}(1+|R|)^{p-2}\big(RE\otimes E + E\otimes RE\big)\,, \\ N &= (\beta_{31}+\beta_{33}|E|^2)(1+|L|)^{p-2}L + \beta_{51}(1+|L|)^{p-2}\big(LE\otimes E + E\otimes LE\big)\,, \end{aligned} \tag{7.3}$$

with $p \in (1,\infty)$ and constants $\alpha_{31}, \alpha_{33}, \alpha_{71}, \beta_{31}, \beta_{33} \geq 0$. The constants $\alpha_{51}, \alpha_{91}, \beta_{51}$ have to satisfy certain restrictions (cf. [24], [46]), which ensure the validity of the second law of thermodynamics. In (7.3) we used the notation $D = \mathcal{D}u$, $R = R(u,\omega) := Wu + \boldsymbol{\varepsilon}\cdot\omega$ and $L = \nabla\omega$.

This model for micropolar electrorheological fluids is rather general and contains as special cases the models for generalized Newtonian fluids ($E = 0$, $\alpha_{31} = 0$), electrorheological fluids with constant exponents ($\alpha_{71} = \alpha_{91} = \beta_{31} = \beta_{33} = \beta_{51} = 0$) and micropolar fluids ($E = $ const., $\alpha_{51} = \alpha_{91} = \beta_{33} = \beta_{51} = 0$). Consequently the results presented in this section also apply to these models either directly or by an easy adaptation.

We refrain from considering concrete constitutive relations for the stress tensors, but we make general assumptions covering prototypical situations:

The continuous mapping $\mathrm{S} : \mathbb{M}^{3\times3}_{\mathrm{sym}} \times \mathbb{M}^{3\times3}_{\mathrm{skew}} \times \mathbb{R}^3 \to \mathbb{M}^{3\times3}$ satisfies for some $p\in(1,\infty)$ and $\kappa \geq 0$ and all $D, P \in \mathbb{M}^{3\times3}_{\mathrm{sym}}$, $R, Q \in \mathbb{M}^{3\times3}_{\mathrm{skew}}$, $E \in \mathbb{R}^3$ the following properties:

(S.1) $|\mathrm{S}^{\mathrm{sym}}(D,R,E)| \leq \alpha_0(1+|E|^2)\,(\kappa+|D|)^{p-2}|D| + \alpha_1$ for ($\alpha_0 > 0$, $\alpha_1 \geq 0$);
$|\mathrm{S}^{\mathrm{skew}}(D,R,E)| \leq \beta_0|E|^2(\kappa+|R|)^{p-2}|R| + \beta_1$ for ($\beta_0 > 0$, $\beta_1 \geq 0$).

(S.2) $\mathrm{S}(D,R,E) : D \geq c_0(1+|E|^2)\,(\kappa+|D|)^{p-2}|D|^2 - c_1$ for ($c_0 > 0$, $c_1 \geq 0$);
$\mathrm{S}(D,R,E) : R \geq c_2|E|^2(\kappa+|R|)^{p-2}|R|^2 - c_3$ for ($c_2 > 0$, $c_3 \geq 0$).

(S.3) $(\mathrm{S}(D,R,E) - \mathrm{S}(P,Q,E)) : (D-P+R-Q) \geq 0$.

The continuous mapping $\mathrm{N} : \mathbb{M}^{3\times3} \times \mathbb{R}^3 \to \mathbb{M}^{3\times3}$ satisfies for the same $p\in(1,\infty)$ and $\kappa\geq 0$ and all $L, K \in \mathbb{M}^{3\times3}$, $E \in \mathbb{R}^3$ the following properties:

(N.1) $|\mathrm{N}(L,E)| \leq \gamma_0(1+|E|^2)\,(\kappa+|L|)^{p-2}|L| + \gamma_1$ for ($\gamma_0 > 0$, $\gamma_1 \geq 0$).

(N.2) $\mathrm{N}(L,E) : L \geq c_3(1+|E|^2)\,(\kappa+|L|)^{p-2}|L|^2 - c_4$ for ($c_3 > 0$, $c_4 \geq 0$).

(N.3) $(\mathrm{N}(L,E) - \mathrm{N}(K,E)) : (L-K) \geq 0$.

Concerning the electric field $\boldsymbol{E}$ solving (7.2) we assume that the boundary data E_0 are regular enough to ensure that

(E.1) $\boldsymbol{E}$ belongs to the space $L^\infty(I, L^\infty(\Omega))$ and a.e. in $I\times\Omega$ there holds $|\boldsymbol{E}| > 0$.

To treat problem (7.1) we define for $p > \frac{6}{5}$ the function spaces

$$\begin{aligned} X &:= W^{1,p}_0(\Omega)^3 \times W^{1,p}_0(\Omega)^3, & Y &:= L^2(\Omega)^3 \times L^2(\Omega)^3, \\ V &:= W^{1,p}_{0,\mathrm{div}}(\Omega) \times W^{1,p}_0(\Omega)^3, & H &:= L^2_{\mathrm{div}}(\Omega)\times L^2(\Omega)^3, \end{aligned}$$

and the families of operators $S(t), N(t), B : X \to X^*$ for all $(u,\omega)^\top, (v,\eta)^\top \in X$ via[8]

$$\begin{aligned}\langle S(t)(u,\omega)^\top,(v,\eta)^\top\rangle_X &:= \int_\Omega \mathrm{S}(\mathcal{D}u, R(u,\omega), \boldsymbol{E}(t)) : (\mathcal{D}v + R(v,\eta))\,dx\,,\\ \langle N(t)(u,\omega)^\top,(v,\eta)^\top\rangle_X &:= \int_\Omega \mathrm{N}(\nabla\omega, \boldsymbol{E}(t)) : \nabla\eta\,dx\,,\\ \langle B(u,\omega)^\top,(v,\eta)^\top\rangle_X &:= \int_\Omega u\otimes u : \nabla v + \omega\otimes u : \nabla\eta\,dx\,,\end{aligned}$$

and set $A(t) := S(t) + N(t) + B : X \to X^*$, $t \in I$. Then, (7.1) for $\mathcal{U}_0 := (u_0,\omega_0)^\top \in H$ and $\boldsymbol{\mathcal{F}} := (\boldsymbol{f},\boldsymbol{\ell})^\top \in L^{p'}(I,X^*)$ can be re-written as the abstract evolution equation for $\boldsymbol{\mathcal{U}} := (\boldsymbol{u},\boldsymbol{\omega})^\top$

$$\begin{aligned}\frac{d\boldsymbol{\mathcal{U}}}{dt}(t) + A(t)(\boldsymbol{\mathcal{U}}(t)) &= \boldsymbol{\mathcal{F}}(t) \quad &&\text{in } V^*,\\ \boldsymbol{\mathcal{U}}(0) &= \mathcal{U}_0 &&\text{in } H.\end{aligned}$$

In [8] it is shown that, under appropriate assumption on the data, there exists a weak solution of the problem (7.1) for $p > \frac{6}{5}$ provided S satisfies (S.1)–(S.3), N satisfies (N.1)–(N.3), and E satisfies (E.1).

Due to the presence of B in the definition of the operator family $A(t) : X \to X^*$, $t \in I$, the condition (A.3) is not satisfied. Thus, we modify this family and define the operator family $\hat{A}(t) : X \to X^*$ via $\hat{A}(t) := S(t) + N(t) + \hat{B}$, $t \in I$, where $\hat{B}$ is given for all $(u,\omega)^\top, (v,\eta)^\top \in X$ via

$$\langle \hat{B}(u,\omega)^\top,(v,\eta)^\top\rangle_X := \frac{1}{2}\int_\Omega v\otimes u : \nabla u - u\otimes u : \nabla v + \eta\otimes u : \nabla\omega - \omega\otimes u : \nabla\eta\,dx\,.$$

The operator $\hat{B}$ is a symmetrized extension of B, as $\langle \hat{B}(u,\omega)^\top,(v,\eta)^\top\rangle_X = \langle B(u,\omega)^\top,(v,\eta)^\top\rangle_X$ for all $(u,\omega)^\top$, $(v,\eta)^\top \in X$, which in contrast to B fulfils $\langle \hat{B}(u,\omega)^\top,(u,\omega)^\top\rangle_X = 0$ for all $(u,\omega)^\top \in X$. Thus, we have the following result:

Proposition 7.4 For $p > \frac{11}{5}$ the operator family $\hat{A}(t) : X \to X^*$, $t \in I$, satisfies (A.1)–(A.4).

Proof The assertion is essentially proved in [8]. For the sake of completeness we sketch the main arguments. Let us first consider $S(t) + N(t) : X \to X^*$, $t \in I$. From (S.1), (N.1), (S.2), (N.2) and (E.1) in conjunction with the standard theory of Nemytskiĭ operators (cf. [44, Theorem 1.43]) we deduce for almost every $t \in I$ the well-definiteness and continuity of $S(t) + N(t) : X \to X^*$, as well as condition (A.3). Condition (A.2) follows in a standard way by using (S.1), (N.1), (E.1), Pettis' and Fubini's theorem, since X is separable. (S.3) and (N.3) certainly imply for almost every $t \in I$ the monotonicity of $S(t) + N(t) : X \to X^*$. Hence, $S(t) + N(t) : X \to X^*$ is for almost every $t \in I$ pseudo-monotone, i.e., condition (A.1) is satisfied. Eventually, it can readily be seen by exploiting (S.1) and (N.1) that $S(t) : X \to X^*$, $t \in I$, satisfies (A.4).

Next, we treat the more delicate part $\hat{B} : X \to X^*$. One can verify by the standard theory of Nemytskiĭ operators and Rellich's compactness theorem that $\hat{B} : X \to X^*$ is bounded and pseudo-monotone, i.e., satisfies (A.1) and (A.2). As already pointed out above we have $\langle \hat{B}(u,\omega),(u,\omega)\rangle_X = 0$ for all $(u,\omega) \in X$, i.e., $\hat{B}$ satisfies (A.3). To verify that $\hat{B}$ satisfies (A.4) we note that it is sufficient to treat the last two terms, since the same estimates apply for the first two if one replaces ω and η by u and v, respectively. Thus, we define for fixed $u \in W_0^{1,p}(\Omega)^3$ the operator $\tilde{B}$ via

[8] To formulate the operator S we used that $\mathrm{S} : \nabla u + (\boldsymbol{\varepsilon} : \mathrm{S}) \cdot \omega = \mathrm{S} : \big(\mathcal{D}u + R(u,\omega)\big)$.

$\langle \tilde{B}\omega, \eta\rangle_{W_0^{1,p}} := \frac{1}{2}\int_\Omega \eta \otimes u : \nabla\omega - \omega \otimes u : \nabla\eta \, dx$. By Hölder's inequality there holds for every $\eta, \omega \in W_0^{1,p}(\Omega)^3$

$$|\langle \tilde{B}\omega, \eta\rangle| \leq \|u\|_{L^{2p'}} \|\eta\|_{L^{2p'}} \|\omega\|_{W_0^{1,p}} + \|\omega\|_{L^{2p'}} \|u\|_{L^{2p'}} \|\eta\|_{W_0^{1,p}} . \tag{7.5}$$

For $p \geq 3$, the Sobolev embedding provides $\|v\|_{L^{2p'}} \leq c\|v\|_{W_0^{1,p}}$ for every $v \in W_0^{1,p}(\Omega)^3$. Thus, a twofold application of the ε–Young inequality yields for every $\varepsilon > 0$

$$\begin{aligned}
|\langle \tilde{B}\omega, \eta\rangle| &\leq c\,\|u\|_{W_0^{1,p}} \|\omega\|_{W_0^{1,p}} \|\eta\|_{W_0^{1,p}} \\
&\leq \varepsilon \|u\|_{W_0^{1,p}}^p + c_\varepsilon (\|\omega\|_{W_0^{1,p}} \|\eta\|_{W_0^{1,p}})^{p'} \\
&\leq \varepsilon \|u\|_{W_0^{1,p}}^p + \varepsilon \|\omega\|_{W_0^{1,p}}^p + c_\varepsilon \|\eta\|_{W_0^{1,p}}^{\frac{p}{p-2}} \\
&\leq \varepsilon \|u\|_{W_0^{1,p}}^p + \varepsilon \|\omega\|_{W_0^{1,p}}^p + c_\varepsilon (1 + \|\eta\|_{W_0^{1,p}}^p),
\end{aligned}$$

where we exploited for the last inequality that $a^{\frac{p}{p-2}} \leq (1+a)^{\frac{p}{p-2}} \leq (1+a)^p \leq c(p)(1+a^p)$, valid for all $a \geq 0$ and $p \geq 3$. Therefore, (A.4) is satisfied for $\varepsilon > 0$ sufficiently small.

If $p \in (\frac{11}{5}, 3)$, then by interpolation with $\frac{1}{\rho} = \frac{1-\theta}{p^*} + \frac{\theta}{2}$, where $\rho = p\frac{5}{3}$, $\theta = \frac{2}{5}$ and $p^* = \frac{3p}{3-p}$, we obtain for all $v \in W_0^{1,p}(\Omega)^3$

$$\|v\|_{L^\rho} \leq \|v\|_{L^2}^{\frac{2}{5}} \|v\|_{L^{p^*}}^{\frac{3}{5}} \leq \|v\|_{L^2}^{\frac{2}{5}} \|v\|_{W_0^{1,p}}^{\frac{3}{5}}. \tag{7.6}$$

Hence, since $\rho \geq 2p'$, we further conclude from (7.6) in (7.5) that for all $\omega, \eta \in W_0^{1,p}(\Omega)^3$

$$\begin{aligned}
|\langle \tilde{B}\omega, \eta\rangle| &\leq c\,\|u\|_{L^2}^{\frac{2}{5}} \|u\|_{W_0^{1,p}}^{\frac{3}{5}} \|\eta\|_{L^2}^{\frac{2}{5}} \|\eta\|_{W_0^{1,p}}^{\frac{3}{5}} \|\omega\|_{W_0^{1,p}} + c\,\|u\|_{L^2}^{\frac{2}{5}} \|u\|_{W_0^{1,p}}^{\frac{3}{5}} \|\omega\|_{L^2}^{\frac{2}{5}} \|\omega\|_{W_0^{1,p}}^{\frac{3}{5}} \|\eta\|_{W_0^{1,p}} \\
&\leq \varepsilon \|u\|_{W_0^{1,p}}^p + \varepsilon \|\omega\|_{W_0^{1,p}}^p + c_\varepsilon \|\eta\|_{W_0^{1,p}}^p + c_\varepsilon \|u\|_{L^2}^{\frac{4p}{5p-11}} + c_\varepsilon \|\omega\|_{L^2}^{\frac{4p}{5p-11}} + c_\varepsilon \|\eta\|_{L^2}^{\frac{4p}{5p-11}},
\end{aligned}$$

where we applied an appropriate version of the ε-Young's inequality with exponents $\frac{10(p-1)}{5p-11}$, $\frac{5(p-1)}{3}$, $\frac{10(p-1)}{5p-11}$, $\frac{5(p-1)}{3}$ and p. Thus, (A.4) is satisfied for $\varepsilon > 0$ sufficiently small.

Altogether, $\hat{B} : X \to X^*$ satisfies (A.4) with $q = \frac{4p}{5p-11}$. As a result, $\hat{A}(t) : X \to X^*$, $t \in I$, satisfies (A.1)–(A.4). □

Let us now define the discrete setup. For given $m, \ell, k \in \mathbb{N}_0$, we define $X_h := X_h^u \times X_h^\omega$, where $X_h^u \subset \mathcal{P}_m(\mathcal{T}_h)^3 \cap W_0^{1,p}(\Omega)^3$, $X_h^\omega \subset \mathcal{P}_k(\mathcal{T}_h)^3 \cap W_0^{1,p}(\Omega)^3$ are appropriate finite element spaces, both equipped with the $W_0^{1,p}(\Omega)^3$–norm. In addition, we define for $h > 0$ and an appropriate finite element space $Z_h \subset \mathcal{P}_\ell(\mathcal{T}_h) \cap Z$ equipped with the Z–norm, where $Z := L^{p'}(\Omega)$, the space

$$V_h := \Big\{ (u_h, \omega_h)^\top \in X_h \,\Big|\, \int_\Omega \operatorname{div} u_h \, \eta_h \, dx = 0 \text{ for all } \eta_h \in Z_h \Big\}.$$

For a null sequence $(h_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$, we set $V_n := V_{h_n}$, $n \in \mathbb{N}$. To ensure that the spaces $(V_n)_{n\in\mathbb{N}}$ are a quasi non-conforming approximation of V in X we make the following assumption:

Assumption 7.7 (Projection operators) For every $h > 0$ it holds $\mathcal{P}_1(\mathcal{T}_h)^3 \subset X_h^u$, $\mathcal{P}_1(\mathcal{T}_h)^3 \subset X_h^\omega$, $\mathbb{R} \subset Z_h$, and that there exist linear interpolation operators $\Pi_h^{\mathrm{div}} : W_0^{1,p}(\Omega)^3 \to X_h^u$, $\Pi_h^\omega : W_0^{1,p}(\Omega)^3 \to X_h^\omega$ and $\Pi_h^Z : Z \to Z_h$ with the following properties:

(i) **Divergence preservation of Π_h^{div} in Z_h^*:** It holds for all $u \in W_0^{1,p}(\Omega)^3$ and $\eta_h \in Z_h$

$$\int_\Omega \operatorname{div} u \, \eta_h \, dx = \int_\Omega \operatorname{div} \Pi_h^{\mathrm{div}} u \, \eta_h \, dx \, .$$

(ii) **local $W^{1,1}$-stability of Π_h^{div}:** There exists a constant $c > 0$, independent of $h > 0$, such that for every $u \in W_0^{1,p}(\Omega)^3$ and $K \in \mathcal{T}_h$[9]

$$\|\Pi_h^{\mathrm{div}} u\|_{L^1(K)} \le c \, \|u\|_{L^1(S_K)} + c \, h_K \, \|\nabla u\|_{L^1(S_K)}.$$

(iii) **local L^1-stability of Π_h^Z:** There exists a constant $c > 0$, independent of $h > 0$, such that for every $\eta \in Z$ and $K \in \mathcal{T}_h$

$$\|\Pi_h^Z \eta\|_{L^1(K)} \le c \, \|\eta\|_{L^1(S_K)}.$$

(iv) **local $W^{1,1}$-stability of Π_h^ω:** There exists a constant $c > 0$, independent of $h > 0$, such that for every $\omega \in W_0^{1,p}(\Omega)^3$ and $K \in \mathcal{T}_h$

$$\|\Pi_h^\omega \omega\|_{L^1(K)} \le c \, \|\omega\|_{L^1(S_K)} + c \, h_K \, \|\nabla \omega\|_{L^1(S_K)}.$$

Certainly, the existence of such operators depends on the choice of X_h and Z_h. It is shown in [14], [29], [31], [9], [30], [21], [48] that Π_h^{div} exists for a variety of spaces X_h and Z_h, which, e.g., include the Taylor-Hood, the spatially conforming Crouzeix–Raviart, and the MINI element in dimension two and three. Projection operators Π_h^Z satisfying Assumption 7.7 (iii) are e.g. the Clément interpolation operator (cf. [17]) and a version of the Scott-Zhang interpolation operator (cf. [49]). The standard Scott-Zhang interpolation operator (cf. [49]) satisfies Assumption 7.7 (iv). The abstract assumptions allow for an easy extension of our results to other choices of X_h and Z_h.

The next proposition shows that the approximation of divergence-free Sobolev functions through discretely divergence-free finite element spaces perfectly fits into the framework of quasi non-conforming approximations.

Proposition 7.8 Let $p > \frac{6}{5}$ and let Assumption 7.7 be satisfied. Then, the sequence $(V_n)_{n\in\mathbb{N}}$ forms a quasi non-conforming approximation of V in X.

Proof Clearly, (V, H, id_V) and (X, Y, id_X) form evolution triples satisfying $V \subseteq X$ with $\|\cdot\|_V = \|\cdot\|_X$ in V and $H \subseteq Y$ with $(\cdot,\cdot)_H = (\cdot,\cdot)_Y$ in $H \times H$. So, let us verify that $(V_n)_{n\in\mathbb{N}}$ satisfies (QNC.1) and (QNC.2):

ad (QNC.1) Due to their finite dimensionality, the spaces $(V_n)_{n\in\mathbb{N}}$ are closed. We set $C := \mathcal{V} \times C_0^\infty(\Omega)^3$, where $\mathcal{V} = \{v \in C_0^\infty(\Omega)^3 \mid \operatorname{div} v = 0\}$. Let $(u, \omega) \in C$. Then, owing to standard estimates for polynomial projection operators (cf. [54, Lemma 2.25]), the sequence $(u_n, \omega_n)^\top := (\Pi_{h_n}^{\mathrm{div}} u, \Pi_{h_n}^\omega \omega)^\top \in V_n$, $n \in \mathbb{N}$, satisfies

$$\|(u, \eta)^\top - (u_n, \omega_n)^\top\|_X \le c \, h_n \, \|(u, \omega)^\top\|_{W^{2,p}(\Omega)^3} \quad \to \quad 0 \quad (n \to \infty).$$

ad (QNC.2) Let $(\boldsymbol{u}_n, \boldsymbol{\omega}_n)^\top \in L^p(I, V_{m_n})$, $n \in \mathbb{N}$, where $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $m_n \to \infty$ $(n \to \infty)$, be such that $(\boldsymbol{u}_n, \boldsymbol{\omega}_n)^\top \rightharpoonup (\boldsymbol{u}, \boldsymbol{\omega})^\top$ in $L^p(I, X)$ $(n \to \infty)$. Since the second component of V and X coincide we only have to show that $\boldsymbol{u} \in L^p(I, W_{0,\mathrm{div}}^{1,p}(\Omega)^3)$. Let $\eta \in C_0^\infty(\Omega)$ and $\varphi \in C_0^\infty(I)$. As

[9] The neighbourhood S_K of a simplex $K \in \mathcal{T}_h$ is defined via $S_K :=$ interior $\bigcup_{\{\overline{K'} \in \mathcal{T}_h \,|\, \overline{K'} \cap \overline{K} \neq \emptyset\}} \overline{K'}$.

in the previous step we infer that the sequence $\eta_n := \Pi^Z_{m_n}\eta \in Z_{h_{m_n}}$, $n \in \mathbb{N}$, satisfies $\eta_n \to \eta$ in Z $(n \to \infty)$. On the other hand, in view of the definition of V_{m_n} there holds $\langle \operatorname{div} \boldsymbol{u}_n(t), \eta_n\rangle_Z = 0$ for almost every $t \in I$ and $n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$ we have

$$\int_I \int_\Omega \operatorname{div} \boldsymbol{u}_n(s)\, \eta_n \, dx\, \varphi(s)\, ds = 0. \tag{7.9}$$

By passing in (7.9) for $n \to \infty$, we obtain for every $\eta \in C_0^\infty(\Omega)$ and $\varphi \in C_0^\infty(I)$

$$\int_I \int_\Omega \operatorname{div} \boldsymbol{u}(s)\, \eta \, dx\, \varphi(s)\, ds = 0,$$

i.e., $\boldsymbol{u} \in L^p(I, W^{1,p}_{0,\operatorname{div}}(\Omega)^3)$. $\square$

Remark 7.10 From the proof of Proposition 7.8 it is clear that instead of Assumption 7.7 it is sufficient to require that there exist dense subsets $\mathcal{X}$ of $W_0^{1,p}(\Omega)^3$ and $\mathcal{Z}$ of $L^{p'}(\Omega)$ and linear interpolation operators $\Pi_h^{\operatorname{div}} : \mathcal{X} \to X_h^u$, $\Pi_h^\omega : \mathcal{X} \to X_h^\omega$ and $\Pi_h^Z : \mathcal{Z} \to Z_h$ which have a global approximation property, i.e., for every $(u, \omega) \in \mathcal{X} \times \mathcal{X}$ there holds $\|(u,\omega)^\top - (\Pi_h^{\operatorname{div}} u, \Pi_h^\omega \omega)^\top\|_X \to 0$ for $h \to 0$, as well as for every $\eta \in \mathcal{Z}$ there holds $\|\eta - \Pi_h^Z \eta\|_Z \to 0$ for $h \to 0$.

Let us summarize our setup for the treatment of problem (7.1) describing the motion of micropolar electrorheological fluids.

Assumption 7.11 Let $\Omega \subseteq \mathbb{R}^3$ be a bounded polygonal Lipschitz domain, $I := (0, T)$, $T < \infty$, and $p > \frac{11}{5}$. We make the following assumptions:

(i) The stress tensors and the electric field satisfy (S.1)–(S.3), (N.1)–(N.3), and (E.1).
(ii) (V, H, id), (X, Y, id) and $(V_n)_{n\in\mathbb{N}}$ are defined as in Proposition 7.8.
(iii) $\mathcal{U}_0 := (u_0, \omega_0)^\top \in H$ and $\mathcal{U}_n^0 := (u_n^0, \omega_n^0)^\top \in V_n$, $n \in \mathbb{N}$, such that $\mathcal{U}_n^0 \to \mathcal{U}_0$ in Y $(n \to \infty)$ and $\sup_{n\in\mathbb{N}} \|\mathcal{U}_n^0\|_Y \le \|\mathcal{U}_0\|_H$.
(iv) $\boldsymbol{\mathcal{F}} := (\boldsymbol{f}, \boldsymbol{\ell})^\top \in L^{p'}(I, X^*)$.
(v) $\widehat{A}(t) : X \to X^*$, $t \in I$, is defined as in Proposition 7.4.

Furthermore, we denote by $e := (\mathrm{id}_V)^* R_H : V \to V^*$ the canonical embedding with respect to the evolution triple (V, H, id). Thus, the quasi non-conforming Rothe–Galerkin scheme in this setup reads:

Algorithm 7.12 Let Assumption 7.11 be satisfied. For given $K, n \in \mathbb{N}$ the sequence of iterates $\mathcal{U}_n^k := (u_n^k, \omega_n^k)^\top \in V_n$, $k = 0, \ldots, K$ is given solving the implicit Rothe–Galerkin scheme for $\tau = \frac{T}{K}$ and $k = 1, \ldots, K$

$$(d_\tau \mathcal{U}_n^k, \mathcal{W}_n)_Y + \langle [\widehat{A}]_k^\tau \mathcal{U}_n^k, \mathcal{W}_n\rangle_X = \langle [\boldsymbol{\mathcal{F}}]_k^\tau, \mathcal{W}_n\rangle_X \quad \text{ for all } \mathcal{W}_n \in V_n. \tag{7.13}$$

By means of Proposition 6.6, Proposition 6.8, Theorem 6.16 and the observations already made in Proposition 7.4 and Proposition 7.8, we can immediately conclude the following results.

Theorem 7.14 (Well-posedness, stability and weak convergence of (7.13))
Let Assumption 7.11 be satisfied. Then, it holds:

(I) **Well-posedness:** For every $K, n \in \mathbb{N}$ there exist iterates $(\mathcal{U}_n^k)_{k=0,\ldots,K} \subseteq V_n$, solving (7.13), without any restrictions on the step-size.

(II) **Stability:** The corresponding piece-wise constant interpolants $\overline{\boldsymbol{\mathcal{U}}}_n^\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$, $K, n \in \mathbb{N}$ with $\tau = \frac{T}{K}$, are bounded in $L^p(I,X) \cap L^\infty(I,Y)$.

(III) **Weak convergence:** If $(\overline{\boldsymbol{\mathcal{U}}}_n)_{n\in\mathbb{N}} := (\overline{\boldsymbol{\mathcal{U}}}_{m_n}^{\tau_n})_{n\in\mathbb{N}}$, where $\tau_n = \frac{T}{K_n}$ and $K_n, m_n \to \infty$ $(n \to \infty)$, is an arbitrary diagonal sequence of the piece-wise constant interpolants $\overline{\boldsymbol{\mathcal{U}}}_n^\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$, $K, n \in \mathbb{N}$ with $\tau = \frac{T}{K}$, then there exists a not relabelled subsequence and a weak limit $\overline{\boldsymbol{\mathcal{U}}} \in L^p(I,V) \cap L^\infty(I,H)$ such that

$$\begin{aligned} \overline{\boldsymbol{\mathcal{U}}}_n &\rightharpoonup \overline{\boldsymbol{\mathcal{U}}} \quad \text{in } L^p(I,X), \\ \overline{\boldsymbol{\mathcal{U}}}_n &\overset{*}{\rightharpoondown} \overline{\boldsymbol{\mathcal{U}}} \quad \text{in } L^\infty(I,Y), \end{aligned} \qquad (n \to \infty).$$

Furthermore, it follows that $\overline{\boldsymbol{\mathcal{U}}} \in \boldsymbol{\mathcal{W}}_e^{1,p,p}(I,V,H) \cap L^\infty(I,H)$ satisfies $\overline{\boldsymbol{\mathcal{U}}}(0) = \mathcal{U}_0$ in H and for all $\boldsymbol{\phi} \in L^p(I,V)$

$$\int_I \Big\langle \frac{d_e\overline{\boldsymbol{\mathcal{U}}}}{dt}(t), \boldsymbol{\phi}(t) \Big\rangle_V \, dt + \int_I \langle A(t)(\overline{\boldsymbol{\mathcal{U}}}(t)), \boldsymbol{\phi}(t)\rangle_X \, dt = \int_I \langle \boldsymbol{\mathcal{F}}(t), \boldsymbol{\phi}(t)\rangle_X \, dt.$$

Proof **ad (I)/(II)** The assertions follow immediately from Proposition 6.6 and Proposition 6.8, since the operator family $\hat{A}(t) := S(t) + N(t) + \hat{B} : X \to X^*$, $t \in I$, satisfies (A.1)–(A.4) with $c_1 = 0$ due to Proposition 7.4.

ad (III) The assertions follow from Theorem 6.16. To be more precise, Theorem 6.16 initially yields that $\overline{\boldsymbol{\mathcal{U}}} \in \boldsymbol{\mathcal{W}}_e^{1,p,q}(I,V,H)$, where $q > 1$ is specified in the proof of Proposition 7.4, satisfies $\overline{\boldsymbol{\mathcal{U}}}(0) = \mathcal{U}_0$ in H and for all $\boldsymbol{\phi} \in C_0^1(I,V)$

$$\int_I \Big\langle \frac{d_e\overline{\boldsymbol{\mathcal{U}}}}{dt}(t), \boldsymbol{\phi}(t) \Big\rangle_V \, dt + \int_I \langle \hat{A}(t)(\overline{\boldsymbol{\mathcal{U}}}(t)), \boldsymbol{\phi}(t)\rangle_X \, dt = \int_I \langle \boldsymbol{\mathcal{F}}(t), \boldsymbol{\phi}(t)\rangle_X \, dt$$

Since $\langle \hat{B}(\overline{\boldsymbol{\mathcal{U}}}(t)), \boldsymbol{\phi}(t)\rangle_X = \langle B(\overline{\boldsymbol{\mathcal{U}}}(t)), \boldsymbol{\phi}(t)\rangle_X$ for almost every $t \in I$ and all $\boldsymbol{\phi} \in C_0^1(I,V)$ as well as $B(\overline{\boldsymbol{\mathcal{U}}}(\cdot)) \in L^{p'}(I,V^*)$ (cf. [8, Theorem 2.30]), as $\overline{\boldsymbol{\mathcal{U}}}(t) \in L^p(I,V) \cap L^\infty(I,H)$, we actually proved that $\overline{\boldsymbol{\mathcal{U}}} \in \boldsymbol{\mathcal{W}}_e^{1,p,p}(I,V,H) \cap L^\infty(I,H)$, is such that for all $\boldsymbol{\phi} \in C_0^1(I,V)$

$$\int_I \Big\langle \frac{d_e\overline{\boldsymbol{\mathcal{U}}}}{dt}(t), \boldsymbol{\phi}(t) \Big\rangle_V \, dt + \int_I \langle A(t)(\overline{\boldsymbol{\mathcal{U}}}(t)), \boldsymbol{\phi}(t)\rangle_X \, dt = \int_I \langle \boldsymbol{\mathcal{F}}(t), \boldsymbol{\phi}(t)\rangle_X \, dt. \qquad \square$$

Remark 7.15 The results in Theorem 7.14 for micropolar electrorheological fluids are completely new. There are some previous results for the various subcases. Let us mention some of them. For the special subcase of generalized Newtonian fluids the result in Theorem 7.14 is, among others, already contained in [54] (cf. [51]). Theorem 7.14 extends the convergence of a conforming implicit fully discrete Rothe–Galerkin scheme of an evolution problem with Bochner pseudo-monotone operators, proved in [7], to the quasi non-conforming setting. Convergence results with optimal rates for the unsteady p-Navier-Stokes equations and related problems can be found e.g. in [10], [25], [12] and [11]. For the special subcase of micropolar fluids with $p = 2$ optimal convergence rates for strong solutions are proved in [41], [43]. The convergence of a fully discrete approximation towards a mollified problem for electrorheological fluids with variable exponents is proved in [15].

7.2 A modified Smagorinsky model

We consider the following modified[10] version of the Smagorinsky model for turbulent flows

$$\begin{aligned} \partial_t \boldsymbol{u} - \operatorname{div}\big(\delta\,|\mathcal{D}\boldsymbol{u}|\mathcal{D}\boldsymbol{u}\big) + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla \boldsymbol{q} &= \boldsymbol{f} && \text{in } I\times\Omega,\\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } I\times\Omega,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } I\times\partial\Omega,\\ \boldsymbol{u}(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{7.16}$$

In these equations $\boldsymbol{u}$ denotes the velocity, $\boldsymbol{q}$ is the pressure, $\boldsymbol{f}$ is the mechanical body force and $\delta(\cdot) := \operatorname{dist}(\cdot,\partial\Omega)$ the distance from the boundary $\partial\Omega$. This modification (with a position-dependent eddy viscosity) is intended to improve some of the weakness of the original Smagorinsky model, which is considered to be too dissipative in laminar regimes and close to walls, and thus does not work satisfactorily for the computation of boundary layers and of the transition to turbulence (cf. [16], [47], [2]). The introduction of models similar to (7.16) dates back to Obukhov and van Driest, at least in the case of a channel flow, see [16]. Improved (as well as adaptive and selective) eddy viscosity models are mostly used in numerical computations, while the basic analytical problems are still mainly open.

For the functional setting we make use of the standard theory of weighted Lebesgue spaces $L^p(\Omega;\sigma)$ and Sobolev spaces $W^{1,p}(\Omega;\sigma)$ (cf. [32], [36]) with a weight σ belonging to the Muckenhoupt class $\mathcal{A}_p$. The norm in $L^p(\Omega;\sigma)$ is defined as $\|v\|_{L^p(\Omega;\sigma)} := \big(\int_\Omega |v|^p\,\sigma\,dx\big)^{\frac{1}{p}}$. The dual space $(L^p(\Omega;\sigma))^*$ can be identified with $L^{p'}(\Omega;\sigma')$, where $\sigma' := \sigma^{\frac{-1}{p-1}}$. In particular we have

$$\Big|\int_\Omega f g\,dx\Big| \le \|f\|_{L^p(\Omega;\sigma)}\|g\|_{L^{p'}(\Omega;\sigma')}, \tag{7.17}$$

if $f\in L^p(\Omega;\sigma)$, $g\in L^{p'}(\Omega;\sigma')$. Note that $\sigma\in\mathcal{A}_p$ iff $\sigma'\in\mathcal{A}_{p'}$. The space $W_0^{1,p}(\Omega;\sigma)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega;\sigma)} := \|\cdot\|_{L^p(\Omega;\sigma)} + \|\nabla\cdot\|_{L^p(\Omega;\sigma)}$. The norm $\|\nabla\cdot\|_{L^p(\Omega;\sigma)}$ is an equivalent norm on $W_0^{1,p}(\Omega;\sigma)$. Let us summarize some facts needed for our analysis. The distance function δ belongs to the Muckenhoupt class $\mathcal{A}_p$ for $p>2$ (cf. [23]). The embedding $W_0^{1,3}(\Omega;\delta)$ into $L^q(\Omega;\delta)$ is compact for all $q\in[1,6)$ (cf. [27, Theorem 2.6]). Moreover, $W_0^{1,3}(\Omega;\delta)$ embeds into $L^3(\Omega;\delta^\beta)$ if $\beta\ge -2$ (cf. [23]).

To treat problem (7.16) we define the function spaces

$$\begin{aligned} X &:= W_0^{1,3}(\Omega;\delta)^3, & Y &:= L^2(\Omega)^3,\\ V &:= W_{0,\operatorname{div}}^{1,p}(\Omega;\delta), & H &:= L^2_{\operatorname{div}}(\Omega), \end{aligned}$$

and the operators $S,B: X\to X^*$ for all $u,v\in X$ via

$$\langle Su,v\rangle_X := \int_\Omega \delta\,|\mathcal{D}u|\mathcal{D}u : \mathcal{D}v\,dx\,,\qquad \langle Bu,v\rangle_X := \int_\Omega u\otimes u : \nabla v\,dx\,,$$

and set $A(t) := S+B : X\to X^*$, $t\in I$. Then, (7.16) for $u_0\in H$ and $\boldsymbol{f}\in L^{p'}(I,X^*)$ can be re-written as the abstract evolution equation

$$\begin{aligned} \frac{d\boldsymbol{u}}{dt}(t) + A(t)(\boldsymbol{u}(t)) &= \boldsymbol{f}(t) && \text{in } V^*,\\ \boldsymbol{u}(0) &= u_0 && \text{in } H. \end{aligned}$$

[10] Models of this type are known in literature as improved eddy viscosity models.

As in the previous section we modify the operator family $A(t)$, $t \in I$, and define $\hat{A}(t) : X \to X^*$ via $\hat{A}(t) := S + \hat{B}$, $t \in I$, where $\hat{B}$ is given for all $u, v \in X$ via

$$\langle \hat{B}u, v\rangle_X := \frac{1}{2}\int_\Omega v \otimes u : \nabla u - u \otimes u : \nabla v \, dx \,.$$

The operator $\hat{B}$ is a symmetrized extension of B, as $\langle \hat{B}u, v\rangle_X = \langle Bu, v\rangle_X$ for all $u, v \in X$ and fulfils $\langle \hat{B}u, v\rangle_X = 0$ for all $u \in X$. Thus, we have the following result:

Proposition 7.18 The operator family $\hat{A}(t) : X \to X^*$, $t \in I$, satisfies (A.1)–(A.4) with $p = 3$.

Proof Let us first consider $S : X \to X^*$. From a straightforward modification of the theory of Nemytskiĭ operators to weighted spaces we deduce for almost every $t \in I$ the well-definiteness and continuity of $S : X \to X^*$, as well as $\langle Su, u\rangle_X = \|\mathcal{D}u\|^3_{L^3(\Omega;\delta)}$, which is condition (A.3). Condition (A.2) is obviously satisfied, since the operator S does not depend on time. The monotonicity of $S : X \to X^*$ follows in the same way as for the operator $\langle \tilde{S}u, v\rangle_{W_0^{1,3}(\Omega)} := \int_\Omega |\mathcal{D}u|\mathcal{D}u : \mathcal{D}v \, dx$ as the arguments to prove this are pointwise. Hence, $S : X \to X^*$ is pseudo-monotone, i.e., condition (A.1) is satisfied. Condition (A.4) follows from the pointwise ε-Young inequality.

Next, we treat $\hat{B} : X \to X^*$. Again a straightforward modification of theory of Nemytskiĭ operators to weighted spaces and the above compact embedding yield that $\hat{B} : X \to X^*$ is bounded and pseudo-monotone, i.e., satisfies (A.1). Condition (A.2) is obvious, since the operator $\hat{B}$ does not depend on time. As already pointed out above we have $\langle \hat{B}u, u\rangle_X = 0$ for all $u \in X$, i.e., $\hat{B}$ satisfies (A.3). We use Hölder's inequality (7.17), the above stated embeddings and the ε-Young inequality to verify that for every $u, v \in X$ there holds

$$\begin{aligned} |\langle \hat{B}u, v\rangle| &\le \|v\|_{L^3(\Omega;\delta^{-\frac{1}{2}})}\|u\|_{L^3(\Omega;\delta^{-\frac{1}{2}})}\|\nabla u\|_{L^3(\Omega;\delta)} + \|u\|^2_{L^3(\Omega;\delta^{-\frac{1}{2}})}\|\nabla v\|_{L^3(\Omega;\delta)} \\ &\le \varepsilon\|\nabla u\|^3_{L^3(\Omega;\delta)} + c_\varepsilon\|\nabla v\|^3_{L^3(\Omega;\delta)}, \end{aligned}$$

i.e., (A.4) for $\varepsilon > 0$ sufficiently small.

Altogether, $\hat{A}(t) : X \to X^*$, $t \in I$, satisfies (A.1)–(A.4). □

Let us now define the discrete setup. For $p \in (1, \infty)$ and a Muckenhoupt weight $\sigma \in \mathcal{A}_p$ we define $Z := L^{p'}(\Omega; \sigma')$. For given $m, \ell \in \mathbb{N}_0$ we denote by $X_h \subset \mathcal{P}_m(\mathcal{T}_h)^3 \cap W_0^{1,p}(\Omega;\sigma)^3$ and $Z_h \subset \mathcal{P}_\ell(\mathcal{T}_h) \cap Z$ suitable finite element spaces equipped with the $W_0^{1,p}(\Omega;\sigma)^3$-norm and Z-norm, respectively. In addition, we define for $h > 0$ the space

$$V_h := \Big\{ u_h \in X_h \;\Big|\; \int_\Omega \operatorname{div} u_h \, \eta_h \, dx = 0 \text{ for all } \eta_h \in Z_h \Big\}.$$

For a null sequence $(h_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ we set $V_n := V_{h_n}$, $n \in \mathbb{N}$. To ensure that the spaces $(V_n)_{n\in\mathbb{N}}$ are a quasi non-conforming approximation of V in X we make the following assumption:

Assumption 7.19 (Projection operators) For every $h > 0$ there exist linear interpolation operators $\Pi_h^{\mathrm{div}} : W_0^{1,p}(\Omega;\sigma)^3 \to X_h$ and $\Pi_h^Z : Z \to Z_h$ with the following properties:

(i) **Divergence preservation of Π_h^{div} in Z_h^*:** It holds for all $u \in W_0^{1,p}(\Omega;\sigma)^3$ and $\eta_h \in Z_h$

$$\int_\Omega \operatorname{div} u \, \eta_h \, dx = \int_\Omega \operatorname{div} \Pi_h^{\mathrm{div}} u \, \eta_h \, dx \,.$$

(ii) **global $W^{1,p}(\Omega;\sigma)$-approximability of Π_h^{div}:** There exists a constant $c>0$, independent of $h>0$, such that for every $u\in W_0^{1,p}(\Omega;\sigma)^3\cap W^{2,p}(\Omega;\sigma)^3$

$$\|u-\Pi_h^{\mathrm{div}}u\|_{L^p(\Omega;\sigma)}+h\,\|\nabla u-\nabla\Pi_h^{\mathrm{div}}u\|_{L^p(\Omega;\sigma)}\le c\,h^2\,\|\nabla^2 u\|_{L^p(\Omega;\sigma)}.$$

(iii) **global $L^{p'}(\Omega;\sigma')$-approximability of Π_h^Z:** There exists a constant $c>0$, independent of $h>0$, such that for every $\eta\in Z\cap W^{1,p'}(\Omega;\sigma')$

$$\|\eta-\Pi_h^Z\eta\|_{L^{p'}(\Omega;\sigma')}\le c\,h\,\|\nabla\eta\|_{L^{p'}(\Omega;\sigma')}.$$

Remark 7.20 Since interpolation operators in weighted spaces are not so common in the literature, we discuss them in some detail.

(i) The Clément interpolation operator (cf. [17]) satisfies Assumption 7.19 (iii). Indeed, the proof of [4, Theorem 4.2], using a duality argument and a local Poincaré inequality, also works in the setting of weighted spaces in view of the local Poincaré inequality in weighted spaces (cf. [20, Theorem 5.1]).

(ii) The existence of an operator Π_h^{div} satisfying Assumptions 7.19 (i), (ii) depends on the choice of X_h and Z_h. The general strategy from [14, Section VI.4] can be adapted to the weighted setting (cf. [22] for a similar approach). To do so one needs a projection operator $\Pi_h:W_0^{1,p}(\Omega,\sigma)^3\to\mathcal{P}_k(\mathcal{T}_h)^3\cap W_0^{1,1}(\Omega)^3$, where $k\in\mathbb{N}$ is such that $\mathcal{P}_k(\mathcal{T}_h)^3\subset X_h$, which satisfies a local approximation property, i.e., for every $v\in W_0^{1,p}(\Omega,\sigma)^3\cap W^{2,p}(\Omega;\sigma)^3$ and $K\in\mathcal{T}_h$

$$\|v-\Pi_h v\|_{L^p(K;\sigma)}+h\,\|\nabla v-\nabla\Pi_h v\|_{L^p(K;\sigma)}\le c\,h^2\,\|\nabla^2 v\|_{L^p(S_K;\sigma)}. \tag{7.21}$$

The existence of such an operator is proved in [40, Theorem 5.2, 5.3]. Moreover, one needs a correction operator $\Pi_h^{\mathrm{cor}}:W_0^{1,1}(\Omega)^3\to X_h$ which is locally $W^{1,1}$-stable, i.e., there exists a constant $c>0$, independent of $h>0$, such that for every $v\in W_0^{1,1}(\Omega)^3$ and $K\in\mathcal{T}_h$

$$\|\Pi_h^{\mathrm{cor}}v\|_{L^1(K)}\le c\,\|v\|_{L^1(S_K)}+c\,h_K\,\|\nabla v\|_{L^1(S_K)}.$$

This inequality implies that there exists a constant $c>0$, independent of $h>0$, such that for every $v\in W_0^{1,p}(\Omega;\sigma)^3$ and $K\in\mathcal{T}_h$

$$\|\Pi_h^{\mathrm{cor}}v\|_{L^p(K;\sigma)}\le c\,\|v\|_{L^p(S_K;\sigma)}+c\,h_K\,\|\nabla v\|_{L^p(S_K;\sigma)}. \tag{7.22}$$

The proof of this assertion just uses Hölder's inequality, the equivalence $\|g\|_{L^\infty(K)}\sim\fint_K|g|\,dx$ valid for all polynomials $g\in\mathcal{P}_m(K)$, and $\sigma\in\mathcal{A}_p$. From (7.21), (7.22) one easily deduces that

$$\Pi_h^{\mathrm{div}}(v):=\Pi_h(v)+\Pi_h^{\mathrm{cor}}(v-\Pi_h v)$$

satisfies Assumptions 7.19 (i), (ii). Consequently, at least for the MINI element we proved that Assumptions 7.19 is satisfied. The abstract assumptions allow for an easy extension of our results to other choices of X_h and Z_h.

Proceeding in the same way as in the proof of Proposition 7.8 one can show:

Proposition 7.23 Let Assumption 7.19 be satisfied for $p=3$ and $\sigma=\delta$. Then, the sequence $(V_n)_{n\in\mathbb{N}}$ forms a quasi non-conforming approximation of V in X.

Let us summarize our setup for the treatment of problem (7.16).

Assumption 7.24 Let $\Omega \subseteq \mathbb{R}^3$ be a bounded polygonal Lipschitz domain, $I := (0,T)$, and $T < \infty$. We make the following assumptions:

(i) (V, H, id), (X, Y, id) and $(V_n)_{n\in\mathbb{N}}$ are defined as in Proposition 7.23.
(ii) $u_0 \in H$ and $u_n^0 \in V_n$, $n \in \mathbb{N}$, are such that $u_n^0 \to u_0$ in Y $(n\to\infty)$ and $\sup_{n\in\mathbb{N}} \|u_n^0\|_Y \le \|u_0\|_H$.
(iii) $\boldsymbol{f} \in L^{p'}(I, X^*)$.
(iv) $\hat{A}(t) : X \to X^*$, $t \in I$, is defined as in Proposition 7.18.

Furthermore, we denote by $e := (\mathrm{id}_V)^* R_H : V \to V^*$ the canonical embedding with respect to the evolution triple (V, H, id). Thus, the quasi non-conforming Rothe–Galerkin scheme in this setup reads:

Algorithm 7.25 Let Assumption 7.24 be satisfied. For given $K, n \in \mathbb{N}$ the sequence of iterates $u_n^k \in V_n$, $k = 0, \dots, K$ is given solving the implicit Rothe–Galerkin scheme for $\tau = \frac{T}{K}$ and $k = 1, \dots, K$

$$(d_\tau u_n^k, v_n)_Y + \langle [\widehat{A}]_k^\tau \, u_n^k, v_n \rangle_X = \langle [\boldsymbol{f}]_k^\tau, v_n \rangle_X \qquad \text{for all } v_n \in V_n. \tag{7.26}$$

By means of Proposition 6.6, Proposition 6.8, Theorem 6.16 and the observations already made in Proposition 7.18 and Proposition 7.23, we can conclude in the same way as in Theorem 7.14 the following results:

Theorem 7.27 (Well-posedness, stability and weak convergence of (7.26))
Let Assumption 7.24 be satisfied. Then, it holds:

(I) **Well-posedness:** For every $K, n \in \mathbb{N}$ there exist iterates $(u_n^k)_{k=0,\dots,K} \subseteq V_n$, solving (7.26), without any restrictions on the step-size.
(II) **Stability:** The corresponding piece-wise constant interpolants $\overline{\boldsymbol{u}}_n^\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$, $K, n \in \mathbb{N}$ with $\tau = \frac{T}{K}$, are bounded in $L^p(I, X) \cap L^\infty(I, Y)$.
(III) **Weak convergence:** If $(\overline{\boldsymbol{u}}_n)_{n\in\mathbb{N}} := (\overline{\boldsymbol{u}}_{m_n}^{\tau_n})_{n\in\mathbb{N}}$, where $\tau_n = \frac{T}{K_n}$ and $K_n, m_n \to \infty$ $(n \to \infty)$, is an arbitrary diagonal sequence of the piece-wise constant interpolants $\overline{\boldsymbol{u}}_n^\tau \in \boldsymbol{\mathcal{S}}^0(\mathcal{I}_\tau, X)$, $K, n \in \mathbb{N}$ with $\tau = \frac{T}{K}$, then there exists a not relabelled subsequence and a weak limit $\overline{\boldsymbol{u}} \in L^p(I, V) \cap L^\infty(I, H)$ such that

$$\begin{aligned} \overline{\boldsymbol{u}}_n &\rightharpoonup \overline{\boldsymbol{u}} \quad \text{in } L^p(I, X), \\ \overline{\boldsymbol{u}}_n &\overset{*}{\rightharpoondown} \overline{\boldsymbol{u}} \quad \text{in } L^\infty(I, Y), \end{aligned} \qquad (n \to \infty).$$

Furthermore, it follows that $\overline{\boldsymbol{u}} \in \boldsymbol{\mathcal{W}}_e^{1,p,p}(I, V, H) \cap L^\infty(I, H)$ satisfies $\overline{\boldsymbol{u}}(0) = u_0$ in H and for all $\boldsymbol{\phi} \in L^p(I, V)$

$$\int_I \Big\langle \frac{d_e \overline{\boldsymbol{u}}}{dt}(t), \boldsymbol{\phi}(t) \Big\rangle_V \, dt + \int_I \langle A(t)(\overline{\boldsymbol{u}}(t)), \boldsymbol{\phi}(t) \rangle_X \, dt = \int_I \langle \boldsymbol{f}(t), \boldsymbol{\phi}(t) \rangle_X \, dt.$$

This result is to the best of the authors' knowledge the first one proving the convergence of a fully discrete approximation of problem (7.16). Moreover, it is even the first existence proof of weak solutions for the problem (7.16) at all.

8 Numerical experiments

To conclude, we want to present some numerical experiments with data having low regularity that perfectly suit the framework of this article. All numerical experiments were conducted employing the finite element software FEniCS [37]. All graphics are generated using the Matplotlib library [33].

We consider for $\Omega := (-1,1)^2 \subseteq \mathbb{R}^2$, $T := 0.1$, $Q_T := I \times \Omega$ and $p := \frac{11}{5}$, the system describing the unsteady motion of micropolar electrorheological fluids in two dimensions, i.e.,[11]

$$\begin{aligned} \partial_t \boldsymbol{u} - \operatorname{div} \mathrm{S} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \boldsymbol{q} &= \boldsymbol{f} && \text{in } I \times \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } I \times \Omega, \\ \partial_t \boldsymbol{\omega} - \operatorname{div} \mathrm{N} + \operatorname{div}(\boldsymbol{\omega} \boldsymbol{u}) &= \boldsymbol{\ell} - \boldsymbol{\varepsilon} : \mathrm{S} && \text{in } I \times \Omega, \\ \boldsymbol{u} = \boldsymbol{0}, \qquad \boldsymbol{\omega} &= \boldsymbol{0} && \text{on } I \times \partial\Omega, \\ \boldsymbol{u}(0) = u_0, \qquad \boldsymbol{\omega}(0) &= \omega_0 && \text{in } \Omega, \end{aligned} \tag{8.1}$$

where $\boldsymbol{\omega}\colon I \times \Omega \to \mathbb{R}$ is the scalar micro-rotation, $\boldsymbol{u}\colon I \times \Omega \to \mathbb{R}^2$ the velocity and $\boldsymbol{q}\colon I \times \Omega \to \mathbb{R}$ the pressure. The system (8.1) differs from its three-dimensional counterpart mainly in equation $(8.1)_3$, which is now a scalar equation that involves a scalar micro-rotation $\boldsymbol{\omega}$ (cf. [38] for the case $p = 2$). The analogue to Theorem 7.14 in this setting holds for $p > 2$. Moreover, for the electric field $\boldsymbol{E}$, solving the two-dimensional quasi-static Maxwell's equations (7.2), we make the particular choice

$$\boldsymbol{E}(t,x) := (t + x_2, t + x_1)^\top$$

for all $(t,x)^\top = (t,x_1,x_2)^\top \in Q_T$. We assume that the stress tensor $\mathrm{S} : \mathbb{M}^{2\times 2}_{\mathrm{sym}} \times \mathbb{M}^{2\times 2}_{\mathrm{skew}} \times \mathbb{R}^2 \to \mathbb{M}^{2\times 2}$ and the couple stress tensor $\mathrm{N} : \mathbb{M}^{2\times 2} \times \mathbb{R}^2 \to \mathbb{M}^{2\times 2}$, for any $D \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$, $R \in \mathbb{M}^{2\times 2}_{\mathrm{skew}}$ and $E, L \in \mathbb{R}^2$, have the form

$$\begin{aligned} \mathrm{S}(D,R,E) &:= (1 + |E|^2)(\kappa + |D|)^{p-2} D + |E|^2 (\kappa + |R|)^{p-2} R, \\ N(L,E) &:= (1 + |E|^2)(\kappa + |L|)^{p-2} L, \end{aligned}$$

with $\kappa = 0.001$, where we used the notation $D = \mathcal{D}u$, $R = R(u,\omega) := Wu + \boldsymbol{\varepsilon}\omega = Wu + \left(\begin{smallmatrix} 0 & \omega \\ -\omega & 0 \end{smallmatrix}\right)$. We treat solutions with a point singularity at the origin in the velocity and the micro-rotation. More precisely, we assume that for every $(t,x)^\top = (t,x_1,x_2)^\top \in Q_T$, there holds[12]

$$\boldsymbol{u}(t,x) := (t,t)^\top + |x|^{\alpha-1}(x_2,-x_1)^\top, \quad \boldsymbol{\omega}(t,x) := t + |x|^{\alpha-1} x_1, \quad \boldsymbol{q}(t,x) := 0, \tag{8.2}$$

where $\alpha := \frac{6}{5} - \frac{2}{p} \approx 0.291$. Then, making the choice

$$\begin{aligned} \boldsymbol{f} &:= \partial_t \boldsymbol{u} - \operatorname{div} \mathrm{S} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \in L^{p'}(I, (W^{1,p}_0(\Omega)^2)^*), \\ \boldsymbol{l} &:= \partial_t \boldsymbol{\omega} - \operatorname{div} \mathrm{N} + \operatorname{div}(\boldsymbol{\omega}\boldsymbol{u}) + \boldsymbol{\varepsilon} : \mathrm{S} \in L^{p'}(I, (W^{1,p}_0(\Omega))^*), \end{aligned} \tag{8.3}$$

the functions (8.2) solve (8.1) with right-hand sides (8.3). In particular, note that the parameter α is chosen so small that just $\boldsymbol{u}(t) \in W^{1,p}_{\mathrm{div}}(\Omega)^2$ for every $t \in \overline{I}$, since $|\mathcal{D}\boldsymbol{u}(t,x)| \sim |x|^{\alpha-1} \in L^p(\Omega)$, but simultaneously $\boldsymbol{u}(t) \notin W^{2,1}(\Omega)^2$ for every $t \in \overline{I}$. Similar, this choice guarantees $\mathrm{S}(t) \in L^{p'}(\Omega)^{2\times 2}$ for every $t \in \overline{I}$, but neither that $\mathrm{S}(t) \notin W^{1,p'}(\Omega)^{2\times 2}$, nor that $\operatorname{div} \mathrm{S}(t) \in L^{p'}(\Omega)^2$ for every $t \in \overline{I}$.

[11] Here, $\boldsymbol{\varepsilon} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathbb{M}^{2\times 2}$ denotes the two-dimensional Levi–Civita tensor.

[12] The exact solutions do not satisfy the homogeneous boundary conditions $(8.1)_4$. However, the error is mainly concentrated around the singularity in the origin and hence the small inconsistency with our theoretical setup does not have any influence in the results.

Thus, the right-hand side has just enough regularity, namely $\boldsymbol{f} \in L^{p'}(I,(W_0^{1,p}(\Omega)^2)^*)$, to fall into the framework of our weak convergence result Theorem 7.14. Exactly the same considerations also apply to both $\boldsymbol{\omega}(t) \in W^{1,p}(\Omega)$ and $\boldsymbol{l} \in L^{p'}(I,(W_0^{1,p}(\Omega))^*)$.

The chosen low regularity has the consequence that one still finds convergence of the scheme, i.e., at least weak convergence in the sense of Theorem 6.16, however, no stable convergence rates could be recorded experimentally (cf. Tables 1 to 4).

The spatial discretization of our domain Ω is obtained by a sequence of uniform finite element meshes $(\mathcal{T}_{h_n})_{n\in\mathbb{N}}$ consisting of triangles with straight sides and diameter $h_n := \frac{h_0}{2^n}$, $h_0 := 2\sqrt{2}$, for every $n \in \mathbb{N}$. Beginning with $\mathcal{T}_{h_1}$, see, e.g., Figure 1 the first and the third picture, for every $n \in \mathbb{N}$ with $n \geq 2$, the mesh $\mathcal{T}_{h_n}$ is a refinement of $\mathcal{T}_{h_{n-1}}$ obtained by subdividing each triangle into four, which is based an edge midpoint or regular $1:4$ refinement algorithm.

We consider the MINI element (cf. Table 1 and Table 2) and the spatially conforming Crouzeix–Raviart element (cf. Table 3 and Table 4). Furthermore, we use the time step-sizes $\tau_n := 0.02 \cdot 2^{-n}$, i.e., $K_n := 10 \cdot 2^n$, $n \in \mathbb{N}$. Then, the iterates $((u_n^k,\omega_n^k)^\top)_{k=0,\dots,K_n} \subseteq V_n$ solving the straightforward two-dimensional analog of (1.8) are approximated employing Newton's iteration. Apart from that, let the mapping $F : \mathbb{M}^{2\times 2}_{\mathrm{sym}} \to \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ be defined by $F(A) := (\kappa + |A|)^{\frac{p-2}{2}} A$ for every $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$. Equally, let the mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $G(a) := (\kappa + |a|)^{\frac{p-2}{2}} a$ for every $a \in \mathbb{R}^2$. Then, for $n = 1,\dots,6$, we are interested in the parabolic errors

$$
\begin{aligned}
e^n_{F,\boldsymbol{u}} &:= \bigg(\sum_{k=0}^{K_n} \tau_n \|F(\mathcal{D}\boldsymbol{u}(t_k)) - F(\mathcal{D}u_n^k)\|^2_{L^2(\Omega)^{2\times 2}}\bigg)^{\frac{1}{2}}, \quad e^n_{L^2,\boldsymbol{u}} := \max_{0\leq k\leq K_n} \|\boldsymbol{u}(t_k) - u_n^k\|_{L^2(\Omega)^2},\\
e^n_{G,\boldsymbol{\omega}} &:= \bigg(\sum_{k=0}^{K_n} \tau_n \|G(\nabla\boldsymbol{\omega}(t_k)) - G(\nabla\omega_n^k)\|^2_{L^2(\Omega)^2}\bigg)^{\frac{1}{2}}, \quad e^n_{L^2,\boldsymbol{\omega}} := \max_{0\leq k\leq K_n} \|\boldsymbol{\omega}(t_k) - \omega_n^k\|_{L^2(\Omega)},
\end{aligned}
$$

which can be considered as approximations of $\|F(\mathcal{D}\boldsymbol{u}) - F(\mathcal{D}\overline{\boldsymbol{u}}^\tau_n)\|_{L^2(Q_T)^{2\times 2}}$, $\|\boldsymbol{u} - \overline{\boldsymbol{u}}^\tau_n\|_{L^\infty(I,L^2(\Omega)^2)}$, $\|G(\nabla\boldsymbol{\omega}) - G(\nabla\overline{\boldsymbol{\omega}}^\tau_n)\|_{L^2(Q_T)^2}$ and $\|\boldsymbol{u} - \overline{\boldsymbol{u}}^\tau_n\|_{L^\infty(I,L^2(\Omega))}$. In particular, we are interested in the total parabolic errors $e^n_{\mathrm{tot},\boldsymbol{u}} := e^n_{F,\boldsymbol{u}} + e^n_{L^2,\boldsymbol{u}}$ and $e^n_{\mathrm{tot},\boldsymbol{\omega}} := e^n_{G,\boldsymbol{\omega}} + e^n_{L^2,\boldsymbol{\omega}}$ for $n = 1,\dots,6$. As an estimation of the convergence rates, we employ the experimental order of convergence (EOC):

$$
\texttt{EOC}(e^n) := \frac{\log\big(\frac{e^n}{e^{n-1}}\big)}{\log\big(\frac{h_n+\tau_n}{h_{n-1}+\tau_{n-1}}\big)}, \quad n = 2,\dots,6,
$$

where e^n, $n = 1,\dots,6$, either denote $e^n_{F,\boldsymbol{u}}$, $e^n_{L^2,\boldsymbol{u}}$, $e^n_{\mathrm{tot},\boldsymbol{u}}$, $e^n_{G,\boldsymbol{\omega}}$, $e^n_{L^2,\boldsymbol{\omega}}$, or $e^n_{\mathrm{tot},\boldsymbol{\omega}}$, $n = 1,\dots,6$, resp.

In order to obtain a higher accuracy in the computation of these errors, in particular, with regard to the singularities of the exact solutions around the origin, we interpolate both $\boldsymbol{u}(t_k)$ and u_n^k, or $\boldsymbol{\omega}(t_k)$ and ω_n^k, into polynomial spaces of higher order with respect to a suitably refined mesh, namely into $\mathcal{P}_5(\mathcal{T}'_{h_n})^2$, or $\mathcal{P}_5(\mathcal{T}'_{h_n})$, resp., where $\mathcal{T}'_{h_n}$ is a refinement of $\mathcal{T}_{h_n}$, which is obtained by applying the longest edge bisection method of FEniCS to $\mathcal{T}_{h_{n+1}}$ for all cells $T \in \mathcal{T}_{h_{n+1}}$ that satisfy $\mathrm{dist}(T,0) < 0.25$ and subsequently on the resulting refined mesh $\widetilde{\mathcal{T}}_{h_{n+1}}$ for all cells $T \in \widetilde{\mathcal{T}}_{h_{n+1}}$ that satisfy $\mathrm{dist}(T,0) < 0.1$, see e.g. Figure 1 the second and the fourth picture.

In this manner, we obtain the following results:

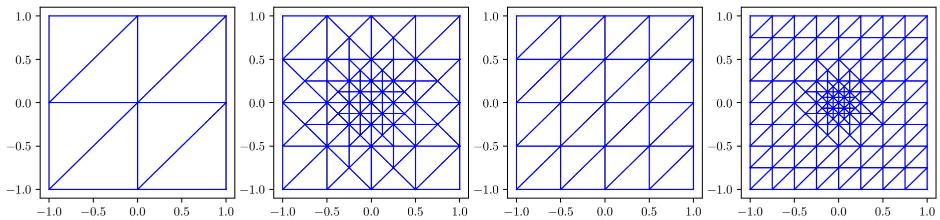

Fig. 1: From the left to the right: snapshots of the meshes $\mathcal{T}_{h_1}$, $\mathcal{T}'_{h_1}$, $\mathcal{T}_{h_2}$ and $\mathcal{T}'_{h_2}$.

n	$h_n = \frac{h_0}{2^n}$	$\tau_n = \frac{0.2T}{2^n}$	$e^n_{L^2,\boldsymbol{u}}$	$\texttt{EOC}(e^n_{L^2,\boldsymbol{u}})$	$e^n_{F,\boldsymbol{u}}$	$\texttt{EOC}(e^n_{F,\boldsymbol{u}})$	$\texttt{EOC}(e^n_{\text{tot},\boldsymbol{u}})$
1	1.414	1.00e-2	0.463	–	0.548	–	–
2	7.07e-1	5.00e-3	0.263	0.81	0.463	0.24	0.48
3	3.54e-1	2.50e-3	0.139	0.92	0.407	0.18	0.41
4	1.77e-1	1.25e-3	0.075	0.89	0.362	0.17	0.32
5	8.84e-2	6.25e-4	0.042	0.86	0.323	0.17	0.26
6	4.42e-2	3.13e-4	0.028	0.57	0.289	0.16	0.20

Table 1: Error analysis with respect to $\boldsymbol{u}$ for the MINI element.

n	$h_n = \frac{h_0}{2^n}$	$\tau_n = \frac{0.2T}{2^n}$	$e^n_{L^2,\boldsymbol{\omega}}$	$\texttt{EOC}(e^n_{L^2,\boldsymbol{\omega}})$	$e^n_{G,\boldsymbol{\omega}}$	$\texttt{EOC}(e^n_{G,\boldsymbol{\omega}})$	$\texttt{EOC}(e^n_{\text{tot},\boldsymbol{\omega}})$
1	1.414	1.00e-2	0.359	–	0.592	–	–
2	7.07e-1	5.00e-3	0.203	0.83	0.527	0.17	0.38
3	3.54e-1	2.50e-3	0.105	0.95	0.465	0.18	0.36
4	1.77e-1	1.25e-3	0.058	0.87	0.410	0.18	0.29
5	8.84e-2	6.25e-4	0.039	0.55	0.363	0.18	0.22
6	4.42e-2	3.13e-4	0.036	0.13	0.324	0.16	0.16

Table 2: Error analysis with respect to $\boldsymbol{\omega}$ for the MINI element.

n	$h_n = \frac{h_0}{2^n}$	$\tau_n = \frac{0.2T}{2^n}$	$e^n_{L^2,\boldsymbol{u}}$	$\texttt{EOC}(e^n_{L^2,\boldsymbol{u}})$	$e^n_{F,\boldsymbol{u}}$	$\texttt{EOC}(e^n_{F,\boldsymbol{u}})$	$\texttt{EOC}(e^n_{\text{tot},\boldsymbol{u}})$
1	1.414	1.00e-2	0.225	–	0.419	–	–
2	7.07e-1	5.00e-3	0.134	0.75	0.367	0.19	0.36
3	3.54e-1	2.50e-3	0.086	0.64	0.329	0.16	0.27
4	1.77e-1	1.25e-3	0.054	0.67	0.299	0.14	0.23
5	8.84e-2	6.25e-4	0.034	0.66	0.272	0.14	0.20
6	4.42e-2	3.13e-4	0.026	0.38	0.249	0.12	0.15

Table 3: Error analysis with respect to $\boldsymbol{u}$ for the spatially conforming Crouzeix–Raviart element.

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n	$h_n = \frac{h_0}{2^n}$	$\tau_n = \frac{0.2T}{2^n}$	$e^n_{L^2,\boldsymbol{\omega}}$	$\texttt{EOC}(e^n_{L^2,\boldsymbol{\omega}})$	$e^n_{G,\boldsymbol{\omega}}$	$\texttt{EOC}(e^n_{G,\boldsymbol{\omega}})$	$\texttt{EOC}(e^n_{\mathrm{tot},\boldsymbol{\omega}})$
1	1.414	1.00e-2	0.359	–	0.591	–	–
2	7.07e-1	5.00e-3	0.203	0.82	0.527	0.17	0.38
3	3.54e-1	2.50e-3	0.106	0.94	0.465	0.18	0.35
4	1.77e-1	1.25e-3	0.058	0.87	0.410	0.18	0.29
5	8.84e-2	6.25e-4	0.040	0.55	0.363	0.18	0.22
6	4.42e-2	3.13e-4	0.036	0.13	0.324	0.16	0.16

Table 4: Error analysis with respect to $\boldsymbol{\omega}$ for the spatially conforming Crouzeix–Raviart element.

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