THE EISENBUD-GREEN-HARRIS CONJECTURE

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Abstract. We survey most of the known results concerning the Eisenbud-Green-Harris Conjecture. Our presentation includes new proofs of several theorems, as well as a unified treatment of many results which are otherwise scattered in the literature. We include a final section with some applications, and examples.

1. An introduction to the conjecture

A very important problem in Commutative Algebra is the study of the growth of the Hilbert function of an ideal in a given degree *if one knows more than one step of* [its] *history*, cit. Mark Green [Gre98]. A classical theorem, due to Macaulay [Mac27], answers this question by providing an estimate on the Hilbert function in a given degree just by knowing its value in the previous one. This result is very useful, but it is far from being optimal. For instance, there is no way of taking into account any additional information about the ideal. The Eisenbud-Green-Harris, henceforth EGH, Conjecture was first raised in [EGH93, EGH96], and precisely addresses this matter. By effectively using the additional data that the given ideal contains a regular sequence, it predicts for instance more accurate growth bounds.

We will now introduce some notation and terminology in order to state the EGH Conjecture. Throughout this article, $A = \bigoplus_{d \geqslant 0} A_d$ will denote a standard graded polynomial ring $K[x_1, \ldots, x_n]$ over a field K, and $\mathfrak{m} = (x_1, \ldots, x_n)$ its homogeneous maximal ideal. We consider A equipped with the lexicographic order $\geqslant$ induced by $x_1 > x_2 > \ldots > x_n$. Given polynomials $g_1, \ldots, g_s \in A$, we will denote by $\langle g_1, \ldots, g_s \rangle$ the K-vector space generated by such elements to distinguish it from the ideal that they generate, which we denote by $(g_1, \ldots, g_s)$. We denote the Hilbert function of a graded module M and its value in d by $H(M)$ and $H(M; d)$, respectively. On the set of Hilbert functions we consider the partial order given by point-wise inequality. Recall that a K-vector space $V \subseteq A_d$ is called *lex-segment* if there exists a monomial $v \in V$ such that $V = \{u \in A_d \mid u \text{ monomial}, u \geqslant v\}$.

The classical Macaulay Theorem states that, given any homogeneous ideal I, if one lets $L_d \subseteq A_d$ be the lex-segment of dimension equal to $H(I; d)$, then $\operatorname{Lex}(I) = \bigoplus_{d \geqslant 0} L_d$ is an ideal, that we call lex-ideal. In order to take into account that I contains a regular sequence, we will introduce the so-called lex-plus-powers ideals.

Given an integer $0 < r \leqslant n$, we let $\underline{a} = (a_1, \ldots, a_r)$ denote an ordered sequence of integers $0 < a_1 \leqslant \ldots \leqslant a_r$, and we call it a *degree sequence*. We call the ideal $\mathfrak{a} = (x_1^{a_1}, \ldots, x_r^{a_r}) \subseteq A$ the *pure-powers ideal of degree* $\underline{a}$. With any homogeneous ideal $I \subseteq A$ which contains an ideal $\mathbf{f}$ generated by a regular sequence $f_1, \ldots, f_r$, of degree $\underline{a} = (a_1, \ldots, a_r)$, we associate the K-vector space

$$\mathrm{LPP}^{\underline{a}}(I) = \bigoplus_{d \geqslant 0} \langle L_d + \mathfrak{a}_d \rangle,$$

where $L_d \subseteq A_d$ is the largest, hence unique, lex-segment which satisfies $H(I;d) = \dim_K \langle L_d + \mathfrak{a}_d \rangle$. As Macaulay Theorem proves that $\mathrm{Lex}(I)$ is an ideal, the EGH Conjecture predicts that $\mathrm{LPP}^{\underline{a}}(I)$ is an ideal, which we call *the lex-plus-powers ideal associated with I with respect to the degree sequence $\underline{a}$*.

Conjecture 1.1 (EGH)**.** Let $I \subseteq A$ be a homogeneous ideal that contains a homogeneous ideal $\mathbf{f}$ generated by regular sequence of degree $\underline{a}$. Then $\mathrm{LPP}^{\underline{a}}(I)$ is an ideal.

Observe that the EGH Conjecture is a generalization of Macaulay Theorem, which corresponds to the case $\mathbf{f} = (f_1)$ with respect to any $0 \neq f_1 \in I$ of degree a_1. Just like lexicographic ideals, lex-plus-powers ideals enjoy several properties of extremality. For example, assuming that the EGH Conjecture is true in general, then one can show that the growth of $\mathrm{LPP}^{\underline{a}}(I)$ in each degree is smaller than that of I. That is, $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I)) \leqslant H(\mathfrak{m}I)$, see Lemma 2.14. This immediately translates into an inequality $\beta_{0j}(\mathrm{LPP}^{\underline{a}}(I)) \geqslant \beta_{0j}(I)$ between minimal number of generators in each degree j. We point out that the more refined version of such inequality, i.e.,

$$\beta_{ij}(\mathrm{LPP}^{\underline{a}}(I)) \geqslant \beta_{ij}(I) \qquad \text{for all } i, j,$$

is currently unknown in general, and goes under the name of LPP-Conjecture, see for instance [Fra04, Ric04, FR07, MPS08, MM11, CS18].

In the following, it will be useful to have several formulations of the EGH Conjecture, which we will use interchangeably at our convenience.

An equivalent way of approaching the conjecture is degree by degree: given a sequence $\underline{a}$, for a non-negative integer d we say that a homogeneous ideal $I \subseteq A = K[x_1, \ldots, x_n]$ *satisfies* $\mathrm{EGH}_{\underline{a}}(d)$ if there exists an $\underline{a}$-lpp ideal J such that $\dim_K(J_d) = \dim_K(I_d)$ and $\dim_K(J_{d+1}) \leqslant \dim_K(I_{d+1})$. We say that I *satisfies* $\mathrm{EGH}_{\underline{a}}$ if it satisfies $\mathrm{EGH}_{\underline{a}}(d)$ for all non-negative integers d. One can readily verify that Conjecture 1.1 holds true if and only if, for every degree sequence $\underline{a}$, every homogeneous ideal containing a regular sequence of degree $\underline{a}$ satisfies $\mathrm{EGH}_{\underline{a}}$, see [CM08].

We conclude this introductory section by recalling a weaker version of the EGH Conjecture, raised in [EGH96]. Let $\underline{a} = (a_1, \ldots, a_n)$ be a degree sequence, and D be an integer such that $a_1 \leqslant D \leqslant \sum_{i=1}^n (a_i - 1)$. Let b the unique integer such that $\sum_{i=1}^b (a_i - 1) \leqslant D < \sum_{i=1}^{b+1} (a_i - 1)$, and set $\delta = \sum_{i=1}^{b+1} (a_i - 1) - D + 1$ if $b < n$, and $\delta = 1$ otherwise.

Conjecture 1.2 (Cayley-Bacharach)**.** Let $\mathbf{f} \subseteq A = K[x_1, \ldots, x_n]$ be an ideal generated by a regular sequence of degree $\underline{a} = (a_1, \ldots, a_n)$, and $g \notin \mathbf{f}$ be a homogeneous element of degree $D \geqslant a_1$. Let $I = \mathbf{f} + (g)$, and e be the multiplicity of A/I. Then

$$e \leqslant \prod_{i=1}^n a_i - \delta \prod_{i=b+1}^n a_i.$$

Conjecture 1.2 has been studied by several researchers, from very different points of view; for instance, see [GKR93, GK13, CDS20a, HLU20]. The validity of the EGH Conjecture in the case $r = n$ for almost complete intersections would imply Conjecture 1.2. For an explicit instance of this, see Example 5.7.

This survey paper is structured as follows: in Section 2 we treat the case when the given ideal already contains a pure-powers ideal, presenting a new proof of the Clements-Lindström Theorem. Section 3 is very brief, and collects some statements from the theory of linkage, together with a result which yields a reduction to the Artinian case. In Section 4 we present proofs of several cases of the conjecture, previously known in the literature. Finally, in Section 5 we collect some applications of the techniques and the results illustrated before, together with several examples.

2. Monomial regular sequences and the Clements-Lindström Theorem

The goal of this section is to prove the Clements-Lindström Theorem [CL69], a more general version of the Kruskal-Katona Theorem [Kru63, Kat68]. The proof presented here relies on recovering a strong hyperplane restriction theorem for strongly-stable-plus-powers and lpp ideals due to Gasharov [Gas98, Gas99], see also [CS16, Theorem 2.2]. Our strategy uses the techniques of [CK13], and is different from the standard one available in the literature [CL69, MP06, MP07].

Recall that a monomial ideal $J \subseteq A = K[x_1, \ldots, x_n]$ is called *strongly stable* if for every monomial $u \in J$ and any variable x_i which divides u, one has that $x_i^{-1}x_j u \in J$ for all $1 \leqslant j \leqslant i$. The ideal J is said to be *$\underline{a}$-strongly-stable-plus-powers*, $\underline{a}$-spp or, simply, spp for short, if there exist a strongly stable ideal S and a pure power ideal $\mathfrak{a}$ of degree $\underline{a}$ such that $J = S + \mathfrak{a}$. Clearly, $\underline{a}$-lpp ideals are $\underline{a}$-spp.

Theorem 2.1. *Let $I \subseteq A$ be a homogeneous ideal that contains a pure-powers ideal $\mathfrak{a}$ of degree $\underline{a}$. Then*

(i) $\mathrm{LPP}^{\underline{a}}(I)$ *is an ideal.*
(ii) If I is $\underline{a}$-spp, then $H(I + (x_n^i)) \geqslant H(\mathrm{LPP}^{\underline{a}}(I) + (x_n^i))$ for all $i > 0$.

We first prove Theorem 2.1 (i) for $n = r = 2$. Since strongly stable ideals in two variables are lex-ideals, $\underline{a}$-spp ideals are automatically $\underline{a}$-lpp in this case.

We start by recalling a few properties of monomial ideals, which are special cases of more general results derived from linkage theory, that we will discuss in Section 3.

Let I be a monomial ideal that contains $\mathfrak{a} = (x_1^{a_1}, x_2^{a_2})$. When we view I as a $K[x_1]$-module, we have a decomposition

$$I = \bigoplus_{i \geqslant 0} x_1^{d_i} K[x_1] \cdot x_2^i; \tag{2.1}$$

observe that, since I is an ideal, one has $d_i \geqslant d_{i+1}$ for all i. Also observe that I is spp if and only if $d_{i+1} + 1 \geqslant d_i$ for all i. Define the link $I^\ell = I^\ell_{\mathfrak{a}}$ of I with respect to the ideal $\mathfrak{a}$ to be the ideal $I^\ell = (\mathfrak{a} :_A I)$. Notice that $I^\ell = (x_1^{a_1-d_0}, x_2^{a_2}) \cap (x_1^{a_1-d_1}, x_2^{a_2-1}) \cap \cdots \cap (x_1^{a_1-d_{a_2-1}}, x_2)$ is an ideal generated by the monomials $x_1^{a_1-d_i} x_2^{a_2-1-i}$, $i = 0, \ldots, a_2-1$, and that as a $K[x_1]$-module can be written as

$$I^\ell = \left(\bigoplus_{i=0}^{a_2-1} x_1^{a_1-d_{a_2-1-i}} K[x_1] \cdot x_2^i \right) \oplus \left(\bigoplus_{i \geqslant a_2} K[x_1] \cdot x_2^i \right). \tag{2.2}$$

Remark 2.2. (1) It is immediate from (2.2) that $(I^\ell)^\ell = I$.
(2) The Hilbert function of I^ℓ is determined by that of I. More precisely, if we let $R = A/\mathfrak{a}$ and $s = a_1 + a_2 - 2$, then $H(R;d) = H(R/IR;d) + H(R/I^\ell R; s-d)$.
(3) The link of an $\underline{a}$-lpp ideal is again an $\underline{a}$-lpp ideal. Thus, we may as well prove that I^ℓ is $\underline{a}$-spp if I is $\underline{a}$-spp. To this end, consider the decomposition of I as in (2.1). Given any monomial $x_1^{b_1}x_2^{b_2} \in I^\ell$ with $1 \leqslant b_2 < a_2$, one just needs to show that $x_1^{b_1+1}x_2^{b_2-1} \in I^\ell$. By (2.2), it is enough to verify that $a_1 - d_i + 1 \geqslant a_1 - d_{i+1}$ for all i, which is equivalent to $d_{i+1} + 1 \geqslant d_i$ for all i. Finally, this is true for all i, because I is spp by assumption.

We are now ready to prove the case $n = 2$ of Theorem 2.1 (i).

Proposition 2.3. *Let* $\underline{a} = (a_1, a_2)$*, and* $I \subseteq A = K[x_1, x_2]$ *be a homogeneous ideal that contains* $\mathfrak{a} = (x_1^{a_1}, x_2^{a_2})$*. Then* $\mathrm{LPP}^{\underline{a}}(I)$ *is an ideal.*

After taking any initial ideal, without loss of generality we may assume that I is monomial. In fact, this operation preserves its Hilbert function, and the initial ideal still contains $\mathfrak{a}$. Next, we give three different proofs of the above proposition.

In the first one, we make use of linkage.

Proof 1. We need to show that the K-vector space $\mathrm{LPP}^{\underline{a}}(I) = \bigoplus_{j \geqslant 0} \langle L_j + \mathfrak{a}_j \rangle$ is indeed an ideal, and we do so by proving that $\mathrm{LPP}^{\underline{a}}(I)$ agrees with an ideal J for all degrees $i \leqslant a_2 - 1$ and it agrees with an ideal J' for all degrees $i \geqslant a_2 - 1$. By Macaulay Theorem, there is a lex-ideal L with the same Hilbert function as I. Consider the $\underline{a}$-lpp ideal $J = L + \mathfrak{a}$. By construction, for all $j = 1, \ldots, a_2 - 1$ one has $H(J;j) = H(L;j) = H(I;j)$.

Now we construct the ideal J' as follows. First consider the link $I^\ell = (\mathfrak{a} :_A I)$. Since $I^\ell \supseteq \mathfrak{a}$, again by Macaulay Theorem there exists a lexicographic ideal L' with the same Hilbert function as I^ℓ. Thus, the $\underline{a}$-lpp ideal $L' + \mathfrak{a}$ has the same Hilbert function as I^ℓ in degrees $j = 0, \ldots, a_1 - 1$. We now let $J' = (L' + \mathfrak{a})^\ell$. By Remark 2.2 (3), J' is an lpp ideal and, by Remark 2.2 (2) its Hilbert function in degrees $j \geqslant a_2 - 1$ coincides with that of I. Therefore J' has the desired properties, and the proof is complete. □

In the second proof we use techniques borrowed from [MM11, Section 3], see also [CK13, Section 4].

Proof 2. The Hilbert function of a monomial ideal is independent of the base field, thus without loss of generality we may assume that $K = \mathbb{C}$. It suffices to construct an $\underline{a}$-spp ideal with the same Hilbert function as I. Let $\xi_1, \ldots, \xi_{a_2}$ the a_2-roots of unity over $\mathbb{C}$, and observe that $x_2^{a_2} - x_1^{a_2} = (x_2 - \xi_1 x_1)(x_2 - \xi_2 x_1)\cdots(x_2 - \xi_{a_2} x_1) \in I$. We consider the distraction $\mathcal{D}$ given by a family of linear forms $\{l_i\}_{i \geqslant 1}$ defined as $l_i = x_2 - \xi_i x_1$, for $i = 1, \ldots, a_2$, and $l_i = x_2$ for all $i > a_2$; see [BCR05] for the theory of distractions. Given a decomposition of $I^{(0)} = I = \bigoplus_{i \geqslant 0} I_{[i]} x_2^i$, we let $J^{(0)}$ be the distracted ideal

$$J^{(0)} = J = \bigoplus_{i \geqslant 0} I_{[i]} \prod_{j=1}^{i} l_j = \bigoplus_{i=0}^{a_2} I_{[i]} \prod_{j=1}^{i} l_j \oplus \bigoplus_{i \geqslant a_2} K[x_1] \cdot x_2^i,$$

which shares with I the same Hilbert function, and the same Betti numbers as well. Observe that the last equality is due to the fact that both $x_1^{a_2}$ and $x_2^{a_2} - x_1^{a_2}$ are in J, and therefore $x_2^{a_2} \in J$. We let $I^{(1)}$ be $\mathrm{in}_>(J^{(0)})$, where $>$ is any monomial order such that $x_1 > x_2$, and $J^{(1)}$

be the ideal obtained by distracting $I^{(1)}$ with $\mathcal{D}$. We construct in this way a sequence $I^{(0)}$, $I^{(1)}, \dots, I^{(h)}$ of ideals with the same Hilbert function, each of which contains $\mathfrak{a}$; we finally want to show that this sequence eventually stabilizes at an ideal, we call it L, which is $\underline{a}$-spp. To this end, observe that for all integers $p \geqslant 0$ we have

$$\begin{aligned} H\left(I^{(h)}_{[0]} \oplus I^{(h)}_{[1]} x_2 \oplus \cdots \oplus I^{(h)}_{[p]} x_2^p\right) &= H\left(\operatorname{in}_>(I^{(h)}_{[0]} \oplus I^{(h)}_{[1]} l_1 \oplus \cdots \oplus I^{(h)}_{[p]} \prod_{j=1}^{p} l_j)\right) \\ &\leqslant H\left(I^{(h+1)}_{[0]} \oplus I^{(h+1)}_{[1]} x_2 \oplus \cdots \oplus I^{(h+1)}_{[p]} x_2^p\right). \end{aligned} \tag{2.3}$$

In the above, we consider three modules whose Hilbert functions are computed as homogenous $K[x_1]$-submodules of the graded $K[x_1]$-module $A = K[x_1, x_2]$, where x_2^d has degree d. Notice that the inequality in (2.3) is due to the inclusion of the second module in the third one. Observe that $I^{(0)}_{[0]} \subseteq I^{(1)}_{[0]} \subseteq \dots$ is an ascending chain of ideals that will eventually stabilize, say at $I^{(h_0)}_{[0]}$. Inductively, assume that for all $i = 0, \dots, p-1$ the ideals in $I^{(h_{i-1})}_{[i]} \subseteq I^{(h_{i-1}+1)}_{[i]} \subseteq \dots$ form a chain that stabilizes, say at h_i. The inclusion of the second into the third module of (2.3), for any $h > \max\{h_0, \dots, h_{p-1}\}$, yields that $I^{(h)}_{[p]} \subseteq I^{(h+1)}_{[p]}$. Thus, for $h \geqslant h_{p-1}$ we have again a chain of ideals which will stabilize, say at h_p. Repeat this process for all $p \leqslant a_2 - 1$, so that for all $h \geqslant h' = \max\{h_1, \dots, h_{a_2-1}\}$ we have $I^{(h)} = I^{(h+1)}$. Let $L = I^{(h')}$. Keeping in mind how L has been constructed, apply (2.3) to L to obtain, for all $p \geqslant 0$

$$\begin{aligned} L_{[0]} \oplus L_{[1]} x_2 \oplus \cdots \oplus L_{[p]} x_2^p &= \operatorname{in}_>\left(L_{[0]} \oplus L_{[1]} l_1 \oplus \cdots \oplus L_{[p]} \prod_{j=1}^{p} l_j\right) \\ &= L_{[0]} \oplus L_{[1]} l_1 \oplus \cdots \oplus L_{[p]} \prod_{j=1}^{p} l_j, \end{aligned} \tag{2.4}$$

where the second equality can be verified by induction on p, using the first equality and the fact that the least monomial with respect to $>$ in the support of $\prod_{j=1}^{p} l_j$ is x_2^p.

Next, we prove that L is $\underline{a}$-spp. By construction $L \supseteq \mathfrak{a}$, since each $I^{(i)}$ and $J^{(i)}$ does; thus, we have to show that $x_1 L_{[p]} \subseteq L_{[p-1]}$ that for all $0 < p \leqslant a_2 - 1$. Again by induction on p, by (2.4) we have $L_{[0]} \oplus L_{[1]} x_2 = L_{[0]} \oplus L_{[1]}(x_2 - x_1)$, which implies $x_1 L_{[1]} \subseteq L_{[0]}$. Moreover, by induction and again by (2.4), $L_{[0]} \oplus L_{[1]} x_2 \oplus \cdots \oplus L_{[p]} x_2^p = L_{[0]} \oplus L_{[1]} x_2 \oplus \cdots \oplus L_{[p-1]} x_2^{p-1} \oplus L_{[p]} \prod_{j=1}^{p} l_j$. Since $l_j = x_2 - \xi_j x_1$ with $j = 1, \dots, p$, we have that $\prod_{j=1}^{p} l_j$ has a full support, i.e., its support contains all of the monomials of degree p. In particular it contains $x_1 x_2^{p-1}$. It follows that $x_1 L_{[p]} \subseteq L_{[p-1]}$, as desired. □

The third proof relies on an application of Gotzmann Persistence Theorem [Got78, Gre89].

Proof 3. Let $\mathrm{LPP}^{\underline{a}}(I) = L + \mathfrak{a}$, where each L_d is the largest lex-segment such that $\dim_K(L_d + \mathfrak{a}_d) = H(I; d)$. In order to show that $\mathrm{LPP}^{\underline{a}}(I)$ is an ideal we have to show that, for every integer $d \geqslant 0$, we have $H(A/(\mathfrak{m}L + \mathfrak{a}); d+1) \geqslant H(A/\mathrm{LPP}^{\underline{a}}(I); d+1)$. For this, without loss of generality we can assume that $(\mathrm{LPP}^{\underline{a}}(I))_j = \mathfrak{a}_j$ for all $j < d$. Let $k = \dim_K \widetilde{L_d}$, where $\widetilde{L_d}$

is the image in $A/\mathfrak{a}$ of the K-vector space $L_d + \mathfrak{a}_d$. If $k = 0$ there is nothing to prove. Let us assume $k > 0$, and study the following three cases separately: $d < a_2 - 1$, $d = a_2 - 1$, and $d \geqslant a_2$. If $d < a_2 - 1$, then $(\mathrm{LPP}^{\underline{a}}(I))_d = L_d$, $(\mathrm{LPP}^{\underline{a}}(I))_{d+1} = L_{d+1}$, and the conclusion follows from Macaulay Theorem.

Now assume $d = a_2 - 1$. If $L_{d+1} = A_{d+1}$, then there is nothing to show, so assume that $L_{d+1} \subsetneq A_{d+1}$. If x_2^{d+1} is a minimal generator of I, then $H(A/(\mathfrak{m}L_d); d+1) \geqslant H(A/\mathfrak{m}I; d+1) \geqslant H(A/I; d+1)+1$. Since $H(A/I; d+1) = H(A/L; d+1)-1$, it follows that $H(A/\mathfrak{m}L; d+1) \geqslant H(A/L; d+1)$, and therefore $\mathfrak{m}_1 L_d \subseteq L_{d+1}$. A fortiori, we have that $\mathfrak{m}_1(\mathrm{LPP}^{\underline{a}}(I))_d \subseteq (\mathrm{LPP}^{\underline{a}}(I))_{d+1}$, and the proof of this case is complete. If x_2^{d+1} is not a minimal generator of I, then $\dim(A/J) = 0$, where $J = I_{\leqslant d}$. In particular, $H(A/I; j) \leqslant H(A/(x_1^d, x_2^d); d) = d$. By Macaulay Theorem we have that $H(A/I; d+1) \leqslant d$. If equality holds, then I has no minimal generators in degree $d + 1$, and thus $H(A/J; d + 1) = d$ as well. By Gotzmann Persistence Theorem applied to J, we have that $H(A/H; j) = d$ for all $j \geqslant d$, which contradicts the fact that $\dim(A/J) = 0$.

Finally, if $d \geqslant a_2$, we first observe that once again $k = H(A/I; d) \leqslant d$, and that $H(A/(\mathfrak{m}L_d) + \mathfrak{a}; d + 1) = k - 1$. Since $k = H(A/I; d)$, to conclude the proof it suffices to show that $k > H(A/I; d+1)$, since the latter is equal to $H(A/\mathrm{LPP}^{\underline{a}}(I); d+1)$. It follows from Macaulay Theorem $H(A/I; d + 1) \leqslant k = H(A/I; d)$, since we have already observed that $k \leqslant d$. If equality holds, then by Gotzmann Persistence Theorem applied to the ideal $J = I_{\leqslant d}$ we would have that $H(A/J; j) = H(A/J; d) = k > 0$ for all $j \geqslant d$. In particular, this would imply that $\dim(A/J) > 0$, in contrast with the fact that J contains $(x_1^{a_1}, x_2^{a_2})$, and hence it is Artinian. ☐

Remark 2.4. (1) Observe that Proof 1 can be adapted to any regular sequence of degree $\underline{a} = (a_1, a_2)$ using properties of linkage analogous to those of Remark 2.2, see Theorem 4.1.
(2) It is easy to see that, in Proof 2, we can also keep track of Betti numbers and prove, in characteristic zero, that they cannot decrease when passing to the lex-plus-powers ideal.
(3) In Proof 3 we do not actually use the fact that the regular sequence is monomial. In fact, the same argument can be used to prove that any ideal which contains a regular sequence of degree $\underline{a} = (a_1, a_2)$ satisfies $\mathrm{EGH}_{\underline{a}}$.

We now move our attention from the case $n = 2$ to the general one.

Proposition 2.5. *Under the same assumptions of Theorem 2.1, there exists an $\underline{a}$-spp ideal with the same Hilbert function as that of I.*

Proof. We define a total order on the set $\mathcal{S}$ of monomial ideals with the same Hilbert function as I, and which contain the pure-powers ideal $\mathfrak{a} = (x_1^{a_1}, \ldots, x_r^{a_r})$. First, given any $J \in \mathcal{S}$, we order the set of its monomials $\{m_i\}$ from lower to higher degrees, and monomials of the same degree lexicographically. Now, given a second ideal $J' \in \mathcal{S}$ and the set of its monomials $\{m'_i\}$, we set $J > J'$ if and only if there exists i such that $m_j = m'_j$ for all $j \leqslant i$ and $m_{i+1} > m'_{i+1}$. Observe that, since J and J' have the same Hilbert function, we are forced to have $\deg m_j = \deg m'_j$ for all j. Let P be the maximal element of $\mathcal{S}$; we claim that P is $\underline{a}$-spp. Assume by contradiction that there exists a monomial $m \in P \smallsetminus \mathfrak{a}$ such that x_i divides m and $x_i^{-1}x_j m \notin P$ for some $j < i$. Write $P = \bigoplus_q P_q \cdot q$, where each $q \in K[x_1, \ldots, \widehat{x_j}, \ldots, \widehat{x_i}, \ldots, x_n]$ is a monomial, and $P_q \subseteq K[x_j, x_i]$ is an ideal. Notice that each P_q contains $(x_j^{a_j}, x_i^{a_i})K[x_j, x_i]$ since $P \in \mathcal{S}$, and that $P_q \subseteq P_{q'}$ whenever q divides q' since P is an ideal. By Proposition 2.3, for every q there exists an (a_j, a_i)-spp ideal $Q_q \subseteq K[x_j, x_i]$ with the same Hilbert function as P_q.

Let now $Q = \bigoplus_q Q_q \cdot q$, and observe that $Q \in \mathcal{S}$. In fact, Q is clearly spanned by monomials, and it contains $\mathfrak{a}$. Moreover, if q divides q' one gets $H(Q_q) = H(P_q) \leqslant H(P_{q'}) = H(Q_{q'})$. Since Q_q and $Q_{q'}$ are both (a_j, a_i)-spp, it follows that $Q_q \subseteq Q_{q'}$, which in turn that Q is an ideal. Since P is not $\underline{a}$-spp, by our choice of the indices i and j there exists q such that P_q is not (a_j, a_i)-spp. In particular, it follows that $Q > P$, which contradicts maximality of P. □

Remark 2.6. As in the case of two variables, see Remark 2.4 (2), in the proof of Proposition 2.5 one can keep track of how the Betti numbers change in order to prove that, in characteristic zero, the Betti numbers of the $\underline{a}$-spp ideal we obtain cannot decrease. This fact is helpful in order to prove the LPP-Conjecture for ideals that contain pure-powers.

We point out that, in all pre-existing proofs of Clements-Lindström Theorem 2.1 [CL69, MP06, MP07], one finds a preliminary reduction step that goes under the name of *compression*. This step consists of assuming that Clements-Lindström Theorem holds in $n-1$ variables in order to construct an $\underline{a}$-spp ideal $J \subseteq A$ in n variables that, for any $i = 1, \ldots, n$, has a decomposition $J = \bigoplus_{j \geqslant 0} J_{[j]} x_i^j$, where $J_{[j]}$ is $(a_1, \ldots, \widehat{a_i}, \ldots, a_r)$-lpp for all j. In our proof, this step corresponds to the reduction provided by Proposition 2.5. Observe that the above ideal J is not necessarily $\underline{a}$-lpp globally in n variables, as the following example shows.

Example 2.7. Let $n \geqslant 4$ and consider the $(2,2)$-spp ideal $I = (x_1^2, x_1x_2, \ldots, x_1x_{n-1}, x_2^2, x_2x_3)$ in $A = K[x_1, \ldots, x_n]$; then I is compressed, but not $(2,2)$-lpp, since the monomial x_1x_n is missing from its generators.

We introduce some notation and terminology, which will be used henceforth in this section. Let $A = K[x_1, \ldots, x_n]$, $\mathfrak{m} = (x_1, \ldots, x_n)$, $\underline{a} = (a_1, \ldots, a_r)$ be a degree sequence, and $\mathfrak{a} = (x_1^{a_1}, \ldots, x_r^{a_r})$ be the corresponding pure-powers ideal. Furthermore, let $\overline{A} = K[x_1, \ldots, x_{n-1}]$, and $\overline{\mathfrak{m}} = (x_1, \ldots, x_{n-1})\overline{A}$. If $r < n$, we let $\overline{\underline{a}} = \underline{a}$ and $\overline{\mathfrak{a}} = (x_1^{a_1}, \ldots, x_r^{a_r})\overline{A}$. Otherwise, if $r = n$, we let $\overline{\underline{a}} = (a_1, \ldots, a_{n-1})$ and $\overline{\mathfrak{a}} = (x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}})\overline{A}$.
Given a K-vector space $V \subseteq A_d$ generated by monomials, we say that V is *$\underline{a}$-lpp* if it is the truncation in degree d of an $\underline{a}$-lpp ideal. Similarly, we say that V is *$\underline{a}$-spp* if it is the truncation in degree d of an $\underline{a}$-spp ideal. Observe that a K-vector subspace $V = \bigoplus_{i=0}^d V_{[d-i]} x_n^i \subseteq A_d$ containing $\mathfrak{a}_d$ is $\underline{a}$-spp if and only if $V_{[i]}$ is $\overline{\underline{a}}$-spp for all i, and $\overline{\mathfrak{m}}_1 V_{[i]} \subseteq V_{[i+1]}$ for all $i \geqslant \max\{d - a_n + 1, 0\}$; we will refer to the latter property as *stability*. Moreover, if $V \subseteq A_d$ is $\underline{a}$-lpp, respectively $\underline{a}$-spp, then $\mathfrak{m}_1 V + \mathfrak{a}_{d+1}$ is also $\underline{a}$-lpp, respectively $\underline{a}$-spp. Finally, if $V, W \subseteq A_d$ are $\underline{a}$-lpp and $\dim_K(V) \leqslant \dim_K(W)$, then $V \subseteq W$.

Let $L \subseteq \overline{A}_d$ be a lex-segment and $V = L + \overline{\mathfrak{a}}_d$. If $V \neq \overline{A}_d$, there exists the largest monomial $u \in \overline{A}_d \smallsetminus V$ with respect to the lexicographic order. In this case, we let $V^+ = V + \langle u \rangle$; otherwise, we let $V^+ = V = A_d$. Either way, V^+ can be written as $L' + \overline{\mathfrak{a}}_d$, where L' is a lex-segment, and therefore it is $\underline{a}$-lpp.
If $V \neq \overline{\mathfrak{a}}_d$ we may write $V = W \oplus \overline{\mathfrak{a}}_d$, with $W \neq 0$ a vector space minimally generated by monomials $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_t$. In this case, we let $V^- = \langle m_1, \ldots, m_{t-1} \rangle + \overline{\mathfrak{a}}_d$; otherwise, we set $V^- = V = \overline{\mathfrak{a}}_d$.

The notion of segment we recall next is extracted from [CK13], and it will be crucial in the proof of Theorem 2.1.

Definition 2.8. Let $V \subseteq A_d$ be a K-vector space, written as $V = \bigoplus_{i=0}^d V_{[d-i]} x_n^i$. Then, V is called an *$\underline{a}$-segment*, or simply a *segment*, if it is $\underline{a}$-spp and, for all i,

(i) $V_{[i]} \subseteq \overline{A}_i$ is $\overline{a}$-lpp, and
(ii) $V_{[i+j]} \subseteq \overline{\mathfrak{m}}_j (V_{[i]})^+ + \overline{\mathfrak{a}}_{i+j}$ for all $1 \leqslant j \leqslant d-i$.

Note that, if $V \subseteq A_d$ is $\underline{a}$-lpp, then it is an $\underline{a}$-segment.

Remark 2.9. If $V \subseteq A_d$ is a segment, it immediately follows from the definition that $\mathfrak{m}_1 V + \mathfrak{a}_{d+1} \subseteq A_{d+1}$ is also an $\underline{a}$-segment.

Lemma 2.10. *Let V and W be two $\underline{a}$-segments in A_d. Then either $V \subseteq W$, or $W \subseteq V$.*

Proof. Write $V = \bigoplus_{i=0}^d V_{[d-i]} x_n^i$ and $W = \bigoplus_{i=0}^d W_{[d-i]} x_n^i$. If the conclusion is false, since V and W are segments we can find $i \neq j$ such that $V_{[i]} \subsetneq W_{[i]}$ and $V_{[j]} \supsetneq W_{[j]}$; say $j < i$. Since $V_{[j]}$ is lpp, $V_{[j]} \supseteq (W_{[j]})^+$, and therefore $V_{[i]} = V_{[i]} + \overline{\mathfrak{a}}_i \supseteq \overline{\mathfrak{m}}_{i-j} V_{[j]} + \overline{\mathfrak{a}}_i \supseteq \overline{\mathfrak{m}}_{i-j} (W_{[j]})^+ + \overline{\mathfrak{a}}_i \supseteq W_{[i]}$, which is a contradiction. ☐

Definition 2.11. Let $V \subseteq A_d$ be a K-vector space, written as $V = \bigoplus_{i=0}^d V_{[d-i]} x_n^i$. We define the dimension sequence $\underline{\delta}(V) = (\dim_K(V_{[d]}), \dim_K(V_{[d]} \oplus V_{[d-1]}), \ldots, \dim_K(V)) \in \mathbb{N}^{d+1}$. On the set of all such sequences, we consider the partial order given by point-wise inequality.

Lemma 2.12. *Let $V \subseteq A_d$ be an $\underline{a}$-spp K-vector space, written as $V = \bigoplus_{i=0}^d V_{[d-i]} x_n^i$. Assume that*

(i) $V_{[i]} \subseteq \overline{A}_i$ is $\overline{a}$-lpp for all i, and
(ii) $\underline{\delta}(V)$ is minimal among all dimension sequences of $\underline{a}$-spp K-vector subspaces $W = \bigoplus_{i=0}^d W_{[d-i]} x_n^i \subseteq A_d$ such that $\dim_K(W) = \dim_K(V)$ and $W_{[i]}$ is $\overline{a}$-lpp for all i.

Then, V is a segment.

Proof. Assume that V is not a segment; then, there exist $i < j$ such that $V_{[i]} \subsetneq \overline{A}_i$ and $\overline{\mathfrak{m}}_{j-i} (V_{[i]})^+ + \overline{\mathfrak{a}}_j \not\supseteq V_{[j]}$, and choose i and j so that $j-i$ is minimal. Observe that necessarily $i \geqslant \max\{d - a_n + 1, 0\}$, since otherwise $V_{[i]} = \overline{A}_i$. Since $V_{[i]}$ and $V_{[j]}$ are $\overline{a}$-lpp, the fact that $\overline{\mathfrak{m}}_{j-i} (V_{[i]})^+ + \overline{\mathfrak{a}}_j$ does not contain $V_{[j]}$ implies that $\overline{\mathfrak{m}}_{j-i} (V_{[i]})^+ + \overline{\mathfrak{a}}_j$ is properly contained in $V_{[j]}$. In particular, the latter properly contains $\overline{\mathfrak{a}}_j$, and we have that

$$\overline{\mathfrak{m}}_{j-i} (V_{[i]})^+ + \overline{\mathfrak{a}}_j \subseteq (V_{[j]})^-. \tag{2.5}$$

Now, define $W = \bigoplus_{k=0}^d W_{[d-k]} x_n^k$, where $W_{[i]} = (V_{[i]})^+$, $W_{[j]} = (V_{[j]})^-$, and $W_{[k]} = V_{[k]}$ for all $k \neq j, i$. We claim that W is an $\underline{a}$-spp vector space.

In fact, let $k \geqslant \max\{d - a_n + 1, 0\}$; by stability, if $k \neq j, j-1, i, i-1$, then $\overline{\mathfrak{m}}_1 W_{[k]} = \overline{\mathfrak{m}}_1 V_{[k]} \subseteq V_{[k+1]} = W_{[k+1]}$; if $k = j$, then $\overline{\mathfrak{m}}_1 W_{[j]} \subseteq \overline{\mathfrak{m}}_1 V_{[j]} \subseteq V_{[j+1]} = W_{[j+1]}$ and if $k = i-1$, then $\overline{\mathfrak{m}}_1 W_{[i-1]} = \overline{\mathfrak{m}}_1 V_{[i-1]} \subseteq V_{[i]} \subseteq W_{[i]}$.

By costruction, we have that $\overline{\mathfrak{m}}_{j-i} W_{[i]} \subseteq W_{[j]}$, see (2.5); therefore, if $j - i = 1$ we are done, again by stability.

Thus, we may assume that $j - i > 1$ and prove next that $\overline{\mathfrak{m}}_{k-i} W_{[i]} + \overline{\mathfrak{a}}_k = W_{[k]}$ for all $i < k < j$. Since $j - i$ is minimal, we have that $\overline{\mathfrak{m}}_{k-i} W_{[i]} + \overline{\mathfrak{a}}_k = \overline{\mathfrak{m}}_{k-i} (V_{[i]})^+ + \overline{\mathfrak{a}}_k \supseteq V_{[k]} =$

$W_{[k]}$. If the containment were strict, then we would have $\overline{\mathfrak{m}}_{k-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_k \supsetneq (V_{[k]})^+$ and, again by minimality, $\overline{\mathfrak{m}}_{j-k}(V_{[k]})^+ + \overline{\mathfrak{a}}_j \supseteq V_{[j]}$; this would in turn imply $\overline{\mathfrak{m}}_{j-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_j = \overline{\mathfrak{m}}_{j-k}\left(\overline{\mathfrak{m}}_{k-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_k\right) + \overline{\mathfrak{a}}_j \supseteq V_{[j]}$, contradicting our initial assumption on i and j.
The only case left to be shown is now $\overline{\mathfrak{m}}_1 W_{[j-1]} \subseteq W_{[j]}$. By applying what we have proved above for $k = j-1$, we have that $\overline{\mathfrak{m}}_1 W_{[j-1]} + \overline{\mathfrak{a}}_j = \overline{\mathfrak{m}}_1\left(\overline{\mathfrak{m}}_{j-1-i}W_{[i]}\right) + \overline{\mathfrak{a}}_j = \overline{\mathfrak{m}}_{j-i}W_{[i]} + \overline{\mathfrak{a}}_j \subseteq W_{[j]}$, as desired.

Thus, W is an $\underline{a}$-spp vector space; furthermore, it is clear from definition that each $W_{[i]}$ is an $\overline{a}$-lpp. Finally, observe that $\underline{\delta}(W) < \underline{\delta}(V)$ by construction, which contradicts the minimality of $\underline{\delta}(V)$, and we are done. ☐

Proposition 2.13. *For every $d \geqslant 0$ and every $D \leqslant \dim_K(A_d)$ there exists a unique segment V with $\dim_K(V) = D$. Moreover, the sequence $\underline{\delta}(V)$ is the minimum of the set of all sequences $\underline{\delta}(W)$ of $\underline{a}$-spp vector spaces $W = \bigoplus_{i=0}^{d} W_{[d-i]}x_n^i \subseteq A_d$ which have dimension D and such that each $W_{[i]}$ is $\overline{a}$-lpp.*

Proof. By Lemma 2.12 we have that any vector space with minimal dimension sequence is a segment, and by Lemma 2.10 any two such segments are comparable, hence equal. ☐

We already mentioned before that, if the EGH Conjecture held in full generality, then $\mathrm{LPP}^{\underline{a}}(I)$ would be the ideal with minimal growth among those containing a regular sequence of degree $\underline{a}$, and with Hilbert function equal to that of I. The proof is easy and we include it here.

Proposition 2.14. *Assume that EGH holds true. Let $I \subseteq A$ be a homogeneous ideal that contains a regular sequence of degree $\underline{a}$. Then $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I)) \leqslant H(\mathfrak{m}I)$.*

Proof. Let $d \geqslant 0$ be an integer, and let $\underline{a}' = (a_1, \dots, a_r)$ be the degree sequence obtained from $\underline{a}$ by considering only the degrees a_i such that $a_i \leqslant d$. Let $J = (I_d)$, and observe that $\mathrm{LPP}^{\underline{a}}(I)_d = \mathrm{LPP}^{\underline{a}'}(I)_d = \mathrm{LPP}^{\underline{a}'}(J)_d$. Moreover, since $J_{d+1} = \mathfrak{m}_1 I_d$, we have $H(\mathrm{LPP}^{\underline{a}'}(J); d+1) = H(J; d+1) = H(\mathfrak{m}I; d+1)$. Since $\mathfrak{m}_1\mathrm{LPP}^{\underline{a}'}(J)_d \subseteq \mathrm{LPP}^{\underline{a}'}(J)_{d+1}$, we finally obtain that $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I); d+1) = H(\mathfrak{m}\mathrm{LPP}^{\underline{a}'}(J); d+1) \leqslant H(\mathfrak{m}I; d+1)$. ☐

We would like to observe that, even if we do not know that EGH holds in general, we can still get an minimal growth statement in a Clements-Lindström ring $A/\mathfrak{a}$, under milder hypotheses.

Lemma 2.15 (Minimal Growth)**.** *Assume that every homogeneous ideal containing $\mathfrak{a}$ satisfies $\mathrm{EGH}_{\underline{a}}$. If $\mathfrak{a} \subseteq I \subseteq A$ is such an ideal, then $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I) + \mathfrak{a}) \leqslant H(\mathfrak{m}I + \mathfrak{a})$.*

Proof. Fix an integer $d \geqslant 0$, and let $J = (I_d) + \mathfrak{a}$. Note that both I and J satisfy the EGH, and $\mathrm{LPP}^{\underline{a}}(I)_d = \mathrm{LPP}^{\underline{a}}(J)_d$. Observe that $J_{d+1} = \mathfrak{m}_1 J_d + \mathfrak{a}_{d+1} = \mathfrak{m}_1 I_d + \mathfrak{a}_{d+1}$, and accordingly $H(\mathrm{LPP}^{\underline{a}}(J); d+1) = H(J; d+1) = H(\mathfrak{m}I + \mathfrak{a}; d+1)$. Now, since $(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(J) + \mathfrak{a})_{d+1} = \mathfrak{m}_1(\mathrm{LPP}^{\underline{a}}(J))_d + \mathfrak{a}_{d+1} \subseteq (\mathrm{LPP}^{\underline{a}}(J))_{d+1}$, we may conclude that $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I) + \mathfrak{a}; d+1) \leqslant H(\mathrm{LPP}^{\underline{a}}(J); d+1) = H(\mathfrak{m}I + \mathfrak{a}; d+1)$. ☐

We are finally in a position to prove the main result of this section. The simple idea underlying the new proof we present here is to demonstrate Clements-Lindström Theorem using Strong Hyperplane Restriction, like Green proved Macaulay Theorem using generic hyperplane section; this also motivates why Part (ii) has been assimilated into the statement.

Proof of Theorem 2.1. By adding sufficiently large powers of the variables $x_{r+1},\dots,x_n$, we may assume that $r=n$. After taking any initial ideal, and by Proposition 2.5, we may assume that I is an $\underline{a}$-spp monomial ideal. By induction, we may also assume that both Part (i) and Part (ii) hold true in polynomial rings with less than n variables, since the case $n=1$ is trivial. In particular, any lpp ideal of $\overline{A}$ has Minimal Growth, see Lemma 2.15.

We write $I=\bigoplus_{i\geqslant 0} I_{[i]}x_n^i$; for all i, we let $J_{[i]}=\mathrm{LPP}^{\underline{a}}(I_{[i]})$, which by induction is an ideal of $\overline{A}$. Next, we prove that

$$J=\bigoplus_{i\geqslant 0} J_{[i]}x_n^i$$

is also an $\underline{a}$-spp ideal. First of all, observe that $I_{[k]}\subseteq I_{[k+1]}$ for all k, since I is an ideal. This implies that $H(J_{[k]})=H(I_{[k]})\leqslant H(I_{[k+1]})=H(J_{[k+1]})$. Since the ideals $J_{[k]}$ and $J_{[k+1]}$ are lpp, it follows that $J_{[k]}\subseteq J_{[k+1]}$, which, in turn, translates into J being an ideal. Since I is $\underline{a}$-spp, for all $i<a_n-1$ we have $\overline{\mathfrak{m}}_1 I_{[i+1]}\subseteq I_{[i]}$ and $\overline{\mathfrak{a}}\subseteq I_{[i]}$; thus

$$H(J_{[i]})=H(I_{[i]})\geqslant H(\overline{\mathfrak{m}}_1 I_{[i+1]}+\overline{\mathfrak{a}})\geqslant H(\overline{\mathfrak{m}}_1 J_{[i+1]}+\overline{\mathfrak{a}}),$$

where the last inequality follows from Lemma 2.15. This yields that that $\overline{\mathfrak{m}}_1 J_{[i+1]}\subseteq J_{[i]}$ for all $i<a_n-1$, and J is $\underline{a}$-spp by stability.

Given an $\underline{a}$-spp vector space $V\subseteq A_d$, denote by $\sigma(V)$ the segment contained in A_d which has the same dimension as V. Let $J=\bigoplus_{d\geqslant 0} J_d$ be the homogeneous ideal we constructed above and let

$$\sigma(J)=\bigoplus_{d\geqslant 0}\sigma(J_d).$$

We claim that $\sigma(J)$ is the $\underline{a}$-lpp ideal we are looking for.

First of all we show that it is an ideal. Fix a degree $d\geqslant 0$, and write $J_d=\bigoplus_{i=0}^d (J_d)_{[d-i]}x_n^i$, $\sigma(J_d)=\bigoplus_{i=0}^d \sigma(J_d)_{[d-i]}x_n^i$; for notational simplicity, in the following we let $\sigma_{[d-i]}=\sigma(J_d)_{[d-i]}$. By stability, we then have

$$\mathfrak{m}_1 J_d+\mathfrak{a}_{d+1}=\begin{cases}\left(\overline{\mathfrak{m}}_1(J_d)_{[d]}+\overline{\mathfrak{a}}_{d+1}\right)\oplus\left(\bigoplus_{i=0}^d (J_d)_{[d-i]}x_n^{i+1}\right), & \text{if } d<a_n-1,\\[2ex] \left(\overline{\mathfrak{m}}_1(J_d)_{[d]}+\overline{\mathfrak{a}}_{d+1}\right)\oplus\left(\bigoplus_{i=0}^{a_n-2}(J_d)_{[d-i]}x_n^{i+1}\right)\oplus\left(\bigoplus_{i=a_n}^d \overline{A}_{d-i}x_n^i\right), & \text{if } d\geqslant a_n-1,\end{cases}$$

and

$$\mathfrak{m}_1\sigma(J_d)+\mathfrak{a}_{d+1}=\begin{cases}\left(\overline{\mathfrak{m}}_1\sigma_{[d]}+\overline{\mathfrak{a}}_{d+1}\right)\oplus\left(\bigoplus_{i=0}^d \sigma_{[d-i]}x_n^{i+1}\right), & \text{if } d<a_n-1\\[2ex] \left(\overline{\mathfrak{m}}_1\sigma_{[d]}+\overline{\mathfrak{a}}_{d+1}\right)\oplus\left(\bigoplus_{i=0}^{a_n-2}\sigma_{[d-i]}x_n^{i+1}\right)\oplus\left(\bigoplus_{i=a_n}^d \overline{A}_{d-i}x_n^i\right), & \text{if } d\geqslant a_n-1.\end{cases}$$

When $d<a_n-1$, we set $\sigma_{[a_n-1]}=(J_d)_{[a_n-1]}=0$. From the above equalities we thus get

$$\begin{aligned}&\dim_K(\mathfrak{m}_1 J_d+\mathfrak{a}_{d+1})-\dim_K(\mathfrak{m}_1\sigma(J_d)+\mathfrak{a}_{d+1})=\\ &\qquad=\left(\dim_K(\overline{\mathfrak{m}}_1(J_d)_{[d]}+\overline{\mathfrak{a}}_{d+1})-\dim_K(\overline{\mathfrak{m}}_1(\sigma_{[d]}+\overline{\mathfrak{a}}_{d+1}))\right)+\\ &\qquad\quad\left(\dim_K(\sigma_{[a_n-1]})-\dim_K((J_d)_{[a_n-1]})\right).\end{aligned}\tag{2.6}$$

Since $\sigma(J_d)$ is a segment, $\sigma_{[d]} \subseteq \overline{A}$ is $\underline{\overline{a}}$-lpp and its dimension sequence $\underline{\delta} = \underline{\delta}(\sigma(J_d))$ is minimal for the Proposition 2.13. Moreover, the $\underline{\overline{a}}$-lpp vector space $L_d \subseteq \overline{A}_d$ with the same Hilbert function as $(J_d)_{[d]}$ has Minimal Growth, and $\sigma_{[d]} \subseteq L_d$ by the minimality of $\underline{\delta}$. Therefore,

$$\dim_K(\overline{\mathfrak{m}}_1(J_d)_{[d]} + \overline{\mathfrak{a}}_{d+1}) \geqslant \dim_K(\overline{\mathfrak{m}}_1 L_d + \overline{\mathfrak{a}}_{d+1}) \geqslant \dim_K(\overline{\mathfrak{m}}_1 \sigma_{[d]} + \overline{\mathfrak{a}}_{d+1}).$$

Recall that the last entry of the dimension sequence is the dimension of the vector space itself; thus, since $\sigma(J_d)$ and J_d have the same dimension and $\underline{\delta}(J_d) \geqslant \underline{\delta}$ we get $\dim_K(\sigma_{[a_n-1]}) \geqslant \dim_K((J_d)_{[a_n-1]})$. An application of (2.6) now yields

$$\dim_K(\mathfrak{m}_1 J_d + \mathfrak{a}_{d+1}) \geqslant \dim_K(\mathfrak{m}_1 \sigma(J_d) + \mathfrak{a}_{d+1}).$$

Since J is an ideal that contains $\mathfrak{a}$, we have that $\mathfrak{m}_1 J_d + \mathfrak{a}_{d+1} \subseteq J_{d+1}$ and, thus,

$$\dim_K(\mathfrak{m}_1 \sigma(J_d) + \mathfrak{a}_{d+1}) \leqslant \dim_K(J_{d+1}) = \dim_K(\sigma(J_{d+1})).$$

By Remark 2.9, we know that $\mathfrak{m}_1 \sigma(J_d) + \mathfrak{a}_{d+1}$ is a segment, and so is $\sigma(J_{d+1})$ by definition; then, it follows that $\mathfrak{m}_1 \sigma(J_d) \subseteq \sigma(J_{d+1})$. We may finally conclude that $\sigma(J)$ is an ideal, which is $\underline{a}$-spp by construction, and has the same Hilbert function as I.

Next, we observe that $\sigma(J)$ satisfies Part (ii) of the theorem, since $H(\sigma(J) + (x_n^i); d)$ is just the i-th entry of $\underline{\delta}(\sigma(J_d))$, $H(J + (x_n^i); d)$ is the i-th entry of $\underline{\delta}(J_d)$, and $\underline{\delta} \leqslant \underline{\delta}(J_d)$.

By construction, $\sigma(J)$ is the ideal with all the required properties, once we have proved the following claim.

Claim. *$\sigma(J)$ is $\underline{a}$-lpp.*

Proof of the Claim: By contradiction, there exists a degree d such that $\sigma(J_d)$ is an $\underline{a}$-spp D-dimensional vector space which is not lpp; thus, we may consider a counterexample of degree d and of minimal dimension D for which the operator σ does not return an $\underline{a}$-lpp vector space of dimension D inside A_d; then, if we apply σ to any $(D-1)$-dimensional $\underline{a}$-spp vector space of A_d, we obtain an $\underline{a}$-lpp vector space, but there is an $\underline{a}$-spp vector space of dimension D which is transformed by σ into an $\underline{a}$-spp vector space $V + \langle v \rangle$ which is not lpp. Thus, V is $\underline{a}$-lpp, $V + \langle v \rangle$ is $\underline{a}$-segment, and we write them as

$$V = \bigoplus_{i=0}^{d} V_{[d-i]} x_n^i, \qquad V + \langle v \rangle = \bigoplus_{i=0}^{d} \widetilde{V}_{[d-i]} x_n^i.$$

Let also w be the monomial such that $V + \langle w \rangle$ is the $\underline{a}$-lpp vector subspace of dimension D of A_d and observe that $w > v$. Write $v = \overline{v} x_n^t$ and $w = \overline{w} x_n^s$, where $\overline{v}, \overline{w}$ are monomials in $\overline{A}$. Since $V + \langle v \rangle$ is a segment, we have that $t \geqslant s$.
If $t = s$ we immediately get a contradiction, since by construction $\overline{v}$ and $\overline{w}$ would both be the largest monomial of degree $d - t$ which is not contained in $V_{[d-t]}$.

Therefore, we may assume that $t > s$, and $a = \deg(\overline{w}) = d - s > d - t = \deg(\overline{v}) = b$. Observe that $\overline{v} \in \widetilde{V}_{[d-t]}$, and that $d - t < a_n$. Moreover $\overline{\mathfrak{m}}_{a-b} \widetilde{V}_{[d-t]} \subseteq \widetilde{V}_{[d-s]}$ holds by stability applied to $V + \langle v \rangle$. We write $\overline{w} = x_{i_1} \cdots x_{i_a}$ and $\overline{v} = x_{j_1} \cdots x_{j_b}$, with $i_1 \leqslant \ldots \leqslant i_a$ and $j_1 \leqslant \ldots \leqslant j_b$. Since $w > v$ we have two cases, either $\overline{v}$ divides $\overline{w}$, or $x_{i_1} \cdots x_{i_b} > \overline{v}$. In both cases, it is easy to see that $\overline{w} \in \overline{\mathfrak{m}}_{a-b} \widetilde{V}_{[d-t]} \subseteq \widetilde{V}_{[d-s]}$, and thus $w \in V + \langle v \rangle$, which is a contradiction. ∎

The proofs of Theorem 2.1 (i) previously available in the literature do not include part (ii), the Strong Hyperplane Section of Gasharov. One advantage of our approach is that, with little additional effort, one can show that the Betti numbers of an $\underline{a}$-spp ideal are at most those of the corresponding $\underline{a}$-lpp ideal; see [Mur08, CK14]. Furthermore, combining this fact with Remark 2.6, one recovers the LPP-Conjecture for ideals containing pure-powers ideals in characteristic zero, which is the main result of [MM11, Section 3]. Note that, in [MM11], the authors also provide a characteristic-free proof that settles the LPP-Conjecture for ideals that contain pure-powers.

3. Artinian reduction and linkage

In this brief section we collect some results which will be useful in what follows. We start with Proposition 10 in [CM08], which offers in many cases a way to prove the EGH Conjecture in the Artinian case only.

Proposition 3.1. *Let $\mathbf{f} \subseteq A = K[x_1, \dots, x_n]$ be an ideal generated by a regular sequence of degree $\underline{a}$, and ℓ be a linear $A/\mathbf{f}$-regular form. Let also $\overline{A} = A/(\ell)$, and $\overline{\mathbf{f}} = \mathbf{f}\overline{A}$. If every homogeneous ideal of $\overline{A}$ containing $\overline{\mathbf{f}}$ satisfies $\mathrm{EGH}_{\overline{a}}$, then every homogeneous ideal of A containing $\mathbf{f}$ satisfies $\mathrm{EGH}_{\underline{a}}$.*

Proof. Without loss of generality, we may assume that K is infinite. Let $I \subseteq A$ be a homogeneous ideal that contains $\mathbf{f}$ and for $i \geqslant 0$ we let $I_i = (I :_A \ell^i) + (\ell)$. By assumption, there exist $\overline{a}$-lpp ideals $J_i \subseteq K[x_1, \dots, x_{n-1}]$ with the same Hilbert function as $I_i/(\ell)$. Now, we define

$$J = \bigoplus_{i \geqslant 0} J_i x_n^i,$$

and we claim that J is an ideal with the same Hilbert function as I; since $\mathfrak{a} \subseteq J_0 \subseteq J$, the conclusion will then follow from Theorem 2.1.
By considering the short exact sequences $0 \longrightarrow A/(I :_A \ell^j)(-1) \xrightarrow{\cdot \ell} A/(I :_A \ell^{j-1}) \longrightarrow \overline{A}/I_{j-1} \longrightarrow 0$ for all j, a straightforward computation yields that $H(J) = H(I)$.

What it is left to be shown is that J is an ideal. Let as before $\overline{\mathfrak{m}} = (x_1, \dots, x_{n-1})$; since J_i is an ideal of $\overline{A}$, we have $\overline{\mathfrak{m}} J_i \subseteq J_i$ for all i and, accordingly, $\overline{\mathfrak{m}} J \subseteq J$. The condition $x_n J \subseteq J$ translates into the containments $J_i \subseteq J_{i+1}$ for all $i \geqslant 0$. Since each J_i is an $\overline{a}$-lpp ideal, it suffices to show that $H(J_i) \leqslant H(J_{i+1})$, which holds true since $I_i \subseteq I_{i+1}$. ☐

We now recall some results from the theory of linkage. In Section 2 we introduced the following notation: given a homogeneous ideal $I \subseteq A = K[x_1, \dots, x_n]$ containing an ideal $\mathbf{f}$ generated by a regular sequence of degree $\underline{a} = (a_1, \dots, a_n)$, we let $I^{\ell}_{\mathbf{f}} = (\mathbf{f} :_A I)$, and call it the *link of I with respect to* $\mathbf{f}$, which is an ideal that contains $\mathbf{f}$. Obviously, the link depends on $\mathbf{f}$; however, when it is clear from the context which $\mathbf{f}$ we consider, we denote $I^{\ell}_{\mathbf{f}}$ simply by I^{ℓ}.

Proposition 3.2. *Let $\underline{a} = (a_1, \dots, a_n)$ and A, I, $\mathbf{f}$ be as above; let also $R = A/\mathbf{f}$ and $s = \sum_{i=1}^n (a_i - 1)$. Then,*

(i) $(I^{\ell})^{\ell} = I$.

(ii) $H(I; d) = H(R; d) - H(I^{\ell}; s - d)$.

(iii) $\operatorname{type}(IR) = \mu(I^\ell R)$*, i.e., the dimension of the socle of* IR *equals the minimal number of generators of its linked ideal.*

In particular, if $I = (\mathbf{f} + (g))$ *is an almost complete intersection, then the ideal* $I^\ell = (\mathbf{f} :_A g)$ *defines a Gorenstein ring, and viceversa. Moreover, if* $\deg(g) = D$*, then* $\operatorname{soc}((\mathbf{f} :_A g)R)$ *is concentrated in degree* $s - D$*.*

Proof. Observe that the functor $(-)^\vee = \operatorname{Hom}_R(-, R)$ is the Matlis dual, since R is Gorenstein Artinian. The statements that we want to prove are a direct consequence of Matlis duality, see [BH93, Sections 3.2 and 3.6]. It is well known that a module and its Matlis dual have the same annihilator. In particular, since $(A/I)^\vee \cong I^\ell/\mathbf{f}$, we obtain that $I = \operatorname{ann}_A(A/I) = \operatorname{ann}_A(I^\ell/\mathbf{f}) = (I^\ell)^\ell$, which proves (i). For (ii), recall that in the graded setting one has $((A/I)^\vee)_d \cong (A/I)_{s-d}$, for all $d \in \mathbb{Z}$. Since $(A/I^\ell)^\vee \cong I/\mathbf{f}$, the claim follows from the graded short exact sequences of K-vector spaces $0 \to (I/\mathbf{f})_d \to (A/\mathbf{f})_d \to (A/I)_d \to 0$. Part (iii) is again a consequence of Matlis duality. □

We conclude this part with an easy lemma.

Lemma 3.3. *Let* $\underline{a} = (a_1, \dots, a_r)$ *and* $\underline{b} = (b_1, \dots, b_r)$ *be degree sequences satisfying* $a_i \leqslant b_i$ *for all* $i = 1, \dots, r$*. If an ideal* I *satisfies* $\mathrm{EGH}_{\underline{a}}$*, then it satisfies* $\mathrm{EGH}_{\underline{b}}$*.*

Proof. By assumption, $J = \mathrm{LPP}^{\underline{a}}(I)$ is a $\underline{a}$-lpp ideal with the same Hilbert function as I. By our assumption on the degree sequences, J also contains the pure-powers ideal $(x_1^{b_1}, \dots, x_r^{b_r})$. Therefore, by Theorem 2.1, $\mathrm{LPP}^{\underline{b}}(J)$ is a $\underline{b}$-lpp ideal with the same Hilbert function as I. □

4. Results on the EGH conjecture

We collect in the following the most relevant cases when EGH is known to be true. We start with a very recent result, Theorem 4.1, proved by the first two authors in [CDS20b, Theorem A], which improves an older result due Maclagan and the first author, [CM08, Theorem 2]. Indeed, Theorem 4.1 covers all of the significant known cases of the EGH Conjecture which take into account only hypotheses on the degree sequence $\underline{a}$ and not on the ideal I. A further generalization can be found in [CDS20b], see Theorem 3.6.

Theorem 4.1. *Let* $I \subseteq A$ *be a homogeneous ideal which contains a regular sequence of degree* $\underline{a} = (a_1, \dots, a_r)$ *and assume that* $a_i \geqslant \sum_{j=1}^{i-1}(a_j - 1)$ *for all* $i \geqslant 3$*; then,* I *satisfies* $\mathrm{EGH}_{\underline{a}}$*.*

Proof. For brevity's sake, we present here only the proof of the weaker statement [CM08, Theorem 2], that is, we will assume that $a_i > \sum_{j=1}^{i-1}(a_j - 1)$ for all $i \geqslant 3$. Observe that, by Proposition 3.1, we may let $r = n$ and work by induction on n. Let $\overline{\underline{a}} = (a_1, \dots, a_{n-1})$; by induction, suppose that every ideal of A containing a regular sequence of degree $\overline{\underline{a}}$ satisfies $\mathrm{EGH}_{\overline{\underline{a}}}(d)$ for all d.

Clearly, for $d < a_n$, we have that $\mathrm{EGH}_{\underline{a}}(d)$ is equivalent to $\mathrm{EGH}_{\overline{\underline{a}}}(d)$. Thus, let $d \geqslant a_n$, so that $s - d < a_n$; by induction, I^ℓ satisfies $\mathrm{EGH}_{\overline{\underline{a}}}$ and the previous case yields that I^ℓ satisfies $\mathrm{EGH}_{\underline{a}}(s - d)$ for all $d \geqslant a_n$. By Proposition 3.2 (ii), we know that $H(I; d) = H(R; d) - H(I^\ell; s - d)$, where $R = A/\mathbf{f}$ and $s = \sum_{i=1}^{n}(a_i - 1)$. It now follows that I satisfies $\mathrm{EGH}_{\underline{a}}(d)$ also for all $d \geqslant a_n$, and the proof is complete. □

One big advantage of Theorem 4.1 is that it can be applied in order to obtain growth bounds for the Hilbert function which are at least as good as the ones given by Macaulay Theorem. This can be done for *any* homogenous ideal, regardless of the degree sequence. The key observation to see this is the following.

Lemma 4.2. *Assume that $|K| = \infty$ and that I contains an ideal $\mathbf{f}$ generated by a regular sequence of degree $\underline{a} = (a_1, \ldots, a_r)$. If $\underline{b} = (b_1, \ldots, b_r)$ is a degree sequence such that $b_i \geqslant a_i$ for all i, then I contains an ideal $\mathbf{g}$ generated by a regular sequence of degree $\underline{b}$.*

Proof. We proceed by induction on $r \geqslant 1$. Let $r = 1$ and observe that $\operatorname{ht}(I) \geqslant 1$, since $f_1 \in I$. Consider the truncation ideal $I_{\geqslant b_1}$ where, by assumption, $b_1 \geqslant a_1$; by avoiding primes, it immediately follows that $\operatorname{ht}(I_{\geqslant b_1}) \geqslant 1$ and there exists a regular element $g_1 \in I$ of degree b_1.

By induction, we have constructed a homogeneous ideal $\mathbf{g}' = (g_1, \ldots, g_{r-1})$, which is unmixed and generated by a regular sequence of degrees $b_1, \ldots, b_{r-1}$. Observe that, since I contains $f_1, \ldots, f_r$, we have that $\operatorname{ht}(I_{\geqslant j}) \geqslant r$ for all $j \geqslant a_r$. In particular, the ideal $\mathbf{g}' + I_{\geqslant b_r}$ has height at least r, since $b_r \geqslant a_r$. Thus, againg by avoiding primes, we find an element $g_r \in I_{b_r}$ which is regular modulo $\mathbf{g}'$ and $\mathbf{g} = (g_1, \ldots, g_r)$ is the ideal we were looking for. □

As another application of the theory of linkage to the EGH Conjecture, we now present a result due to Chong [Cho16], which settles the conjecture for Gorenstein ideals of height three.

Proposition 4.3. *Let I be a homogeneous ideal that contains an ideal $\mathbf{f}$ generated by a regular sequence of degree $\underline{a} = (a_1, \ldots, a_n)$. Assume that $\underline{b} = (b_1, \ldots, b_n)$ is a degree sequence such that $b_i \leqslant a_i$ for all i, and $I^\ell_{\mathbf{f}}$ satisfies $\mathrm{EGH}_{\underline{b}}$; then I satisfies $\mathrm{EGH}_{\underline{a}}$.*

Proof. Let $s = \sum_{i=1}^n (a_i - 1)$ and $I^\ell = I^\ell_{\mathbf{f}}$; by hypothesis there exists a $\underline{b}$-lpp ideal J with the same Hilbert function as I^ℓ that also contains the pure-powers ideal $\mathfrak{a} = (x_1^{a_1}, \ldots, x_r^{a_r})$, since $a_i \geqslant b_i$ for all i. Consider now $J^\ell_{\mathfrak{a}}$; by Proposition 3.2 (ii) for all $d \geqslant 0$ we have

$$H(I; d) = H(A/\mathbf{f}; d) - H(I^\ell; s - d) = H(A/\mathfrak{a}; d) - H(J; s - d) = H(J^\ell_{\mathfrak{a}}; d).$$

Again by Theorem 2.1, there exists an $\underline{a}$-lpp ideal with the same Hilbert function as $J^\ell_{\mathfrak{a}}$, and we are done. □

Observe that in the above proof we used Theorem 2.1 to transform the monomial ideal $J^\ell_{\mathfrak{a}}$ into an $\underline{a}$-lpp ideal. In fact, it can be proved in general that $J^\ell_{\mathfrak{a}}$ is already $\underline{a}$-lpp whenever J is $\underline{a}$-lpp, see for instance [RS08, Theorem 5.7], or [CS18, Proposition 3.2].

Sequentially bounded licci ideals were first introduced in [Cho16], and are those ideals to which Proposition 4.3 can be applied repeatedly in order to prove the EGH Conjecture. We recall the main definitions here.

Definition 4.4. Let $I \subseteq A = K[x_1, \ldots, x_n]$ be a homogeneous ideal, and set $I_0 = I$. We say that I is *linked to a complete intersection*, or *licci* for short, if there exist ideals $I_j = (I_{j-1})^\ell_{\mathbf{f}_j}$ where $\mathbf{f}_1, \ldots, \mathbf{f}_s$ are ideals of the same height as I generated by regular sequences of degrees $\underline{a}_1, \ldots, \underline{a}_s$, such that I_s is generated by a regular sequence of degree $\underline{a}_{s+1}$.
We say that I is *sequentially bounded licci* if the above sequence also satisfies $\underline{a}_1 \geqslant \ldots \geqslant \underline{a}_{s+1}$.

We also recall that I is said to be *minimally licci* if it is licci and, in addition, for each j the regular sequence generating $\mathbf{f}_{j+1}$ can be chosen to be of minimal degree among all the regular sequences contained in I_j. Observe that $\mathbf{f}_j \subseteq I_j$, therefore minimally licci ideals are sequentially bounded licci. It was proved by Watanabe [Wat73] that height three Gorenstein ideals are licci. Later on, Migliore and Nagel show that such ideals are also minimally licci [MN10]. We see next how these facts together, combined with Proposition 4.3, yield the main result of [Cho16].

Theorem 4.5. *Let $I \subseteq A$ be a sequentially bounded licci ideal, where the first link of I is performed with respect to a regular sequence of degree $\underline{a}$; then I satisfies* $\mathrm{EGH}_{\underline{a}}$.

In particular, if I is a Gorenstein ideal of height 3 containing a regular sequence of degree $\underline{a} = (a_1, a_2, a_3)$, then I satisfies $\mathrm{EGH}_{\underline{a}}$.

Proof. We prove the first part only for $n = r$, and we refer the reader to the original paper for the reduction to this case; this is shown in [Cho16, Proposition 10], where the proof runs along the same lines as that of Proposition 3.1.
Since I_s is a complete intersection of degree $\underline{a}_{s+1}$ by assumption, it trivially satisfies $\mathrm{EGH}_{\underline{a}_{s+1}}$; therefore Proposition 4.3 implies that I_{s-1} satisfies $\mathrm{EGH}_{\underline{a}_s}$, and its repeated application to the sequence of linked ideals eventually yields that I satisfies $\mathrm{EGH}_{\underline{a}_1}$, that is $\mathrm{EGH}_{\underline{a}}$. ☐

Remark 4.6. The height 3 Gorenstein case proved by Chong is also related to a previous result due to Geramita and Kreuzer concerning the Cayley-Bacharach Conjecture in $\mathbb{P}^3$ [GK13, Corollary 4.4]. In fact, EGH for a height 3 Gorenstein ideal I is equivalent to EGH for its linked ideal I^ℓ, which is an almost complete intersection by Proposition 3.2 (iii). As pointed out in the introduction, EGH for almost complete intersections implies the Cayley-Bacharach Conjecture 1.2.

A result due to Francisco [Fra04, Corollary 5.2] settles $\mathrm{EGH}_{\underline{a}}(D)$ for almost complete intersections $(\mathbf{f} + (g))$ in the first relevant degree, namely $D = \deg(g)$.

Theorem 4.7. *Let $\mathbf{f} \subseteq A$ be an ideal generated by a regular sequence of degree $\underline{a} = (a_1, \ldots, a_r)$, and let $g \notin \mathbf{f}$ be an element of degree $D \geqslant a_1$ such that $I = \mathbf{f} + (g)$ has height r. Then, I satisfies* $\mathrm{EGH}_{\underline{a}}(D)$.

Proof. We may assume that K is infinite. First, we reduce to the Artinian case by arguing as follows: we choose some $N > D + 1$ and homogeneous elements of degree N such that $f_1, \ldots, f_r, f_{r+1}, \ldots, f_n$ is a full regular sequence of degree $\underline{a}' = (a_1, \ldots, a_r, N, \ldots, N)$. In this way, proving $\mathrm{EGH}_{\underline{a}}(D)$ for I is equivalent to proving $\mathrm{EGH}_{\underline{a}'}(D)$ for $I + (f_{r+1}, \ldots, f_n)$. Thus, for the rest of proof $r = n$ and $A/\mathbf{f}$ is Artinian.

Now, let b be the unique integer such that $\sum_{i=1}^{b}(a_i - 1) \leqslant D < \sum_{i=1}^{b+1}(a_i - 1)$. It is then easy to see that $J = \mathfrak{a} + (h)$, where $h = x_1^{a_1-1} \cdots x_b^{a_b-1} \cdot x_{b+1}^{D-\sum_{i=1}^{b}(a_i-1)}$, is the smallest $\underline{a}$-lpp ideal with $H(J; D) = H(I; D)$.

To conclude the proof, it suffices to show that $H(J; D+1) \leqslant H(I; D+1)$. To this end, let $s = \sum_{i=1}^{n}(a_i - 1)$, and consider the links $I^\ell = I^\ell_{\mathbf{f}} = (\mathbf{f} :_A I)$ and $J^\ell = J^\ell_{\mathfrak{a}} = (\mathfrak{a} :_A J)$. Then, Proposition 3.2 yields the natural graded short exact sequences

$$0 \to A/I^\ell(-D) \to A/\mathbf{f} \to A/I \to 0 \quad \text{and} \quad 0 \to A/J^\ell(-D) \to A/\mathfrak{a} \to A/J \to 0,$$

which in turn show that we only have to prove that $H(J^\ell;1) \geqslant H(I^\ell;1)$. A direct computation shows that

$$J^\ell = \mathfrak{a} : (h) = (x_1, \ldots, x_b, x_{b+1}^{\sum_{i=1}^{b+1}(a_i-1)-D+1}, x_{b+2}^{a_{b+2}}, \ldots, x_n^{a_n}),$$

that is, $H(J^\ell;1) = b$.

Suppose, by contradiction, that I^ℓ contains c linear forms, with $c > b$; then, by Prime Avoidance we can find a homogeneous ideal $\mathbf{g} \subseteq I^\ell$ generated by a regular sequence of degree $(1, \ldots, 1, a_{c+1}, \ldots, a_n)$ such that the socle degree of $A/\mathbf{g}$ is $\sum_{i=c+1}^n (a_i - 1) < \sum_{i=b+1}^n (a_i - 1) \leqslant s - D$. Thus, $H(A/I^\ell; s-D) \leqslant H(A/\mathbf{g}; s-D) = 0$ which is not possible, since the ring A/I^ℓ is Gorenstein of socle degree $s - D$ by Proposition 3.2 (iii). □

Remark 4.8. It is easy to see by means of Lemma 4.2 that the condition $D \geqslant a_1$ in the statement of Theorem 4.7 can always be met.

Observe that, again by Proposition 3.2 (ii), the statement of Theorem 4.7 is equivalent to proving $\mathrm{EGH}_{\underline{a}}(s-D-1)$ for the ideal $I^\ell = I^\ell_{\mathbf{f}}$. Since the socle of A/I^ℓ is concentrated in degree $s-D$, this is equivalent to controlling the growth of the Hilbert function of a Gorenstein ring from socle degree minus 1 to the socle degree. For other results of this nature, see for instance [Otw02].

The next result we present is due to Abedelfatah, see [Abe15] and [Abe16]; it can be viewed as a generalization of the Clements-Lindström Theorem to ideals that contain a regular sequence generated by products of linear forms. Below we provide the proof of the general version, cf. [Abe16, Theorem 3.4].

Theorem 4.9. *Let $\mathbf{f} \subseteq A$ be an ideal generated by a regular sequence of degree $\underline{a} = (a_1, \ldots, a_r)$. Assume that $\mathbf{f} \subseteq P$, where P is an ideal generated by products of linear forms. Then, any ideal $I \subseteq A$ that contains P satisfies $\mathrm{EGH}_{\underline{a}}$.*

Proof. By induction we may assume that the claim is true for ideals in polynomial rings with less than n variables, since the base case $n = 1$is trivial.
Let s be the smallest degree of a minimal generator p of P. Since $s \leqslant a_1$, by Lemma 3.3 it suffices to show that I satisfies $\mathrm{EGH}_{\underline{a}'}$, where $\underline{a}' = (s, a_2, \ldots, a_r)$. Moreover, by Theorem 2.1, it is enough to prove that, for every degree $d \geqslant 0$, there exists a monomial ideal J that contains $(x_1^s, x_2^{a_2}, \ldots, x_r^{a_r})$ such that $H(I;d) = H(J;d)$ and $H(I;d+1) = H(J;d+1)$.

We write $p = \ell_1 \cdots \ell_s$, where ℓ_i are linear forms which we order as follows:
For $k = 1, \ldots, s$, let $I_k^{(0)}$ denote the image ideal of I in $A/(\ell_k)$ and choose ℓ_1 so that $H(I_1^{(0)};d) = \min_k\{H(I_k^{(0)};d)\}$.
Inductively, given $\ell_1, \ldots, \ell_j$, for $k = j+1, \ldots, s$ we let $I_k^{(j)}$ denote the image ideal of $(I :_A (\ell_1 \cdots \ell_j))$ in $A/(\ell_k)$ and choose ℓ_{j+1} so that $H(I_{j+1}^{(j)}; d-j) = \min_k\{H(I_k^{(j)}; d-j)\}$.

Now, with some abuse of notation, we let $A_k = A/(\ell_k)$ for $k = 1, \ldots, s$; for notational simplicity, we also set $I_j = I_{j+1}^{(j)}$ for $j = 0, \ldots, s-1$. By construction, we thus have

$$H(I_j; d-j) \leqslant H(I_{j+1}; d-j) \quad \text{for all } j = 0, \ldots, s-1. \tag{4.1}$$

Moreover, for all $j = 1, \ldots, s-1$, the short exact sequences

$$0 \longrightarrow A/(I :_A (\ell_1 \cdots \ell_j))(-1) \longrightarrow A/(I :_A (\ell_1 \cdots \ell_{j-1})) \longrightarrow A_j/I_{j-1} \longrightarrow 0$$

provide that

$$H(A/I;i) = \sum_{j=0}^{s-1} H(A_{j+1}/I_j; i-j), \text{ for all } i. \tag{4.2}$$

Let $\underline{\tilde{a}} = (a_2, \dots, a_n)$ and $\tilde{A} = K[x_2, \dots, x_n]$. Observe that $A_k \cong \tilde{A}$ for all k, thus, by induction, we can find $\underline{\tilde{a}}$-lpp ideals $J_{[j]}$ in $\tilde{A}$ with the same Hilbert function as I_j, for $j = 0, \dots, s-1$. Consider now $J = \bigoplus_{j=0}^{s-1} J_{[j]} x_1^j \oplus A x_1^s$, and let J_d denote the degree d component of J. If we show, and we shall do, that $\mathfrak{m}_1 J_d \subseteq J_{d+1}$, that is, J is closed under multiplication from degree d to degree $d+1$, then the proof is complete, since $H(A/J;i) = H(A/I;i)$ for all i by (4.2).
To this end, we clearly have that $(x_2, \dots, x_n)_1 (J_{[j]})_{d-j} \subseteq (J_{[j]})_{d-j+1}$, since each $J_{[j]}$ is an ideal in $\tilde{A}$. It is left to show that $x_1 J_d \subseteq J_{d+1}$, which translates into $(J_{[j]})_{d-j} \subseteq (J_{[j+1]})_{d-j}$ for all $j = 0, \dots, s-1$; since such ideals are both $\underline{\tilde{a}}$-lpp, this is yielded by (4.1). □

Corollary 4.10. *The EGH Conjecture is true for monomial ideals.*

Another interesting case, of different nature, when the EGH Conjecture is known in general is when the regular sequence that defines $\mathbf{f}$ is a Gröbner basis with respect to some monomial order. In fact, in this situation, the initial forms of the sequence form a regular sequence of monomials.

Proposition 4.11. *Let $\underline{a} = (a_1, \dots, a_r)$ be a degree sequence and let $\mathbf{f}$ be an ideal of A generated by a regular sequence $f_1, \dots, f_r$ of degree $\underline{a}$, such that $\{f_1, \dots, f_r\}$ is a Gröbner basis with respect to some monomial order $\succcurlyeq$. Then, every homogeneous ideal of A containing $\mathbf{f}$ satisfies* $\mathrm{EGH}_{\underline{a}}$.

Proof. Let I be a homogeneous ideal that contains $\mathbf{f}$. Let us consider the set $\mathscr{S}$ of all homogeneous ideals of A with the same Hilbert function as I that contain a monomial regular sequence $g_1, \dots, g_r$ of degree $\underline{a}$. Observe that $\mathscr{S}$ is not empty since, by assumption, the initial ideal of I contains the regular sequence of monomials given by the initial forms of $f_1, \dots, f_r$, which has degree $\underline{a}$.

Since the monomials $g_1, \dots, g_r$ are pairwise coprime, we may write $g_i = \prod_{j \in B_i} x_j^{b_{ij}}$, for some subsets B_i's of $\{1, \dots, n\}$ with $B_i \cap B_{i'} = \emptyset$ if $i \neq i'$, and we let $|g_1, \dots, g_r| = \sum_{i=1}^r |B_i|$ denote the cardinality of the support of $g_1, \dots, g_r$.

Now, we choose an element J of $\mathscr{S}$ which contains a regular sequence $h_1, \dots, h_r$ with minimal support and we will show that

$$|h_1, \dots, h_r| = r;$$

in this way we have that each h_k is the a_k-th power of a variable. After reordering the variables if necessary, we may assume that $h_k = x_k^{a_k}$, and the conclusion follows by Theorem 2.1.

Clearly $|h_1, \dots, h_r| \geqslant r$, and assume by way of contradiction that the inequality were strict, i. e., there would exist $i \in \{1, \dots, r\}$ and $1 \leqslant j < j' \leqslant n$ such that $x_j x_{j'} \mid h_i$. Consider then the change of coordinates φ defined by

$$x_k \mapsto x_k, \quad \text{for all } k \neq j', \qquad \text{and } x_{j'} \mapsto x_j + x_{j'},$$

let $J' = \mathrm{in}_{\geqslant}(\varphi(J)$, where $\geqslant$ denotes the lexicographic order, and let $h'_k = \mathrm{in}_{\geqslant}(\varphi(h_k)) \in J'$ for $k = 1, \dots r$. It is immediate to see that $h'_1, \dots, h'_r$ is still a monomial regular sequence of degree $\underline{a}$; since J' has the same Hilbert function as I, it belongs to $\mathscr{S}$. However, $h'_k = h_k$ for all $k \neq i$, whereas h'_i has one less variable than h_i in its support. In particular, $|h'_1, \dots, h'_r| < |h_1, \dots, h_r|$, which contradicts the minimality of the support of $h_1, \dots, h_r$, and we are done. □

Clearly, one can generalize the above by using a weight order ω, as long as the given regular sequence form a Gröbner basis with respect to the induced order $\geqslant_\omega$; therefore, any ideal that contains the initial forms of the sequence satisfies the EGH Conjecture.

Corollary 4.12.

Contrary to the "special" case in which the regular sequence $f_1, \dots, f_r$ is a Gröbner basis, as far as we know the "generic" version of the conjecture is still open. We record this fact as a question.

Question 4.13. Let $\underline{a} = (a_1, \dots, a_r)$ be a degree sequence. Does there exist a non-empty Zariski open set $U \subseteq \mathbb{P}^{n-1}(A_{a_1}) \times \mathbb{P}^{n-1}(A_{a_2}) \times \cdots \times \mathbb{P}^{n-1}(A_{a_r})$ of general forms of degree $\underline{a}$ such that, for every $[f_1, \dots, f_r] \in U$, any ideal I containing $\mathbf{f} = (f_1, \dots, f_r)$ satisfies $\mathrm{EGH}_{\underline{a}}$?

In [HP98, Proposition 4.2], Herzog and Popescu show that, once a regular sequence of degree $\underline{a} = (2, 2 \dots, 2)$ is fixed, then any generic ideal generated by quadrics that contains it satisfies $\mathrm{EGH}_{\underline{a}}$. We would like to warn the reader that Question 4.13 addresses a different kind of "genericity". In fact, we are not fixing the regular sequence beforehand, but we are asking whether the EGH Conjecture holds for any ideal containing a general regular sequence.

Remark 4.14. (1) When $\mathbf{f}$ is a general complete intersection, then the set of monomials of A which do not belong to $\mathbf{f}$???forms a K-basis of $A/\mathbf{f}$, and this is well-known.
(2) It is currently not known, though, whether or not, after a general change of coordinates $\varphi : A \to A$ the set of monomials of A which does not belong to $\varphi(\mathbf{f})$ is a K-basis of $A/\varphi(\mathbf{f})$, when $\mathbf{f}$ is a complete intersection. A positive answer in this matter would make Question 4.13 even more interesting. In fact, in light of the first part of the remark, it would provide a strategy to attack the EGH Conjecture at once.

There are some other very special cases when EGH is known to hold that can be found in the literature; we complete this section with two of them mmh

A special case of interest is when I contains a regular sequence of quadrics, and this is the assumption on I in the original statement of the conjecture. In this case, EGH is known to be true in low dimension; for $n \leqslant 4$, it can be proven by a direct application of linkage; see also [Che12]. The validity of the conjecture for $n = 5$ was first claimed in [Ric04], but a proof was never provided until recently, when Güntürkün and Hochster finally settle the case of five quadrics in [GH19, Theorem 4.1]. We present an alternative, much shorter proof of their result which relies on the techniques we used so far.

Theorem 4.15. *$I \subseteq A = K[x_1, \dots, x_n]$ be a homogeneous ideal containing a regular sequence of degree $\underline{a} = (2, 2, 2, 2, 2)$; then, I satisfies* $\mathrm{EGH}_{\underline{a}}$.

Proof. We may assume that $K = \overline{K}$. By Proposition 3.1 we may assume that $n = 5$, A/I is Artinian and $\mathbf{f} \subseteq I$ is an ideal generated by a regular sequence of five quadrics; notice that the socle degree s of $A/\mathbf{f}$ is $s = 5$.
By Proposition 3.2 (ii) it suffices to show that I satisfies $\mathrm{EGH}_{\underline{a}}(j)$ for $j = 0, 1, 2$; this is clearly true for $j = 0, 1$ and we are left with the case $j = 2$.

If $H(I; 2) = 6$, then we are done by Theorem 4.7. Since the locus of reducible elements in $\mathbb{P}(\mathrm{Sym}^2(A_1))$ has dimension $2n - 2 = 8$, if $H(I; 2) \geqslant 7$ then I must contain a reducible quadric $Q = \ell_1\ell_2$. Proceeding as in the proof of Theorem 4.9, we construct ideals $J_{[0]}$ and $J_{[1]}$ in $\tilde{A} = K[x_2, \ldots, x_5]$ such that $J = J_{[0]} \oplus J_{[1]}x_1 \oplus Ax_1^2$ is a monomial vector space which contains $\mathfrak{a} = (x_1^2, \ldots, x_5^2)$, $\mathfrak{m}_1 J_2 \subseteq J_3$, $H(A/J; i) = H(A/I; i)$ for all i, and the conclusion follows from an application of Theorem 2.1. □

In [Coo12], Cooper proves some cases of EGH when r is small, including $\underline{a} = (a_1, a_2, a_3)$ with $a_1 = 2, 3$, and with $a_2 = a_3$, and with $\underline{a} = (3, a, a)$. We present a proof of the latter one, which is again based on the techniques of [CDS20b].

Proposition 4.16. *Let $I \subseteq A = K[x_1, \ldots, x_n]$ be a homogeneous ideal containing a regular sequence of degree $\underline{a} = (3, a, a)$. Then, I satisfies $\mathrm{EGH}_{\underline{a}}$.*

Proof. By Proposition 3.1, we may assume that $n = 3$, and that K is infinite. Let $\mathbf{f} = (f_1, f_2, f_3) \subseteq I$ be an ideal generated by a regular sequence of degree $\underline{a}$. By Proposition 3.2 (ii), we only have to show that $\mathrm{EGH}_{\underline{a}}(d)$ holds for all $d \leqslant a$. Let $\overline{\underline{a}} = (3, a)$, and observe that I satisfies $\mathrm{EGH}_{\overline{\underline{a}}}$ by Theorem 4.1. In particular, since $\mathrm{EGH}_{\underline{a}}(d)$ is equivalent to $\mathrm{EGH}_{\overline{\underline{a}}}(d)$ for all $d < a$, we only have to show that $\mathrm{EGH}_{\underline{a}}(a-1)$ holds. Let $\{v_1, \ldots, v_c\}$ be the pre-image in A of a K-basis of $(I/\mathbf{f})_{a-1}$, and let $Q = (f_1, f_2, v_1, \ldots, v_c)$. First, assume that $f_3 \notin Q$. By assumption, we have that Q satisfies $\mathrm{EGH}_{\overline{\underline{a}}}(a-1)$ so that, if $\overline{J}$ is the smallest $\overline{\underline{a}}$-lpp ideal such that $H(Q; a-1) = H(\overline{J}; a-1)$, then $H(Q; a) \geqslant H(J; a)$. Observe that $J = J' + (x_3^a)$ is an $\underline{a}$-lpp ideal such that $H(J; a-1) = H(\overline{J}; a-1)$ and $H(J; a) = H(\overline{J}; a) + 1$. We then have that $H(I; a) = H(Q + (f_3); a) = H(Q; a) + 1 \geqslant H(\overline{J}; a) + 1 = H(J; a)$, and we are done. If $f_3 \in Q$ then $\mathrm{ht}(Q) = 3$, and by homogeneous prime avoidance we may assume that f_1, v_c, f_2 forms a regular sequence of degree $\underline{a}' = (3, a-1, a)$ (or $(a-1, 3, a)$ in case $a = 3$). Either way, I satisfies $\mathrm{EGH}_{\underline{a}'}$ by Theorem 4.1, and therefore there exists a $\underline{a}'$-lpp ideal J with the same Hilbert function as I. In particular, since $\underline{a} \geqslant \underline{a}'$, the monomial ideal J also contains $\mathfrak{a} = (x_1^3, x_2^a, x_3^a)$, and we conclude by Theorem 2.1. □

5. Applications and examples

In this section, we present some applications of the EGH Conjecture, supported by several examples. For our computations, it is convenient to introduce the following integers.

Definition 5.1. Let $\underline{a} = (a_1, \ldots, a_r)$ be a degree sequence, and k, d be non-negative integers. For $r < i \leqslant k$, we let $a_i = \infty$ and $x_i^{a_i} = 0$. Also, we let

$$\begin{bmatrix} k \\ d \end{bmatrix}_{\underline{a}} = \begin{cases} \dim_K \left(\dfrac{K[x_{n-k+1}, \ldots, x_n]}{(x_i^{a_i} \, : \, n-k+1 \leqslant i \leqslant n)} \right)_d & \text{if } k \geqslant 1; \\ 0 & \text{if } k = 0. \end{cases}$$

Whenever $\underline{a}$ is clear from the context, we will omit it from the notation.

Remark 5.2. Notice that $\begin{bmatrix} k \\ d \end{bmatrix}_{\underline{a}}$ actually depends on n: for instance $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(2)} = \begin{cases} 0 \text{ if } n = 1; \\ 1 \text{ otherwise.} \end{cases}$

The next definition mimics the Macaulay representation of a number [BH93, Section 4.2], but also takes into account the additional information of a degree sequence $\underline{a}$.

Definition 5.3. An $(\underline{a}, n)$-*Macaulay representation* of a non-negative integer k in base d is the expression of k as sum

$$k = \begin{bmatrix} k_d \\ d \end{bmatrix} + \begin{bmatrix} k_{d-1} \\ d-1 \end{bmatrix} + \ldots + \begin{bmatrix} k_1 \\ 1 \end{bmatrix},$$

where $k_d \geqslant k_{d-1} \geqslant \ldots \geqslant k_1 \geqslant 0$. Whenever n is understood from the context, we just called the above an $\underline{a}$-Macaulay representation of k in base d.

As for the standard Macaulay representation, which corresponds to the choice $a_i = \infty$ for all i, the $(\underline{a}, n)$-Macaulay representation of k in base d exists, and it is unique. Moreover, if $V \subseteq A_d$ is the $\underline{a}$-lpp vector space with $\dim_K(V) = k$, then there is a vector space decomposition $V = \bigoplus_{j=0}^{d} V_{[d-j]} u_j$, where u_j is a monomial of degree j and $V_{[d-j]}$ can be identified with $\left(\frac{K[x_{n-k_{d-j}-1}, \ldots, x_n]}{(x_i^{a_i} | n-k_{d-j}-1 \leqslant i \leqslant n)} \right)_{d-j}$. In particular, $\dim_K(V_{[d-j]}) = \begin{bmatrix} k_{d-j} \\ d-j \end{bmatrix}$.

Given the $\underline{a}$-Macaulay representation of k in base d we let

$$k_{\underline{a}}^{\langle d \rangle} = \begin{bmatrix} k_d \\ d+1 \end{bmatrix} + \begin{bmatrix} k_{d-1} \\ d \end{bmatrix} + \ldots + \begin{bmatrix} k_1 \\ 2 \end{bmatrix}.$$

The following enhanced version of Macaulay Theorem is a direct consequence of the proof of Theorem 2.1. For instance, see [RS08, CR09].

Theorem 5.4. *Let $\underline{a} = (a_1, \ldots, a_r)$ be a degree sequence,$\mathfrak{a}$ be the corresponding plus-powers ideal, and set $R = A/\mathfrak{a}$. Let $H \colon \mathbb{N} \longrightarrow \mathbb{N}$ be a numerical function. Then H is the Hilbert function of R/I for some homogeneous ideal $I \subseteq R$ if and only if $H(d+1) \leqslant H(d)_{\underline{a}}^{\langle d \rangle}$.*

Proof. If $J \subseteq A$ is any $\underline{a}$-lpp ideal, and we set $k = H(A/J; d)$, then one can check that $H(A/\mathfrak{m}J; d+1) = k_{\underline{a}}^{(d)}$. Since $\mathfrak{m}J_d \subseteq J_{d+1}$, the value $H(A/J; d)_{\underline{a}}^{(d)}$ is therefore the maximal growth in degree $d+1$ of the quotient by an $\underline{a}$-lpp ideal which has Hilbert function equal to k in degree d.

Let $I \subseteq R$ be a homogeneous ideal. We lift it to A, and still call it I. By assumption, $\mathrm{LPP}^{\underline{a}}(I)$ is an ideal with the same Hilbert function as I, and from the fact that $\mathfrak{m}I_d \subseteq I_{d+1}$ by the above observation we get that $H(A/I;d+1) \leqslant H(A/I;d)_{\underline{a}}^{(d)}$.

Conversely, let H be a numerical function that satisfies the hypothesis, and let $V \subseteq A_d$ be an $\underline{a}$-lpp K-vector space such that $\dim_K(A_d/V) = H(d)$. Consider the $\underline{a}$-lpp ideal $J = (V) + \mathfrak{a}$. By the above observation, $H(d)_{\underline{a}}^{(d)}$ coincides with the dimension of $(A/J)_{d+1}$ which, by assumption, is at least $H(d+1)$. By adding appropriate monomials to J_{d+1}, we can assume that J is an $\underline{a}$-lpp K-ideal such that $\dim_K((A/J)_{d+1}) = H(d+1)$. Arguing in this way for all integers d, we obtain a monomial ideal I containing $\mathfrak{a}$ (in fact, an $\underline{a}$-lpp ideal) such that $H(A/I) = H$. □

There are some implementations of these results in software systems such as Macaulay2. For instance, the one authored by White [Whi].

Example 5.5. Let $A = K[x_1,x_2,x_3]$, and let I be a homogeneous ideal which contains a regular sequence of degree $\underline{a} = (3,3,4)$. Suppose that, regarding its Hilbert function, we only know that $H(A/I;5) = 5$, and that we want to get an estimate on $H(A/I;6)$. Classically, this can be done by means of Macaulay Theorem, which in this case gives that $H(A/I;6) \leqslant 5$. However, since $\mathrm{EGH}_{\underline{a}}$ holds in this case by Theorem 4.1, we know that $H(I) = H(\mathrm{LPP}^{\underline{a}}(I))$ and, by Theorem 5.4, we conclude that $H(A/I;6) \leqslant 5_{\underline{a}}^{(5)} = 2$.

The following result was observed by Liang.

Proposition 5.6. *Let $I \subseteq A = K[x_1,x_2,x_3]$ be an ideal which contains an ideal $\mathbf{f}$ generated by a regular sequence of degree (a_1,a_2). The minimal number of generators of I is bounded above by $a_1 \cdot a_2$.*

Proof. Observe that any ideal containing $\mathbf{f}$ satisfies $\mathrm{EGH}_{(a_1,a_2)}$ by Theorem 4.1. By Lemma 2.15 we have that $H(I/\mathfrak{m}I) \leqslant H(J/\mathfrak{m}J)$, where $J = \mathrm{LPP}^{(a_1,a_2)}(I)$, and thus we may just bound the number of generators of J. As J is monomial, we can find a minimal monomial generating set. Notice that, if $u = x_1^i x_2^j x_3^k$ is a minimal generator of J, then $0 \leqslant i < a_1$ and $0 \leqslant j < a_2$, since J contains $\mathfrak{a} = (x_1^{a_1}, x_2^{a_2})$. Moreover, if $v = x_1^{i'} x_2^{j'} x_3^{k'}$ is another minimal monomial generator of J, then $i' \neq i$ or $j' \neq j$, otherwise u and v would be one multiple of the other. Therefore, there are at most $a_1 \cdot a_2$ possible choices for i and j, which proves the desired bound. □

Proposition 5.6 can be applied to bound the number of defining equations of curves in $\mathbb{P}^3$. In fact, such a curve can is defined by a homogeneous height two prime ideal $P \subseteq K[\mathbb{P}^3] = K[x_0,x_1,x_2,x_3]$, which then contains a regular sequence of some degree (a_1,a_2). We may assume that K is infinite, and we pick a general linear form ℓ which is regular modulo P. Let $\overline{P}$ be the image of P inside $\overline{A} = A/\ell \cong K[x_1,x_2,x_3]$. The minimal number of generators of $\overline{P}$ in $\overline{A}$ coincides with that of P in A, and we can now apply the previous proposition to the ideal $\overline{P}$, which still contains a regular sequence of degree (a_1,a_2).

Another application of the EGH conjecture is the Cayley-Bacharach Theorem, Conjecture 1.2. Its most classical version states that if a cubic $\mathscr{C} \subseteq \mathbb{P}^2$ contains eight points that lie on the intersection of two cubics, then it must contain the ninth point as well. This has been extended and generalized in several ways. We illustrate the relation with the EGH conjecture in the following example.

Example 5.7. Let $X \subseteq \mathbb{P}^3$ be a complete intersection of degree $(3,3,3)$. If a cubic hypersurface Y contains at least 22 of the 27 points of X, then it must contain X. To see this, let $\mathbf{f} = (f_1, f_2, f_3)$ be an ideal of definition of X, generated by three cubics. Let g be a cubic defining Y, and let $I = \mathbf{f} + (g)$. Our assumptions guarantee that $\mathrm{e}(A/I) \geqslant 22$. By way of contradiction, assume that $g \notin \mathbf{f}$. We may assume that K is infinite and, after a general change of coordinates, we may assume that $I^{\mathrm{sat}} = (I :_A x_4^\infty)$, and x_4 is a regular element modulo I^{sat}. Let $\overline{\mathbf{f}}, \overline{I}, \overline{g}$ and $\overline{I^{\mathrm{sat}}}$ denote the images of $\mathbf{f}, I, g$ and I^{sat} in $A/(x_4) \cong \overline{A}$.

First, assume that $g \in \mathbf{f} + (x_4)$. In this case, there exists $g' \in (I :_A x_4) \subseteq I^{\mathrm{sat}}$ of degree at most 2. If $g' \in \mathbf{f} + (x_4)$, we repeat the process, to find that I^{sat} actually contains a linear form ℓ. At this point $\ell \notin \mathbf{f} + (x_4)$ is forced by our assumption that x_4 is regular modulo I^{sat}. Either way, we found an element of degree less than 3 which belongs to I^{sat} but not to $\mathbf{f} + (x_4)$. Multiplying such an element by an appropriate power of a general linear form, we obtain a form of degree 3 which still belongs to I^{sat}, but does not belong to $\mathbf{f} + (x_4)$. By abuse of notation, we still call it g, and will assume henceforth that $g \notin \mathbf{f} + (x_4)$, that is $\overline{g} \notin \overline{\mathbf{f}}$. We have that $\mathrm{e}(A/I) = \mathrm{e}(\overline{A}/\overline{I^{\mathrm{sat}}}) \leqslant \mathrm{e}(\overline{A}/J)$, where $J = \overline{\mathbf{f}} + (\overline{g}) \subseteq \overline{A}$. By Proposition 4.16, $\mathrm{LPP}^{(3,3,3)}(J)$ is an ideal with the same Hilbert function as J. Moreover, since $\overline{g} \notin \overline{\mathbf{f}}$, the ideal $\mathrm{LPP}^{(3,3,3)}(J)$ must contain the monomial $x_1^2 x_2$. In particular, $\mathrm{e}(\overline{A}/J) = \mathrm{e}(A/\mathrm{LPP}^{(3,3,3)}(J)) \leqslant \mathrm{e}(\overline{A}/(x_1^3, x_1^2x_2, x_2^3, x_3^3)) = 21$. However, our assumptions guarantee that $\mathrm{e}(A/I) \geqslant 22$, a contradiction.

We conclude the paper by illustrating how the combinatorial Kruskal-Katona Theorem [Kru63, Kat68] is related to the EGH conjecture and, in fact, recovered by the Clements-Lindström Theorem 2.1 in the case $\underline{a} = (2, 2, \ldots, 2)$. For additional details on what follows, see for instance [HH11, Section 6.4].

The Kruskal-Katona Theorem characterizes all the possible f-vectors of simplicial complexes Δ. Recall that the f-vector $f(\Delta) = (f_0, \ldots, f_{r-1})$ of an $(r-1)$-dimensional simplicial complex Δ simply records in the i-th entry the number of faces of Δ of dimension i. Given non-negative integers f, d, there is a unique Macaulay representation

$$f = \binom{f_d}{d} + \binom{f_{d-1}}{d-1} + \ldots + \binom{f_1}{1},$$

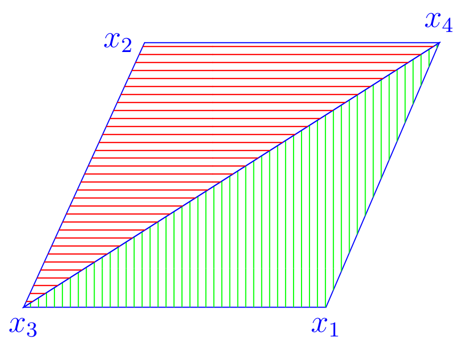
where $f_d \geqslant f_{d-1} \geqslant \ldots \geqslant f_1 \geqslant 0$. If we set

$$f^{(d)} = \binom{f_d}{d+1} + \binom{f_{d-1}}{d} + \ldots + \binom{f_1}{2},$$

then the Kruskal-Katona Theorem states that $(f_0, \ldots, f_{r-1})$ is the f-vector of a simplicial complex of dimension $r-1$ if and only if $f_{d+1} \leqslant f_d^{(d+1)}$ for each $d = 0, \ldots, r-2$.

If $K[\Delta]$ denotes the Stanley-Reisner ring associated to the simplicial complex Δ, then $K[\Delta] = K[x_1, \ldots, x_n]/J$, where $n = f_0$ and J is a square-free monomial ideal. Letting $R = K[x_1, \ldots, x_n]/I$, where $I = J + (x_1^2, \ldots, x_n^2)$, then it is easy to see that $H(R; i) = f_{i-1}$ for all $i \geqslant 0$, where we set $f_{-1} = 1$. On the other hand, given any monomial ideal $I \subseteq A = K[x_1, \ldots, x_n]$ containing $\mathfrak{a} = (x_1^2, \ldots, x_n^2)$, we can write it uniquely as $I = J + \mathfrak{a}$, where J is a square-free monomial ideal. The simplicial complex Δ associated to the Stanley-Reisner ring A/J has f-vector $f = (f_0, \dots, f_{r-1})$, where $f_i = H(A/I; i+1)$ for $i = 0, \dots, r-1$. Finally, we observe that if $\underline{a} = (2, 2, \dots, 2)$, then $\begin{bmatrix} k \\ d \end{bmatrix}_{\underline{a}} = \binom{k}{d}$. Therefore, the numerical condition of Theorem 5.4 can be restated as $f_d = H(R; d+1) \leqslant H(R; d)^{(d)}_{\underline{a}} = H(R; d)^{(d)} = f^{(d)}_{d-1}$ for all $d \geqslant 1$, which is precisely the condition of the Kruskal-Katona Theorem.

Example 5.8. Let $f = (4, 5, 2)$, and let us construct a simplicial complex Δ such that $f(\Delta) = f$. Consider the numerical function $H : \mathbb{N} \to \mathbb{N}$ defined as $H(0) = 1$, $H(1) = 4$, $H(2) = 5$, $H(3) = 2$, and $H(d) = 0$ for $d > 3$. By Theorem 5.4, it can be checked that there exists a $(2, 2, 2, 2)$-lpp ideal I with Hilbert function equal to H, namely, $I = (x_1x_2) + (x_1^2, x_2^2, x_3^2, x_4^2)$. If we let $J = (x_1x_2)$, then the Stanley-Reisner ring $K[x_1, x_2, x_3, x_4]/J$ is associated to the following 2-dimensional simplicial complex, which has f-vector precisely equal to f:



Example 5.9. Let $f = (4, 5, 3)$, and let us show that there is no simplicial complex Δ with such f-vector. To show this, by Theorem 5.4 there is no $(2, 2, 2, 2)$-lpp ideal of $K[x_1, x_2, x_3, x_4]$ which has Hilbert function H satisfying $H(2) = 5$ and $H(3) = 3 > H(2)^{(2)}_{(2,2,2,2)} = 2$.

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