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START-UP/SHUT-DOWN MINLP FORMULATIONS FOR THE UNIT COMMITMENT WITH RAMP CONSTRAINTS

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Abstract

In [1] the first MIP exact formulation was provided that describes the convex hull of the solutions satisfying all the standard operational constraints for the thermal units: minimum up- and down-time, minimum and maximum power output, ramp (including start-up and shut-down) limits, general history-dependent start-up costs, and nonlinear convex power production costs. That formulation contains a polynomial, but large, number of variables and constraints. We present two new formulations with fewer variables defined on the shut-down period and computationally test the trade-off between reduced size and possibly weaker bounds.

Key words: Unit Commitment problem, Ramp Constraints, MIP Formulations, Dynamic Programming, Convex Costs

1. Introduction

The Unit Commitment (UC) problem is a fundamental problem in power industries. It requires to coordinate the production of a set of power generation units by finding a feasible schedule—satisfying complex operational constraints—of each, over some time period, in order to minimize operational costs while satisfying system-wide constraints. The latter usually comprise the satisfaction of the energy demand, the provision of different types of reserve, and the handling of the transmission network. Operational constraints depend on the type of generation units. Despite the significant increase of contribution of Renewable Energy Sources (RES) units (wind, solar, ...), most power systems are still mainly based on thermal units (comprised nuclear ones) and hydro units. Indeed, these are necessary if only to be able to cope with the uncertainty in the production output that is typical of most RES units, which lead to highly complex uncertain (robust and/or stochastic) UC variants [15]. Thus, thermal units remain at the heart of basically every UC model of practical interest. In the last decade, the advances in Mixed-Integer (linear and convex) Programming (MIP) solvers have made MIP approaches an attractive option for solving UC, either as a whole or for specific sub-problems in the context of decomposition approaches (e.g., [14, 15, 2, 13]. This motivated a significant amount of research on the strong combinatorial structure of operational constraints of thermal units. In [10] many of the different types of formulations that appeared in the literature have been surveyed and compared with a large computational experience.

In [1] we gave the first MIP description of the convex hull of the solutions satisfying *all* the standard operational constraints for the thermal units: minimum up- and down-time, minimum and maximum power output, ramp (including start-up and shut-down) limits, general history-dependent start-up costs, and nonlinear convex power production costs. This formulation is inspired by a Dynamic Programming algorithm [5], and contains a polynomial number of variables and constraints. However, the number of variables grows cubically with the number of instants in the time horizon, making the formulation somehow impractical. This is why we also presented two additional MIP formulations which trade a weaker bound for fewer variables. We mention that three independent groups obtained a similar result restricted to linear objective function: the first was [6, 7] and then [8, 9] also appeared with very similar structure but with different proof techniques.

In this paper we continue and extend this line of research by deriving two new MIP formulations for UC that investigates complementary options to reduce the number of variables. In [1] one of the presented formulations was based on variables defining the power produced by a unit when the start-up time has been fixed. Here, we present a nearly-symmetric formulation based on variables defining the power produced by a unit when the shut-down time has been fixed. Despite the near symmetry, the two formulation behave somewhat differently, as our computational results show. Finally, we present and test a further formulation that combines both the "start-up" and the "shut-down" approach.

The structure of the paper is as follows. In Section 2 we recall the present most popular UC formulation restricted to thermal units. In Section 3 we recall the results in the recent paper [1] on the new formulations based on the DP algorithm in [5]. In Section 4 we present the new formulation based on shut-down power variables and the combined one. In Section 5 we present some preliminary computational experiments to characterize the placement of the new formulations within the state-of-the-art of MIP formulations for UC. Finally, in Section 6 we sum up the results, and draw some possible lines for future research on the topic.

2. The thermal Unit Commitment problem

Here we briefly recall the MIP formulation of the thermal Unit Commitment problem that became more and more popular in the last years, as it is one of the main innovations which made UC solvable by standard MIP solvers. It has been introduced in [12] and, independently, in [11], and it is usually referred to as the "*3-bin formulation*" from the number of vectors of binary variables that are considered.

Let I be the set of thermal generators, with $m = |I|$, and $T = \{1, \dots, n\}$ be the set of time periods in the planning horizon. Given two time instants t' and t'', we will denote by $T(t', t'')$ the set of all the periods from t' to t'', extremes included. For each $i \in I$ and $t \in T$, let p_{it} (the *power variables*) be the power level of unit i at period t, and x_{it} (the *commitment variables*) be the binary variable denoting the on/off state of unit i at period t. If $x_{it} = 1$ ("on" state), then the power p_{it} may be nonzero and subject to some technical constraints specified in the following. If $x_{it} = 0$ ("off" state), then $p_{it} = 0$. The 3-bin formulation requires two additional sets of variables: *start-up variables* v_{it} denoting if unit i has been started up at period t (i.e., x_{it}=1 and $x_{i,t-1}$=0) and *shut-down variables* w_{it} denoting if i has been shut down at t (i.e., x_{it}=0 and $x_{i,t-1}$=1). The basic version of the 3-bin formulation is

$$\min \ \textstyle\sum_{i\in I}\sum_{t\in T}\big(x_{it} f_i(p_{it}/x_{it}) + c_i x_{it} + s_i v_{it}\big) \tag{1}$$

$$\textstyle\sum_{i\in I} p_{it} = d_t \qquad t \in T \tag{2}$$

$$l_i x_{it} \le p_{it} \le u_i x_{it} \qquad i \in I \ , \ t \in T \tag{3}$$

$$\textstyle\sum_{s\in T(t-\tau^i_+ +1, t)} v_{is} \le x_{it} \qquad i \in I \ , \ t \in T(\tau_i^+ , n) \tag{4}$$

$$\textstyle\sum_{s\in T(t-\tau^i_- +1, t)} w_{is} \le 1 - x_{it} \qquad i \in I \ , \ t \in T(\tau_i^- , n) \tag{5}$$

$$x_{it} - x_{i,t-1} = v_{it} - w_{it} \qquad i \in I \ , \ t \in T \tag{6}$$

$$p_{it} - p_{i,t-1} \le \Delta_i^+ x_{i,t-1} + \bar{l}_i v_{it} \qquad i \in I \ , \ t \in T \tag{7}$$

$$p_{i,t-1} - p_{it} \le \Delta_i^- x_{it} + \bar{u}_i w_{it} \qquad i \in I \ , \ t \in T \tag{8}$$

$$x_{it} \ , \ v_{it} \ , \ w_{it} \in \{0,1\} \qquad i \in I \ , \ t \in T \tag{9}$$

The objective function (1) is composed of three parts: the variable generation costs evaluated as the Perspective Reformulation [5] of the quadratic function $f_i(p_{it}) = a_i p_{it}^2 + b_i$, the fixed generation costs $c_i x_{it}$, and the start-up costs $s_i v_{it}$. For simplicity, in this formulation we consider only fixed start-up costs; history-dependent start-up costs can be included with some complication [10], and are handled basically "for free" by the DP-based formulations examined here (cf. [1] for details). The *demand constraints* (2) are the simplest version of system-wide constraints, where d_t is the (forecast) total energy demand at period t; other types may relate reserves and the distribution network (e.g. [10]), but we only consider (2) since our focus is on the description of the individual thermal units, which is logically independent from system-wide constraints. *Minimum and maximum power output constraints* are imposed by (3), where l_i and u_i are the extreme values for the generated power for each unit $i \in I$ (when on). In order to limit the technical stress due to frequent start-up and shut-down operations, *minimum up- and down-time constraints* (4)-(5) establish a minimum number of periods that unit i has to be in on and off state, τ_i^+ and τ_i^-, respectively; for simplicity we have omitted the obvious constraint that may fix the on/off status of the unit depending on its state prior to the beginning of the planning horizon. Constraints (6) establishes the relation among state, start-up, and shut-down variables. *Ramp-up and ramp-down constraints* (7)-(8) limit the maximum increase Δ_i^+ or decrease Δ_i^-, respectively, of the power produced by unit i in two consecutive time instants. These are usually related with *start-up and shut-down limits*, that is the maximum power $\bar{l}_i$ when the unit is started-up and the maximum power $\bar{u}_i$ before the unit is shut-down. For consistency, it must be $l_i \le \bar{l}_i \le u_i$ and $l_i \le \bar{u}_i \le u_i$.

The above formulation, minus (7)-(8), is known to be exact only when no ramp-up/down limits are imposed. The question if it is possible to write an exact formulation for UC restricted to a single thermal unit (1UC) in the variable space of the 3-bin formulation is still unsolved. However, 1UC is known to be an easy problem: indeed, in [5] a Dynamic Programming (DP) algorithm was proposed that can solve 1UC with all the above constraints in $O(n^3)$ (and that can be generalized to more complex objectives). Based on that, in [1] we gave the first exact formulation for 1UC that considers all the above mentioned technical features, which is recalled in the next section.

3. DP formulations

For the description of the DP algorithm we drop the unit index i for notational simplicity. We then define a state-space graph $G = (N, A)$. The nodes in N are of two types: ON_t and OFF_t for each $t \in T$, plus two special nodes, the source s and the sink d. The arcs in A are of two types: *ON-arcs* (OFF_h, ON_k), denoting that the unit is turned on at the beginning of period h and unit remains on until the end of period k, and *OFF-arcs* (ON_k, OFF_r), denoting that the unit is off from period $k+1$ to period $r-1$. Both on- and off-arcs are only constructed, obviously, if they satisfy the minimum (respectively) up- and down-time constraints. Moreover, there are the connections between the source node s and the ON and OFF nodes defined according to the initial state of the unit. That is, if the unit is on since τ^0 periods, then there is an on-arc from s to each node ON_k such that $k + \tau^0 \geq \tau^+$. If, instead, the unit is off since $-\tau^0$ periods, then there is an off-arc from s to each node OFF_h such that $h - \tau^0 - 1 \geq \tau^-$. ON-arcs (OFF_h, ON_k) are labeled with costs γ_{ON}^{hk} computed as the fixed cost c_i multiplied by $(k - h + 1)$; OFF-arcs are labeled with γ_{OFF}^{kr} corresponding to the start-up cost. All nodes are then connected to the sink node d: OFF-arcs (ON_t, d) and ON-arcs (OFF_t, d). Finally, the single arc (s, d) means that the unit remains with the same status for all the time horizon, and it is an ON- or OFF-arc according to the fact that the unit, is, respectively, on or off at time 0. The details about the (efficient) computation of the arcs costs within the DP algorithm are provided in [5].

The formulation inspired by the DP algorithm for (1UC) consists of two parts:

- the shortest path formulation based on the state-space graph G;
- new power variables and the linking constraints with the previous part.

The shortest path formulation is straightforward: one just introduces the node-arcs incidence matrix of the state-space graph and writes the obvious system of inequalities. Then we can then simply write this part of the formulation as:

$$E^i y_i = \delta^i \ , \ y_i \geq 0 \ , \tag{10}$$

where E^i is the node-arcs incidence matrix of $G^i = (N^i, A^i)$ (here we reintroduced the unit index $i \in I$), y_i is the vector of arc flow variables, and δ^i is the vector with all zero entries except $\delta_s^i = -1$ and $\delta_d^i = 1$. Within the vector y_i, we denote with y_i^{hk} the variable associated with an ON-arc $(OFF_h, ON_k) \in A^i$. For short, we define as A_{ON}^i as the subset of such ON-arcs, and we denote them simply as "(h, k)" (as the type of the nodes is obvious). For each $(h, k) \in A_{ON}^i$ and $t \in T(h, k)$ we construct a variable p_t^{hk} to denote the power level for each time instant if the unit i is started-up at time h and shut-down at time k. The following result is proven in [1]:

Theorem 3.1. *[1] The following is an exact formulation for (1UC):*

$$min\ \gamma_i^T y_i + \sum_{(h,k)\in A^i_{ON}} \sum_{t\in T(h,k)} y_i^{hk} f_i(p_{it}^{hk}/y_i^{hk})) \tag{11}$$

(10)

$$\left.\begin{array}{ll} ly^{hk} \le p_h^{hk} \le \bar{l}y^{hk} & \\ ly^{hk} \le p_t^{hk} \le uy^{hk} & t\in T(h+1,k-1) \\ ly^{hk} \le p_t^{hk} \le \bar{u}y^{hk} & \\ p_{t+1}^{hk} \le p_t^{hk} + y^{hk}\Delta^+ & t\in T(h,k-1) \\ p_t^{hk} \le p_{t+1}^{hk} + y^{hk}\Delta^- & t\in T(h,k-1) \end{array}\right\} (h,k)\in A^i_{ON} \tag{12}$$

Constraints (12) express the Economic Dispatch conditions associated with an ON-arc (OFF_h, ON_k) and the objective function (11) is the Perspective Reformulation [4] of the original objective function f_i. This immediately yields the *DP formulation* for the complete UC problem

$$\begin{array}{lll} \min & \sum_{i\in I} \left(\gamma_i^T y_i + \sum_{(h,k)\in A^i_{ON}} \sum_{t\in T(h,k)} y_i^{hk} f_i(p_{it}^{hk}/y_i^{hk})\right) & \\ & \sum_{i\in I}\sum_{(h,k)\,:\,t\in T(h,k)} p_{it}^{hk} = d_t & t\in T \\ & (10)\ ,\ (12) & i\in I \end{array} \tag{13}$$

The number of binary and continuous variables in (13) is, respectively, $O(n^2|I|)$ and $O(n^3|I|)$. Although the formulation provides (as expected) a strong bound, its size grows quickly, in particular due to the number of continuous variables. Because of this, in [1] two other formulations were introduced which are also based on the DP approach, but achieve different trade-offs between size and tightness. When restricted to 1UC, both are less tight than the exact formulation (10)–(12). The first one uses the original $O(n|I|)$ power variables p_{it} of the 3-bin formulation, while the second one presents a new type of power variables whose cardinality is intermediate between 3-bin and DP formulations.

Given a unit i, consider the commitment variable x_{it}, the start-up/shut-down variables v_{it}/w_{it} and the set of variables y_i^{hk} for $(h,k)\in A^i_{ON}$. It is easy to see that these variables are related by the following equations:

$$x_{it} = \sum_{(h,k):t\in T(h,k)} y_i^{hk}\ ,\ \ v_{it} = \sum_{k\ge t} y_i^{tk}\ ,\ \ w_{it+1} = \sum_{h\le t} y_i^{ht}\ . \tag{14}$$

Consequently, the ramp-up/down constraints assume, respectively, the following form:

$$p_{it} - p_{it-1} \le \Delta_i^+ \sum_{(h,k)\,:\,t-1\in T(h,k-1)} y_i^{hk} + \bar{l}_i \sum_{k\,:\,k\ge t} y_i^{tk} - l_i \sum_{h\,:\,h\le t-1} y_i^{ht-1} \tag{15}$$

$$p_{it-1} - p_{it} \le \Delta_i^- \sum_{(h,k)\,:\,t-1\in T(h,k-1)} y_i^{hk} + \bar{u}_i \sum_{h\,:\,h\le t-1} y_i^{ht-1} - l_i \sum_{k\,:\,k\ge t} y_i^{tk} \tag{16}$$

Note that, in case the unit is on at the beginning of time horizon ($\tau_i^0 > 0$), the initial ramp-up/down conditions have to be set by

$$p_{i1} \le (\Delta_i^+ + p_{i0}) \sum_{k\,:\,1\le k} y_i^{0k}\ ,\ \ -p_{i1} \le (\Delta_i^- - p_{i0}) \sum_{k\,:\,1\le k} y_i^{0k} \tag{17}$$

Then minimum and maximum power output constraints can be rewritten as follows:

$$l_i \sum_{(h,k)\,:\,t\in T(h,k)} y_i^{hk} \le p_{it} \le u_i \sum_{(h,k)\,:\,t\in T(h,k)} y_i^{hk} \tag{18}$$

The right-hand side of constraints (18) can be reinforced as follows. Assuming that $\tau_i^+ \ge 2$, if a unit i is switched on at time t then $\sum_{k\,:\,k\ge t} y_i^{tk} = 1$ and the power p_{it} is bounded by $\bar{l}_i$. If the unit is switched off at time t then $\sum_{h\,:\,h\le t} y_i^{ht} = 1$ and the power p_{it} must not exceed $\bar{u}_i$. In case the unit does not turn on or off but it is committed at time t then $\sum_{(h,k)\,:\,h<t<k} y_i^{hk} = 1$ holds. Consequently, there exists (h,k) such that $h < t < k$ and $y_i^{hk} = 1$. Because of the maximum power output and the ramp-up/down constraints, the power p_{it} is bounded by $\psi_{it}^{hk} = \min\{\, u_i\,,\, \bar{l}_i + \Delta_i^+(t-h)\,,\, \bar{u}_i + \Delta_i^-(k-t)\,\}$. Furthermore, if the unit is initially committed ($\tau_i^0 > 0$) then $\sum_{k\,:l1\le k} y_i^{0k} = 1$ and we have to set $\psi_{it}^{0k} = \min\{\, u_i\,,\, p_{i0} + \Delta_i^+ \cdot t\,,\, \bar{u}_i + \Delta_i^-(k-t)\,\}$. Hence, if $\tau_i^+ \ge 2$ then (18) can be reinforced as

$$p_{it} \le \bar{l}_i \sum_{k\,:\,k\ge t} y_i^{tk} + \bar{u}_i \sum_{h\,:\,h\le t} y_i^{ht} + \sum_{(h,k)\,:\,h<t<k} \psi_{it}^{hk} y_i^{hk} \tag{19}$$

whereas if $\tau_i^+ = 1$ and $y_i^{tt} = 1$, the power p_{it} is bounded by the minimum between $\bar{l}$ and $\bar{u}$, which means that (18) rather becomes

$$p_{it} \le \bar{l}_i \sum_{k\,:\,k>t} y_i^{tk} + \bar{u}_i \sum_{h\,:\,h<t} y_i^{ht} + \sum_{(h,k)\,:\,h<t<k} \psi_{it}^{hk} y_i^{hk} + \min\{\, \bar{l}_i\,,\, \bar{u}_i\,\} y_i^{tt} \tag{20}$$

This finally yields the *p_t-model*

$$\begin{array}{ll} \min & \sum_{i\in I} \big(\gamma_i^T y_i + \sum_{t\in T} \sum_{(h,k)\,:\,t\in T(h,k)} y_i^{hk} \big) f_i(p_{it}/(\sum_{(h,k)\,:\,t\in T(h,k)} y_i^{hk})\big) \\ & (2)\ ,\ (10)\ ,\ (15)\text{–}(20) \end{array} \tag{21}$$

The last formulation introduced in [1] is rather centered on defining variables p_{it}^h denoting the power produced by unit i if committed at time t and if it has been turned on at time instant h; differently from the variable p_{it}^{hk}, in this case the time when the unit will be turned off is not fixed. The relation between p_{it} and p_{it}^h variables is

$$p_{it} = \sum_{h\,:\,h\le t} p_{it}^h \ . \tag{22}$$

The ramp-up/down constraints are then reformulated as

$$p_{it}^h - p_{it-1}^h \le \Delta_i^+ \sum_{k\,:\,k\ge t} y_i^{hk} - l_i y_i^{ht-1} \qquad h\in T(1,n-1)\ ,\ t\in T(h+1,n) \tag{23}$$

$$p_{it-1}^h - p_{it}^h \le \Delta_i^- \sum_{k\,:\,k\ge t} y^{hk} + \bar{u}_i y_i^{ht-1} \qquad h\in T(1,n-1)\ ,\ t\in T(h+1,n) \tag{24}$$

If $\tau_i^0 > 0$, the initial ramp-up/down conditions can be imposed by

$$p_{i1}^0 \le (\Delta^+ + p_0) \sum_{k\,:\,1\le k} y_i^{0k} \quad , \quad -p_{i1}^0 \le (\Delta^- - p_0) \sum_{k\,:\,1\le k} y_i^{0k} \tag{25}$$

and minimum/maximum power output constraints take the form

$$l_i \sum_{k\,:\,k\ge t} y_i^{hk} \le p_{it}^h \le u_i \sum_{k\,:\,k\ge t} y_i^{hk} \qquad h\in T(0,n)\ ,\ t\in T(h,n) \tag{26}$$

However, for $t = h$ the rightmost inequality in (26) can be substituted by

$$p_{ih}^h \le \bar{l}_i \sum_{k\,:\,k>h} y_i^{hk} + \min\{\, \bar{l}_i\,,\, \bar{u}_i\,\} y_i^{hh} \tag{27}$$

while for $t > h$ one can rather use

$$p_{it}^h \le \bar{u}_i y_i^{ht} + \sum_{k\,:lk>t} \psi_{it}^{hk} y_i^{hk} \tag{28}$$

In conclusion, the *Start-Up formulation* (SU) is

$$\begin{array}{lll} \min & \sum_{i\in I} \big(\gamma_i^T y_i + \sum_{t\in T} \sum_{h\,:\,t\ge h} (\sum_{k\,:\,k\ge t} y_i^{hk}) f_i(p_{it}^h/(\sum_{k\,:\,k\ge t} y_i^{hk}))\big) & \qquad (29) \\ & (10)\ ,\ (23)\text{–}(28) & i\in I \\ & \sum_{i\in I} \sum_{h\,:\,h\le t} p_{it}^h = d_t & t\in T \qquad (30) \end{array}$$

This has $O(n^2|I|)$ power variables, compared to $O(n|I|)$ of the p_t-model and $O(n^3|I|)$ of the original DP formulation, with a bound to match [1].

4. Two new formulations for UC

Mirroring the derivation of (29)–(30), we can construct a nearly symmetric formulation, that we name *Shut-Down formulation* (SD). This is based on variables $\tilde{p}^k_{it}$ denoting the power produced at time t by a unit i that will be turned *off* at time instant k, i.e.,

$$p_{it} = \sum_{k\,:lk \geq t} \tilde{p}^k_{it} \tag{31}$$

All the constraints can be derived by using (31); in particular, the ramp-up/down constraints become

$$\tilde{p}^k_{it} - \tilde{p}^k_{it-1} \leq \quad \bar{l}_i y^{tk}_i + \Delta^+_i \sum_{h\,:\,h \leq t-1} y^{hk}_i \qquad k \in T(2\,,n) \ , \ t \in T(2\,,k) \tag{32}$$

$$\tilde{p}^k_{it-1} - \tilde{p}^k_{it} \leq -l_i y^{tk}_i + \Delta^-_i \sum_{h\,:\,h \leq t-1} y^{hk}_i \qquad k \in T(2\,,n) \ , \ t \in T(2\,,k) \tag{33}$$

If $\tau^0_i > 0$, the initial ramp-up/down conditions can be imposed by

$$\tilde{p}^k_{i1} \leq (\Delta^+ + p_0) y^{0k}_i \quad , \quad -\tilde{p}^k_{i1} \leq (\Delta^- - p_0) y^{0k}_i \tag{34}$$

The minimum/maximum power output constraints take the form

$$l_i \sum_{h\,:\,h \leq t} y^{hk}_i \leq \tilde{p}^k_{it} \leq u_i \sum_{h\,:\,h \leq t} y^{hk}_i \qquad k \in T \ , \ t \in T(1\,,k) \tag{35}$$

which for can $t = k$ be strengthened by

$$\tilde{p}^k_{ik} \leq \bar{u}_i \sum_{h\,:\,h<k} y^{hk}_i + \min\{\, \bar{l}_i \,,\, \bar{u}_i \,\} y^{kk}_i \tag{36}$$

while for $t < k$ by

$$\tilde{p}^k_{it} \leq \bar{l}_i y^{tk}_i + \sum_{h\,:\,h<t} \psi^{hk}_{it} y^{hk}_i \tag{37}$$

due to the fact that the unit could be turned on at time t ($y^{tk}_i = 1$) or not ($\sum_{h\,:\,h<t} y^{hk}_i = 1$). All in all, the SD formulation is

$$\begin{aligned}
\min \ & \textstyle\sum_{i \in I} \big(\gamma^T_i y_i + \sum_{t \in T} \sum_{k:t \leq k} (\sum_{h:h \leq t} y^{hk}_i) f_i(\tilde{p}^k_{it} / (\sum_{h:\,h \leq t} y^{hk}_i)) \big) && && (38) \\
& (10) \ , \ (32)\text{–}(37) && i \in I && \\
& \textstyle\sum_{i \in I} \sum_{k\,:\,k \geq t} \tilde{p}^k_{it} = d_t && t \in T && (39)
\end{aligned}$$

It is now natural to define the *Start-Up/Shut-Down formulation* (SUSD) by basically combining the previous two:

$$\begin{aligned}
\min \ & \textstyle\sum_{i \in I} \big(\gamma^T_i y_i + \sum_{t \in T} \theta_{it} \big) && && (40) \\
& \theta_{it} \geq (\textstyle\sum_{k\,:\,k \geq t} y^{hk}_i) f_i(p^h_{it} / (\sum_{k\,:\,k \geq t} y^{hk}_i)) && i \in I \ , \ t \in T && (41) \\
& \theta_{it} \geq (\textstyle\sum_{h\,:\,h \leq t} y^{hk}_i) f_i(\tilde{p}^k_{it} / (\sum_{h\,:\,h \leq t} y^{hk}_i)) && i \in I \ , \ t \in T && (42) \\
& (10) \quad , \quad (30) \quad , \quad (23)\text{-}(28) \quad , \quad (32)\text{-}(37) && i \in I && \\
& \textstyle\sum_{h\,:\,h \leq t} p^h_{it} = \sum_{k\,:\,k \geq t} \tilde{p}^k_{it} && t \in T && (43)
\end{aligned}$$

Basically, (41) and (42) guarantee that the objective function (40) represents the maximum between these of the SU and the SD formulations. The constraints, and in particular (43), enforce the intersection between the feasible solutions of the two formulations. Thus, the lower bound provided by the continuous relaxation of the SUSD formulation has to be at least as good as the ones of both the SU and the SD models, at the cost of having roughly twice the number of variables of each (but still $O(n^2|I|)$ as opposed to the $O(n^3|I|)$ ones of the original DP formulation).

	3-bin		DP		p_t		*SU*		*SD*		*SUSD*	
units	time	gap	time	gap	time	gap	time	gap	time	gap	time	gap
10	0.12	1.522	16.40	0.671	0.59	0.931	2.83	0.916	4.29	0.678	44.84	0.671
20	0.28	1.435	59.07	0.512	1.46	0.775	8.53	0.758	11.09	0.515	139.52	0.512
50	0.96	0.871	300.53	0.076	4.32	0.301	22.42	0.287	33.64	0.079	529.96	0.076

Table 1: Root node gaps of the 3-bin, DP, p_t, SU, SD, and SUSD formulations.

5. Computational Results

In this section we provide some preliminary computational experiments to compare the new formulations presented with the DP-based ones introduced in [1] as well as the reference 3-bin model. For our tests, we considered standard benchmark realistic instances with 10, 20 and 50 units and $n = 24$ time periods, available at

`http://www.di.unipi.it/optimize/Data/UC.html`

All the experiments were conducted on a PC with 2.2 GHz Intel Xeon Gold 5120 CPUs and 64 GB of RAM, under a GNU/Linux Ubuntu 18.04.3 LTS operating system. We used CPLEX 12.9 as optimization tool. For a given number of units, we considered five instances and the average of the results thus obtained are reported.

In Table 1 we compare the formulations by evaluating the lower bound by their continuous relaxations. In particular, for each instance size and for each model, "time" denotes the average time for solving the linear relaxation while "gap" is the average gap (in percentage) between the optimal solution and the value of the linear relaxation.

As it can be expected from the theory, the DP formulation is the one that provides the best gaps; however, it being the largest one, it also has a high computing time for the linear relaxation. In general, a clear trend exists between having a larger number of variables, and therefore a larger cost, and having a stronger bound. The interesting comparison is between the SU and SD models that have basically identical size. On the test instances, the SD formulation is somewhat more costly to solve, but it provides a better bound. This is somewhat unexpected due to the high degree of symmetry between the two, and worth further investigation. A separate analysis is to be done for the SUSD formulation. It shows the same gaps of the produced by the DP model; it is unclear whether this happens by chance or if it can be proven theoretically. Surprisingly, it runs with the highest computing times even if it has a size that is between SU and SD models on the one hand and the DP model on the other hand. Finally, we remark that the 3-bin formulation provides the worst lower bounds although in small computing times.

Table 2 show some computational experiments for solving the integer formulation with the B&C approach in CPLEX. We set a required relative gap of 0.01% and a time limit of 10000 seconds. For each model, column "time" reports the average total time, "opt" the number of instances, over five, solved within the time limit, "nodes" the number of nodes explored during the B&C, and "gap" the average final gap (in percentage).

The results show that the SD formulation is indeed somewhat more effective (surprisingly) than the SU one, and its performances are comparable with the p_t-model, which is the best performing one among the formulations presented in [1]. Hence, the trade-off between the higher cost and the tighter bound (cf. Table 1) is positive, at least on these instances. This is not the case for the SUSD model, that has the worst results in general.

	3-bin				DP				p_t			
units	time	opt	nodes	gap	time	opt	nodes	gap	time	opt	nodes	gap
10	28	5	275	0.01	832	5	599	0.01	5	5	41	0.01
20	7036	2	3561	0.08	7902	2	1961	0.05	1066	5	1234	0.01
50	10000	0	1619	0.12	10000	0	695	0.14	8095	1	2303	0.03

	SU				SD				SUSD			
units	time	opt	nodes	gap	time	opt	nodes	gap	time	opt	nodes	gap
10	152	5	591	0.01	251	5	473	0.01	8033	2	3187	0.17
20	6694	3	3996	0.02	4884	4	2281	0.02	10000	0	1824	0.20
50	8471	1	2669	0.08	8150	2	1617	0.07	10000	0	671	0.20

Table 2: Computational results with gap 10^{-4} and time limit 10000 seconds.

6. Conclusions

We have introduced two new formulations for the UC problem with convex cost function. Both models are based on the DP formulation introduced in [1]. In particular, the Shut-Down is a nearly symmetric formulation of the Start-Up model already introduced in [1], while the Start-Up/Shut-Down is a combination of the two.

The results of our computational experiments show that the Shut-Down formulation is surprisingly more effective that its closest sibling, the Start-Up one. The reason is not completely clear, and surely worth further investigation. On the other hand, the Start-Up/Shut-Down model so far does not seem effective for solving UC. However, it can be further investigated in at least two aspects. First, the trade-off between size and bound quality is inherently tied with the algorithm that is used to solve the continuous relaxation. A column-and-rows generation approach, such as the Structured Dantzig-Wolfe Decomposition [3], may considerably shift the balance in favour of models that would not be effective using standard linear programming approaches. Second, the experimental results show that the value of the linear relaxation of the Start-Up/Shut-Down model is equal to the one provided by the DP formulation, that is the strongest model in this sense. It may be interesting to consider whether this equivalence can be proven theoretically, since the Start-Up/Shut-Down model has considerably less variables than the DP one.

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