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Perturbative BF theory

Enore Guadagnini *, Federico Rottoli

Dipartimento di Fisica E. Fermi dell'Università di Pisa, and INFN Sezione di Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy

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Abstract

We consider a superrenormalisable gauge theory of topological type, in which the structure group is equal to the inhomogeneous group $ISU(2)$. The generating functional of the correlation functions of the gauge fields is derived and its connection with the generating functional of the Chern-Simons theory is discussed. The complete renormalisation of this model defined in $\mathbb{R}^3$ is presented. The structure of the $ISU(2)$ conjugacy classes is determined. Gauge invariant observables are defined by means of appropriately normalised traces of $ISU(2)$ holonomies associated with oriented, framed and coloured knots. The perturbative evaluation of the Wilson lines expectation values is investigated and the up-to-third-order contributions to the perturbative expansion of the observables, which correspond to knot invariants, are produced. The general dependence of the knot observables on the framing is worked out.

1. Introduction

Among the quantum field theory models of topological type [1], the so-called BF theory [2–5] has been proposed in order to describe several different phenomena.

The gauge structure group of the BF theory naturally suggests possible connections with (2+1) gravity [6–19], and applications of the BF formalism in the context of loop quantum gravity have also been studied [20–26]. Generalisations of the BF models in higher dimensions have been considered [27–42]. Quite recently, the use of the BF field theory has been envisaged for the description of topological effects in condensed matter [43–51].

* Corresponding author.
E-mail address: enore.guadagnini@unipi.it (E. Guadagnini).

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Various BF quantisation procedures have been examined [52–61] and the renormalisability of the theory has been proved by power counting and general arguments [62–67]. The BF model is actually a superrenormalisable theory; nevertheless, the explicit renormalisation —with specified normalisation conditions— has never been produced. One of the purposes of the present article is precisely to provide the complete renormalisation of the nonabelian BF theory in $\mathbb{R}^3$.

The definition and computation of topological invariants [68–77] are central issues in the BF model. The observables that we propose have not been considered in literature. We shall demonstrate that the appropriately normalised traces of the expectation values of the holonomies —for the inhomogeneous group $ISU(2)$— associated with oriented framed knots in $\mathbb{R}^3$ are well defined. The first three orders of the perturbative computation of these observables are presented.

Let us recall that the solution of the abelian BF theory in generic closed oriented 3-manifolds has been produced by Mathieu and Thuillier [78–80]. In the present paper we shall concentrate on the perturbative approach to the nonabelian BF theory in $\mathbb{R}^3$ with structure group $ISU(2)$.

The Lie algebra of the inhomogeneous group $ISU(2)$ can be interpreted as a particular extension of the $SU(2)$ algebra which, in the quantum mechanics description of one particle moving in $\mathbb{R}^3$, is obtained by the introduction of the three components P^a of the momentum in addition to the three components J^a of the angular momentum. The corresponding $ISU(2)$ connection has then six components $\mathcal{A}_\mu = A^a_\mu(x) J^a + B^a_\mu(x) P^a$. The most general action in $\mathbb{R}^3$ which is $ISU(2)$ gauge invariant and metric-independent contains two different terms: the first term $\int B^a \wedge F^a(A)$ —where $F^a(A)$ are the angular momentum components of the curvature— gives the name to the model and the second term $\int \mathrm{Tr}(A \wedge dA + i\frac{2}{3} A \wedge A \wedge A)$ coincides with the Chern-Simons action for the $SU(2)$ subgroup.

Section 2 contains the fundamentals of the perturbative approach for the computation of the BF correlation functions of the connection in the Landau gauge. The general structure of the connected Feynman diagrams is worked out. The computation of the generating functional of the connected correlation functions to all orders of perturbation theory is presented and its Chern-Simons relationship is discussed in Section 3. The complete renormalisation of the BF theory is given in Section 4. It is shown that the theory is superrenormalisable, and only six one-loop diagrams need to be examined. These one-particle-irreducible diagrams concern the two-point function and the three-point proper vertex of the connection. It is shown that, as in the case of the Chern-Simons theory, the two-point function of the connection does not receive loop corrections and therefore the bare propagator coincides with the dressed propagator.

In order to introduce Wilson line observables in the BF model, certain unitary representations of $ISU(2)$ are described in Section 5. Since the group $ISU(2)$ is noncompact, these nontrivial representations are infinite dimensional. Wilson line operators are defined by means of normalised traces of the $ISU(2)$ holonomies associated with oriented knots. For completeness, the classical traces of the $ISU(2)$ conjugacy classes are described in Section 6. The proof that the BF expectation values of the Wilson line operators are well defined is contained in Section 7. It is shown that, since the correlation functions of the connection are invariant under global $ISU(2)$ transformations, the expectation value of a knot holonomy is a function of the Casimir operators of $ISU(2)$. This implies that the BF mean values of the Wilson line operators are well defined and describe topological invariants for oriented and framed knots in $\mathbb{R}^3$.

The perturbative computation of the knot observables up to the third order in powers of $\hbar$ is described in Section 8. The knot invariants that are found at first and second order correspond to the knot invariants that also appear in the Chern-Simons theory. While, at the third order of perturbation theory, the BF and Chern-Simons knot invariants differ. A proof that the entire framing dependence of the knot observables is completely determined by an overall multiplicative factor is given. This factor is the exponential of the linking number between the knot and its framing multiplied by the combination of the quadratic Casimir operators which is determined by the two point function of the connection. Section 9 contains the conclusions.

2. Fields, lagrangian and diagrams

The fundamental fields of the so-called BF theory [1–5,9] are given by the components of the $ISU(2)$ connection

$$\mathcal{A} = \mathcal{A}_\mu(x) dx^\mu = \left\{ A^a_\mu(x) J^a + B^a_\mu(x) P^a \right\} dx^\mu \; , \tag{2.1}$$

where the generators (J^a, P^a) (with $a = 1, 2, 3$) of the algebra of $ISU(2)$ satisfy the commutation relations

$$[J^a, J^b] = i\epsilon^{abc} J^c \quad , \quad [J^a, P^b] = i\epsilon^{abc} P^c \quad , \quad [P^a, P^b] = 0 \, . \tag{2.2}$$

Let us consider the BF model defined in $\mathbb{R}^3$. Gauge transformations act as

$$\mathcal{A} \longrightarrow \mathcal{A}^\Omega = \Omega^{-1} \mathcal{A} \Omega - i \Omega^{-1} d\Omega \, , \tag{2.3}$$

where $\Omega : \mathbb{R}^3 \to ISU(2)$. When $\Omega \simeq 1 + i\beta^a J^a + i\eta^a P^a$, the infinitesimal gauge transformations take the form

$$\begin{aligned} A^a_\mu &\to A^a_\mu + \Delta A^a_\mu \quad , \quad \Delta A^a_\mu = \partial_\mu \beta^a - \epsilon^{abc} A^b_\mu \beta^c \\ B^a_\mu &\to B^a_\mu + \Delta B^a_\mu \quad , \quad \Delta B^a_\mu = \partial_\mu \eta^a - \epsilon^{abc} A^b_\mu \eta^c - \epsilon^{abc} B^b_\mu \beta^c \, . \end{aligned} \tag{2.4}$$

The components of the curvature are given by

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= -i [\partial_\mu + i\mathcal{A}_\mu, \partial_\nu + i\mathcal{A}_\nu] \\ &= F^a_{\mu\nu}(A) J^a + \big(D_\mu(A) B_\nu - D_\nu(A) B_\mu \big)^a \, P^a \; , \end{aligned} \tag{2.5}$$

in which

$$F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \epsilon^{abc} A^b_\mu A^c_\nu \; , \tag{2.6}$$

and

$$\big(D_\mu(A) B_\nu \big)^a = \partial_\mu B^a_\nu - \epsilon^{abc} A^b_\mu B^c_\nu \, . \tag{2.7}$$

The action of the BF theory in $\mathbb{R}^3$ is the sum of the two metric-independent terms which are separately invariant under $ISU(2)$ transformations (2.4)

$$S = \int d^3x \, \epsilon^{\mu\nu\lambda} \left\{ \tfrac{1}{2} \, B^a_\mu F^a_{\nu\lambda}(A) + g \left[\tfrac{1}{2} A^a_\mu \partial_\nu A^a_\lambda - \tfrac{1}{6} \epsilon^{abc} \, A^a_\mu A^b_\nu A^c_\lambda \right] \right\} \, . \tag{2.8}$$

Without loss of generality, the overall normalisation of the first term in expression (2.8) can be taken to be $(1/2)$, because the $ISU(2)$ generators P^a can be rescaled without any modification of the Lie algebra commutation relations (and consequently B^a_μ also can be rescaled). The real parameter g is a dimensionless coupling constant which multiplies the Chern-Simons lagrangian term

$$S_{CS}[A] = \int d^3x \, \epsilon^{\mu\nu\lambda} \left[\tfrac{1}{2} A^a_\mu \partial_\nu A^a_\lambda - \tfrac{1}{6} \epsilon^{abc} \, A^a_\mu A^b_\nu A^c_\lambda \right] \, . \tag{2.9}$$

When $g = (k/4\pi)$ with integer k, one also recovers invariance under large gauge transformations, which anyway play no role in the perturbative approach to the theory. Note that, in order to discuss the renormalisation of any gauge theory model, all the possible lagrangian terms which are gauge invariant must be taken into account. This is why the renormalisation of the BF model requires that both lagrangian terms —shown in expression (2.8)— must be included in the action.

2.1. Gauge fixing

The gauge fixing procedure is implemented according to the BRST method [81,82]. The BRST transformations [83] are given by

$$\begin{gathered}\delta A^a_\mu = \partial_\mu c^a - \epsilon^{abc} A^b_\mu c^c \;\; , \; \delta B^a_\mu = \partial_\mu \xi^a - \epsilon^{abc} A^b_\mu \xi^c - \epsilon^{abc} B^b_\mu c^c \, , \\ \delta c^a = \tfrac{1}{2}\epsilon^{abc} c^b c^c \quad , \quad \delta \overline{c}^a = M^a \;\; , \; \delta \xi^a = \epsilon^{abc} \xi^b c^c \quad , \quad \delta \overline{\xi}^a = N^a \, , \\ \delta M^a = 0 \;\; , \; \delta N^a = 0 \, ,\end{gathered} \tag{2.10}$$

where $\{\xi^a, \overline{\xi}^a, c^a, \overline{c}^a\}$ is the set of anticommuting ghosts and antighosts fields, whereas M^a, N^a represent the commuting auxiliary fields. In the Landau gauge, the gauge-fixing and ghosts action terms are given by

$$\begin{aligned} S_{\phi\pi} = \int d^3x \Big\{ & M^a \partial^\mu A^a_\mu + N^a \partial^\mu B^a_\mu + \partial^\mu \overline{c}^a (\partial_\mu c^a - \epsilon^{abc} A^b_\mu c^c) \\ & + \partial^\mu \overline{\xi}^a (\partial_\mu \xi^a - \epsilon^{abc} A^b_\mu \xi^c - \epsilon^{abc} B^b_\mu c^c) \Big\} \, , \end{aligned} \tag{2.11}$$

where the flat euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$ of $\mathbb{R}^3$ has been introduced in order to contract the vector indices. The total action $S_{TOT} = S + S_{\phi\pi}$ is invariant under BRST transformations.

In order to recognise the structure constants of the $ISU(2)$ Lie algebra in the gauge-fixing procedure, it is convenient to introduce the ghost field $\mathcal{C} = c^a J^a + \xi^a P^a$, the antighost field $\overline{\mathcal{C}} = \overline{\xi}^a J^a + \overline{c}^a P^a$ and the auxiliary field $\mathcal{N} = N^a J^a + M^a P^a$. The BRST transformations take the form

$$\delta \mathcal{A}_\mu(x) = [D_\mu(\mathcal{A}), \mathcal{C}] \quad , \quad \delta \mathcal{C} = \tfrac{-i}{2}\{\mathcal{C}, \mathcal{C}\} \quad , \quad \delta \overline{\mathcal{C}} = \mathcal{N} \quad , \quad \delta \mathcal{N} = 0 \, ,$$

and $S_{\phi\pi}$ can be written as

$$S_{\phi\pi} = \delta \int d^3x \, \langle \overline{\mathcal{C}} \, \partial^\mu \mathcal{A}_\mu \rangle^{JP} \, ,$$

where the bracket $\langle \cdot\cdot \rangle^{JP}$ denotes the non-degenerate bilinear form [9] on the $ISU(2)$ algebra

$$\langle J^a P^b \rangle^{JP} = \delta^{ab} \quad , \quad \langle J^a J^b \rangle^{JP} = 0 = \langle P^a P^b \rangle^{JP} \, .$$

2.2. Propagators

The Green functions of the differential operators acting on the fields —and entering the quadratic parts of S_{TOT} in powers of the fields— determine the form of the fields propagators. As far as the bosonic fields are concerned, the nonvanishing components of the propagators are given by

$$
\begin{aligned}
&\overbracket{A^a_\mu(x)B}{}^b_\nu(y) = \delta^{ab} \int \frac{d^3k}{(2\pi)^3}\, e^{ik(x-y)}\, \epsilon_{\mu\nu\lambda} \frac{k^\lambda}{k^2} = -i\delta^{ab}\epsilon_{\mu\nu\lambda}\partial^\lambda \Delta(x-y)\,, \\
&\overbracket{B^a_\mu(x)B}{}^b_\nu(y) = -g\,\delta^{ab} \int \frac{d^3k}{(2\pi)^3}\, e^{ik(x-y)}\, \epsilon_{\mu\nu\lambda} \frac{k^\lambda}{k^2} = ig\delta^{ab}\epsilon_{\mu\nu\lambda}\partial^\lambda \Delta(x-y)\,,
\end{aligned}
\tag{2.12}
$$

and

$$
\begin{aligned}
&\overbracket{A^a_\mu(x)M}{}^b(y) = \delta^{ab} \int \frac{d^3k}{(2\pi)^3}\, e^{ik(x-y)}\, \frac{k_\mu}{k^2} = -i\delta^{ab}\partial_\mu \Delta(x-y)\,, \\
&\overbracket{B^a_\mu(x)N}{}^b(y) = \delta^{ab} \int \frac{d^3k}{(2\pi)^3}\, e^{ik(x-y)}\, \frac{k_\mu}{k^2} = -i\delta^{ab}\partial_\mu \Delta(x-y)\,.
\end{aligned}
\tag{2.13}
$$

For the anticommuting fields one gets

$$
\overbracket{c^a(x)\,\overline{c}}{}^b(y) = \overbracket{\xi^a(x)\,\overline{\xi}}{}^b(y) = i\delta^{ab} \int \frac{d^3k}{(2\pi)^3}\, e^{ik(x-y)}\, \frac{1}{k^2} = i\delta^{ab}\Delta(x-y)\,. \tag{2.14}
$$

2.3. *Structure of the diagrams*

The Feynman diagrams of the BF theory, which is defined by the action $S_{TOT} = S + S_{\phi\pi}$ in $\mathbb{R}^3$, have quite peculiar properties that we shall now discuss. Let us consider the generating functional $W[J,K]$ of the connected correlation functions of the components of the connection

$$
e^{iW[J,K]} = \langle e^{i\int d^3x (J^a_\mu A^a_\mu + K^a_\mu B^a_\mu)} \rangle = \frac{\int D(\text{fields})\, e^{iS_{TOT}}\, e^{i\int d^3x (J^a_\mu A^a_\mu + K^a_\mu B^a_\mu)}}{\int D(\text{fields})\, e^{iS_{TOT}}}\,, \tag{2.15}
$$

where $J^a_\mu(x)$ and $K^a_\mu(x)$ are classical sources. We are interested in $W[J,K]$ because in the following sections we shall consider the expectation values of observables which are functions of the fields A^a_μ and B^a_μ exclusively. In this section we consider the combinatorial structure —which is determined by the Wick contractions— of the Feynman diagrams contributing to $W[J,K]$. The renormalisation will be treated in Section 4. The first issue to be discussed is an extension of the Oda and Yahikozawa observation presented in [84].

Proposition 1. *The entire generating functional $W[J,K]$ is given by the sum of connected Feynman diagrams with no loops and with one loop only,*

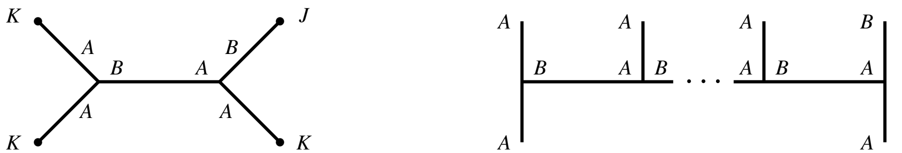
$$
W[J,K] = W_0[J,K] + W_1[J,K]\,. \tag{2.16}
$$

The contribution $W_0[J,K]$ of the tree-level Feynman diagrams can be decomposed into the sum of two terms,

$$
W_0[J,K] = U[K] + \int d^3x\, J^a_\mu(x) H^a_\mu[K](x)\,, \tag{2.17}
$$

in which $U[K]$ and $H^a_\mu[K](x)$ only depend on K^a_μ. The term $U[K]$ is linear in g whereas $H^a_\mu[K](x)$ does not depend on g. The contribution $W_1[J,K]$ of the one-loop diagrams does not depend on g and does not depend on J^a_μ,

$$
W_1[J,K] = W_1[K]\,. \tag{2.18}
$$



(*a*) (*b*)

Fig. 1. (*a*) Example of basic diagram. (*b*) Branch diagram.

(*a*) (*b*)

Fig. 2. (*a*) Example of diagram with one $\overbracket{BB}$ propagator. (*b*) Diagram with one AAA vertex.

Proof. Let us first consider the connected tree level diagrams which contribute to $W[J, K]$. Diagrams which do not contain interaction vertices obviously satisfy equation (2.17) because $\overbracket{AA} = 0$ and the nonvanishing components of the propagators are shown in equation (2.12). So let us now concentrate on diagrams which contain interaction vertices, which are of the type BAA or of the type AAA; it is convenient to recover all these diagrams in three steps.

1. The diagrams constructed with BAA interaction vertices and $\overbracket{AB}$ propagators exclusively are called the basic diagrams; these are the diagrams that remain in the $g \to 0$ limit. They contain one power of the field J^a_μ and may contain an arbitrary number bigger than unit of K^a_μ fields. One example is shown in Fig. 1(a). Indeed, each tree diagram is obtained by combining subdiagrams called "branches". One branch is a one-dimensional ordered sequence of vertices connected by propagators, as shown in Fig. 1(b). Note that the external lines of Fig. 1(b) correspond to field components and do not represent propagators; in particular, one branch diagram necessarily has external legs corresponding to one B^a_μ field and several A^a_μ fields.
2. By adding the possibility of using also $\overbracket{BB}$ propagators, the new diagrams only contain one extra $\overbracket{BB}$ propagator —with respect to the basic diagrams of the previous case— and then they are of first order in powers of g and do not depend on J^a_μ. See for instance Fig. 2(a). The $\overbracket{BB}$ propagator may correspond to one internal line in the Feynman diagrams, or to an external leg of the diagrams.
3. Finally, by incorporating the additional possibility of including also vertices of the AAA type, the new diagrams only contain one extra AAA vertex with respect to the basic diagrams, they are linear in g and do not depend on J^a_μ, as shown in the example of Fig. 2(b).

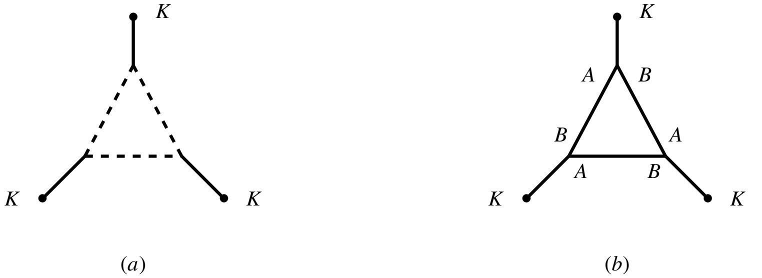


Fig. 3. (a) Diagram with one ghost loop. (b) One-loop diagram without ghost propagators.

As a result, the set of all the connected tree-level diagrams contains diagrams which are linear in g and do not depend on J^a_μ and diagrams which linearly depend on J^a_μ and do not depend on g. This concludes the proof of equation (2.17).

Let us now consider the one-loop connected diagrams entering $W[J, K]$. As shown in the example of Fig. 3(a), connected diagrams with one loop of ghosts only depend on the source field K^a_μ because of the particular structure of the ghosts couplings (2.11). As far as the diagrams without a ghost loop are concerned, by cutting one internal propagator of each one-loop diagram one can open the loop and obtain a connected zero-loop diagram. In view of the result (2.17), the broken propagator was necessary of the AB type. Consequently, also each one-loop diagram with no ghost propagators does not depend on the J^a_μ field and does not depend on g, see the example of Fig. 3(b). This concludes the proof of equation (2.18).
Finally, there are no connected diagrams with two or more loops contributing to $W[J, K]$ because all the one-loop diagrams have external legs corresponding to the A field and the component $\overbracket{A\,A}$ of the propagator is vanishing. □

As a final remark, consider the contributions to $W[J, K]$ of the diagrams containing ghost loops. Since only one-loop diagrams enter $W[J, K]$, all the corresponding possible subdiagrams that have external ghost fields are tree-level diagrams (which are well defined and finite). Consequently, in discussing the renormalisation of $W[J, K]$, the diagrams with external ghost fields can be ignored.

Let $X[A, B]$ be a function of the field components A^a_μ and B^a_μ. In the perturbative computation of the expectation value $\langle X[A, B]\rangle$,

$$\langle X[A,B]\rangle = \frac{\int D(\text{fields})\, e^{iS_{TOT}}\; X[A,B]}{\int D(\text{fields})\; e^{iS_{TOT}}} \,, \tag{2.19}$$

the ghosts contributions are described by diagrams with ghost loops. As shown in equation (2.14), the nonvanishing components of the ghosts propagator are of the type $\overbracket{c\,\overline{c}}$ or $\overbracket{\xi\,\overline{\xi}}$; therefore the lagrangian term $\epsilon^{abd}\partial^\mu\overline{\xi}^a(x)B^b_\mu(x)c^d(x)$ —contained in $S_{TOT} = S + S_{\phi\pi}$— does not contribute to $\langle X[A, B]\rangle$.

3. Generating functionals and Chern-Simons relationship

In order to complete the description of the BF diagrams, in this section we derive the BF generating functional of the connected correlation functions and discuss its relationship with the generating functional of the Chern-Simons theory.

3.1. Connected diagrams

In the computation of the path integral which appears in the numerator of expression (2.15), it is convenient to make the linear change of variables

$$A^a_\mu \longrightarrow A^a_\mu + \widehat{A}^a_\mu \quad , \quad B^a_\mu \longrightarrow B^a_\mu + \widehat{B}^a_\mu \quad , \tag{3.1}$$

in which A^a_μ and B^a_μ are called the quantum components, whereas the classical components $\widehat{A}^a_\mu$ and $\widehat{B}^a_\mu$ satisfy the equations of motion in the presence of the sources

$$\frac{\delta S[\widehat{A}, \widehat{B}]}{\delta \widehat{B}^a_\mu(x)} + K^a_\mu(x) = 0 \quad , \quad \frac{\delta S[\widehat{A}, \widehat{B}]}{\delta \widehat{A}^a_\mu(x)} + J^a_\mu(x) = 0 \, , \tag{3.2}$$

together with the gauge-fixing constraints

$$\partial^\mu \widehat{A}^a_\mu(x) = 0 = \partial^\mu \widehat{B}^a_\mu(x) \, . \tag{3.3}$$

Because of equations (3.2), the classical components $\widehat{A}^a_\mu$ and $\widehat{B}^a_\mu$ are functions of J^a_μ and K^a_μ, (and, for localised J^a_μ and K^a_μ, both components $\widehat{A}^a_\mu$ and $\widehat{B}^a_\mu$ vanish in the $|x| \to \infty$ limit as $\sim 1/|x|^2$). One then finds

$$\begin{aligned} S_{TOT}[A + \widehat{A}, B + \widehat{B}, ...] + \int d^3x \left[J^a_\mu (A^a_\mu + \widehat{A}^a_\mu) + K^a_\mu (B^a_\mu + \widehat{B}^a_\mu) \right] = \\ = S[\widehat{A}, \widehat{B}] + \int d^3x \left[J^a_\mu \widehat{A}^a_\mu + K^a_\mu \widehat{B}^a_\mu \right] + \widetilde{S}[A, B, ...] \, , \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \widetilde{S}[A, B, ...] = S_{TOT}[A, B, M, N, \xi, \overline{\xi}, c, \overline{c}] \\ - \int d^3x \, \epsilon^{\mu\nu\lambda} \epsilon^{abc} \left[B^a_\mu \widehat{A}^b_\nu A^c_\lambda + \tfrac{1}{2} \widehat{B}^a_\mu A^b_\nu A^c_\lambda + \tfrac{1}{2} g A^a_\mu \widehat{A}^b_\nu A^c_\lambda \right] \\ - \int d^3x \, \epsilon^{abc} \left[\partial^\mu \overline{c} \, \widehat{A}^b_\mu c^c + \partial^\mu \overline{\xi} \, \widehat{A}^b_\mu \xi^c + \partial^\mu \overline{\xi}^a \, \widehat{B}^b_\mu c^c \right] \, . \end{aligned} \tag{3.5}$$

Note that $\widetilde{S}[A, B, ...]$ represents the resulting action for the quantum components A^a_μ and B^a_μ of the fields in which

- the linear terms in the quantum fields are missing. Indeed, as a consequence of equations (3.2) and (3.3), $\widehat{A}^a_\mu$ and $\widehat{B}^a_\mu$ satisfy the classical gauge-fixing constraint and represent a stationary point of the action in the presence of the source terms;
- the lagrangian vertices for the quantum fields —which are contained in $\widetilde{S}[A, B, ...]$— depend on the J^a_μ and K^a_μ through the classical components $\widehat{A}^a_\mu$ and $\widehat{B}^a_\mu$.

Therefore the generating functional $W[J, K]$ satisfies

$$e^{iW[J,K]} = e^{iS[\widehat{A},\widehat{B}] + i\int d^3x \left[J^a_\mu \widehat{A}^a_\mu + K^a_\mu \widehat{B}^a_\mu \right]} \frac{\int D(\text{fields}) \, e^{i\widetilde{S}}}{\int D(\text{fields}) \, e^{iS_{TOT}}} \, . \tag{3.6}$$

This expression shows that $W[J, K]$ can be written as the sum of two parts, $W = W_0 + W_1$, in which

- the connected tree-level Feynman diagrams entering W_0 are described by a Legendre transformation of the classical action,

$$W_0[J,K] = S[\widehat{A},\widehat{B}] + \int d^3x \left[J^a_\mu \widehat{A}^a_\mu + K^a_\mu \widehat{B}^a_\mu \right] \; ; \tag{3.7}$$

- the connected diagrams containing loops —described by W_1— are obtained by computing the vacuum-to-vacuum diagrams of the quantum field components. These diagrams are determined by the lagrangian terms contained in the resulting action $\widetilde{S}$, with the normalisation given by the vacuum-to-vacuum diagrams computed in the absence of sources, *i.e.*, when $\widehat{A}^a_\mu$ and $\widehat{B}^a_\mu$ vanish.

Proposition 2. *The function $W_0[J,K]$ is given by*

$$W_0[J,K] = g\, S_{CS}[\widehat{A}] + \int d^3x\, J^a_\mu(x)\widehat{A}^a_\mu(x) \; , \tag{3.8}$$

where the Chern-Simons action $S_{CS}[A]$ is shown in equation (2.9); $\widehat{A}^a_\mu$ is a classical field which only depends on K^a_μ, it satisfies $\partial^\mu \widehat{A}^a_\mu(x) = 0$ and

$$\frac{\partial S_{CS}[\widehat{A}]}{\delta \widehat{A}^a_\mu(x)} = -K^a_\mu(x) \; . \tag{3.9}$$

Proof. Since the BF action (2.8) can be written as

$$S[A,B] = \int d^3x\, B^a_\mu(x) \frac{\delta S_{CS}[A]}{\delta A^a_\mu(x)} + g\, S_{CS}[A] \; , \tag{3.10}$$

the first of equations (3.2) coincides with equation (3.9). This means that $\widehat{A}^a_\mu(x)$ only depends on K^a_μ and does not depend on J^a_μ and g. Finally, the action $S[A,B]$ is a linear function of B^a_μ. Therefore, in the Legendre transform (3.7), the two terms which are linear in $\widehat{B}^a_\mu$ cancel, and one obtains precisely expression (3.8). □

Equation (3.8) is in agreement with expression (2.17), and shows that when $K^a_\mu = (1/g)J^a_\mu$, the functional $W_0[J,(1/g)J]$ satisfies

$$W_0[J,(1/g)J] = W_{0,CS}[J] \; , \tag{3.11}$$

where $W_{0,CS}[J]$ denotes the generating functional of the tree-level connected diagrams of the Chern-Simons theory, which is defined by the action $g S_{CS}[A]$,

$$W_{0,CS}[J] = g\, S_{CS}[\widehat{A}] + \int d^3x\, J^a_\mu(x)\widehat{A}^a_\mu(x) \; , \quad \text{with} \quad g\frac{\partial S_{CS}[\widehat{A}]}{\delta \widehat{A}^a_\mu(x)} = -J^a_\mu(x) \; . \tag{3.12}$$

Let us now consider diagrams with loops.

Proposition 3. *The whole set of the vacuum-to-vacuum connected diagrams for the quantum field components is equal to the set $i W_1[K]$ of the one-loop connected diagrams which only depend on K^a_μ,*

$$e^{iW_1[K]} = \langle e^{-i\int d^3x \epsilon^{abc}\{\epsilon^{\mu\nu\lambda} B^a_\mu \widehat{A}^b_\nu A^c_\lambda + \partial^\mu \overline{c}^a\, \widehat{A}^b_\mu c^c + \partial^\mu \overline{\xi}^a\, \widehat{A}^b_\mu \xi^c\}} \rangle \; . \tag{3.13}$$

Proof. The field propagators that are derived from the S_{TOT} are shown in equations (2.12) and (2.13); in particular, it turns out that $\overbracket{A^a_\mu(x) A}{}^b_\nu(y) = 0$ and $\overbracket{\xi(x)\overline{c}}(y) = 0 = \overbracket{c(x)\overline{\xi}}(y)$. Consequently, the only connected source-dependent diagrams containing loops are the one-loop connected diagrams entering equation (3.13). □

The result (3.13) is in agreement with the statements of Proposition 1 and shows that, when $K^a_\mu = (1/g) J^a_\mu$, the functional $W_1[(1/g)J]$ verifies

$$W_1[(1/g)J] = 2 W_{1,CS}[J] \, , \tag{3.14}$$

where the factor 2 is due to the combinatorics and the presence of two ghost fields, and $W_{1,CS}[J]$ denotes the generating functional of the one-loop connected diagrams in the Chern-Simons theory,

$$e^{i W_{1,CS}[J]} = \langle e^{-i \int d^3x \, \epsilon^{abc} \{ (g/2) \epsilon^{\mu\nu\lambda} A^a_\mu \widehat{A}^b_\nu A^c_\lambda + \partial^\mu \overline{c} \, \widehat{A}^b_\mu c^c \}} \rangle \Big|_{CS} \, . \tag{3.15}$$

3.2. Connected one-loop diagrams

As a consequence of equation (3.13), the functional $W_1[K]$ can be written as

$$W_1[K] = W_1^{(v)}[K] + W_1^{(g)}[K] \, , \tag{3.16}$$

where $W_1^{(v)}[K]$ corresponds to the sum of the connected diagrams with one loop of the vector fields, whereas $W_1^{(g)}[K]$ denotes the sum of the connected diagrams with one loop of the ghost fields. In Schwinger notations [85], the $\overbracket{AB}$ propagator (2.12) reads

$$\overbracket{A^a_\mu(x) B}{}^b_\nu(y) = \langle x\,; a, \mu | \, i \, \frac{\epsilon_{\mu\lambda\nu} \partial^\lambda}{\partial^2} \, | y\,; b, \nu \rangle \, , \tag{3.17}$$

and then

$$\begin{aligned} i \, W_1^{(v)}[K] &= \sum_{n=1}^{\infty} \frac{i^n}{n} \left[\sum \int d^3x_1 \dots d^3x_n \, \langle x_1 | \, \overbracket{A \; B} \; \widehat{A} \, | x_2 \rangle \cdots \langle x_n | \, \overbracket{A \; B} \; \widehat{A} \, | x_1 \rangle \right] \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \, \mathrm{Tr} \left[\frac{1}{\partial^2} \, \widehat{M} \right]^n \, , \end{aligned} \tag{3.18}$$

where

$$\left(\widehat{M} \right)^{a,c\,;\,\nu}_{\;\;\mu} = \epsilon_{\mu\tau\sigma} \, \partial^\tau \epsilon^{abc} \epsilon^{\sigma\lambda\nu} \widehat{A}^b_\lambda \, , \tag{3.19}$$

and Tr denotes the trace in the colour indices, vector indices and orbital indices

$$\mathrm{Tr} \, (Q) = \sum_{a,\mu} \int d^3x \, \langle x\,; a, \mu | \, Q \, | x\,; a, \mu \rangle \, . \tag{3.20}$$

The connected diagrams with one loop of the ghost fields give the contribution

$$i \, W_1^{(g)}[K] = -2 \sum_{n=1}^{\infty} \frac{1}{n} \, \mathrm{tr} \left[\frac{1}{\partial^2} \, \widehat{N} \right]^n \, , \tag{3.21}$$

in which

$$\big(\widehat{N}\big)^{a,c} = \partial^{\lambda}\, \epsilon^{abc}\, \widehat{A}^{b}_{\lambda} \; , \tag{3.22}$$

and

$$\mathrm{tr}\,(Q) = \sum_{a} \int d^3x \, \langle x \, ; a |\, Q \,| x \, ; a \rangle \; . \tag{3.23}$$

Some diagrams contained in $W_1[K]$ have values which are not well defined because of possible ultraviolet divergences; these diagrams will be renormalised in the Section 4.

4. Renormalisation

Since the observables that we shall consider only depend on A^a_μ and B^a_μ, and since the corresponding BF connected diagrams have zero loops or one loop only, in order to discuss the relevant aspects of the renormalisation we need to consider the functional

$$\Gamma = S_{TOT}[A, B, M, N, \xi, \overline{\xi}, c, \overline{c}] + \Gamma_1[A, B] \; , \tag{4.1}$$

where $i\Gamma_1[A, B]$ denotes to the sum of the one-particle-irreducible diagrams with one loop, in which A^a_μ and B^a_μ represent the external legs [86–88]. In other words, $\Gamma_1[A, B]$ is the sum of the one-loop proper vertices for the field components A^a_μ and B^a_μ. Indeed, as it has been shown in Section 2 and in Section 3, in the BF theory the contributions to the proper vertices which are described by diagrams with two or more loops are absent. The zero-loop component of the proper vertices coincides with the lagrangian and the one-loop component only contains primitive divergences. Therefore, in the renormalisation procedure, diagrams with external ghost fields can be ignored.

Equations (2.18), (3.18) and (3.21) imply that $\Gamma_1[A, B]$ nontrivially depends on A^a_μ only,

$$\Gamma_1[A, B] = \Gamma_1[A] \; . \tag{4.2}$$

Each term of the expansion of $\Gamma_1[A]$ in powers of the fields A^a_μ is well defined apart from the terms with two and three fields. The corresponding six diagrams are not well defined a priori; they possibly have ultraviolet divergences. Since only a finite number of diagrams need to be renormalised, the BF model is a superrenormalisable field theory.

4.1. Normalisation conditions

As there are no gauge anomalies in three dimensions, it is possible to define a renormalised Γ which is BRST invariant. Let us define

$$\left.\frac{\delta^2 \Gamma}{\delta A^b_\nu(y)\delta A^a_\mu(x)}\right|_{A=0,B=0} = \int \frac{d^3k}{(2\pi)^3} e^{ik(x-y)}\, \Pi^{ab}_{\mu\nu}(k) \; , \tag{4.3}$$

$$\left.\frac{\delta^2 \Gamma}{\delta A^b_\nu(y)\delta B^a_\mu(x)}\right|_{A=0,B=0} = \int \frac{d^3k}{(2\pi)^3} e^{ik(x-y)}\, \Sigma^{ab}_{\mu\nu}(k) \; . \tag{4.4}$$

In addition to the BRST invariance of Γ, the normalisation conditions are taken to be

$$\lim_{k\to 0} \Pi^{ab}_{\mu\nu}(k) = i g \delta^{ab} \epsilon^{\mu\lambda\nu} k_\lambda \; , \tag{4.5}$$

(a) (b)

Fig. 4. One loop contributions to the two points function.

and

$$\lim_{k\to 0} \Sigma^{ab}_{\mu\nu}(k) = i\delta^{ab}\epsilon^{\mu\lambda\nu}k_\lambda \, . \tag{4.6}$$

Equations (4.5) and (4.6) establish the normalisation of the fields and specify the value of the coupling constant g. Since the one-loop contributions contained in $\Gamma_1[A]$ do not depend on the field B^a_μ, equation (4.6) —which is valid at the tree-level— remains valid to all orders of perturbations theory. Consequently, only equation (4.5) needs to be considered; in renormalised perturbation theory [87], equation (4.5) controls the one-loop counterterms. Let us consider the renormalisation procedure [86,89–93] in the space of the coordinates x^μ. Of course, the final result coincides with the result obtained by means of the renormalisation procedure in momentum space.

4.2. One-loop two points function

$\Gamma_1[A]$ can be expanded in powers of the fields A^a_μ; the quadratic term is given by the sum of the contribution $\Gamma^{(v)}_1[A]$, corresponding to the one-loop diagram of Fig. 4(a), and $\Gamma^{(g)}_1[A]$ which is obtained by adding the two equal amplitudes which are described by the diagram of Fig. 4(b) containing one loop of the two types of ghosts.
One has

$$\begin{aligned} i\Gamma^{(v)}_1[A] &= \frac{(-i)^2}{2!}\int d^3x\, d^3y\, A^a_\mu(x)A^b_\nu(y)\epsilon^{cad}\epsilon^{ebh}\epsilon^{\lambda\mu\tau}\epsilon^{\sigma\nu\alpha}\, A^d_\tau(x)B^e_\sigma(y)\;\, A^h_\alpha(y)B^c_\lambda(x) \\ &= -2\int d^3x\, d^3y\, A^a_\mu(x)A^b_\nu(y)\,\delta^{ab}\,\partial^\mu_x\Delta(x-y)\,\partial^\nu_y\Delta(y-x)\, , \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} i\Gamma^{(g)}_1[A] &= -(-i)^2\int d^3x\, d^3y\, A^a_\mu(x)A^b_\nu(y)\epsilon^{cad}\epsilon^{ebh}\; c^d(x)\partial^\nu\overline{c}^e(y)\;\, c^h(y)\partial^\mu\overline{c}^c(x) \\ &= 2\int d^3x\, d^3y\, A^a_\mu(x)A^b_\nu(y)\,\delta^{ab}\,\partial^\mu_x\Delta(x-y)\,\partial^\nu_y\Delta(y-x)\, . \end{aligned} \tag{4.8}$$

Precisely like in the Chern-Simons theory [94,95], the sum of the two contributions $\Gamma^{(v)}_1[A] + \Gamma^{(g)}_1[A]$ formally vanishes, indeed

$$\Gamma^{(v)}_1[A] + \Gamma^{(g)}_1[A] = 2i\int d^3x\, d^3y\, A^a_\mu(x)A^b_\nu(y)\,\delta^{ab}\,H^{\mu\nu}(x,y)\, , \tag{4.9}$$

where

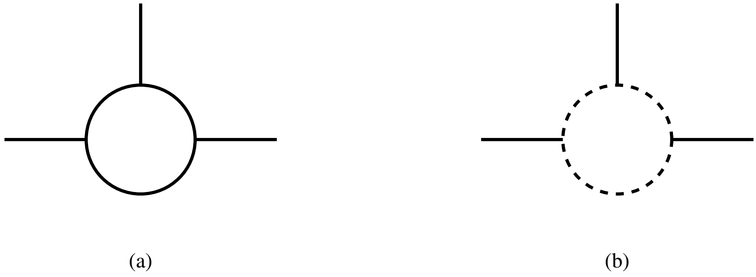


Fig. 5. One loop contributions to the three points function.

$$H^{\mu\nu}(x,y) = \partial_x^\mu \Delta(x-y)\, \partial_y^\nu \Delta(y-x) - \partial_x^\mu \Delta(x-y)\, \partial_y^\nu \Delta(y-x)\,. \tag{4.10}$$

The amplitude

$$\partial_x^\mu \Delta(x-y)\, \partial_y^\nu \Delta(y-x) = \frac{(x-y)^\mu (y-x)^\nu}{(4\pi)^2\, |x-y|^6}\,, \tag{4.11}$$

which appears in equation (4.10), is well defined for $x \neq y$. Consequently "the nonlocal component" of $\Gamma_1^{(v)}[A] + \Gamma_1^{(g)}[A]$ is well defined and vanishes because

$$H^{\mu\nu}(x,y)\Big|_{x\neq y} = 0\,. \tag{4.12}$$

When $x = y$ expression (4.11) is not well defined, so one has to specify the value of $H^{\mu\nu}(x,y)$ in the case $x = y$. In facts, since "the nonlocal component" of $\Gamma_1^{(v)}[A] + \Gamma_1^{(g)}[A]$ vanishes, the entire renormalisation of $\Gamma_1^{(v)}[A] + \Gamma_1^{(g)}[A]$ consists [86] precisely in specifying the value of "the local component" of $\Gamma_1^{(v)}[A] + \Gamma_1^{(g)}[A]$, which is defined by $H^{\mu\nu}(x,y)$ for $x = y$. This value is uniquely determined by the normalisation condition (4.5), which requires

$$\left(\Gamma_1^{(v)}[A] + \Gamma_1^{(g)}[A]\right)\Big|_{\text{renormalised}} = 0\,. \tag{4.13}$$

It should be noted that the renormalised value (4.13) of $\Gamma_1^{(v)}[A] + \Gamma_1^{(g)}[A]$ is also in agreement with the point-splitting procedure, that we shall use in the definition of the composite Wilson line operators. Indeed, the point-splitting definition of $H^{\mu\nu}(x,y)$ for $x = y$ gives

$$H^{\mu\nu}(x,y)\Big|_{x=y} \equiv \lim_{x\to y} H^{\mu\nu}(x,y)\Big|_{x\neq y} = 0\,, \tag{4.14}$$

which implies precisely equation (4.13).

From equation (4.13) it follows that the BF vacuum polarisation vanishes and the Feynman propagators (2.12) coincide with the dressed propagators.

4.3. One-loop three points function

The term of $\Gamma_1[A]$ which contains three powers of the field A_μ^a is the sum of $\widetilde{\Gamma}_1^{(v)}[A]$, which is described by the Feynman diagram of Fig. 5(a), and $\widetilde{\Gamma}_1^{(g)}[A]$ which is specified by the one-loop contributions of Fig. 5(b) induced by the two kinds of ghosts.

One finds

$$
\begin{aligned}
i\widetilde{\Gamma}_1^{(v)}[A] &= \frac{2(-i)^3}{3!}\int d^3x\, d^3y\, d^3z\, A^b_\nu(x) A^e_\rho(y) A^h_\tau(z)\, \epsilon^{abc}\epsilon^{def}\epsilon^{ghi} \\
&\qquad \epsilon^{\mu\nu\lambda}\epsilon^{\sigma\rho\gamma}\epsilon^{\alpha\tau\beta}\; A^c_\lambda(x) B^d_\sigma(y)\;\; A^f_\gamma(y) B^g_\alpha(z)\;\; A^i_\beta(z) B^a_\mu(x) \\
&= \frac{1}{3}\int d^3x\, d^3y\, d^3z\, \epsilon^{abc}\, A^a_\mu(x) A^b_\nu(y) A^c_\lambda(z)\, T^{\mu\nu\lambda}_{\tau\rho\sigma} \\
&\qquad \partial^\tau_x \Delta(x-y)\, \partial^\rho_y \Delta(y-z)\, \partial^\sigma_z \Delta(z-x)\,, \qquad (4.15)
\end{aligned}
$$

where

$$
T^{\mu\nu\lambda}_{\tau\rho\sigma} = \delta^\mu_\tau \delta^\nu_\sigma \delta^\lambda_\rho + \delta^\mu_\sigma \delta^\nu_\rho \delta^\lambda_\tau + \delta^\mu_\rho \delta^\nu_\tau \delta^\lambda_\sigma - \delta^\mu_\rho \delta^\nu_\sigma \delta^\lambda_\tau\,. \qquad (4.16)
$$

Moreover

$$
\begin{aligned}
i\widetilde{\Gamma}_1^{(g)}[A] &= -4\frac{(-i)^3}{3!}\int d^3x\, d^3y\, d^3z\, A^b_\mu(x) A^e_\nu(y) A^h_\lambda(z)\, \epsilon^{abc}\epsilon^{def}\epsilon^{ghi} \\
&\qquad c^c(x)\partial^\nu \overline{c}^d(y)\;\; c^f(y)\partial^\lambda \overline{c}^g(z)\;\; c^i(z)\partial^\mu \overline{c}^a(x) \\
&= -\frac{1}{3}\int d^3x\, d^3y\, d^3z\, \epsilon^{abc}\, A^a_\mu(x) A^b_\nu(y) A^c_\lambda(z)(\delta^\mu_\sigma \delta^\nu_\tau \delta^\lambda_\rho + \delta^\mu_\tau \delta^\nu_\rho \delta^\lambda_\sigma) \\
&\qquad \partial^\tau_x \Delta(x-y)\, \partial^\rho_y \Delta(y-z)\, \partial^\sigma_z \Delta(z-x)\,. \qquad (4.17)
\end{aligned}
$$

Therefore

$$
i\widetilde{\Gamma}_1^{(v)}[A] + i\widetilde{\Gamma}_1^{(g)}[A] = \frac{1}{3}\int d^3x\, d^3y\, d^3z\, \epsilon^{abc}\, A^a_\mu(x) A^b_\nu(y) A^c_\lambda(z)\, V^{\mu\nu\lambda}(x,y,z)\,, \qquad (4.18)
$$

in which

$$
V^{\mu\nu\lambda}(x,y,z) = \epsilon^{\mu\nu\lambda}_{\tau\rho\sigma}\, \partial^\tau_x \Delta(x-y)\, \partial^\rho_y \Delta(y-z)\, \partial^\sigma_z \Delta(z-x)\,, \qquad (4.19)
$$

and

$$
\epsilon^{\mu\nu\lambda}_{\tau\rho\sigma} = \delta^\mu_\tau \delta^\nu_\sigma \delta^\lambda_\rho + \delta^\mu_\sigma \delta^\nu_\rho \delta^\lambda_\tau + \delta^\mu_\rho \delta^\nu_\tau \delta^\lambda_\sigma - \delta^\mu_\rho \delta^\nu_\sigma \delta^\lambda_\tau - \delta^\mu_\sigma \delta^\nu_\tau \delta^\lambda_\rho - \delta^\mu_\tau \delta^\nu_\rho \delta^\lambda_\sigma\,. \qquad (4.20)
$$

When $x \neq y$, $x \neq z$ and $y \neq z$, the amplitude

$$
\partial^\tau_x \Delta(x-y)\, \partial^\rho_y \Delta(y-z)\, \partial^\sigma_z \Delta(z-x) = \frac{(x-y)^\tau (y-z)^\rho (x-z)^\sigma}{(4\pi)^3\, |x-y|^3 |y-z|^3 |z-x|^3} \qquad (4.21)
$$

is well defined and, when it is multiplied by the completely antisymmetric tensor $\epsilon^{\mu\nu\lambda}_{\tau\rho\sigma}$, it vanishes,

$$
V^{\mu\nu\lambda}(x,y,z)\Big|_{x\neq y\neq z} = 0\,. \qquad (4.22)
$$

Therefore, as in the case of the two points functions, "the nonlocal component" of $\widetilde{\Gamma}_1^{(v)}[A] + \widetilde{\Gamma}_1^{(g)}[A]$ is vanishing. In order to specify the renormalised value of $\widetilde{\Gamma}_1^{(v)}[A] + \widetilde{\Gamma}_1^{(g)}[A]$ we need to define [89–93] the value of the "diagonal local component" of $V^{\mu\nu\lambda}(x,y,z)$, corresponding to the case in which the external fields are defined at coincident points $x = y = z$. This is in agreement with the general fact that, in one-loop diagrams, the possibly divergent (not well defined) contribution is local or, to be more precise, the introduction of appropriate local counterterms makes the diagrams well defined.

The renormalised value of $\widetilde{\Gamma}_1^{(v)}[A] + \widetilde{\Gamma}_1^{(g)}[A]$ is determined by the normalisation conditions and by symmetry arguments. Indeed the BRS invariance of Γ requires that the value of the local component of the one-loop contribution to the 3-point proper vertex must be $(1/6)$ the value of the one-loop contribution to the dressed propagator, which vanishes. Therefore relation (4.13) and BRST invariance imply

$$\left(\widetilde{\Gamma}_1^{(v)}[A] + \widetilde{\Gamma}_1^{(g)}[A]\right)\Big|_{\text{renormalised}} = 0 \,. \tag{4.23}$$

The result (4.23) can also be obtained by means of the point-splitting procedure, according to which

$$V^{\mu\nu\lambda}(x,y,z)\Big|_{x=y=z} = \lim_{x\to y}\,\lim_{y\to z}\, V^{\mu\nu\lambda}(x,y,z)\Big|_{x\neq y\neq z} = 0 \,. \tag{4.24}$$

The point-splitting procedure also shows that each "partially local component", say $x = y \neq z$, is vanishing because

$$V^{\mu\nu\lambda}(x,y,z)\Big|_{x=y\neq z} = \lim_{x\to y}\, V^{\mu\nu\lambda}(x,y,z)\Big|_{x\neq y\neq z} = 0 \,.$$

In renormalisable field theories, the "partially local components" of the diagrams are possibly related with the (overlapping) sub-divergences. In the connected diagrams of the BF theory, there are no subdivergences to deal with because the connected diagrams have at most one loop.

Since all the remaining diagrams contributing to Γ are finite, this concludes the renormalisation of the BF theory in $\mathbb{R}^3$. This means that, by taking into account equations (4.13) and (4.23), the expectation values

$$\langle A^{a_1}_{\mu_1}(x_1) A^{a_2}_{\mu_2}(x_2)\cdots A^{a_n}_{\mu_n}(x_n) B^{c_1}_{\nu_1}(y_1) B^{c_2}_{\nu_2}(y_2)\cdots B^{c_m}_{\nu_m}(y_m)\rangle \,, \tag{4.25}$$

when the fields are defined at not coincident points, are well defined. In the computation of the BF observables, we shall need to remove certain ambiguities of the expectation values which appear in a specific limit in which two fields are defined in the same point. This issue, which is related to the introduction of a framing for the knots, will be discussed in Section 7.

5. Wilson line observables

Similarly to the case of the Chern-Simons gauge field theory, the gauge invariant observables that we shall consider correspond to appropriately normalised traces of the expectation values of the gauge holonomies which are associated with oriented framed knots in $\mathbb{R}^3$ in a given representation of $ISU(2)$.

5.1. Representations of $ISU(2)$

We shall consider linear unitary representations of $ISU(2)$ in which $\{P^a\}$ are nontrivially represented and which are specified by the values of the two quadratic Casimir operators P^aP^a and J^aP^a. More precisely, if $|\varphi\rangle$ denotes a vector transforming according to the irreducible (Λ, r) representation, it must satisfy

$$P^aP^a|\varphi\rangle = \Lambda^2|\varphi\rangle \,, \tag{5.1}$$

and

$$J^a P^a |\varphi\rangle = r \Lambda |\varphi\rangle \ , \tag{5.2}$$

with fixed positive Λ and fixed semi-integer r (*i.e.*, $2r \in \mathbb{Z}$). In this article we shall concentrate on the "scalar" $(\Lambda, 0)$ representation and the "fundamental" $(\Lambda, 1/2)$ representation.

In order to describe these representations, let us first consider the quantum mechanics states space of a spinless particle moving in three dimensional euclidean space. Let P^a represent the cartesian components of the momentum operator and let L^a denote the components of the orbital angular momentum of the particle,

$$L^a = \epsilon^{abc} Q^b P^c \ , \tag{5.3}$$

in which $[Q^a, P^b] = i\delta^{ab}$. The operators $\{J^a = L^a, P^a\}$ satisfy the commutation relations (2.2).

5.1.1. Scalar representation

The plane wave

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\boldsymbol{r}} \tag{5.4}$$

verifies

$$P^a \, \psi_{\boldsymbol{k}}(\boldsymbol{r}) = k^a \, \psi_{\boldsymbol{k}}(\boldsymbol{r}) \ . \tag{5.5}$$

When the value of the first Casimir operator $P^a P^a$ of $ISU(2)$ is chosen to be Λ^2, one needs to consider the linear space $\mathcal{H}_{(\Lambda,0)}$ which is generated by all the vectors

$$\{\psi_{\boldsymbol{k}}(\boldsymbol{r})\} \text{ with } \boldsymbol{k}\boldsymbol{k} = \Lambda^2 \ . \tag{5.6}$$

In this case, the possible eigenvalues $\boldsymbol{k}$ of the momentum belong to a spherical surface in $\mathbb{R}^3$ of radius equal to Λ. The set of all the plane waves $\{\psi_{\boldsymbol{k}}(\boldsymbol{r})\}$ with $\boldsymbol{k}\boldsymbol{k} = \Lambda^2$ is left invariant by the action of the $SU(2)$ group which is generated by the orbital angular momentum components (5.3). Therefore the linear space $\mathcal{H}_{(\Lambda,0)}$ is invariant under the transformations generated by $\{J^a = L^a, P^a\}$. Since $L^a P^a = 0$, the $ISU(2)$ action on $\mathcal{H}_{(\Lambda,0)}$ which is implemented by the transformations $\exp\{i\beta^a J^a + i\eta^a P^a\}$ defines the scalar $(\Lambda, 0)$ representation of $ISU(2)$.

The commutation relations of the operators $\{J^a = L^a, Q^a\}$ also coincide with the commutation relations of the $ISU(2)$ algebra. Thus an alternative interpretation of this $ISU(2)$ representation can be obtained by considering the quantum mechanics states of one particle moving on the surface of a 2-sphere in $\mathbb{R}^3$. For the purposes of the present article, we don't need to discuss the rigged Hilbert space structure [96] associated with $\mathcal{H}_{(\Lambda,0)}$.

5.1.2. Fundamental representation

Let us now examine the fundamental $(\Lambda, 1/2)$ representation of $ISU(2)$. Let $\mathcal{H}_{spin}$ denote the two dimensional space of the spin states of a spin $(1/2)$ nonrelativistic particle, and let S^a represent the components of the spin operator,

$$S^a = \tfrac{1}{2}\sigma^a \ , \tag{5.7}$$

where σ^a denote the Pauli sigma matrices. The operators S^a act on the vectors which belong to $\mathcal{H}_{spin}$. In the tensor product $\mathcal{H}_{(\Lambda,0)} \otimes \mathcal{H}_{spin}$, one can put

$$J^a = L^a + S^a \ . \tag{5.8}$$

In addition to the constraint $\boldsymbol{kk} = \Lambda^2$, the specification of the value $(1/2)\Lambda$ of the second Casimir operator $J^a P^a$ selects the states in $\mathcal{H}_{(\Lambda,0)} \otimes \mathcal{H}_{spin}$ of positive helicity. Let π_+ denote the projector on the positive helicity states,

$$\pi_+ = \tfrac{1}{2}\left(1 + \frac{\boldsymbol{P\sigma}}{\Lambda}\right) . \tag{5.9}$$

Let $\mathcal{H}_{(\Lambda,1/2)}$ be the linear space which is generated by the vectors

$$\left\{ \pi_+ |\chi\rangle \right\} \text{ in which } |\chi\rangle \in \mathcal{H}_{(\Lambda,0)} \otimes \mathcal{H}_{spin} . \tag{5.10}$$

The $ISU(2)$ action on $\mathcal{H}_{(\Lambda,1/2)}$, which is carried out by the transformations generated by $\{J^a = L^a + S^a, P^a\}$, defines the $(\Lambda, 1/2)$ representation. One can easily verify that the projector π_+ commutes with the generators of $ISU(2)$.

A generic (Λ, r) representation could be constructed by means of a procedure which is similar to the procedure that has been illustrated in the case of the $(\Lambda, 1/2)$ representation. Each representation (Λ, r), with $r = 0$ or $r = 1/2$, is irreducible and infinite dimensional.

5.2. Holonomies

Let us consider a classical gauge configuration which is described by the components $A^a_\mu(x)$ and $B^a_\mu(x)$. Given an oriented path γ in $\mathbb{R}^3$, which connects the starting point x_1 to the final point x_2, the corresponding $ISU(2)$ holonomy $h_\gamma \in ISU(2)$ is defined by

$$h_\gamma = \mathrm{P} e^{i \int_\gamma dx^\mu (A^a_\mu(x) J^a + B^a_\mu(x) P^a)} , \tag{5.11}$$

where the symbol P denotes the path-ordering of the $\{J^a, P^b\}$ operators along the direction specified by the orientation of γ. Under a gauge transformation (2.3), h_γ transforms as

$$h_\gamma \to \Omega^{-1}(x_1)\, h_\gamma\, \Omega(x_2) . \tag{5.12}$$

Thus for each non intersecting closed path C —that is, for each oriented knot $C \subset \mathbb{R}^3$— with a given starting and final point x_0, the associated holonomy h_C transforms covariantly under gauge transformations,

$$h_C \to \Omega^{-1}(x_0)\, h_C\, \Omega(x_0) . \tag{5.13}$$

Therefore any function, which is defined on the $ISU(2)$ conjugacy classes, determines a classical gauge invariant observable. We shall describe the conjugacy classes of the group $ISU(2)$ in Section 6. For the moment, let us recall the normal construction of classical gauge invariant observables for finite dimensional representations of the structure group. Let $[h_C]_\rho$ be the representative of the element $h_C \in ISU(2)$ in the representation ρ of the gauge group. If the representation ρ is finite dimensional, the cyclic property of the trace implies that $\mathrm{Tr}[h_C]_\rho$ is gauge invariant. Really, in the BF theory we are interested in the $ISU(2)$ representations (Λ, r), with $r = 0$ or $r = 1/2$, which are not finite dimensional. In this case, the ordinary traces of the holonomies in the representation spaces $\mathcal{H}_{(\Lambda,0)}$ and $\mathcal{H}_{(\Lambda,1/2)}$ need to be improved in order to specify a well defined observable.

5.3. Trace of holonomies

Let us consider the standard method which is used in physics —for instance in particle physics and in statistical mechanics— to describe the sum over the one-particle quantum states. One can introduce appropriately normalised plane waves

$$|\boldsymbol{k}) = \frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\boldsymbol{r}} \ , \tag{5.14}$$

where $V = L^3$ is the volume of a cubic box in which the particle can propagate; then one must consider the $V \to \infty$ limit. From the definition (5.14) it follows

$$(\boldsymbol{k}|\boldsymbol{k}') = \frac{(2\pi)^3}{V} \delta^3(\boldsymbol{k} - \boldsymbol{k}') \ , \tag{5.15}$$

and

$$(\boldsymbol{k}|\boldsymbol{k}) = 1 \ . \tag{5.16}$$

With periodic boundary conditions, for instance, the possible values of the momenta are given by $\boldsymbol{k} = (2\pi/L)\,\boldsymbol{n}$, with $n_j \in \mathbb{Z}$. Therefore, in the large L limit, the sum over the eigenstates of the momentum is given by the integral $[L^3/(2\pi)^3] \int d^3k$, which also coincides with the counting of the number of quantum states in the semiclassical limit by means of the integral $\int d^3p\, d^3q/(2\pi)^3$ in classical phase space. With this notation, the trace of a given operator O_p in the linear space of the one-particle orbital states takes the form

$$\mathrm{Tr}(O_p) = \int \frac{V\, d^3k}{(2\pi)^3} (\boldsymbol{k}|\, O_p \,|\boldsymbol{k}) \ , \tag{5.17}$$

which can easily be controlled in the $V \to \infty$ limit because of the presence of the overall multiplicative V factor.

The states of the $(\Lambda, 0)$ representation are characterised by values of the momentum which belong to the 2-dimensional surface $\boldsymbol{k}^2 = \Lambda^2$ in momentum space. In order to make contact with the $\int d^3p\, d^3q/(2\pi)^3$ expression for the counting of states in $\mathcal{H}_{(\Lambda,0)}$, one can introduce a small thickness Δ_P to the $\boldsymbol{k}^2 = \Lambda^2$ surface. If, for instance, the relation $L\Delta_P/(2\pi) = 1$ is satisfied, then the $\Delta_P \to 0$ limit is recovered in the $L \to \infty$ limit. According to this prescription, the trace of a given operator O_p in the space $\mathcal{H}_{(\Lambda,0)}$ of the $(\Lambda, 0)$ representation of $ISU(2)$ reads

$$\begin{aligned} \mathrm{Tr}\left(O_p\right)\Big|_{(\Lambda,0)} &= \frac{L^3}{(2\pi)^3} \int \left[d^3k\right]_{\boldsymbol{k}^2 \to \Lambda^2} (\boldsymbol{k}|\, O_p \,|\boldsymbol{k}) \\ &= \frac{L^2\Lambda^2}{(2\pi)^2} \int d\omega\, (\boldsymbol{k}|\, O_p \,|\boldsymbol{k}) \quad , \quad \left(\text{with } \boldsymbol{k}\boldsymbol{k} = \Lambda^2\right) \ , \end{aligned} \tag{5.18}$$

where $d\omega = \sin\theta\, d\theta\, d\phi$ refers to the solid angle which is defined by the direction of the vector $\boldsymbol{k}$,

$$\boldsymbol{k} = \Lambda(\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta) \ . \tag{5.19}$$

Note that the presence of the product $L^2\Lambda^2$ in equation (5.18) is required by dimensional reasons. Whereas different prescriptions for the $\Delta_P \to 0$ limit may lead to the presence of different adimensional multiplicative factors. These factors play no role because the Wilson line operators will correspond to appropriately normalised traces.

In the definition of the normalised trace of the holonomy h_C, the multiplicative factor $L^2\Lambda^2/\pi$ in front of expression (5.18) can be removed. So (in the $L \to \infty$ limit) we define the Wilson line operator W_C in the $(\Lambda, 0)$ representation by means of the normalised trace

$$W_C \Big|_{(\Lambda,0)} = \int \frac{d\omega}{4\pi} \, (\boldsymbol{k}| \, h_C \, |\boldsymbol{k}) \quad , \quad \Big(\text{with } \boldsymbol{k}\boldsymbol{k} = \Lambda^2 \Big) \; . \tag{5.20}$$

Let us denote the quantum state vectors of a nonrelativistic spin 1/2 particle moving inside a box by $|\boldsymbol{k})|s) = |\boldsymbol{k}, s)$, where $s = \pm 1/2$ refers to the value of one component of the spin. The normalised trace of the holonomy h_C in the $(\Lambda, 1/2)$ representation is defined by

$$W_C \Big|_{(\Lambda,1/2)} = \sum_s \int \frac{d\omega}{4\pi} \, (\boldsymbol{k}, s| \, h_C \, \pi_+ \, |\boldsymbol{k}, s) \quad , \quad \Big(\text{with } \boldsymbol{k}\boldsymbol{k} = \Lambda^2 \Big) \; . \tag{5.21}$$

The proof that the BF expectation values of expressions (5.20) and (5.21) are well defined is reported in Section 7.

6. $ISU(2)$ conjugacy classes

The set of the conjugacy classes of the inhomogeneous group $ISU(2)$ has rather peculiar properties that show up also in the values of the corresponding classical characters.

6.1. *Classes of conjugated elements*

A generic element $\mathcal{G} \in ISU(2)$ can be written as

$$\mathcal{G} = \exp\left[i \left(\Theta^a J^a + X^a P^a\right)\right] = \exp\left[i \left(\boldsymbol{\Theta} \boldsymbol{J} + \boldsymbol{X} \boldsymbol{P}\right)\right] \; , \tag{6.1}$$

with real parameters $\boldsymbol{\Theta}$ and $\boldsymbol{X}$, in which $0 \leq |\boldsymbol{\Theta}| < 2\pi$ whereas there are no restrictions on the value of $\boldsymbol{X}$. Under conjugation with an element of the subgroup $SU(2)$ of $ISU(2)$, the commutation relations (2.2) give

$$\mathcal{G} \longrightarrow e^{-i\beta^a J^a} \, \mathcal{G} \, e^{i\beta^a J^a} = \exp\left[i \left(\boldsymbol{\Theta}' \boldsymbol{J} + \boldsymbol{X}' \boldsymbol{P}\right)\right] \; , \tag{6.2}$$

where $\boldsymbol{\Theta}'$ and $\boldsymbol{X}'$ denote the rotated vectors

$$\left(\Theta'\right)^a = R^{ab}(\beta)\Theta^b \quad , \quad \left(X'\right)^a = R^{ab}(\beta) X^b \; , \tag{6.3}$$

which are obtained according to the adjoint representation of $SU(2)$, *i.e.* $R^{ab}(\beta) \in SO(3)$. Therefore, the conjugacy class of $\mathcal{G}$ is possibly labelled by the rotation invariants $|\boldsymbol{\Theta}|$, $|\boldsymbol{X}|$ and $\boldsymbol{\Theta}\boldsymbol{X} = \Theta^a X^a$. On the other hand, under conjugation with a translation element of $ISU(2)$

$$\mathcal{G} \longrightarrow e^{-i\eta^a P^a} \, \mathcal{G} \, e^{i\eta^a P^a} = \exp\left[i \left(\widetilde{\boldsymbol{\Theta}} \boldsymbol{J} + \widetilde{\boldsymbol{X}} \boldsymbol{P}\right)\right] \; , \tag{6.4}$$

one finds

$$\widetilde{\Theta}^a = \Theta^a \quad , \quad \widetilde{X}^a = X^a + \epsilon^{abc} \eta^b \Theta^c \; . \tag{6.5}$$

Equation (6.5) shows that the parameter $\boldsymbol{\Theta}$ is not modified and

- when $\boldsymbol{\Theta} = 0$, $\boldsymbol{X}$ is not modified;
- when $\boldsymbol{\Theta} \neq 0$, the component of $\boldsymbol{X}$ which is orthogonal to $\boldsymbol{\Theta}$ can be arbitrarily modified. While the component of $\boldsymbol{X}$ along the direction of $\boldsymbol{\Theta}$ is not modified.

Consequently, equations (6.3) and (6.5) show that the conjugacy classes of $ISU(2)$ can be labelled by two real numbers (r_1, r_2) with $r_1 = |\boldsymbol{\Theta}|$ and

- $r_2 = |\boldsymbol{X}|$, when $r_1 = 0$;
- $r_2 = \boldsymbol{\Theta}\boldsymbol{X}$, when $r_1 \neq 0$.

The set of variables $\{(r_1, r_2)\}$ does not parametrise a two dimensional manifold because of the singularity at $r_1 = 0$.

6.2. Classical traces

Let $\mathrm{Tr}\,(\mathcal{G})\big|_{(\Lambda,r)}$ be the trace of $\mathcal{G} \in ISU(2)$ in the (Λ, r) representation of $ISU(2)$ (with $r = 0, 1/2$),

$$\mathrm{Tr}\,(\mathcal{G})\Bigg|_{(\Lambda,r)} = \begin{cases} \frac{L^3}{(2\pi)^3} \int \left[d^3k\right]_{\boldsymbol{k}^2 \to \Lambda^2} (\boldsymbol{k}|\,\mathcal{G}\,|\boldsymbol{k}) & \text{when } \; r = 0\,; \\ \\ \frac{L^3}{(2\pi)^3} \sum_s \int \left[d^3k\right]_{\boldsymbol{k}^2 \to \Lambda^2} (\boldsymbol{k}, s|\,\mathcal{G}\,\pi_+\,|\boldsymbol{k}, s) & \text{when } \; r = 1/2\,. \end{cases} \tag{6.6}$$

By means of equations (5.15), (5.16) and (5.18) one finds

(1) When $\boldsymbol{\Theta} = 0$ and $\boldsymbol{X} = 0$,

$$\mathrm{Tr}\,(\mathcal{G})\Bigg|_{(\Lambda,r)} = \frac{L^2}{(2\pi)^2}\, 4\pi\,\Lambda^2\,. \tag{6.7}$$

(2) When $\boldsymbol{\Theta} = 0$ and $\boldsymbol{X} \neq 0$,

$$\mathrm{Tr}\,(\mathcal{G})\Bigg|_{(\Lambda,r)} = \frac{L^2}{(2\pi)^2}\, 4\pi\,\Lambda^2\, \frac{\sin(\Lambda|\boldsymbol{X}|)}{\Lambda|\boldsymbol{X}|}\,. \tag{6.8}$$

(3) When $\boldsymbol{\Theta} \neq 0$ and $\boldsymbol{X} = 0$, let $|\boldsymbol{k}') = e^{i\,\Theta^a J^a}\,|\boldsymbol{k})$. One has

$$(\boldsymbol{k}|\,e^{i\,\Theta^a J^a}\,|\boldsymbol{k}) = \frac{(2\pi)^3}{V}\,\delta\left(\boldsymbol{k} - \boldsymbol{k}'\right)\,. \tag{6.9}$$

Since $(\boldsymbol{k}|\boldsymbol{k}') = (\boldsymbol{k}|\,e^{i\,\Theta^a J^a}\,|\boldsymbol{k})$ is vanishing unless the vector $\boldsymbol{k}$ is directed as $\pm\boldsymbol{\Theta}$, with $\boldsymbol{\Theta} = (\Theta^1, \Theta^2, \Theta^3)$, one obtains

$$\mathrm{Tr}\,(\mathcal{G})\Bigg|_{(\Lambda,0)} = 2\,, \tag{6.10}$$

which is in agreement with the Frobenius fixed point theorem [97] since any nontrivial rotation of a spherical surface in $\mathbb{R}^3$ has just two fixed points. In the case of the $(\Lambda, 1/2)$ representation, one finds

$$\mathrm{Tr}\,(\mathcal{G})\Bigg|_{(\Lambda,1/2)} = 2\,\cos\left(|\boldsymbol{\Theta}|/2\right)\,. \tag{6.11}$$

(4) When $\boldsymbol{\Theta} \neq 0$ and $\boldsymbol{X} \neq 0$,

$$\mathrm{Tr}\,(\mathcal{G})\Bigg|_{(\Lambda,r)} = 2\,\cos\left(\Lambda\boldsymbol{X}\widehat{\boldsymbol{\Theta}} + r|\boldsymbol{\Theta}|\right)\,, \tag{6.12}$$

where the unit vector $\widehat{\boldsymbol{\Theta}}$ is defined by $\widehat{\boldsymbol{\Theta}} = \boldsymbol{\Theta}/|\boldsymbol{\Theta}|$.

The observed discontinuity of the classical trace of $\mathcal{G}$ at $\Theta = 0$ matches the structure of the set of $ISU(2)$ conjugacy classes discussed in Section 6.1.

7. Expectation values

Let us concentrate on the BF topological invariants which are associated with oriented framed coloured knots in $\mathbb{R}^3$. A knot C in $\mathbb{R}^3$, with a specified irreducible $ISU(2)$ representation, is called a coloured knot. The invariant $\langle W_C \rangle$ which is associated with the knot C is defined by the BF expectation value of the Wilson line operator

$$\langle W_C \rangle = \frac{\int D(\text{fields})\, e^{iS_{TOT}}\, W_C}{\int D(\text{fields})\, e^{iS_{TOT}}} \,, \tag{7.1}$$

where W_C corresponds to the normalised trace of the holonomy h_C shown in equations (5.20) and (5.21). In perturbation theory, the determination of $\langle W_C \rangle$ is obtained by means of the following steps: (1) expansion of the holonomy h_C in powers of the gauge fields, (2) computation of the vacuum expectation values of the products of the gauge fields, and (3) evaluation of the normalised trace of the $ISU(2)$ generators.

In the quantum BF field theory, the holonomy h_C is a composite operator and its expansion in powers of the connection $\mathcal{A}$ contains product of fields at coincident points. As in the case of the quantum Chern-Simons field theory, the ambiguities of the mean value (7.1), which are due to the presence of fields at coincident points, are removed by means of the point-splitting limit procedure [98,99] which is based on the introduction of a framing of the knot C. So, the invariant (7.1) is really defined for framed knots.

The perturbative computation of $\langle W_C \rangle$ is based on the expansion of h_C in powers of the fields

$$\begin{aligned} h_C = 1 + i \int_C \mathcal{A}_\mu(x) dx^\mu + i^2 \int_C dx^\mu \int_{x_0}^{x} dy^\nu\, \mathcal{A}_\nu(y)\mathcal{A}_\mu(x) \\ + i^3 \int_C dx^\mu \int_{x_0}^{x} dy^\nu \int_{x_0}^{y} dz^\lambda\, \mathcal{A}_\lambda(z)\mathcal{A}_\nu(y)\mathcal{A}_\mu(x) + \cdots \end{aligned} \tag{7.2}$$

where $\mathcal{A}_\mu(x) = A^a_\mu(x) J^a + B^a_\mu(x) P^a$ and x_0 denotes a given base point on the oriented knot C. In expression (7.2), it is understood that the generators $\{J^a, P^b\}$ are multiplied according to the order shown in the formula. More precisely, if $\{J^a, P^b\}$ are collectively denoted by $\{T^\alpha\}$, one has $\mathcal{A}_\mu(x) = \mathcal{A}^\alpha_\mu(x) T^\alpha$ and in equation (7.2) the products of connections mean, for instance,

$$\begin{aligned} \left[\mathcal{A}_\nu(y)\mathcal{A}_\mu(x)\right]_{ij} &= \mathcal{A}^\beta_\nu(y)\mathcal{A}^\alpha_\mu(x)\, T^\beta_{ik} T^\alpha_{kj} \,, \\ \left[\mathcal{A}_\lambda(z)\mathcal{A}_\nu(y)\mathcal{A}_\mu(x)\right]_{ij} &= \mathcal{A}^\gamma_\lambda(z)\mathcal{A}^\beta_\nu(y)\mathcal{A}^\alpha_\mu(x)\, T^\gamma_{i\ell} T^\beta_{\ell k} T^\alpha_{kj} \,. \end{aligned}$$

When the $ISU(2)$ generators are not multiplied, they can be understood as elements of a tensor product in colour space; so, it is convenient to introduce the notation

$$\begin{aligned} \mathcal{A}_\mu(x) \otimes \mathcal{A}_\nu(y) &= \mathcal{A}^\alpha_\mu(x)\mathcal{A}^\beta_\nu(y)\, T^\alpha_{ij} T^\beta_{k\ell} \\ \mathcal{A}_\mu(x) \otimes \mathcal{A}_\nu(y) \otimes \mathcal{A}_\lambda(z) &= \mathcal{A}^\alpha_\mu(x)\mathcal{A}^\beta_\nu(y)\mathcal{A}^\gamma_\lambda(z)\, T^\alpha_{ij} T^\beta_{k\ell} T^\gamma_{mn} \;, \ldots \text{etc.} \end{aligned} \tag{7.3}$$

According to equation (7.2), for each $ISU(2)$ representation (Λ, r) with $r = 0$ or $r = 1/2$, the normalised trace of h_C in the colour space takes the form of a sum of normalised traces of product of generators J^a and P^b. It should be noted that, since the representations (Λ, r) are infinite dimensional, the cyclic property of the trace is no more valid; consequently, the classical gauge invariance of the trace of h_C is not guaranteed. What saves the day is that the field theory expectation values of connection's products are invariant under global $ISU(2)$ transformations.

Proposition 4. *The* BF *expectation values computed by means of the total action $S_{TOT} = S + S_{\phi\pi}$ satisfy*

$$\langle \mathcal{A}_\mu(x_1) \otimes \mathcal{A}_\nu(x_2) \otimes \cdots \otimes \mathcal{A}_\lambda(x_n) \rangle = \\ = \langle \Big(\mathcal{G}^{-1} \mathcal{A}_\mu(x_1) \mathcal{G} \Big) \otimes \Big(\mathcal{G}^{-1} \mathcal{A}_\nu(x_2) \mathcal{G} \Big) \otimes \cdots \otimes \Big(\mathcal{G}^{-1} \mathcal{A}_\lambda(x_n) \mathcal{G} \Big) \rangle \, , \tag{7.4}$$

for any $\mathcal{G} \in ISU(2)$.

Proof. The proof is made of two parts. First it shown that equation (7.4) is satisfied in the case in which $\mathcal{G} = e^{i\beta^a J^a}$, and then it is demonstrated that equality (7.4) is satisfied for $\mathcal{G} = e^{i\eta^a P^a}$.

When $\mathcal{G} = e^{i\beta^a J^a}$, one has

$$\mathcal{G}^{-1} \mathcal{A}_\mu(x) \mathcal{G} = A'^a_\mu(x) J^a + B'^a_\mu(x) P^a \, , \tag{7.5}$$

where

$$A'^a_\mu(x) = R^{ab}(\beta) A^b_\mu(x) \quad , \quad B'^a_\mu(x) = R^{ab}(\beta) B^b_\mu(x) \, , \tag{7.6}$$

with $R^{ab}(\beta) \in SO(3)$. Under the change of variables $A^a_\mu(x) \to A'^a_\mu(x)$, $B^a_\mu(x) \to B'^a_\mu(x)$ and

$$\begin{aligned} M^a(x) &\to R^{ab}(\beta) M^b(x) \quad , \quad N^a(x) \to R^{ab}(\beta) N^b(x) \, , \\ c^a(x) &\to R^{ab}(\beta) c^b(x) \quad , \quad \overline{c}^a(x) \to R^{ab}(\beta) \overline{c}^b(x) \, , \\ \xi^a(x) &\to R^{ab}(\beta) \xi^b(x) \quad , \quad \overline{\xi}^a(x) \to R^{ab}(\beta) \overline{\xi}^b(x) \, , \end{aligned} \tag{7.7}$$

the total action $S_{TOT} = S + S_{\phi\pi}$ is invariant. Therefore equation (7.4) is fulfilled when $\mathcal{G} = e^{i\beta^a J^a}$.

In the case $\mathcal{G} = e^{i\eta^a P^a}$, one gets

$$\mathcal{G}^{-1} \mathcal{A}_\mu(x) \mathcal{G} = \widetilde{A}^a_\mu(x) J^a + \widetilde{B}^a_\mu(x) P^a \, , \tag{7.8}$$

where

$$\widetilde{A}^a_\mu(x) = A^a_\mu(x) \quad , \quad \widetilde{B}^a_\mu(x) = B^a_\mu(x) + \epsilon^{abc} \eta^b A^a_\mu(x) \, . \tag{7.9}$$

Under the change of variables $A^a_\mu(x) \to \widetilde{A}^a_\mu(x)$, $B^a_\mu(x) \to \widetilde{B}^a_\mu(x)$ and

$$\begin{aligned} M^a(x) &\to M^a(x) - \epsilon^{abc} N^b \eta^c \quad , \quad N^a(x) \to N^a(x) \, , \\ c^a(x) &\to c^a(x) \quad , \quad \overline{c}^a(x) \to \overline{c}^a(x) - \epsilon^{abc} \overline{\xi}^b \eta^c \, , \\ \xi^a(x) &\to \xi^a(x) - \epsilon^{abc} c^b \eta^c \quad , \quad \overline{\xi}^a(x) \to \overline{\xi}^a(x) \, , \end{aligned} \tag{7.10}$$

the total action $S_{TOT} = S + S_{\phi\pi}$ is invariant as a consequence of the Jacobi identity. Thus equation (7.4) is satisfied for $\mathcal{G} = e^{i\eta^a P^a}$.

To sum up, equation (7.4) is satisfied when $\mathcal{G} = e^{i\beta^a J^a}$ with arbitrary β^a and also when $\mathcal{G} = e^{i\eta^a P^a}$ with arbitrary η^a. Therefore equality (7.4) holds for any $\mathcal{G} \in ISU(2)$. □

A first consequence of equation (7.4) is that the two-points function $\langle A^a_\mu(x)A^b_\nu(y)\rangle$ must vanish because there is not an $ISU(2)$ invariant which is quadratic in J^a.

In the expansion (7.2) of h_C in powers of the fields, the generators of $ISU(2)$ are multiplied; hence equation (7.4) implies

$$\langle h_C\rangle = \mathcal{G}^{-1}\langle h_C\rangle\mathcal{G}\ , \qquad \forall\mathcal{G}\in ISU(2)\ . \tag{7.11}$$

Thus, as in the case of the Chern-Simons theory, the expectation value of the holonomy associated with a knot C —with colour given by an irreducible representation of the gauge group— is proportional to the identity in colour space or, more precisely, it is a function of the Casimir operators of the structure group. This means that $\langle W_C\rangle$, which is the normalised trace of $\langle h_C\rangle$ in the $ISU(2)$ representations $(\Lambda, 0)$ and $(\Lambda, 1/2)$, is well defined, it is gauge invariant and it does not depend on the choice of the base point on C.

Finally, since the holonomy h_C does not depend on the metric of $\mathbb{R}^3$ and the only dependence of the total action on the metric is contained in the gauge fixing terms, the expectation value (7.1) corresponds to a topological invariant of oriented framed coloured knots in $\mathbb{R}^3$.

8. Perturbative expansion of the observables

The value of the observable $\langle W_C\rangle$, which is given by the normalised trace of the expectation value of the holonomy associated with the knot $C \subset \mathbb{R}^3$,

$$\langle W_C\rangle\bigg|_{(\Lambda,0)} = \int \frac{d\omega}{4\pi}\,(\boldsymbol{k}|\,\langle h_C\rangle\,|\boldsymbol{k})\quad ,\quad \Big(\text{with } \boldsymbol{k}\boldsymbol{k} = \Lambda^2\Big)\ , \tag{8.1}$$

$$\langle W_C\rangle\bigg|_{(\Lambda,1/2)} = \sum_s \int \frac{d\omega}{4\pi}\,(\boldsymbol{k}, s|\,\langle h_C\rangle\,\pi_+\,|\boldsymbol{k}, s)\quad ,\quad \Big(\text{with } \boldsymbol{k}\boldsymbol{k} = \Lambda^2\Big)\ , \tag{8.2}$$

can be obtained by computing the expectation value $\langle h_C\rangle$ by means of an expansion of h_C in powers of the fields. It is important to note that, in the evaluation of $\langle h_C\rangle$, the presence of a base point x_0 in the knot C must be taken into account. Thus, $\langle W_C\rangle$ takes the form of a sum of an infinite number of perturbative contributions.

The invariant $\langle W_C\rangle$ can be approximated by considering only a finite number of terms, but the truncation of the perturbative series cannot be introduced arbitrarily. In order to obtain topological invariants, one needs to sum all the diagrams which are necessary to ensure the gauge invariance of the result. This can be achieved by summing all the diagrams which are of the same order in powers of the parameters which multiply the gauge invariant lagrangian terms. The BF action is the sum of two terms which are separately invariant under gauge transformations; so, two independent parameters are required. One parameter can be chosen to be $\hbar$, and the second parameter can be taken to be the coupling constant g.

In the previous sections, the convention $\hbar = 1$ has been used. In this section, the dependence of the Feynman diagrams on $\hbar$ is made explicit. Let us recall that a given Feynman diagram with $\mathcal{P}$ propagators and $\mathcal{V}$ interaction vertices is of order $\hbar^{\mathcal{P}-\mathcal{V}}$. The dependence of a diagram on the coupling constant g can easily be determined because g multiplies the BB component of the propagator, equation (2.12), and the AAA interaction lagrangian term.

Note that the possible values of the group generators J^a and P^b represent "colour quantum numbers" that have vanishing field theory dimensions. If one wishes to give a physical interpretation to the vectors of the $ISU(2)$ representations as particle state vectors, one can imagine that the eigenvalues of "momentum" P^a refer to a given momentum scale, so that Λ is dimensionless.

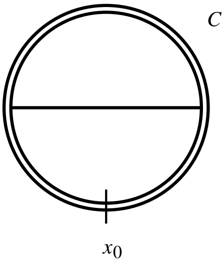


Fig. 6. First order contribution to $\langle W_C \rangle$.

In what follows, the perturbative contributions to $\langle W_C \rangle$ of order $\hbar^n$ with $n = 0, 1, 2, 3$ are in order. The contribution of order $\hbar^n$ is indicated by $\langle W_C \rangle^{(n)}$ and contains all the nonvanishing components which are labelled by powers of g. The colour of the knot is specified by the (Λ, r) representation of $ISU(2)$ with $r = 0, 1/2$.

8.1. Lowest order

With the chosen normalisation of the traces shown in equations (8.1) and (8.2), the component of $\langle W_C \rangle$ of order $\hbar^0$ is just the unit

$$\langle W_C \rangle^{(0)} = 1 \, . \tag{8.3}$$

8.2. First order

The contributions of order $\hbar$ are given by the integration of the two components of the field propagator along the knot C, as sketched in Fig. 6. The double line of Fig. 6 generically indicates a framed knot C with its base point x_0 pointed out. The embedding of C in $\mathbb{R}^3$ is not shown. A simple line represents a gauge field propagator (2.12).
In this case, the point-splitting procedure, which is defined by means of the framing C_f of the knot C, is used. Since the AB component of the propagator is of order $\hbar$ and the BB component of the propagator is of order $\hbar g$, one finds

$$\langle W_C \rangle^{(1)} = -i \left(\frac{\hbar}{2} \right) \ell k(C, C_f) \left(2\Lambda r - g\Lambda^2 \right) , \tag{8.4}$$

where $\ell k(C, C_f)$ denotes the linking number of C and its framing C_f. Indeed, the linking number of two oriented knots C_1 and C_2 can be expressed [100] by means of the Gauss integral

$$\ell k(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} dx^\nu \oint_{C_2} dy^\sigma \epsilon_{\nu\sigma\lambda} \frac{(x-y)^\lambda}{|x-y|^3} \, . \tag{8.5}$$

8.3. Second order

The nonvanishing contributions of order $\hbar^2$ to $\langle W_C \rangle$ are related with diagrams with two field propagators, shown in Fig. 7, and diagrams with one vertex and three field propagators shown in Fig. 8. As shown in Section 4, diagrams with one loop give vanishing results of order $\hbar^2$.

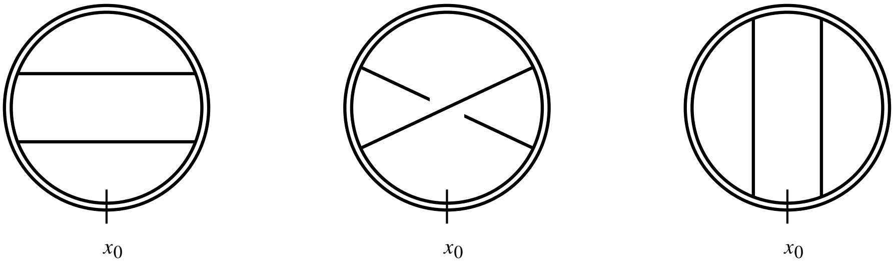


Fig. 7. Second order contribution to $\langle W_C \rangle$ with two field propagators.

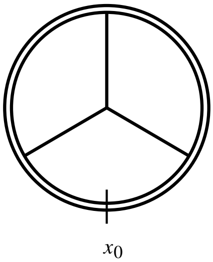


Fig. 8. Second order contribution to $\langle W_C \rangle$ with one vertex.

In the computation of $\langle h_C \rangle$, diagrams with two field propagators give contributions which are proportional to the combinations of Casimir operators: $(JP)^2 = (J^a P^a)^2$, $(JP)(P^2) = (J^a P^a)(P^b P^b)$ and $(P^2)^2 = (P^a P^a)^2$. Moreover, from the diagrams of the type shown in the second picture of Fig. 7, one gets an additional contribution which is proportional to the Casimir operator $P^a P^a$. This is a consequence of the identity

$$P^a J^b J^a P^b = (J^a P^a)^2 - 2(P^a P^a) , \tag{8.6}$$

which follows from the structure of the $ISU(2)$ algebra.

The contributions to $\langle h_C \rangle$ coming from the diagrams of Fig. 7 are

$$\begin{aligned} &-\frac{\hbar^2}{2} \left(\oint_C dx^\nu \oint_{C_f} dy^\sigma \epsilon_{\nu\sigma\lambda} \frac{(x-y)^\lambda}{4\pi |x-y|^3} \right)^2 \left[(JP)^2 - g(JP)P^2 + \frac{g^2}{4}(P^2)^2 \right] + \\ &+ 2P^2\hbar^2 \oint_C dx^\mu \int_{x_0}^{x} dy^\nu \int_{x_0}^{y} dz^\lambda \int_{x_0}^{z} dw^\sigma \, \frac{\epsilon_{\nu\sigma\tau}\epsilon_{\lambda\mu\rho}(y-w)^\tau (z-x)^\rho}{16\pi^2 |y-w|^3 |z-x|^3} \, . \end{aligned} \tag{8.7}$$

The nonvanishing contribution to $\langle h_C \rangle$ coming from the diagram of Fig. 8 is proportional to the Casimir operator P^2, as a consequence of the identity

$$\epsilon^{abc} P^b J^a P^c = -2i P^a P^a , \tag{8.8}$$

and takes the form

$$2P^2\hbar^2 \int d^3x \oint_C dz^\sigma \int_{x_0}^{z} du^\tau \int_{x_0}^{u} dv^\rho \, \epsilon^{\mu\nu\lambda} \epsilon_{\mu\tau\xi}\epsilon_{\nu\rho\beta}\epsilon_{\lambda\sigma\alpha} \frac{(x-u)^\xi (x-v)^\beta (x-z)^\alpha}{64\pi^3 |x-u|^3 |x-v|^3 |x-z|^3} \, . \tag{8.9}$$

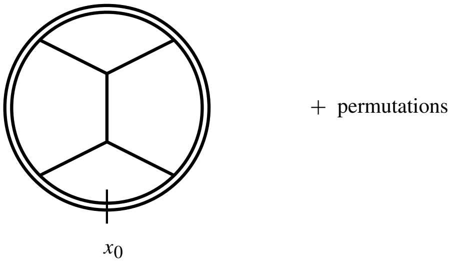


Fig. 9. Third order diagrams with two vertices.

The sum of all the terms of order $\hbar^2$ is given by

$$\langle W_C \rangle^{(2)} = -\frac{1}{2}\left(\frac{\hbar}{2}\right)^2 \left[\ell k(C, C_f)\right]^2 \left(2\Lambda r - g\Lambda^2\right)^2 + \hbar^2 \Lambda^2 \rho(C)\,, \tag{8.10}$$

where $\rho(C)$ is the knot invariant that has been found [98] in the study of the knot polynomials which are derived from the Chern-Simons field theory,

$$\rho(C) = \oint_C dx^\mu \int_{x_0}^{x} dy^\nu \int_{x_0}^{y} dz^\lambda \int_{x_0}^{z} dw^\sigma \, \frac{\epsilon_{\nu\sigma\tau}\epsilon_{\lambda\mu\rho}(y-w)^\tau (z-x)^\rho}{8\pi^2 |y-w|^3 |z-x|^3}$$
$$+ \oint_C dz^\sigma \int_{x_0}^{z} du^\tau \int_{x_0}^{u} dv^\rho \, \epsilon^{\mu\nu\lambda} \epsilon_{\mu\tau\xi}\epsilon_{\nu\rho\beta}\epsilon_{\lambda\sigma\alpha}\, \partial_u^\xi \, \partial_v^\beta \, \mathcal{I}^\alpha\,, \tag{8.11}$$

where

$$\mathcal{I}^\alpha = \frac{|v-u| + |z-u| - |v-z|}{16\pi^2 \left(|v-u|\,|z-u| + (v-u)(z-u)\right)} \left[\frac{(v-u)^\alpha}{|v-u|} + \frac{(z-u)^\alpha}{|z-u|}\right]. \tag{8.12}$$

The $\rho(C)$ knot invariant [98] gives the analytic expression of the second coefficient of the Alexander-Conway polynomial [100–102].

8.4. Third order

The value of $\langle W_C \rangle^{(3)}$ is given by the sum of the amplitudes which are associated with diagrams containing 3, 4 and 5 field propagators (2.12). In the computation of $\langle h_C \rangle$ at order $\hbar^3$, diagrams with one loop produce vanishing results. The contributions corresponding to the diagrams with 5 propagators and two lagrangian vertices, shown in Fig. 9, are vanishing as a consequence of the algebra structure (2.2) of the $ISU(2)$ generators.
Diagrams with 4 propagators contain one vertex and are of the type shown in Fig. 10. The corresponding amplitudes contain the combinations $(JP)P^2$ and $(P^2)^2$ of the Casimir operators. The sum of these contributions to $\langle h_C \rangle$ is given by

$$P^2 \hbar^3 \int d^3w \oint_C dz^\sigma \int_{x_0}^{z} du^\tau \int_{x_0}^{u} dv^\rho \, \epsilon^{\mu\nu\lambda}\epsilon_{\mu\tau\xi}\epsilon_{\nu\rho\beta}\epsilon_{\lambda\sigma\alpha} \frac{(w-u)^\xi (w-v)^\beta (w-z)^\alpha}{|w-u|^3 |w-v|^3 |w-z|^3} \times$$
$$\times \frac{-i}{128\pi^4}\left[(JP) - \frac{g}{2}P^2\right] \oint_C dx^\mu \oint_{C_f} dy^\nu \epsilon_{\mu\nu\lambda} \frac{(x-y)^\lambda}{|x-y|^3}\,. \tag{8.13}$$

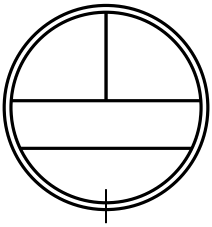


Fig. 10. Third order diagrams with four propagators.

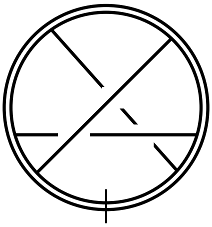


Fig. 11. Third order diagrams with three propagators.

Diagrams with 3 propagators are sketched in Fig. 11. The combinations of Casimir operators that one finds in this case are $(JP)^3$, $(JP)^2P^2$, $(JP)(P^2)^2$, $(P^2)^3$, $(JP)P^2$ and $(P^2)^2$. The resulting $\langle h_C \rangle$ amplitude which is associated with the diagrams of Fig. 11 is given by

$$
\begin{aligned}
&\frac{i\hbar^3}{6}\left[(JP) - \frac{g}{2}P^2\right]^3 \left(\oint_C dx^\nu \oint_{C_f} dy^\sigma \epsilon_{\nu\sigma\lambda} \frac{(x-y)^\lambda}{4\pi|x-y|^3} \right)^3 + \\
&+ \int d^3x \oint_C dz^\sigma \int_{x_0}^{z} du^\tau \int_{x_0}^{u} dv^\rho \, \epsilon^{\mu\nu\lambda} \epsilon_{\mu\tau\xi} \epsilon_{\nu\rho\beta} \epsilon_{\lambda\sigma\alpha} \frac{(x-u)^\xi (x-v)^\beta (x-z)^\alpha}{64\pi^3 |x-u|^3 |x-v|^3 |x-z|^3} \times \\
&\times \left(-i2\hbar^3\right) \left[(JP) - \frac{g}{2}P^2\right] P^2 \left(\oint_C dx^\nu \oint_{C_f} dy^\sigma \epsilon_{\nu\sigma\lambda} \frac{(x-y)^\lambda}{4\pi|x-y|^3} \right) . \qquad (8.14)
\end{aligned}
$$

Finally, the sum of all the contributions of order $\hbar^3$ takes the form

$$
\begin{aligned}
\langle W_C \rangle^{(3)} = &\frac{i}{6}\left(\frac{\hbar}{2}\right)^3 \left(2\Lambda r - g\Lambda^2\right)^3 \left[\ell k(C, C_f)\right]^3 + \\
&-i\frac{\hbar^3}{2}\left(2\Lambda r - g\Lambda^2\right) \Lambda^2 \left[\ell k(C, C_f)\right] \rho(C) . \qquad (8.15)
\end{aligned}
$$

8.5. Chern-Simons comparison

The knot invariants contained in $\langle W_C \rangle^{(1)}$ and $\langle W_C \rangle^{(2)}$ are precisely the invariants that one also finds in the Chern-Simons field theory (multiplying different Casimir operators, of course). At the third order, the knot invariants of the BF and of the Chern-Simos theory differ significantly. Indeed, the third order term $\langle W_C \rangle^{(3)}$ in the Chern-Simons theory —which has been computed correctly by Hirshfeld and Sassenberg [103]— contains a new knot invariant ρ^{III} that does not appear in the BF theory. This seems to be caused by the special structure of the commutation algebra of the $ISU(2)$ generators.

8.6. Framing dependence

Up to terms of order $\hbar^3$, the normalised trace of the expectation value of the knot holonomy in the BF theory is given by the sum $\sum_{n=0}^{3} \langle W_C \rangle^{(n)}$ and can be written as

$$\langle W_C \rangle = e^{-i\hbar \ell k(C,C_f)[\Lambda r-(g/2)\Lambda^2]} \Big[1 + \hbar^2 \Lambda^2 \rho_C \Big] + \mathcal{O}(\hbar^4) \, . \tag{8.16}$$

Expression (8.16) is in agreement with the general structure of the BF knot invariant, in which the whole dependence of $\langle W_C \rangle$ on the framing C_f of the knot C is given by the overall multiplicative factor

$$\text{BF framing factor } = e^{-i\hbar \ell k(C,C_f)[\Lambda r-(g/2)\Lambda^2]} \, . \tag{8.17}$$

Let us recall that, in the Chern-Simons theory, the framing factor [98,99] of the knot invariants is given by

$$\text{CS framing factor } = e^{-i \frac{\hbar}{2g} \ell k(C,C_f) C_2(R)} \, , \tag{8.18}$$

where $C_2(R)$ denotes the value of the quadratic Casimir operator in the R representation —of the structure group— which is associated with the knot, and $g = (k/4\pi)$ is the CS coupling constant [95] which multiplies the Chern-Simons action.

The framing dependence of the knot observables has a common origin in both the BF and the CS theories.

Proposition 5. *The* BF *knot invariant* $\langle W_C \rangle$ *of the framed knot* C *has the form*

$$\langle W_C \rangle = e^{-i\hbar \ell k(C,C_f)[\Lambda r-(g/2)\Lambda^2]} Q_C \, , \tag{8.19}$$

where Q_C *does not depend on the framing* C_f *of the knot* C.

Proof. Let us recall that the framing of the knot C can be defined by means of a knot C_f which belongs to the boundary of a tubular neighbourhood of C. If C is oriented, the orientation of C_f is chosen to agree with the orientation of C.

It should be noted that the choice of a framing of a knot $C \subset \mathbb{R}^3$ is equivalent to the specification of a trivialisation [100] of a tubular neighbourhood N of C. The space $N \subset \mathbb{R}^3$ is a solid torus, in which C is the core of N and $C_f \subset \partial N$. Let us define the standard solid torus V as the product $V = S^1 \times D^2$, where the two-dimensional disc D^2 is represented by the unit disc in the complex plane with coordinates $\{re^{i\theta}\}$ in which $0 \leq r \leq 1$ and $0 < \theta \leq 2\pi$. Let $\{e^{i\phi}, re^{i\theta}\}$ be coordinates of V; the standard longitude λ of V is the curve on the boundary ∂V of coordinates

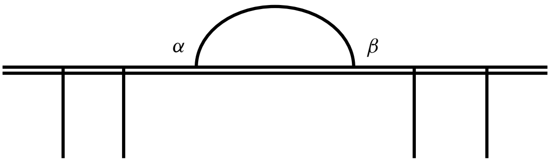


Fig. 12. Part of a diagram with one propagator associated with two consecutive generators.

$\{e^{i\phi}, 1\}$ with $0 < \phi \leq 2\pi$. A framing for C is a homeomorphism $f : V \to N$, and the image of λ is precisely the knot C_f.

Up to ambient isotopy, the homeomorphism $f : V \to N$ is uniquely specified by the linking number of C and C_f. This means that, in the quantum field theory context of the BF or CS theories, the whole dependence of $\langle W_C \rangle$ on the framing is given precisely by the sum of all the perturbative contributions which are proportional to the linking number $\ell k(C, C_f)$.

The linking number $\ell k(C, C_f)$ is given the integral along C and C_f of the corresponding Gauss density which appears in the expression (2.12) of the components of the propagator for the connection. The propagator corresponds to the two-point function of the connection fields

$$\langle A^a_\mu(x) B^b_\nu(y) \rangle = \frac{i\delta^{ab}}{4\pi} \epsilon_{\mu\nu\lambda} \frac{(x-y)^\lambda}{|x-y|^3} \quad , \quad \langle B^a_\mu(x) B^b_\nu(y) \rangle = \frac{-ig\delta^{ab}}{4\pi} \epsilon_{\mu\nu\lambda} \frac{(x-y)^\lambda}{|x-y|^3} \, , \tag{8.20}$$

that, in the BF and CS theories, receives no loop corrections (see Section 4 and [98,95]). When the components of the connection are coupled with classical sources $J^a_\mu(x)$ and $K^a_\mu(x)$, the set of the corresponding Feynman diagrams is described by the generating functional

$$\langle e^{i \int d^3x (J^a_\mu A^a_\mu + K^a_\mu B^a_\mu)} \rangle$$

and, since the two-point function is connected, the sum of all the contributions containing the linking number $\ell k(C, C_f)$ is precisely the exponential of the two-point function [87,88,104]. This means that, by neglecting the commutators between the generators J^a and P^a, the entire framing dependence of $\langle W_C \rangle$ is given by the overall multiplicative factor which is just the exponential of $\ell k(C, C_f)$ multiplied by the quadratic Casimir operator which is defined by the two-point function of the connection

$$\text{framing factor} = e^{-i\hbar \ell k(C,C_f)[PJ-(g/2)P^2]} \, . \tag{8.21}$$

Let us now take into account the fact that the generators $\{J^a, P^b\}$ do not generally commute. The holonomy h_C is defined by means of the path-ordered exponential and, in the perturbative expansion (7.2) of h_C in powers of the fields, the path-ordering determines the precise position of the J^a and P^b operators in the product of the group generators along the knot C. Let us consider the Feynman diagrams —contributing to $\langle W_C \rangle$— in which a $\mathcal{A}\mathcal{A}$ propagator connects two points of the knot C. There are only two possibilities: (a) the associated group generators are placed in consecutive positions in the path-ordering, or (b) the associated generators are nonconsecutive. In the case (a), sketched in Fig. 12, the two-point function is proportional to the contraction $\overbracket{T^\alpha T^\beta}$ which is equal to the Casimir operators JP or P^2, which commute with all the remaining generators and therefore behave as classical numbers (or classical sources).

In case (b), depicted in Fig. 13, the generators T^α and T^β which are associated with the propagator are nonconsecutive, and one has, for instance, the sequence $T^\alpha T^\sigma T^\gamma T^\beta$; this product can be written as

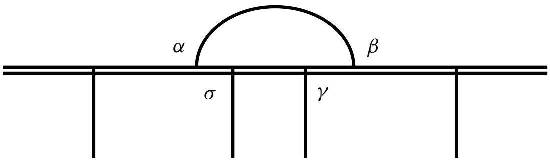


Fig. 13. Part of a diagram with one propagator associated with nonconsecutive generators.

$$T^{\alpha}T^{\sigma}T^{\gamma}T^{\beta} = T^{\sigma}T^{\gamma}\,T^{\alpha}T^{\beta} + \left[T^{\alpha}, T^{\sigma}T^{\gamma}\right]T^{\beta}\ . \tag{8.22}$$

The first term on the r.h.s. of expression (8.22) contains the quadratic Casimir operator entering $\overline{T^{\alpha}T^{\beta}}$ (which is equal to JP or PP) and, when one combines all the terms of this type with the terms coming from case (a), one gets precisely the exponentiation shown in equation (8.21).

Since the set of all the perturbative contributions to $\langle W_C \rangle$ takes the form of a sum of knot invariants, if one extract the knot invariant $\ell k(C, C_f)$ the remaining terms necessarily represent knot invariants. Thus the remaining contributions, which contain the commutator appearing in expression (8.22), combine to produce knot invariants, which necessarily are not proportional to the linking number $\ell k(C, C_f)$ because they do not contain the complete line integral along C and C_f of the Gauss density.
Therefore the framing dependence of $\langle W_C \rangle$ is given by an overall factor which is precisely the exponential of $\ell k(C, C_f)$ multiplied by the quadratic Casimir operator which is defined by the two-point function of the connection. In the CS theory, the quadratic Casimir operator is exactly $T^b T^b = c_2(R)$, whereas in the BF theory the two points function gives the combination $[\,PJ - (g/2)PP\,]$ of Casimir operators. □

9. Conclusions

The gauge theory of topological type which is usually called the BF theory is a superrenormalisable quantum field theory in $\mathbb{R}^3$. We have described the structure of the Feynman diagrams which enter the perturbative expansion of the correlation functions of the connection, the corresponding generating functional has been computed and the relationship with the Chern-Simons theory has been produced. We have presented the complete renormalisation of the BF theory, which involves the two-points function and three-points function of the connection. By means of the renormalisation procedure in the space of coordinates —which is in complete agreement with the renormalisation procedure in momentum space— one finds that, as in the case of the Chern-Simons theory, the two-points function of the connection does not receive loop corrections and therefore the bare propagator coincides with the dressed propagator.

We have defined gauge invariant observables by means of appropriately normalised traces of the holonomies which are associated with oriented, framed and coloured knots in $\mathbb{R}^3$. The colour of a knot is specified by a given unitary irreducible representation of the structure group $ISU(2)$. We have described the unitary $ISU(2)$ representations with Casimir operators $P^2 = \Lambda^2$ and $JP = r\Lambda$ —with $r = 0, 1/2$— and the $ISU(2)$ conjugacy classes have been determined. It has been shown that the expectation value of a knot holonomy is a function of the Casimir operators of the gauge group, so the expectation value of the normalised trace of knot holonomies are well defined and are gauge invariant.

The perturbative computation of the observables has been successfully achieved up to the third order in powers of $\hbar$. The knot invariants that we have found at first and second order correspond to the knot invariants that also appear in the Chern-Simons theory. Whereas the BF and CS knot invariants differ at the third order of perturbation theory. We have shown that the entire framing dependence of the knot observables is completely determined by an overall multiplicative factor which is the exponential of the linking number between the knot and its framing multiplied by the combination of the quadratic Casimir operators which is determined by the two point function of the connection.

In the present article, we have described the fundamentals of the perturbative approach to the BF theory in the case of structure group $ISU(2)$. The extensions to more complicated groups appear to be quite natural. In particular, our results admit rather simple generalisations to the case of gauge group $ISO(2,1)$, which is related to a gravitational model in $(2+1)$ dimensions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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