DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **14**, Number **5**, May **2021**

doi:10.3934/dcdss.2020449

pp. **1673–1692**

THE ALGORITHMIC NUMBERS IN NON-ARCHIMEDEAN NUMERICAL COMPUTING ENVIRONMENTS

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Abstract. There are many natural phenomena that can best be described by the use of infinitesimal and infinite numbers (see e.g. [1, 5, 13, 23]. However, until now, the Non-standard techniques have been applied to theoretical models. In this paper we investigate the possibility to implement such models in numerical simulations. First we define the field of Euclidean numbers which is a particular field of hyperreal numbers. Then, we introduce a set of families of Euclidean numbers, that we have called altogether *algorithmic numbers*, some of which are inspired by the IEEE 754 standard for floating point numbers. In particular, we suggest three formats which are relevant from the hardware implementation point of view: the Polynomial Algorithmic Numbers, the Bounded Algorithmic Numbers and the Truncated Algorithmic Numbers. In the second part of the paper, we show a few applications of such numbers.

1. Introduction.

There are many natural phenomena that can best be described by the use of infinitesimal and infinite numbers. We refer to [1, 5, 13, 23] and to the references contained therein for an analysis of these facts. In this work we investigate how we can deal with numerical computations. In particular we describe subsets of a Non-Archimedean field that can be treated numerically with the use of a suitably programmed computer. What follows is a step towards the implementation of a Matlab-like environment for performing number crunching with infinite or infinitesimal numbers.

The theoretical usefulness of infinitesimal numbers in the description of natural phenomena has been debated for centuries. Russel and Peano have demonstrated that infinitesimal numbers are not necessary to rigorously formalize Analysis. On the contrary, their opinion was that: i) infinitesimal numbers, as formalized at their time, was not rigorous enough; ii) their introduction was useless, since the central concepts in Analysis can be rigorously formalized without them. Things have changed since the introduction of Non-Standard Analysis, done by Robinson in the 60's. In NSA the concept of infinitesimal number has been rigorously formalized.

2020 *Mathematics Subject Classification.* Primary: 68U01; Secondary: 26E35, 12J25, 14B10, 46N10, 34M03.

Key words and phrases. Infinitesimal numbers, nonstandard analysis, euclidean numbers, non-archimedean scientific computing, fixed-length formats, hardware-friendly representations.

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Our approach to Non-Archimedean Mathematics (NAM) presents two main differences with the classical NSA: the first one in the methods and the second one in the aims. As far as the methods are concerned, we introduce a Non-Archimedean field via the notion of limit, and a large use of the number α [3, 7]. This approach allows us to avoid the technicalities of formal logic. In particular, the use of limits allows the reader to recover most of the language, tools and results of standard analysis in a more straightforward way. Another difference between our approach and NSA is that we do not assume the existence of two distinct mathematical universes. Let us examine the difference in the aims. Usually NSA is considered a theory which makes use of infinitesimals as a device to simplify the proves of theorems; on the contrary, we think that infinitesimals are mathematical entities which have the same status of the others and can be used to build models as any other mathematical entity. Actually, in our opinion, the advantages of a theory which includes infinitesimals rely more on the possibility of making and manipulating new models of the physical reality rather than in the techniques used to obtain the proofs (for instance, see [5, 13, 15, 16, 18, 19, 21]).

2. **Euclidean numbers.** The field of Euclidean numbers is a field which enlarge the field of real numbers and hence it contains infinite and infinitesimal numbers. It belongs to the class of hyperreal fields which are the fields used for Nonstandard Analysis and it satisfies some peculiar properties which characterize it (see section 2.4 and 2.5; for details see [10] and [12]).

We introduce the Euclidean numbers via an algebraic approach as in [4]. An elementary presentation of (part of) this theory can be found in [6]. In [20] and [11], the reader can find several other approaches to NSA and an analysis of them.

2.1. **Non-Archimedean fields.** Here, we recall the basic definitions and facts regarding Non-Archimedean fields. In the following, $\mathbb{K}$ will denote an ordered field; its elements will be called numbers. We recall that such a field contains (a copy of) the rational numbers.

Definition 1. Let $\mathbb{K}$ be an ordered field. Let $\xi \in \mathbb{K}$. We say that:

- ξ is infinitesimal if, for all positive $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;
- ξ is finite if there exists $n \in \mathbb{N}$ such that $|\xi| < n$;
- ξ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if ξ is not finite).

Definition 2. An ordered field $\mathbb{K}$ is called Non-Archimedean if it contains an infinitesimal $\xi \neq 0$.

It's easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number, and that the inverse of a nonzero infinitesimal number is infinite.

It is easy to show that any field which enlarge $\mathbb{R}$ contains infinitesimal and infinite numbers. Infinitesimal numbers can be used to formalize a new notion of "closeness":

Definition 3. We say that two numbers $\xi, \zeta \in \mathbb{K}$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case we write $\xi \sim \zeta$.

Clearly, the relation "$\sim$" of infinite closeness is an equivalence relation.

Theorem 4. *If $\mathbb{K} \supset \mathbb{R}$, every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$, called the the* ***standard part*** *of ξ.*

Given a finite number ξ, we denote it standard part by $st(\xi)$, and we put $st(\xi) = +\infty$ ($st(\xi) = -\infty$) if $\xi \in \mathbb{K}$ is a positive (negative) infinite number.

2.2. **Definition of the Euclidean numbers.** Let Λ be an infinite set containing $\mathbb{R}$ and let $\mathfrak{L}$ be the family of finite subsets of Λ. A function $\varphi : \mathfrak{L} \to E$ will be called *net* (with values in E). The set of such nets is denoted by $\mathfrak{F}(\mathfrak{L}, \mathbb{R})$. Such a set is a real algebra equipped with the natural operations

$$\begin{aligned}(\varphi + \psi)(\lambda) &= \varphi(\lambda) + \psi(\lambda);\\ (\varphi \cdot \psi)(\lambda) &= \varphi(\lambda) \cdot \psi(\lambda);\end{aligned}$$

and the partial order relation:

$$\varphi \geq \psi \Leftrightarrow \forall \lambda \in \mathfrak{L},\ \varphi(\lambda) \geq \psi(\lambda).$$

Definition 5. The set of Euclidean numbers $\mathbb{E} \supset \mathbb{R}$ is a field such that there is a surjective homomorphism

$$J : \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \to \mathbb{E}$$

or, more exactly a map which satisfies the following properties:

- $J(\varphi + \psi) = J(\varphi) + J(\psi)$;
- $J(\varphi \cdot \psi) = J(\varphi) \cdot J(\psi)$;
- *if* $\varphi(\lambda) \geq r$, then $J(\varphi) \geq r$.

The proof of the existence a field which satisfy the above definition is an easy consequence of the Krull-Zorn theorem. It can be found, e.g. in [4, 5, 6, 13].

The number $J(\varphi)$ is called the Λ limit of the net φ and will be denoted by

$$J(\varphi) = \lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$$

The reason of this name/notation is that the operation

$$\varphi \mapsto \lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$$

satisfies many of the properties of the usual limit, more exactly it satisfies the following properties:

- (Λ-1) **Existence.** *Every net* $\varphi : \mathfrak{L} \to \mathbb{R}$ *has a unique limit* $L \in \mathbb{E}$.
- (Λ-2) **Constant**. *If* $\varphi(\lambda)$ *is eventually constant*, *namely* $\exists \lambda_0 \in \mathfrak{L}, r \in \mathbb{R}$ such that $\forall \lambda \supset \lambda_0,\ \varphi(\lambda) = r$, *then*

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = r.$$

- (Λ-3) **Sum and product**. *For all* $\varphi, \psi : \mathfrak{L} \to \mathbb{R}$*:*

$$\begin{aligned}\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) + \lim_{\lambda \uparrow \Lambda} \psi(\lambda) &= \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) + \psi(\lambda));\\ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \cdot \lim_{\lambda \uparrow \Lambda} \psi(\lambda) &= \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) \cdot \psi(\lambda)).\end{aligned}$$

Now let us see the main differences between the usual limit (which we will call Cauchy limit) and the Λ-limit. We recall the definition of Cauchy limit (as formalized by Weierstrass):

$$L = \lim_{\lambda \to \Lambda} \varphi(\lambda)$$

if and only if, $\forall \varepsilon \in \mathbb{R}^+$, $\exists \lambda_0 \in \mathfrak{L}$, such that $\forall \lambda \supset \lambda_0$,

$$|\varphi(\lambda) - L| \leq \varepsilon$$

The classical example of Cauchy limit of a net is provided by the definition of the Cauchy integral:

$$\int_a^b f(x)dx = \lim_{\lambda\to\Lambda} \sum_{x\in[a,b]\cap\lambda} f(x)(x^+ - x); \quad x^+ = \min\{y \in \mathbb{R}\cap\lambda \mid y > x\}$$

Notice that in order to distinguish the two kind of limits we have used the symbols "$\lambda \uparrow \Lambda$" and "$\lambda \to \Lambda$" respectively. Since also the Cauchy limit (when it exists) satisfies (Λ-2) and (Λ-3) the only difference between the the Cauchy limit and the Λ-limit is that the latter always exists. This fact implies that $\mathbb{E}$ must be larger than $\mathbb{R}$ since otherwise a diverging net cannot have a limit.

In the case in which the Cauchy limit exists the relation between the two limits is given by the following identity:

$$\lim_{\lambda\to\Lambda} \varphi(\lambda) = st\left(\lim_{\lambda\uparrow\Lambda} \varphi(\lambda)\right) \tag{1}$$

2.3. **Numerosities and the number α.** In order to give a feeling of the Euclidean numbers, we will describe a possible interpretation of some of them. If $E \subset \Lambda$, we set

$$\mathfrak{n}\,(E) = \lim_{\lambda\uparrow\Lambda}\ |E\cap\lambda| \tag{2}$$

where $|F|$ denotes the number of elements of the finite set F. Notice that $E\cap\lambda$ is a finite set since $\lambda \in \mathfrak{L}$. Then the above limit makes sense since for every $\lambda \in \mathfrak{L}$, $|E\cap\lambda| \ \in \mathbb{N}$. If E is a finite set, the sequence is eventually constant, namely, $\forall\lambda \supset E$, $|E\cap\lambda| \ = |E|$ and hence $\mathfrak{n}\,(E) = |E|$. If E is an infinite set, the above limit gives an infinite number. Hence $\mathfrak{n}\,(E)$ extends the "measure of the size of a set" to infinite sets. The Euclidean number $\mathfrak{n}\,(E)$ is called *numerosity* of E. For example, the number α defined by

$$\alpha := \lim_{\lambda\uparrow\Lambda}\ |\mathbb{N}\cap\lambda| \tag{3}$$

is the numerosity of $\mathbb{N}$ and it represents a measure of the size of $\mathbb{N} = \{1,2,3,...\}$.

Definition 6. The set of numerosities is defined as follows:

$$\mathbf{Num} := \{\nu \in \mathbb{E} \mid \exists A \subset \Lambda,\ \nu = \mathfrak{n}(A)\}$$

The theory of numerosity can be considered as an extension of the Cantorian theory of cardinal and ordinal numbers. The reader interested to the details and the developments of this theory is referred to [2, 7, 9, 10] and the papers quoted therein.

In all this paper the number α will play a special role, and, in order to simplify the notation, we will use also a special symbol, namely η, to denote its inverse.

2.4. **Basic properties of the Euclidean numbers.** The main feature of the Euclidean numbers is that every real function $f : \mathbb{R} \to \mathbb{R}$ can be extended to a function f^* defined on all the field $\mathbb{E}$; in fact given a point $\xi \in \mathbb{E}$, such that

$$\xi = \lim_{\lambda\uparrow\Lambda}\ x_\lambda$$

we set

$$f^*(\xi) = \lim_{\lambda\uparrow\Lambda}\ f\,(x_\lambda)$$

It is possible to prove that the value of $f^*(\xi)$ is independent of the choice of the approximating net $\{x_\lambda\}$. This peculiarity implies that expressions such as

$\sin(\eta)$, $\exp(\alpha^2)$, etc. make sense and can be manipulated following the usual rules of the real numbers. For example if

$$x = \lim_{\lambda \uparrow \Lambda} x_\lambda \quad \text{and} \quad y = \lim_{\lambda \uparrow \Lambda} y_\lambda$$

are Euclidean numbers, we have that

$$\exp(x+y) = \exp(x) \cdot \exp(y).$$

In fact

$$\begin{aligned}
\exp(x+y) &= \exp\left(\lim_{\lambda \uparrow \Lambda} (x_\lambda + y_\lambda)\right) = \lim_{\lambda \uparrow \Lambda} \exp(x_\lambda + y_\lambda) \\
&= \lim_{\lambda \uparrow \Lambda} \left[\exp(x_\lambda) \cdot \exp(y_\lambda)\right] = \lim_{\lambda \uparrow \Lambda} \left[\exp(x_\lambda)\right] \cdot \lim_{\lambda \uparrow \Lambda} \left[\exp(y_\lambda)\right] \\
&= \left[\exp\left(\lim_{\lambda \uparrow \Lambda} x_\lambda\right)\right] \cdot \left[\exp\left(\lim_{\lambda \uparrow \Lambda} y_\lambda\right)\right] = \exp(x) \cdot \exp(y).
\end{aligned}$$

There are other properties of the Euclidean numbers which depend on the choice of the set Λ and the homomorphism J. They will not be used in this paper but we will mention them for completeness, without proofs and details, but with the appropriate references for the interested reader. If we choose Λ and a suitable homomorphism J properly, then the following properties hold.

Property 1. *If $q \in \mathbb{Q}$, then*

$$\alpha^q \in \mathbb{N}^*$$

where

$$\mathbb{N}^* = \left\{ \lim_{\lambda \uparrow \Lambda} x_\lambda \mid x_\lambda \in \mathbb{N} \right\}$$

*is (called) the set of **hypernatural** numbers.*

See [2], [4] or [8]. By this property, it follows that $\alpha/2, \alpha/3, .., \sqrt{\alpha}, \sqrt[3]{\alpha}, ..$ etc. are the numerosities of suitable sets; for example $\alpha/2$ is the numerosity of the even numbers; $\sqrt{\alpha}$ is the numerosity of the set of the square numbers $\{1, 4, 9, 16, ...\}$ etc.

In order to state another interesting property, we need the following definition:

Definition 7. A set is called **ordinary** if its cardinality is less than the first inaccessible cardinal number κ.

Then, $\mathbb{E}$ satisfies the following properties:

Property 2. ***κ-saturation.*** *Given two ordinary subsets, $A, B \subset \mathbb{E}$, such that*

$$a \in A, b \in B \Rightarrow a < b$$

then $\exists c \in \mathbb{E}$, $\forall a \in A$, $\forall b \in B$,

$$a < c < b.$$

Property 3. $\mathbb{E}$ *is the **unique** real closed field κ-saturated and any ordered field of cardinality less or equal to κ is isomorphic to a subfield of $\mathbb{E}$.*

Property 4. $\mathbb{E}$ *is the unique hyperreal field having cardinality κ and it is isomorphic to the Keisler field.*

See [10] for details or [12] for a divulgative exposition.

2.5. **Euclidean numbers versus other classes of numbers.** This section is independent of the rest of the paper. It is dedicated to the reader who wants to orient himself in the jungle of infinite numbers present in the mathematical literature.

Probably the best known infinite numbers are the cardinal numbers. Since we do not want to work with classes which make life hard, we will consider only ordinary cardinal numbers (namely the numbers smaller than the first inaccessible cardinal number κ), and we will denote this set by **Card**. It is well known that any cardinal number can be identified with the smallest ordinal number of a given cardinality. Thus we can directly consider the set **Ord** of ordinal numbers (smaller than the first inaccessible cardinal).

Now let us compare the set of ordinal numbers with **Num**, the set of numerosities (see Def. 6). Then the set of ordinal numbers **Ord** can be identified with the subset of **Num** characterized by the following property:

$$\gamma \in \mathbf{Ord} \Longleftrightarrow \gamma = \mathfrak{n}(S_\gamma)$$

where

$$S_\gamma = \{x \in \mathbf{Ord} \mid x < \gamma\}.$$

Clearly $\mathbf{Ord} \neq \varnothing$ since $0 \in \mathbf{Ord}$. Moreover it is easy to see that the natural numbers are ordinal numbers since, it is easy to prove by induction that

$$n = \mathfrak{n}(\{0, ..., n-1\})$$

is an ordinal number. On the other hand there are numerosities which are not ordinal numbers such as α defined by (3); to see this we argue indirectly; if we assume that α is an ordinal numbers, we get the following contradiction:

$$\alpha = \mathfrak{n}(\,\{x \in \mathbf{Ord} \mid x < \alpha\}) = \mathfrak{n}(\mathbb{N}_0) = \mathfrak{n}(\mathbb{N} \cup \{0\}) = \mathfrak{n}(\mathbb{N}) + 1 = \alpha + 1.$$

With a similar argument, we can prove that $\alpha - 1, \alpha - 2, .., \alpha/2, \alpha/3, ..\sqrt{\alpha}$ are not ordinal numbers. On the contrary,

$$\alpha + 1 = \mathfrak{n}(\mathbb{N}_0) = \{x \in \mathbf{Ord} \mid x < \alpha + 1\}$$

is an ordinal number; clearly this number is the smallest infinite ordinal number usually denoted by ω.

In conclusion, we have the following situation:

$$\mathbf{Card} \subset \mathbf{Ord} \subset \mathbf{Num} \subset \mathbb{N}_0^* = \mathbb{N}^* \cup \{0\}.$$

Another theory which uses infinite numbers is the *Grossone Methodology* of Sergeyev [25]. Also these numbers can be easily embedded in the set of Euclidean number identifying the number grossone ① with α.

However, when using this identification, we have to be very carefully since the author uses a rather problematic notation. In fact he uses the symbol $\mathbb{N}$ to denote the set

$$\{x \in \mathbb{N}^* \mid x \leq \text{①}\}$$

and he call these numbers *natural numbers*. So, in his theory, the “finite” natural numbers are looking for a name having lost the one used for more than 2000 years and this fact might cause some confusion. Undoubtedly Grossone Methodology has the merit of having being successfully used to solve several applied problems (see [25] and references therein).

A beautiful theory of non-Archimedean numbers is the theory of surreal numbers created by Conway. The set of surreal numbers[1] **No** is a field having cardinality κ and hence it is a field isomorphic to $\mathbb{E}$ (see property (3)). Moreover there is a natural identification

$$\mathbf{Ord} \subset \mathbf{No}.$$

This implies that it is possible to identify a large class of surreal numbers with the corresponding Euclidean numbers. However, as far as we know, a satisfactory isomorphism (or homomorphism) between these two field has not been established and this seems to be a rather involved problem.

A other important Non-Archimedean field is the Levi-Civita field which will be discussed in the next section (see section 3 and in particular Remark 10).

3. Algorithmic numbers (ANs).

The algorithmic numbers (ANs) form a subset of the Euclidean numbers which can be easily represented and manipulated with a computer. As we will see, the full set of these numbers is still too large to allow efficient numerical computations on personal computer. However, it is useful since it contains sets of numerically friendly numbers which will be presented below.

Following Levi-Civita [22], we will call monosemium an Euclidean number of the form

$$r\alpha^p$$

where $r \in \mathbb{R}$ and $p \in \mathbb{Q}$; p will be called the order of the monosemium.

Definition 8. An Euclidean number ξ is called algorithmic if it can be represented as a finite sum of monosemia, namely

$$\xi = \sum_{k=0}^{\ell} r_k \alpha^{s_k}; \quad r_k \in \mathbb{R},\ s_k \in \mathbb{Q};\ s_k > s_{k+1}. \tag{4}$$

s_0 will be called *"order of ξ"* and $r_0\alpha^{s_0}$ will be called *leading term* or *leading monosemium* of ξ. We recall that the Levi-Civita field is defined as the set of numbers obtained by an infinite sum of monosemia; this sum is well defined provided that the s_k's do not have accumulation points (see Remark 10). Thus the algorithmic numbers form a subset of the Levi-Civita field.

In some situations, it is useful to reduce an AN "ξ" to its *normal form*:

Definition 9. The normal form of a AN "ξ" is given by

$$\xi = \alpha^p P\left(\eta^{\frac{1}{m}}\right) \tag{5}$$

where $p \in \mathbb{Q}$, $m \in \mathbb{N}$ and $P(x)$ is a polynomial with real coefficients such that $P(0) = r_0 \neq 0$.

It is easy to see that an AN can be reduced to a normal form; indeed

$$\xi = \sum_{k=0}^{\ell} r_k \alpha^{s_k} = \sum_{k=0}^{\ell} r_k \alpha^{\frac{n_k}{m}} = \alpha^{\frac{n_0}{m}} \sum_{k=0}^{\ell} r_k \alpha^{\frac{n_k - n_0}{m}}$$

where m is the least common denominator of the s_k's; then

$$\xi = \alpha^{\frac{n_0}{m}} \sum_{k=0}^{\ell} r_k \left(\eta^{\frac{1}{m}}\right)^{n_0 - n_k}$$

[1] Actually he uses the Gödel-Bernays theory of classes but here, to make life easier, we consider **No** to be a set having cardinality κ.

has the form (5).

Let us show an example. Suppose to have the following AN number:

$$1.63024\alpha^{13/2} - 1.03469\alpha^{11/2} + 0.488894\alpha^{7/2}$$

its normalized version is:

$$\alpha^{13/2}\left(1.63024\left(\eta^{1/2}\right)^{13-13} - 1.03469\left(\eta^{1/2}\right)^{13-11} + 0.488894\left(\eta^{1/2}\right)^{13-7}\right)$$

i.e.:

$$\alpha^{13/2}\left(1.63024 - 1.03469\left(\eta^{1/2}\right)^{2} + 0.488894\left(\eta^{1/2}\right)^{6}\right)$$

The algorithmic numbers can be represented and manipulated in actual computations; in fact any algorithmic number can be coded by a triple (n_0, m, P) where $n_0 \in \mathbb{Z}$; $m \in \mathbb{N}$ and P is a polynomial with real coefficients.

In practical applications, however, the ANs present two fundamental problems:

- the inverse of an AN is not an AN; for example $(\alpha + 1)^{-1}$ is not an AN; hence they form a commutative ring with identity and not a field;
- after few computations they fill the memory (this is due to the fact that AN numbers have a variable length coding, and this length tends to increase during the execution of an iterative algorithm).

This fact is not surprising since the same phenomenon occurs when we try to represent the real numbers in a (finite) decimal notation. The inverse of a number written in a finite decimal notation cannot be written in the same way: e.g. $3^{-1} = 0.3333...$ cannot be written by a finite number of digits; moreover after few multiplications the computer will go in overflow, since every multiplication increases the number of decimal digits.

Remark 10. The Levi-Civita field can be defined as the Cantor closure of ANs. Namely, a number in the Levi-Civita field is an equivalence class of Cauchy sequences of algorithmic numbers. It possible to show that such a number can be univocally represented by an infinite sum $\sum_{k=0}^{\infty} r_k \alpha^{s_k}$ where the s_k's form a sequence of decreasing rational numbers without accumulation points (i.e., either finite or diverging to $-\infty$).

3.1. **Approximated inverse of algorithmic numbers.** The inverse of algorithmic numbers can be performed with the same spirit which we have when we divide two decimal numbers; since the division, in general, produces infinitely many decimal digits, we have to decide the degree of precision, namely when to stop the iteration procedure.

For this reason, we need a "truncation" operator. Given a polynomial

$$P_m(x) = a_0 + .. + a_n x^n + a_{n+1}x^{n+1} + .. + a_m x^m$$

we set

$$\mathfrak{tr}_n P_m(x) = \mathfrak{tr}_n\left[a_0 + .. + a_n x^n + a_{n+1}x^{n+1} + .. + a_m x^m\right] = a_0 + ... + a_n x^n$$

namely $\mathfrak{tr}_n P(x)$ is a polynomial of degree $\leq n$ which, for small x's, approximates $P(x)$.

Definition 11. Given an algorithmic number $\xi = \alpha^p P\left(\eta^{\frac{1}{m}}\right)$, the n-truncated inverse is an algorithmic number

$$\zeta = \alpha^{-p} Q_n\left(\eta^{\frac{1}{m}}\right)$$

such that

$$\mathfrak{tr}_n\left[P\left(\eta^{\frac{1}{m}}\right)\cdot Q_n\left(\eta^{\frac{1}{m}}\right)\right]=1$$

Let us see how the polynomial Q_n can be constructed. Given

$$\alpha^p P\left(\eta^{\frac{1}{m}}\right)=\alpha^p\left(p_0+\sum_{k=1}^{\ell}p_k\eta^{\frac{k}{m}}\right),$$

we will rewrite it in the following form:

$$\alpha^p P\left(\eta^{\frac{1}{m}}\right)=p_0\alpha^p\left(1-\varepsilon\right)$$

where

$$\varepsilon=-\sum_{k=1}^{\ell}\frac{p_k}{p_0}\eta^{\frac{k}{m}}$$

Now, by observing that the Taylor expansion of $(1-\varepsilon)$ is:

$$(1-\varepsilon)^{-1}=(1+\varepsilon+\varepsilon^2+\varepsilon^3+...)$$

we set

$$Q_n\left(\eta^{\frac{1}{m}}\right)=\frac{1}{p_0}\cdot\mathfrak{tr}_n\left[1+\varepsilon+\varepsilon^2+...+\varepsilon^n\right]$$

It is easy to see that $\alpha^{-p}Q_n\left(\eta^{\frac{1}{m}}\right)$ is the n-truncated inverse of $\alpha^p P\left(\eta^{\frac{1}{m}}\right)$; indeed:

$$\begin{aligned}\mathfrak{tr}_n\left[P\left(\eta^{\frac{1}{m}}\right)\cdot Q_n\left(\eta^{\frac{1}{m}}\right)\right] &= \mathfrak{tr}_n\left[(1-\varepsilon)\cdot\left(1+\varepsilon+\varepsilon^2+...+\varepsilon^n\right)\right]\\ &= \mathfrak{tr}_n\left[1-\varepsilon^{n+1}\right]=1\end{aligned}$$

Using the same method we we can compute the n-approximated r-th root of a AN. We write the number in the form (3.1) and then we apply the Newton formula

$$(1+x)^r=1+\binom{r}{1}x+\binom{r}{2}x^2+\binom{r}{2}x^2+...+\binom{r}{n}x^n+O(x^{n+1})$$

where

$$\binom{r}{k}=\frac{r(r-1)(r-2)...(r-k+1)}{k!}$$

As an example, the approximated square root of the generic AN number

$$\alpha^p P\left(\eta^{\frac{1}{m}}\right)=p_0\alpha^p\left(1-\varepsilon\right)$$

can be computed as:

$$\sqrt{\alpha^p P\left(\eta^{\frac{1}{m}}\right)}=\sqrt{p_0}\alpha^{\frac{p}{2}}\left(1-\varepsilon\right)^{\frac{1}{2}}\cong\sqrt{p_0}\alpha^{\frac{p}{2}}\mathfrak{tr}_n\left(1-\frac{1}{2}\varepsilon-\frac{1}{8}\varepsilon^2-\frac{1}{16}\varepsilon^3-\frac{5}{128}\varepsilon^4+...\right).$$

3.2. **Polynomial algorithmic numbers (PANs).** In order to perform computations with algorithmic numbers, we will adopt the same strategy used with real numbers: we will use suitable subclasses of approximate ANs and we will perform approximate operations with them which make such classes closed. Clearly, the choice of the class depends on the problem and on the degree of precision needed.

The approximate sum and product will be denoted by $\oplus$ and $\odot$ and the approximate division by $\oslash$. The simplest class of algorithmic number is provided by the following definition:

Definition 12. A polynomial algorithmic number (PAN) of accuracy n is an Euclidean number ξ which can be represented as follows

$$\xi = S(\alpha)\eta^n \tag{6}$$

where S is a polynomial with real coefficients.

Let us discuss some features of this class of numbers. The polynomial algorithmic numbers are the analogous of the decimal number with a fixed number of decimal digits (*fixed-point representation*); indeed such a number can be written as follows

$$P\left(10\right)\left(\frac{1}{10}\right)^n$$

where P is a polynomial with coefficients in $\{0,1,2,...,9\}$. Example: the number $527,36$ is represented by the polynomial

$$S\left(x\right) = 5x^4 + 2x^3 + 7x^2 + 3x + 6$$

and $n = 2$. While a real number is approximated up to 10^{-n}, a Euclidean numbers is approximated up to η^n, namely up to an infinitesimal of order n.

Examples of PANs having accuracy $n = 2$.

$$\xi = \left(5\alpha^4 + 2\alpha^3 + 3\alpha^2 + 7\alpha^1 + 6\right)\eta^2;$$

$$\zeta = \left(-\frac{1}{7}\alpha^3 - 5\right)\eta^2.$$

3.3. **Approximated operations on PANs.** The approximated operations between two PAN correspond to the usual operations up to an approximation of infinitesimals of order n. One of the characteristic of this class of numbers is that we can operate with them just performing operations with polynomials in α:

sum:

$$P(\alpha)\eta^n \oplus Q(\alpha)\eta^n = \left[P(\alpha) + Q(\alpha)\right]\eta^n;$$

no error occurs in this case.

In order to define the approximated product and the approximated quotient we recall the definition of quotient between two polynomials: given two polynomials $A(x)$ and $Q(x)$ the usual polynomial division gives the following result

$$\frac{A(x)}{Q(x)} = C(x) + \frac{R(x)}{Q(x)}$$

where $C(x)$ is the quotient polynomial and the remainder term $R(x)$ is a polynomial of lower degree than $Q(x)$; we will use the following notation:

$$C(x) = \mathsf{quo}\left[\frac{A(x)}{Q(x)}\right]$$

- **approximated product:**

$$P(\alpha)\ \eta^n \otimes Q(\alpha)\eta^n = \mathfrak{quo}\left[\frac{P(\alpha)\cdot Q(\alpha)}{\alpha^n}\right]\eta^n$$

- **approximated division:**

$$P(\alpha)\eta^n \oslash Q(\alpha)\eta^n = \mathfrak{quo}\left[\frac{\alpha^n P(\alpha)}{Q(\alpha)}\right]\eta^n$$

Example. Given $\xi = \left(4\alpha^2 + 3\alpha\right)\eta^2$ and $\zeta = \left(\alpha^3 - 5\right)\eta^2$

$$\begin{aligned} \xi \otimes \zeta &= \mathfrak{quo}\left[\frac{\left(4\alpha^2+3\alpha\right)\cdot\left(\alpha^3-5\right)}{\alpha^2}\right]\eta^2 = \mathfrak{quo}\left[\frac{4\alpha^5+3\alpha^4-20\alpha^2-15\alpha}{\alpha^2}\right]\eta^2 \\ &= (4\alpha^3+3\alpha^2-20)\eta^2 \end{aligned}$$

$$\begin{aligned} \xi \oslash \zeta &= \mathfrak{quo}\left[\frac{4\alpha^4+3\alpha^3}{\alpha^3-5}\right]\eta^2 = \mathfrak{quo}\left[4\alpha+3+\frac{20\alpha+15}{\alpha^3-5}\right]\eta^2 \\ &= (4\alpha+3)\,\eta^2. \end{aligned}$$

PANs like numbers have been implicitly used, for instance, in [14, 16], to transform a lexicographic multi-objective optimization problem into a single-objective one, where the cost function is a vector of PANs. More precisely, to solve an r-objective optimization problem, PANs of degree r and accuracy r have been used. As an example, when $r + 1 = 5$ objectives, the numbers are of the form:

$$\xi = \left(c_0\alpha^4 + c_1\alpha^3 + c_2\alpha^2 + c_3\alpha^1 + c_4\right)\eta^4.$$

3.4. **Bounded algorithmic numbers (BANs).** The PANs are very simple to implement in a computer, but they present several problems. First, similarly to fixed-point real numbers, they are suitable to go in overflow since every multiplication increases the degree of the representative polynomial and in any case they require a lot of memory. Moreover, if you get a number of accuracy less than n the computer gives 0 and this fact, in some class of problems, is not acceptable. In the case of real numbers this problem has been solved using the scientific notation. A (positive) approximated real number in scientific notation has the following form:

$$P_n\left(10^{-1}\right)\cdot 10^p$$

where P_n is a polynomial with coefficients in $\{0, 1, 2, ..., 9\}$, and $p \in \mathbb{Z}$.

Example. the number 0.00271 is represented as follows:

$$2.71\cdot 10^{-3} = \left(2 + 7\cdot 10^{-1} + 1\cdot 10^{-2}\right)\cdot 10^{-3}$$

Definition 13. A bounded algorithmic number (BAN) of accuracy n is an Euclidean number ξ which can be represented as follows

$$\alpha^p P_n(\eta) \tag{7}$$

where P_n is a polynomial with real coefficients of degree n such that $P_n(0) \neq 0$ and p is an integer number; the set of BANs of accuracy n will be denoted by $\mathbb{E}_n$.

For each n the set $\mathbb{E}_n$ of the algorithmic numbers of degree n is a set with the four approximated elementary operations ($\oplus$, $\odot$ and their inverses) which approximate the euclidean numbers so that they can be used more efficiently in the numerical calculations.

3.5. **Approximated operations on BANs.** Now we can define "approximate" field operations in such a way that $\mathbb{E}_n$ is closed:

- **approximated sum**: reduce the minor number (of order q) to the order of the major (p) and the resulting polynomials are added:

$$\alpha^p P_n(\eta) \oplus \alpha^q Q_n(\eta) = \alpha^p P_n(\eta) \oplus \alpha^p \left[Q_n(\eta)\,\eta^{p-q}\right] = \alpha^p \left(P_n(\eta) + \mathfrak{tr}_n\left[Q_n(\eta)\,\eta^{p-q}\right]\right)$$

- **approximated product:**

$$P_n(\eta)\,\alpha^p \odot Q_n(\eta)\,\alpha^q = \alpha^{p+q}\mathfrak{tr}_n\left[P_n(\eta)\cdot Q_n(\eta)\right]$$

- **approximated division**: given $\alpha^p P_n(\eta)$ and $\alpha^q Q_n(\eta)$ we rewrite:

$$\alpha^q Q_n(\eta) = \alpha^q \left(q_0 + \sum_{k=1}^{n} q_k \eta^k\right)$$

in the following form:

$$\alpha^q Q_n(\eta) = q_0 \alpha^q (1-\varepsilon)$$

where

$$\varepsilon = -\sum_{k=1}^{n} \frac{q_k}{q_0}\eta^k;$$

Now, we pose

$$P_n(\eta)\,\alpha^p \oslash Q_n(\eta)\,\alpha^q = \alpha^{p-q}\mathfrak{tr}_n\left[\frac{P_n(\eta)}{q_0}\left(1+\varepsilon+\varepsilon^2+\dots+\varepsilon^n\right)\right]$$

$$= \alpha^{p-q}\left[\frac{P_n(\eta)}{q_0} + \mathfrak{tr}_n\left[\varepsilon\frac{P_n(\eta)}{q_0}\right] + \mathfrak{tr}_n\left[\varepsilon^2\frac{P_n(\eta)}{q_0}\right] + \dots + \mathfrak{tr}_n\left[\varepsilon^n\frac{P_n(\eta)}{q_0}\right]\right]$$

In particular, the approximated inverse of $\xi = Q_n(\eta)\,\alpha^q$ is given by

$$\begin{aligned} 1 \oslash Q_n(\eta)\,\alpha^q =& \alpha^{-q}\left[\frac{1}{q_0} - \mathfrak{tr}_n\left[\frac{1}{q_0}\cdot\sum_{k=1}^{n}\frac{q_k}{q_0}\eta^k\right] + \dots + (-1)^n\mathfrak{tr}_n\left[\frac{1}{q_0}\cdot\frac{q_1^n}{q_0^n}\eta^n\right]\right] = \\ =& \frac{\alpha^{-q}}{q_0}\left[1 - \mathfrak{tr}_n\left[\sum_{k=1}^{n}\frac{q_k}{q_0}\eta^k\right] + \dots + (-1)^n\mathfrak{tr}_n\left[\frac{q_1^n}{q_0^n}\eta^n\right]\right] \end{aligned}$$

Let us see an example of division:

$$\begin{aligned} \frac{1}{\alpha^5\left(2+3\eta-2\eta^2\right)} &= \left[2\alpha^5\left(1+\frac{3}{2}\eta-\eta^2\right)\right]^{-1} \\ &= \frac{1}{2}\alpha^{-5}\left(1+\frac{3}{2}\eta-\eta^2\right)^{-1} \\ &\cong \frac{1}{2}\alpha^{-5}\mathfrak{tr}_2\left[1-\left(\frac{3}{2}\eta-\eta^2\right)+\left(\frac{3}{2}\eta-\eta^2\right)^2\right] \\ &= \frac{1}{2}\alpha^{-5}\left[\left(1-\frac{3}{2}\eta+\eta^2\right)+\mathfrak{tr}_2\left[\left(\frac{3}{2}\eta-\eta^2\right)^2\right]\right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\alpha^{-5}\left[1-\frac{3}{2}\eta+\eta^2+\left(\frac{3}{2}\eta\right)^2\right] \\
&= \frac{1}{2}\alpha^{-5}\left(1-\frac{3}{2}\eta+\frac{13}{4}\eta^2\right) \\
&= \left(\frac{1}{2}-\frac{3}{4}\eta+\frac{13}{8}\eta^2\right)\alpha^{-5}
\end{aligned}$$

Let us check if the answer is correct:

$$\begin{aligned}
&\left(2+3\eta-2\eta^2\right)\alpha^5\cdot\left(\frac{1}{2}-\frac{3}{4}\eta+\frac{13}{8}\eta^2\right)\alpha^{-5} \\
= \; & \mathfrak{tr}_2\left[\left(2+3\eta-2\eta^2\right)\left(\frac{1}{2}-\frac{3}{4}\eta+\frac{13}{8}\eta^2\right)\right] \\
= \; & 1
\end{aligned}$$

Let us consider another example in $\mathbb{E}_3$:

$$\begin{aligned}
1\oslash\left(1+\eta+\eta^2\right) &\cong 1-\mathfrak{tr}_2\left(\eta+\eta^2\right)+\mathfrak{tr}_2\left(\eta+\eta^2\right)^2 \\
&= 1-\eta-\eta^2+\eta^2\cong 1-\eta;
\end{aligned}$$

in fact

$$\begin{aligned}
\left(1+\eta+\eta^2\right)\odot(1-\eta) &= \mathfrak{tr}_2\left[\left(1+\eta+\eta^2\right)(1-\eta)\right] \\
&= \mathfrak{tr}_2\left[1-\eta^3\right]=1.
\end{aligned}$$

3.6. **Truncated algorithmic numbers (TANs).** BANs are useful in all numerical simulations when the four elementary operations are sufficient; for instance, in lexicographic multi-objective linear programming [16].

However, when we need non-linear operations the BANs might not be sufficient. For example, the solutions of the equation

$$x^2=\alpha$$

cannot be approximated by BANs, being $\sqrt{\alpha}$ greater than any number of order 0 and lower than any number of order 1. In general, the square root, of the BAN (7), if p is an odd number, cannot be approximated by a BAN. Hence, for certain problems it is necessary to use the AN and to truncate the polynomial P to a suitable degree.

Definition 14. A truncated algorithmic number (TAN) of degree n and type m is an Euclidean number ξ which can be represented as follows

$$\alpha^p P_n\left(\eta^{\frac{1}{m}}\right)$$

where P_n is a polynomial with real coefficients of degree n, $p\in\mathbb{Q}$ and m is a positive integer. The set of TANs of degree n and type m will be denoted by $\mathbb{E}_{n,m}$.

The operations with TANs are described in sections 3 and 3.1; of course, if we want to get a TAN we need to write the result in the normal form and to truncate the result to the degree n .

Example. Let us study the following equation in $\mathbb{E}_{2,2}$:

$$x^2+2x-4\alpha=0$$

In this case

$$\sqrt{\frac{\triangle}{4}} = \sqrt{1+4\alpha} = 2\alpha^{\frac{1}{2}}\sqrt{1+\frac{1}{4}\eta} \cong 2\alpha^{\frac{1}{2}}\left(1+\frac{1}{8}\eta\right)$$

then

$$\begin{aligned} x_1 &= -1+\sqrt{1+4\alpha} \cong 2\alpha^{\frac{1}{2}}\left(1-\frac{1}{2}\eta^{\frac{1}{2}}+\frac{1}{8}\left(\eta^{\frac{1}{2}}\right)^2\right) \\ x_2 &= -1-\sqrt{4\alpha+1} \cong -2\alpha^{\frac{1}{2}}\left(1+\frac{1}{2}\eta^{\frac{1}{2}}+\frac{1}{8}\left(\eta^{\frac{1}{2}}\right)^2\right) \end{aligned}$$

Example. Let us see what happens making an infinitesimal singular perturbation to a simple linear equation

$$3x+5=0$$

namely, we consider the second order equation

$$\eta x^2+3x+5=0.$$

The exact solutions of this equation are given by

$$x_1 = \frac{-3+\sqrt{\triangle}}{2\eta};\ \ x_2 = \frac{-3-\sqrt{\triangle}}{2\eta};\ \ \triangle = 9-20\eta$$

Then

$$x_1 = -\frac{3}{2}\alpha+\frac{1}{2}\alpha\sqrt{9-20\eta};\ \ x_2 = -\frac{3}{2}\alpha-\frac{1}{2}\alpha\sqrt{9-20\eta}.$$

In this case the TANs are not necessary. If fact, we can estimate the solution with a BAN. Let us work in $\mathbb{E}_2$. We need to evaluate $\sqrt{9-20\eta}$; since

$$(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\mathcal{O}\left(x^3\right)$$

we get

$$\sqrt{9-20\eta} = 3\sqrt{1-\frac{20}{9}\eta} \cong 3\left(1-\frac{10}{9}\eta+\frac{50}{81}\eta^2\right) = 3-\frac{10}{3}\eta+\frac{50}{27}\eta^2$$

then

$$\begin{aligned} x_1 &= -\frac{3}{2}\alpha+\frac{1}{2}\alpha\sqrt{9-20\eta} \cong -\frac{3}{2}\alpha+\frac{1}{2}\alpha\left(3-\frac{10}{3}\eta+\frac{50}{27}\eta^2\right) = -\frac{5}{3}+\frac{25}{54}\eta \\ x_2 &\cong -\frac{3}{2}\alpha-\frac{1}{2}\alpha\left(3-\frac{10}{3}\eta+\frac{50}{27}\eta^2\right) = -3\alpha+\frac{5}{3}-\frac{25}{27}\eta = -\alpha\left(3-\frac{5}{3}\eta+\frac{25}{27}\eta^2\right) \end{aligned}$$

Notice that x_1 is an infinitesimal perturbation of the solution of the linear problem; x_2 is a new solution which appears at $-\infty$ (or, to be more precise, close to -3α).

3.7. **Beyond the Levi-Civita field.** One of the strong limitations of the Levi-Civita field is that is not possible to define transcendental functions for all the numbers, not even by a suitable approximation. This is one of the reasons that make NSA and the hyperreal fields necessary. For example the number $\log\alpha$, which in the world of Euclidean numbers is well defined by the formula

$$\log\alpha = \lim_{\lambda\uparrow\Lambda}\ \log\left(|\lambda\cap\mathbb{N}|\right)$$

cannot be approximated by a Levi-Civita number. In fact it is a "small" infinite number in the sense that, for every $m\in\mathbb{N}$,

$$\log\alpha \ll \alpha^{\frac{1}{m}}$$

where $\xi \ll \zeta$ means that ξ/ζ is an infinitesimal. So if we want to manipulate also these kind of numbers we need a more sophisticated strategy.

Definition 15. We say that the set of euclidean numbers

$$\{\beta_{-m}, .., \beta_{-1}, \beta_0, \beta_1, ..., \beta_n\}$$

is a transcendent basis if $\beta_0 = 1$, and

$$h < k \Rightarrow \forall r \in \mathbb{R},\ \alpha^{r\beta_h} \ll \alpha^{\beta_k}$$

The real vector space V generated by such a basis will be called the order group. An Euclidean number of the form

$$r\alpha^v,\ r \in \mathbb{R},\ v \in V$$

will be called V-monosemium. An Euclidean number of the form

$$\xi = \sum_{k=0}^{\ell} r_k \alpha^{v_k};\quad v_k > v_{k+1}$$

is called V-algorithmic number [2].

The V-algorithmic numbers generalize the notion of algorithmic numbers. It is necessary to say that they are not very suitable for numeric computations since they do not have a normal form defined by a polynomial such as (5). We introduced them just for completeness since they might be used in some particular problem. In any case they form a ring and they have an approximate inverse. It is possible to prove that there exists an infinite-dimensional real vector pace $V_{\mathbb{E}}$ such that any Euclidean number can be approximated by a transcendental AN with an order group $V \subset V_{\mathbb{E}}$.

Example. Take

$$V = \{\beta_0, \beta_1\} = \left\{1, \frac{\alpha}{\log \alpha}\right\}$$

The number $e^{2\alpha+3} + 5\alpha^2$ can be represented as follows:

$$e^{2\alpha+3} + 5\alpha^2 = e^3 e^{2\alpha} + 5\alpha^2 = e^3 \alpha^{\frac{2\alpha}{\log \alpha}} + 5\alpha^2 = e^3 \alpha^{2\beta_1} + 5\alpha^{2\beta_0}.$$

4. **Some applications.**

4.1. **Application to linear dynamic systems.**

Control Theory of Linear Dynamic Systems [24] is a good candidate to show the powerfulness of the proposed algorithmic numbers. In particular, we will consider the theory of linear systems, i.e., systems described by ordinary differential equations. Such systems can be fruitfully analyzed by means of the well-known Laplace transform [24].

4.1.1. *A second-order system with one of the two poles at infinity, approximated as a first-order one.*

Suppose to have the continuous, time invariant linear dynamic system described by the following ordinary differential equation:

$$\begin{cases} 3\dot{y} + 5y = 0 \\ \quad y(0) = 1 \end{cases}$$

The dynamical system above (along with its initial condition) is completely described by its Laplace transform [24]:

$$G(s) = \frac{1}{3s+5}.$$

[2] Actually, Levi-Civita, in his paper [22], considered also fields of this type.

Being the linear system of the differential equation of the first order, the system is said to be of the first order as well. A first-order system has only one pole (a pole is defined as one of the roots of the denominator of the $G(s)$ transform). In the considered case, the pole p is located in $-\frac{5}{3}$. Any system having all negative poles is known to be asymptotically stable. The linear system above can be excited (i.e., forced) using inputs such as the Dirac delta (see below), the Heaviside step function (see below) and so on. In particular, when the input is the Dirac delta, it is known that the output $y(t)$ of the linear dynamical system is the inverse Laplace transform of the $G(s)$. In this case, it is:

$$y(t) = e^{-\frac{5}{3}t}.$$

We can ask our self what happens if the true system had an infinitesimal component of the second order:

$$\begin{cases} \eta\ddot{y} + 3\dot{y} + 5y = 0 \\ y(0) = 1 \end{cases}$$

$$G(s) = \frac{1}{\eta s^2 + 3s + 5}$$

The question is: what is the impact of neglecting the second order component? In general, in system identification it is of paramount importance to correctly identify the order of a system, or, at least, to use a "valid" (i.e., acceptable) lower-order approximation.

As we have seen in 3.6, the two poles can be easily found using BAN numbers:

$$\begin{aligned} p_1 &\cong -\frac{5}{3} + \frac{25}{54}\eta \\ p_2 &\cong -\alpha\left(3 - \frac{5}{3}\eta + \frac{25}{27}\eta^2\right) \end{aligned}$$

Again, p_1 is an infinitesimal perturbation of the original pole of the first-order linear system; p_2 is a new pole that appears at $-\infty$.

This simple example shows that p_1 is the dominant pole and it is correct to neglect the contribution of the dominated pole as it is usually done.

4.1.2. *Marginally stable systems due to the presence of infinitesimal poles.* From the theory of linear systems, it is well-known how the following system:

$$\begin{cases} \dot{y} = 0 \\ y(0) = 1 \end{cases}$$

$$G(s) = \frac{1}{s}$$

is *marginally* stable [24], due to the presence of one pole on the origin. In fact, when the system is forced with the Dirac impulse (who's Laplace transform is equal to 1) the Laplace transform of the output is:

$$Y(s) = 1 \cdot G(s)$$

which has the anti-transform in time equal to:

$$y(t) = u(t)$$

where $u(t)$ is the Heaviside step function (a function which is zero for $t \leqslant 0$ and 1 for $t > 0$).

We might ask what happens if the pole is not exactly zero, but negative and infinitesimally close to zero. As an example, our system could be:

$$G(s) = \frac{1}{s+\eta}$$

The associated anti-transform is equal to

$$y(t) = e^{-\eta t}u(t)$$

The Taylor expansion of the exponential function in zero is:

$$e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \mathcal{O}\left(x^4\right) \tag{8}$$

Thus

$$y(t) = e^{-\eta t}u(t) \cong \left(1 - \eta t + \frac{1}{2}\eta^2 t^2\right) u(t).$$

It it clear that the system is still marginally stable, or Lyapunov stable, since the output is always bounded (in particular, $y(t)$ is always < 1, being $\eta > 0$ and $t \geqslant 0$).

In addition, we can also compute the value after an infinite amount of time, more precisely, for, instance, when $t = \alpha$:

$$y(t = \alpha) \cong \left(1 - 1 + \frac{1}{2}\right) = \frac{1}{2}.$$

However in this case the approximation above, computed using (8), is too coarse, since $x = \eta\alpha = 1$ is not infinitesimal. We can still obtain the exact value for $y(t = \alpha)$ by using the full expansion of the exponential function:

$$y(t = \alpha) = e^{-\eta\alpha} = e^{-1}.$$

We can now repeat the study in an even more interesting setting, i.e., when the pole is infinitesimal but *positive*. We know from the theory that a system with a pole on the positive half-plane is always unstable, but what happens when it is infinitesimal?

In this case:

$$G(s) = \frac{1}{s-\eta}$$

$$y(t) = e^{\eta t}u(t) \cong \left(1 + \eta t + \frac{1}{2}\eta^2 t^2\right) u(t).$$

Thus, after an infinite amount of time $t = \alpha$, the output is still finite:

$$y(t = \alpha) = e^{\eta\alpha} = e.$$

This is really interesting, because we have found that the system is marginally stable *even when a pole is located on the positive half-plane*, provided that it is *infinitesimal in magnitude*.

A further interesting observation can be deduced. We can compute an accurate approximated value of the output after an infinite amount of time τ, provided that τ is *a sufficiently small infinite*. For example, when τ is $\sqrt{\alpha}$, by using (8) we obtain:

$$y\left(t = \sqrt{\alpha}\right) \cong \left(1 - \eta\sqrt{\alpha} + \frac{1}{2}\eta^2\alpha\right) \cong 1.$$

Finally observe how the Dirac function, which is actually a distribution over $\mathbb{R}$, can be easily defined as an internal function in $\mathbb{E}$:

$$\delta(t) = \begin{cases} e^{\alpha} & \forall t \in \left[-\frac{e^{-\alpha}}{2}, \ \frac{e^{-\alpha}}{2}\right] \\ 0 & \text{otherwise} \end{cases}$$

Using this definition we have that for every continuous function ϕ:

$$\int \phi(x)\delta(x)dx = \phi(0) + \epsilon$$

Here, ϵ is an infinitesimal of order $-\infty$ and hence, in the world of algorithmic numbers, ϵ can be neglected and so we can write:

$$\int \phi(x)\delta(x)dx = \phi(0)$$

Before switching to the next application, observe how we have now all the tools needed to draw the *root locus* [24] for poles that are finite, infinite or infinitesimal.

4.2. **Application to linear programming.** Let us consider a linear programming problem [16, 17], i.e., a problem of the form:

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} \leqslant \mathbf{b}, \end{aligned} \tag{9}$$

where $\mathbf{c}$ is a column vectors $\in \mathbb{R}^n$, $\mathbf{x}$ is a column vector $\in \mathbb{R}^n$, $\mathbf{A}$ is a full-rank matrix $\in \mathbb{R}^{m\times n}$, $\mathbf{b}$ is a column vector $\in \mathbb{R}^m$.

Most LP solvers require that $\mathbf{A}$ is full rank to run. When $\mathbf{A}$ has not full rank, some pre-processing occurs to remove the linear dependent variables (columns), thus the dimension of the problem decreases. Let us see such an example:

$$\begin{aligned} \min \quad & x_2 \\ \text{s. t.} \quad & x_1 \leqslant 1, \\ & x_1 \geqslant -1 \end{aligned} \tag{10}$$

The given problem corresponds to the following one:

$$\mathbf{A} = \begin{bmatrix} +1 & 0 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $rank(\mathbf{A}) = 1$ (< 2).

However, if we perturb matrix $\mathbf{A}$ by infinitesimal normally distributed random noise we can make it again full rank:

$$\mathbf{A} = \begin{bmatrix} 1 + 2.78911\eta & -0.77306\eta \\ -1 + 0.72757\eta & 0.83663\eta \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In particular, it is now possible to compute the inverse of $\mathbf{A}$, which is:

$$\mathbf{A^{-1}} = \begin{bmatrix} 13.1608 & 12.1608 \\ 15.7307\alpha - 728.0511 & 15.7307\alpha - 672.7313 \end{bmatrix}$$

This means that now the generalization of the simplex algorithm described in [16] is able to solve the problem. In addition, having used infinitesimal random noise in place of finite random noise, we are assured that we have not changed the solution of the original problem, or, more precisely, its standard, purely finite, part.

To verify that the computed $\mathbf{A}^{-1}$ is the algorithmic inverse of $\mathbf{A}$, let us compute their product, to verify if it is (numerically close) to the identity matrix:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 - 1.1369\eta \end{bmatrix} \cong \mathbf{I}_2$$

It is worth nothing that the obtained matrix is infinitely close to the 2×2 identity matrix.

In the previous LP example, there is another way to make the matrix $\mathbf{A}$ full rank, i.e., by adding a bounding box constraint. However, in order to not change the problem, such box must be infinite in size. More precisely, we can add the constraint $-\alpha \leqslant x_i \leqslant +\alpha$, $\forall i = 1...n$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ \alpha \\ \alpha \\ \alpha \\ \alpha \end{bmatrix}$$

Matrix $\mathbf{A}$ now has rank 2, a mandatory pre-condition needed by most LP solvers. We can also draw the feasible region, from which it is clear that polyhedron is bounded (although at infinity). This helps even in finding a first feasible basic solution, since now we have four vertex for our feasible region, as shown by Figure 1.

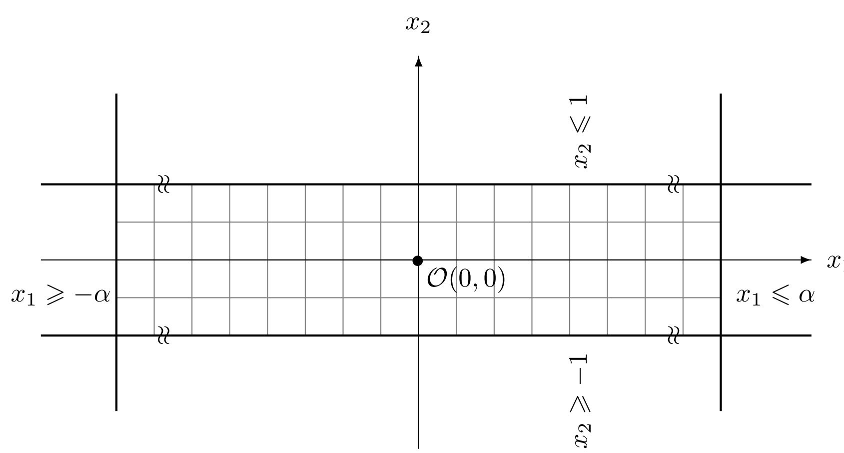

FIGURE 1. Feasible region (the vertical line on the left is positioned at $x_1 = -\alpha$, while the vertical constraint on the right is at $x_1 = \alpha$. The upper horizontal constraint at $x_2 = 1$ and the lower constraint at $x_2 = -1$.

Again, the generalized simplex described in [16] is now able to solve the problem, being the new $\mathbf{A}$ full rank.

REFERENCES

[1] A. Albeverio, J. E. Fenstad, R. Høegh-Krohn and T. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Dover Publications, Princeton, 2009.

[2] V. Benci, *I Numeri e gli Insiemi Etichettati*, Laterza, Bari, Italia, 1995. Conferenze del seminario di matematica dell' Università di Bari, vol. 261, pp. 29.

[3] V. Benci, An algebraic approach to nonstandard analysis, In G. Buttazzo, A. Marino, and M.K.V. Murthy, editors, *Calculus of Variations and Partial Differential Equations: Topics on Geometrical Evolution Problems and Degree Theory*, pages 285–326. Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[4] V. Benci, An algebraic approach to nonstandard analysis, In G. Buttazzo, editor, *Calculus of Variations and Partial differential equations*, volume 4 of *5*, chapter 8, pages 285–326. Springer, Berlin, 2000.

[5] V. Benci, Ultrafunctions and generalized solutions, *Adv. Nonlinear Studies*, **13** (2013), 461–486.

[6] V. Benci, *Alla Scoperta dei Numeri Infinitesimi, Lezioni di Analisi Matematica Esposte in un Campo Non-Archimedeo*, Aracne Editrice, Rome, 2018.

[7] V. Benci and M. Di Nasso, Numerosities of labelled sets: A new way of counting, *Adv. Math.*, **173** (2003), 50–67.

[8] V. Benci and M. Di Nasso, *How to Measure the Infinite: Mathematics with Infinite and Infinitesimal Numbers*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019.

[9] V. Benci, M. Di Nasso and M. Forti, An aristotelian notion of size, *Ann. Pure Appl. Logic*, **143** (2006), 43–53.

[10] V. Benci and M. Forti, The Euclidean numbers, `arXiv:1702.04163v2`, 2018.

[11] V. Benci, M. Forti and M. Di Nasso, The eightfold path to nonstandard analysis, In D. A. Ross N. J. Cutland, M. Di Nasso, editor, *Nonstandard Methods and Applications in Mathematics*, volume 25 of *Lecture Notes in Logic*, pages 3–44. Association for Symbolic Logic, AK Peters, Wellesley, MA, 2006.

[12] V. Benci and P. Freguglia, Alcune osservazioni sulla matematica non archimedea, *Matem. Cultura e Soc., RUMI*, **1** (2016), 105–122.

[13] V. Benci and L. Luperi Baglini, Ultrafunctions and applications, *Discrete Contin. Dyn. Syst. Ser. S*, **7** (2014), 593–616.

[14] M. Cococcioni, A. Cudazzo, M. Pappalardo and Y. D. Sergeyev, Solving the Lexicographic Multi-Objective Mixed-Integer Linear Programming Problem using Branch-and-Bound and Grossone Methodology, *Commun. Nonlinear Sci. Numer. Simul.*, **84** (2020), 105177, 20 pp.

[15] M. Cococcioni and L. Fiaschi, The Big-M method with the Numerical Infinite M, *Optimization Letters*, 2020

[16] M. Cococcioni, M. Pappalardo and Y. D. Sergeyev, Lexicographic Multi-Objective Linear Programming using Grossone Methodology: Theory and Algorithm, *Applied Mathematics and Computation*, **318** (2018), 298–311.

[17] G. B. Dantzig and M. N. Thapa, *Linear Programming 2: Theory and Extensions*, Springer-Verlag, New York, 2003.

[18] L. Fiaschi and M. Cococcioni, Numerical Asymptotic Results in Game Theory using Sergeyev's Infinity Computing, *International Journal of Unconventional Computing*, **14** (2018), 1–25.

[19] L. Fiaschi and M. Cococcioni, Non-Archimedean Game Theory: A Numerical Approach, *Applied Mathematics and Computation*, 2020, 125356.

[20] P. Fletcher, K. Hrbacek, V. Kanovei, M. G. Katz, C. Lobry and S. Sanders, Approaches to analysis following Robinson, Nelson and others, *Real Analysis Exchange*, **42** (2017), 193–251.

[21] L. Lai, L. Fiaschi and M. Cococcioni, Solving Mixed Pareto-Lexicographic Multi-Objective Optimization Problems: The Case of Priority Chains, *Swarm and Evolutionary Computation*, **55** (2020), 100687.

[22] T. Levi-Civita, Sugli infiniti ed infinitesimi attuali quali elementi analitici, *Atti del R. Istituto Veneto di Scienze Lettere ed Arti, Venezia*, Series 7, 1892.

[23] E. Nelson, *Radically Elementary Probability Theory*, Princeton University Press, Princeton, New Jersey, 1987.

[24] K. Ogata, *Modern Control Engineering*, Prentice Hall, New Jersey, 5 edition, 2010.

[25] Y. D. Sergeyev, Numerical infinities and infinitesimals: Methodology, applications, and repercussions on two Hilbert problems, *EMS Surveys in Mathematical Sciences*, **4** (2017), 219–320.

Received February 2020; revised June 2020.

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