Received April 25, 2020, accepted May 4, 2020, date of publication May 12, 2020, date of current version May 27, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2994052

Forming Sequences of Patterns With Luminous Robots

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This work was supported in part by the Progetto, Università di Pisa, Pisa, Italy, under Grant PRA_2018_43, and in part by The Natural Sciences and Engineering Research Council of Canada through the Discovery Grants Program.

ABSTRACT The extensive studies on computing by a team of identical mobile robots operating in the plane in *Look-Compute-Move* cycles have been carried out mainly in the traditional $\mathcal{OBLOT}$ model, where the robots are silent (have no communication capabilities) and oblivious (in a cycle, they have no memory previous cycles). To partially overcome the limits of obliviousness and silence while maintaining some of their advantages, the stronger model of *luminous robots*, $\mathcal{LUMI}$, has been introduced where the robots, otherwise oblivious and silent, carry a visible light that can take a number of different colors; a color can be seen by observing robots, and persists from a cycle to the next. In the study of the computational impact of lights, an immediate concern has been to understand and determine the additional computational strength of $\mathcal{LUMI}$ over $\mathcal{OBLOT}$. Within this line of investigation, we examine the problem of forming a sequence of geometric patterns, PatternSequenceFormation. A complete characterization of the sequences of patterns formable from a given starting configuration has been determined in the $\mathcal{OBLOT}$ model. In this paper, we study the formation of sequences of patterns in the $\mathcal{LUMI}$ model and provide a complete characterization. The characterization is constructive: our universal protocol forms all formable sequences, and it does so asynchronously and without rigidity. This characterization explicitly and clearly identifies the computational strength of $\mathcal{LUMI}$ over $\mathcal{OBLOT}$ with respect to the PatternSequenceFormation problem.

INDEX TERMS Autonomous mobile robots, distributed computing, oblivious, pattern formation, sequence of patterns, visible lights.

I. INTRODUCTION

The control and coordination of systems of autonomous mobile *robots* operating in continuous spaces has been the object of intensive investigations in a variety of fields, including robotics, control, AI, and distributed computing.

A. FRAMEWORK

Within distributed computing, the robots are seen as mobile computational entities operating in the Euclidean space in *Look-Compute-Move* cycles. During a cycle, a robot obtains a snapshot of the positions of the other robots, expressed in its own local coordinate system (*Look*); using the snapshot as an input, it executes an algorithm, the same for all robots, to determine a destination (*Compute*); and it moves towards the computed destination (*Move*); after a cycle, a robot may be inactive for some time. The main focus has been on determining the minimal capabilities the robots need to have to be able to solve a problem, and the extensive literature has almost exclusively focused on simple computational entities operating under strong adversarial conditions. The robots are *anonymous* (without ids or distinguishable features), *autonomous* (without central or external control), *homogeneous* (they execute the same program), *silent* (without explicit means of communication), and *oblivious* (no recollection of computations and observations done in previous cycles).

In this standard model of robots, typically called $\mathcal{OBLOT}$, depending on the activation schedule and timing assumptions, three main sub-models have been studied in the literature: the *asynchronous* model ($\mathcal{A}$sync), where no assumptions are made on synchronization among the robots' cycles

The associate editor coordinating the review of this manuscript and approving it for publication was Yilun Shang.

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nor their duration; and the *fully synchronous* ($\mathcal{F}$SYNC) and *semi-synchronous* ($\mathcal{S}$SYNC) models where the robots operate in synchronous atomic rounds, the only difference being whether all robots are activated in every round or, subject to some fairness condition, a possibly different subset is activated in each round. Extensive research has been carried out to determine, in all three sub-models, the computational boundaries as well as the impact that factors such as presence/absence of common coordinate system, movement rigidity (i.e., being able to always reach the destination), etc., have on computability and complexity (e.g., see [3], [6], [14]–[16], [18], [28], [29], [31], and the recent books and chapters therein [12], [13]). Although this research is highly theoretical, its influence is sometimes felt in more practical settings (e.g., see [5], [19], [20], [22], [27] in the recent literature).

Among all the restricting conditions, being oblivious and silent are the most limiting constraints for the robots, forcing them to rely only on movement and observation of the environment to coordinate and decide their next step. On the other hand, both obliviousness and silence are very desirable characteristics, as the former provides a form of self-stabilization, while the latter allows for applications in environments where communication is impossible or dangerous.

Recently, to partially overcome the limits of obliviousness and silence while maintaining some of their advantages, the stronger model of *luminous robots*, $\mathcal{LUMI}$, has been introduced where robots, otherwise oblivious and silent, carry a visible light that can take (a bounded number of) different colors, can be seen by observing robots, and persists from cycle to cycle [7], [8].

The availability of visible lights clearly changes the computational power of the robots, which have now the means for a bounded amount of persistent memory (by coding information into the colors), and limited communication (by using the change of colors to transmit a bounded amount of information to the other robots). The research is now ongoing to determine the impact that the availability of visible lights has on the computability power of the robots, and how lights can be used to overcome additional limitations (e.g., obstructed visibility). The first crucial result is the existence of an algorithmic transformation that allows any semi-synchronous protocol for luminous robots with k colors to be executed asynchronously with $O(k)$ colors [8]. In other words, when solving a problem for asynchronous luminous robots, it suffices to solve it for semi-synchronous ones; the price to pay is a (small) constant muliplicative factor in the number of colors. Furthermore, since robots in $\mathcal{OBLOT}$ are luminous robots with $k = 1$ color, this result implies that asynchronous luminous robots are at least as powerful as semi-synchronous traditional robots. Since their introduction, a large amount of work has been done on luminous robots ([1], [2], [4], [8], [10], [17], [21], [23]–[26], [30]; see [11] for a recent review). In this paper we continue the investigation on the computational impact of lights, and examine the problem of forming a sequence of geometric patterns.

B. FORMING A SEQUENCE OF PATTERNS

The research on the computational aspects of autonomous mobile robots has in large part focused on the fundamental class of *Geometric Pattern Formation* problems. A *geometric pattern* (or simply *pattern*) P is a set of points in the plane; the robots form the pattern P at time t if the configuration of the robots (i.e., the set of their positions) at time t is similar to P (i.e., coincident with P up to scaling, rotation, translation, and reflection); a pattern P is *formable* if there exists an algorithm that allows the robots to form P within finite time and no longer move.

Determining which patterns are formable under what conditions has been the subject of extensive studies in a variety of settings (e.g., see [16], [18], [28], [29], [31]). A crucial role is played by the notion of *symmetricity* of a set of points, which informally measures the amount of symmetries of the set. Indeed, for a pattern P to be formable, its symmetricity must divide the one of the starting configuration (i.e. initial placement of the robots in the plane). The obtained results indicate that, in most settings, the robots can form single arbitrary geometric patterns in spite of their silence and obliviousness. In other words, apart from the inevitable limitations due to symmetry, obliviousness and silence are not limiting factors to form a single pattern.

The problem of forming more than one pattern in a specified order is clearly much more difficult due to the obliviousness of the robots. In fact, to be able to correctly transition from a pattern to the next in the prescribed order, the robots must create in the environment the mechanism to "remember" and "communicate".

Let $\mathcal{S} = \langle S_0, S_1, \ldots, S_{m-1} \rangle$ be an ordered sequence of patterns. *Forming* sequence $\mathcal{S}$ requires the robots to form the infinite periodic sequence $\mathcal{S}^\infty = \langle S_0, S_1, \ldots, S_{m-1} \rangle^\infty$, called *coreography*; that is, after forming pattern S_i, the robots must form pattern $S_{(i+1)mod\ m}$, where the first formed pattern can be any pattern in the sequence [9]. The problem of determining which coreographies can be formed from a given initial configuration, the PATTERNSEQUENCEFORMATION problem, has been studied in the $\mathcal{OBLOT}$ model in [9], where a complete characterization was given for the set of the sequences formable from the initial configuration $\overline{\Gamma}$ by *semi-synchronous* robots with *rigid* movements (i.e., when moving, a robot always reaches its destination), assuming *chirality* (i.e., clockwise orientation). Specifically, the authors proved that a sequence $\mathcal{S}$ of patterns formable from $\overline{\Gamma}$ can be formed by rigid robots in $\mathcal{S}$SYNC only if (R1) the number of points in each pattern is the same and (R2) no pattern appears more than once in the sequence.

Since traditional robots without lights are the particular case of luminous robots with one color, the results proven in [8] imply that all sequences formable by robots in $\mathcal{S}$SYNC with rigid movements and chirality can be done by asynchronous luminous robots under the same conditions.

This naturally poses several questions on the sequence of patterns formable by asynchronous luminous robots vs those formable by traditional robots. Specifically, are asynchronous luminous robots more powerful than traditional robots in $\mathcal{S}$SYNC? If so, what other sequence of patterns can be formed? how? can they be formed without rigidity? In this paper, we provide a complete answer to these questions.

C. CONTRIBUTIONS

We provide a complete characterization of the sequences of patterns formable by (non-rigid) luminous robots with chirality, starting from an input configuration $\overline{\Gamma}$.

The characterization shows that, in contrast with the case without lights, the sequences $\mathcal{F}(\overline{\Gamma})$ of patterns that can be formed do not suffer from restrictions (R1) and (R2); in fact, formable sequences can contain repeated patterns (*repetitions*), and the number of points in the patterns can be different (*contractions*). The number of colors needed by the robots, and used by the algorithm, depends necessarily on the number of repetitions and contractions in the sequence of patterns; i.e., it is bounded but not by a constant. Furthermore, unlike the case without lights, the set $\mathcal{F}(\overline{\Gamma})$ of sequences formable by luminous robots is the same regardless of (a)synchrony and regardless of rigidity.

The characterization is constructive. We first identify the sequences that cannot be formed even if the robots are fully synchronous and rigid; we then provide a universal protocol that, for any given arbitrary initial configuration $\overline{\Gamma}$ and any given ordered sequence $\mathcal{S} = \langle S_0, \ldots, S_{m-1} \rangle$ of patterns formable from $\overline{\Gamma}$, allows the robots to form $\mathcal{S}$ starting from $\overline{\Gamma}$; furthermore, this is done asynchronously, without requiring rigidity, and always starting from S_0.

This characterization explicitly and clearly identifies the computational strength of $\mathcal{LUMI}$ over $\mathcal{OBLOT}$ with respect to the PATTERNSEQUENCEFORMATION problem.

The paper is organized as follows: in Section II, we introduce some notation and definitions; in Section III, we establish some basic limitations of the robots as well as a lower bound on the number of colors to be employed; in Section IV, we describe an Algorithm to solve the problem and analyze its correctness; in Section V, we establish bounds on the number of colors necessary to solve the problem; finally, in Section VI we draw some conclusions and discuss open research directions.

II. MODEL AND DEFINITIONS

A. THE MODEL

The system is composed of a team $\mathcal{R}$ of mobile entities, called *robots*, each modelled as a computational unit provided with its own local memory and capable of performing local computations.

The robots are placed in a spatial universe, here assumed to be $\mathbb{R}^2$, and they are viewed as points in $\mathbb{R}^2$. We assume that the robots always start from distinct points in the plane, but during the course of the algorithm multiple robots may occupy the same point in $\mathbb{R}^2$. Each robot has its own local coordinate system that might not be consistent with that of the other robots. A robot is endowed with sensorial capabilities and it observes the world by activating its sensors, which return a snapshot of the positions of the other robots in its local coordinate system.

The robots are *identical* and *anonymous*: they are indistinguishable by their appearance, execute the same protocol, and do not have distinct identities. The robots are *autonomous*, without a central control. The robots are *silent*, meaning that a robot cannot directly communicate with another robot (e.g., using wireless communication). The robots can however communicate information indirectly, e.g. using specific movements or using lights, as explained later.

Each robot is endowed with motorial capabilities, and can freely move in the plane.

At any point in time, a robot is either *active* or *inactive*. When *active*, a robot executes a *Look-Compute-Move* (LCM) cycle. In *Look*, a robot obtains an instantaneous snapshot of the positions of all robots (i.e., the set of their coordinates) expressed in its own coordinate system, which might be different in different cycles. In *Compute*, the robot executes its algorithm, using the snapshot as input; the result of the computation is a destination point. In *Move*, the robots moves toward the computed destination; if the destination is the current location, the robot stays still (performs a *null movement*). When *inactive*, a robot is idle. All robots are initially inactive. The amount of time to complete a cycle is assumed to be finite.

The algorithm to be executed, with all the functions it uses, as well as the target to be achieved (in our case, the sequence of patterns) are the only information persistently available to a robot. Otherwise, the robots are *oblivious*: they start each cycle without any information of past observations and computations. The local coordinate systems of the robots may not be consistent with each other, and might differ in different cycles; however, a robot always sees itself at the origin of the system and, as in [9], [18], it shares the same *chirality* (e.g., a clockwise orientation of the plane) with all the other robots.

In a move, a robot might always reach its destination, in which case the move is said to be *rigid*; alternatively, the move might stop before reaching its destination, e.g., because of limits to its motion energy. In the non-rigid case, the distance traveled in a move is neither infinite nor infinitesimally small. More precisely, there exists an (arbitrarily small) constant $\delta > 0$ such that, if the destination point is closer than δ, the robot will reach it; otherwise, it will move towards it a distance of at least δ; the quantity δ might not be known to the robots. In this paper, the proposed algorithms are able to operate in this more difficult general setting of non-rigid movements.

With respect to the activation schedule of the robots and their *Look-Compute-Move* cycles, the most common models are the fully-synchronous, the semi-synchronous, and the asynchronous. In the *fully-synchronous* ($\mathcal{F}$SYNC) model, the activations of all robots can be logically divided into global rounds; in each round, the robots are all activated, obtain the same snapshot, compute and perform their move. Note that this is computationally equivalent to a fully synchronized system in which all robots are activated simultaneously and all operations are instantaneous. The *semi-synchronous* ($\mathcal{S}$SYNC) model is like the fully-synchronous model where however not all robots are necessarily activated in each round. Based on the fairness of the activation scheduler, sub-models can be obviously defined; it is assumed that every robot is activated infinitely often. On the opposite side of the spectrum, in the *asynchronous* ($\mathcal{A}$SYNC) model, the robots do not have a common notion of time; they are activated independently from each other, and the duration of each *Compute*, *Move* and inactivity is finite but unpredictable. As a consequence, robots perceived in a snapshot might be performing any operation and, in particular, they might be moving; moreover, computations might be made based on obsolete observations. In this paper, the proposed algorithm is able to operate in this more difficult but more general model.

A *luminous* robot r is a robot that, in addition to its capabilities, is endowed with a persistent and externally visible state variable $Light[r]$, called *light*, whose values are from a finite set C of states called *colors* (including the color that represents the state when the light is *off*). Note that if $|C| = 1$, then the light is always off; thus, this case corresponds to the traditional setting of robots without lights. In this paper, we assume $|C| > 1$ for (truly) luminous robots. The value of $Light[r]$ (i.e., its color) can be changed in each cycle by r at the end of its *Compute* operation. A light is *externally visible* in the sense that its color at time t is visible to all robots that perform a *Look* operation at that time; we assume *weak chromatic multiplicity detection*: the robots can distinguish the colors of robots co-located at a point, but not their number.[1] A light is *persistent* from one computational cycle to the next; while robot r might be oblivious, forgetting all other information from previous cycles, the color is not automatically reset at the end of a cycle.

B. CONFIGURATIONS, SYMMETRIES, AND PATTERNS

Let V be a set of points in the Euclidean plane, $size(V)$ be the cardinality of V, and $SEC(V)$ be the smallest circle enclosing all the points (refer to Figure 2 for a summary of the used notation). The set V can be decomposed into a set of concentric circles centred in the centre of $SEC(V)$, each containing at least a point of V; let $\sigma(V)$ be their number, and let us denote them as $Cir_1^V, \ldots, Cir_{\sigma(V)}^V$, with Cir_1^V being the smallest one; in the following, when clear from the context, the superscript will be omitted. As well known, there exists a $q \geq 1$ divisor of $size(V)$, such that every Cir_i can be decomposed into a set of q-gons; if there is a point at the centre of $SEC(V)$, this point is, by definition, a (degenerated) 1-gon. The largest q for which this holds is called *symmetricity* of V and denoted by $q(V)$ (refer also to examples in Figure 1). The set of points in each $q(V)$-gon is called a *symmetricity class*, or simply *class*; we denote by $\alpha(V)$ the number of classes in V. We extend

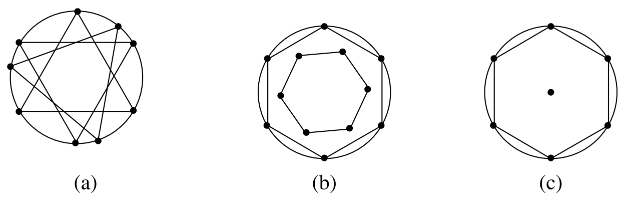


FIGURE 1. (a) A pattern with symmetricity 3. (b) A pattern with symmetricity 6. (c) A pattern with symmetricity 1.

Symbol	Description
V	Set of points
$SEC(V)$	Smallest Enclosing Circle of V
$\sigma(V)$	Number of concentric circles of V
$q(V)$	Symmetricity of V
$\alpha(V)$	Number of symmetricity classes of V
λ	Coloring function
$q_c(V, \lambda)$	Chromatic symmetricity of V w.r.t λ
$\Gamma(t)$	Configuration of robots at time t
$\overline{\Gamma} = \Gamma(t_0)$	Initial configuration
$L(\Gamma(t))$	Set of distinct robots' positions in $\Gamma(t)$
$\mathcal{S}$	Sequence of patterns
$\mathcal{S}^\infty$	Coreography
$\mu(P),\, P \in \mathcal{S}$	Number of occurrences of P in $\mathcal{S}$

FIGURE 2. Summary of notation.

in a natural way the definition of symmetricity also to multisets; in a multiset $\hat{V}$, a point of multiplicity k at the centre of $SEC(\hat{V})$ is, by definition, a (degenerated) k-gon.

For two k-gons A and B placed on two concentric circles C_A and C_B, we denote by $\theta(C_A, C_B)$ the clockwise rotational angle between A and B.

Given a set of distinct *colors* $C = \{c_1, \ldots, c_k\}$ and a set of points V, we define a *coloring* of V with C any function $\lambda : V \to C$. We say that λ is *proper* when $\lambda(x) = \lambda(y)$ iff x and y belong to the same class of symmetricity in V. We denote by $q_c(V, \lambda)$ the *chromatic symmetricity* of V when colored by λ; in the following, when λ is understood, we use just $q_c(V)$. We extend in a natural way the definition of chromatic symmetricity also to multi-sets.

In the following, we will describe the global positions of the robots using a fixed coordinate system (not available to the robots); let $r_i(t)$ denote the position of r_i at time t in this system.

The *configuration* of the robots on the plane at time t is the multi-set $\Gamma(t) = \{(r(t), \lambda(r(t))) : r \in \mathcal{R}\}$ where $\lambda(r(t))$ is the color of robot r at time t; the set of distinct points occupied by the robots in $\Gamma(t)$ is denoted by $L(\Gamma(t))$. A configuration Γ is said to be *central* if there is a point at the center of $SEC(\Gamma)$, not-central otherwise; we denote by Γ^- the configuration where the robots at the center of $SEC(\Gamma)$, if any, are removed. In the *initial configuration* $\overline{\Gamma} = \Gamma(t_0)$, we assume that all robots occupy distinct positions and that all robots have the same color, called OFF.

[1]Note that this requirement corresponds to *absence* of multiplicity detection if the robots are not luminous.

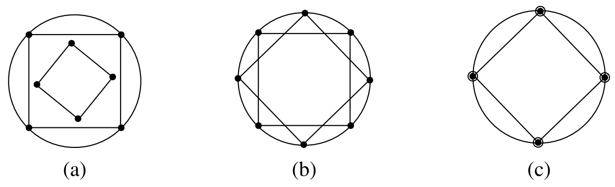


FIGURE 3. **(a) A pattern consisting of two classes. (b) and (c) show the two possible kind of contractions: in (b), the two classes are contracted into just one, where all robots occupy distinct positions; in (c) the two classes are contracted into just one, where points of multiplicity two are created (the circled dots).**

A *pattern* P is a set of distinct points. A pattern P is said to be *central* if there is a point at the center of $SEC(P)$, non-central otherwise. A pattern P_i is said to be *isomorphic* to a pattern P_j, denoted $P_i \equiv P_j$, if P_j can be obtained by a combination of translation, rotation and scaling of pattern P_i. Two patterns that are not isomorphic to each other are said to be *distinct*. We will denote the size of a pattern P by $size(P)$, and denote by P^- the pattern where the point at the center of $SEC(P)$, if any, is removed.

We say that the robots have *formed* the pattern P at time t if $L(\Gamma(t)) \equiv P$.

Let $\mathcal{S} = \langle S_0, \ldots, S_{m-1}\rangle$ be an ordered *sequence* of patterns with $S_i \not\equiv S_{i+1}$. The maximum number of points in the patterns is called *size* of the choreography and it is denoted by $size(\mathcal{S}) = max_i\{size(S_i)\}$. The maximum number of symmetricity classes in $\mathcal{S}$ is defined as $\alpha(\mathcal{S}) = \max_i\{\alpha(S_i)\}$. Without any loss of generality, let $\alpha(S_0) = \alpha(\mathcal{S})$. Given a pattern $P \in \mathcal{S}$, we denote by $\mu(P)$ the number of occurrences of P in $\mathcal{S}$, i.e. $\forall P \in \mathcal{S}$, $\mu(P) = |\{S_i \in \mathcal{S} : S_i \equiv P\}|$. Let $\mu(\mathcal{S}) = \max_i\{\mu(S_i)\}$. We say that the sequence $\mathcal{S}$ has *repetitions*, if $\mu(\mathcal{S}) > 1$; and *contractions*, if there is P in $\mathcal{S}$ such that $\alpha(P) < \alpha(\mathcal{S})$ (refer to Figure 3). Also, if $\alpha(S_i) < \alpha(S_{i+1})$, we will say that S_i *contracts*.

A set of luminous robots executing an algorithm $\mathcal{A}$ starting from a configuration $\Gamma(t_0)$ is said to *form* $\mathcal{S}$ if in every possible execution of $\mathcal{A}$ from configuration $\Gamma(t_0)$, there exist increasing time instances $t'_0, \ldots t'_{m-1}$, with $t_0 \leq t'_0$ such that $L(\Gamma(t'_j)) \equiv S_j$.

Given the sequence $\mathcal{S} = \langle S_0, \ldots, S_{m-1}\rangle$, we call *choreography* the infinite periodic sequence $\mathcal{S}^\infty = \langle S_0, S_2, \ldots, S_{m-1}\rangle^\infty$. We say that a set of robots executing an algorithm $\mathcal{A}$, starting from configuration $\Gamma(t_0)$ *performs* the choreography $\mathcal{S}^\infty = \langle S_0, S_2, \ldots, S_{m-1}\rangle^\infty$ if they repeatedly form $\mathcal{S}$.

In the following, for simplicity, with an abuse of notation, we will refer to a choreography $\mathcal{S}^\infty$ simply as $\mathcal{S}$.

III. CHARACTERIZATION AND LIMITATIONS

A. BASIC LIMITATIONS

We first establish an obvious basic limitation of pattern-formation by luminous robots.

Lemma 1: Consider a set of n luminous robots in configuration Γ with chromatic symmetricity $q_c(\Gamma)$. The robots cannot form a configuration Γ' where $q_c(\Gamma')/q_c(\Gamma)$ is not an integer, even in $\mathcal{F}$SYNC, with rigid movements, and regardless of the number of available colors.

Proof: By contradiction, let $\mathcal{A}$ be an algorithm that, starting from configuration Γ, correctly allows the robots to reach configuration Γ', where $q_c(\Gamma')/q_c(\Gamma)$ is not an integer, and it does so in all possible executions, regardless of the local coordinate systems of the robots. For the $q_c(\Gamma)$ robots of a class not at the center of $SEC(L(\Gamma))$ chirality induces a natural cyclical order; for the $q_c(\Gamma)$ robots of a class at the center of $SEC(L(\Gamma))$, consider an arbitrary cyclic order. Consider now a fully synchronous execution of $\mathcal{A}$ where the coordinate system of each robot is a rotation by $2\pi/q_c(\Gamma)$ of that of the robot before it in the cyclic order of its class. In every step of such an execution, all the robots in the same chromatic class of Γ will obtain the same chromatic snapshot, perform the same computation, choose the same new color, and perform a move which will not change their symmetry; hence, they will remain in the same chromatic class. It is possible that two (or more) distinct classes choose the same color and their movements position them into a symmetric configuration; however, this will cause the size of the chromatic classes to either stay unchanged or to increase by an integer factor. This means that, in each step t of the execution, we have $q_c(\Gamma(t-1)) \geq q_c(\Gamma(t))$ and $q_c(\Gamma(t-1))/q_c(\Gamma(t))$ is an integer. This will hold also when the robots finally reach Γ' at step t'. That is, $\Gamma(t') = \Gamma'$ but $q_c(\Gamma)/q_c(\Gamma(t'))$ is an integer; a contradiction. ■

The basic limit to performing a choreography immediately follows.

Theorem 2: A set of n luminous robots starting from initial configuration $\overline{\Gamma}$ where all robots are in distinct positions and have the same color, *cannot* perform a choreography $\mathcal{S}$, if any of the following holds:

1) $size(S_i) > n$, for some $S_i \in \mathcal{S}$.
2) $q(S_j^-)$ is not a multiple of $q(\overline{\Gamma})$, for some $S_j \in \mathcal{S}$.
3) $\mathcal{S}$ contains both central and non-central patterns, $q(\overline{\Gamma}) > 1$, and $\alpha(\overline{\Gamma}) = \alpha(\mathcal{S})$.

This holds even in $\mathcal{F}$SYNC, with rigid movements, and regardless of the number of available colors.

Proof: We consider each condition separately.

(1) This condition trivially holds since $size(\overline{\Gamma}) = n$ robots cannot occupy more than n distinct positions at any one time.

(2) This condition follows from Lemma 1 observing that, since in the initial configuration $\overline{\Gamma}$ all robots have the same color, $q_c(\overline{\Gamma}) = q(\overline{\Gamma})$.

(3) Let $\mathcal{S}$ contains both central and non-central patterns, $q(\overline{\Gamma}) > 1$, and $\alpha(\overline{\Gamma}) = \alpha(\mathcal{S})$. By contradiction, let $\mathcal{S}$ be formable; that is, there is an algorithm $\mathcal{A}$ that starting from $\overline{\Gamma}$ always correctly performs choreography $\mathcal{S}$ in all possible executions.

By condition (2) of this theorem, for all $S_i \in \mathcal{S}$, $q(S_i^-)$ must be a multiple of $q(\overline{\Gamma})$; thus $q(\mathcal{S}) = max_i\ \{q(S_i^-)\} \geq q(\overline{\Gamma})$. Since $\alpha(\overline{\Gamma}) = \alpha(\mathcal{S})$ by assumption, it follows that $size(\mathcal{S}) = \alpha(\mathcal{S})q(\mathcal{S}) \geq \alpha(\overline{\Gamma})q(\overline{\Gamma}) = size(\overline{\Gamma})$. Since, by condition (1) of this theorem, to be formable $\mathcal{S}$ must have $size(\overline{\Gamma}) \geq size(\mathcal{S})$, it follows that $size(\overline{\Gamma}) = size(\mathcal{S})$; that is, there exists a $S_{max} \in \mathcal{S}$ such that $size(S_{max}) = size(\overline{\Gamma})$.

Consider now a fully synchronous execution of $\mathcal{A}$ where initially the coordinate system of each robot is a rotation by $2\pi/q_c(\Gamma)$ of that of the robot before it in the cyclic order of its class. By assumption, $\mathcal{S}$ contains central and non-central patterns. Let S_k be the first central pattern formed in this execution (i.e., all S_i with $0 \leq i < k$ are non-central), and let Γ_k be the configuration when the robots form S_k for the first time. Since in $\overline{\Gamma}$ the robots are in distinct positions and $q(\overline{\Gamma}) > 1$, to form Γ_k at least one robot must move to the center. In this execution, when a robot moves to the center, all the robots in the same class will necessarily do the same ("class collapse"). It follows that $size(S_k) < size(\overline{\Gamma})$. Continue now the synchronous execution of $\mathcal{A}$ with all robots belonging to the same (chromatic) class located in the center of Γ_k having the same coordinate system; these robots will, from this moment on, move as one robot, always taking the same decision and moving to the same point. In other words, the number of distinct points occupied by robots in any subsequent configuration is less than $size(\overline{\Gamma})$. Recall that there exists a pattern $S_{max} \in \mathcal{S}$ with $size(S_{max}) = size(\overline{\Gamma})$ distinct points; this means that the pattern S_{max} cannot ever be formed again, contradicting the correctness of $\mathcal{A}$. ■

A choreography is *feasible* from initial configuration $\overline{\Gamma}$ if none of the three forbidden conditions stated in the previous theorem holds. Let $\mathcal{F}(\overline{\Gamma})$ denotes the class of all feasible choreographies starting from $\overline{\Gamma}$.

Notice that, as a consequence of Theorem 2, unless the initial configuration is asymmetric or the number of robots is greater than the number of points in the largest pattern, feasible choreographies can only be composed of exclusively non-central patters, or of exclusively central patterns: in the former case, we will say that $\mathcal{S}$ is non-central, and in the latter that $\mathcal{S}$ is central.

For simplicity, in the following, we will use $size(\Gamma(t))$, $q(\Gamma(t))$, and $\alpha(\Gamma(t))$ to denote $size(L(\Gamma(t)))$, $q(L(\Gamma(t)))$, and $\alpha(L(\Gamma(t)))$, respectively, and when no ambiguity arises we will omit the variable t.

IV. THE ALGORITHM

In this section we present an algorithm that, starting from an arbitrary initial configuration $\overline{\Gamma}$, allows the robots to perform any choreography in $\mathcal{F}(\overline{\Gamma})$ even if the execution is asynchronous and the movements are not rigid. In the following, for simplicity, we assume $\alpha(\overline{\Gamma}) = \alpha(\mathcal{S})$; the case $\alpha(\overline{\Gamma}) > \alpha(\mathcal{S})$ is easily handled, as discussed in Section VI.

We will first consider the case when the initial configuration is symmetric (i.e., $q(\overline{\Gamma}) > 1$); for ease of presentation, we will introduce in Sections from IV-A to IV-H a solution that assumes $\mathcal{S}$ non-central; in Section IV-I we will describe how to extend such a solution to handle also the case when $\mathcal{S}$ is central.

The case of asymmetric initial configuration (i.e., $q(\overline{\Gamma}) = 1$), can be solved with a simpler protocol, and will be described in Section IV-J.

A. SYMMETRIC INITIAL CONFIGURATION

Let the initial configuration $\overline{\Gamma}$ be symmetric and $\mathcal{S}$ be non-central; that is, $q(\overline{\Gamma}) > 1$, and all patterns $S_i \in \mathcal{S}$ are non-central.

We start with an informal overview of the algorithm for this case (Algorithm 1).

Algorithm 1 PatternSequence()

```
SetUp(); %Algorithm 2%
PatternIdentification(); %Algorithm 3%

If Γ is ready-to-separate or in-separation Then
    Separation(); %Algorithm 5%
If Γ is separated or in-rotation Then
    Rotation(); %Algorithm 7%
If Γ is rotated or in-formation Then
    Formation(); %Algorithm 8%
```

Whenever the robots have to form the next (initially, the first) pattern in $\mathcal{S}$, from a global point of view, the algorithm makes the robots perform two main phases: a pre-processing phase and an actual formation phase. The pre-processing phase is articulated into a 2-step process:

- *SetUp.* A class is elected as *leader*.
- *Pattern Identification.* The robots identify the pattern S_i of the sequence that has to be formed.

Initially all the robots are colored `OFF`. From this initial configuration, as a preliminary step, a class of robots is elected as *leader* in SetUp(): robots in this class assume a special color that will uniquely identify them from this moment on. At this point the robots start the actual formation process.

The first action undertaken by the robots is to ensure that the next pattern to be formed, say S_i, is recognizable during the entire formation process. This is done in PatternIdentification() by positioning the leader robots on a special circle, centered in the *SEC* of the current configuration, and having a special radius that uniquely identifies S_i; hence, once all the leader robots reach that circle, their position enables the other robots to identify S_i as the next pattern to be formed. The algorithm ensures that, while the leaders move towards that circle, they can understand from the position and colors of the other robots, the radius they have to reach.

Next, the algorithm starts the actual formation phase of S_i, that is articulated into a 3-step process:

- *Separation.* Each class of robots places itself on a distinct concentric circle centred in the center of *SEC*.
- *Rotation.* Each class rotates so that the relative angles of the pattern to be formed are respected.
- *Formation.* The robots move to their final destination, thus forming S_i.

In particular, the algorithm now determines which of these three steps the current configuration of the robots Γ is in. This check is carried out by testing conditions *ready-to-separate*, *in-separation*, *separated*, *in-rotation*, *rotated*, and *in-formation* that will be detailed in Sections IV-E, IV-G, and IV-H: these tests are carried out by considering the positions and the colors of the robots. We will prove that, at each time, at most one of these conditions can be verified.

To start the actual formation of S_i, each class of robots is first placed on a distinct concentric circle, thus *separating* the different classes, and then it is mapped onto a "target" class of S_i (SEPARATION()). The robots of every class then start *rotating* (one class at the time and never leaving their own circle) in order to be at the correct relative rotational angle with respect to their own target class (ROTATION()).

Once all classes, including the one containing the leader robots, are placed at the correct rotational angle, the robots start moving, one class at the time, towards their target positions, *forming* S_i (FORMATION()). As soon as the robots of a class reach their final destination, they assume a proper color, assigned by a coloring function defined in Section IV-C; the last class to be moved in this process is the leader class. Once the process is completed, it starts again to form the next pattern $S_{(i+1) mod\ m}$.

The difficulties obviously derive from the fact that, in this endless process, each robot performs all the required operations in complete asynchrony, without communication or memory (other than the colors), and its movements can be interrupted (they are not rigid).

We now proceed with the detailed description and discussion of the PATTERNSEQUENCE algorithm and of its components. The algorithm is expressed from a robot's point of view, and it is executed, from the start, by a robot every time it performs the *Compute* operation.

B. SET UP AND LEADERS IDENTIFICATION

Since the robots agree on chirality, it follows that, in the initial configuration $\overline{\Gamma}$, the robots can compute and agree on a total ordering of the symmetricity classes of $\overline{\Gamma}$. For instance, one such ordering can be obtained as follows: classes are ordered from the ones lying on the largest concentric circle (i.e., the *SEC*) to the ones on the smallest concentric circle of Γ; if on a concentric circle there is more than one class, these are ordered according to the angles between them in the clockwise direction.

As a consequence, one class can always be elected as leader class. Let IDENTIFYLEADERS(Γ) denote the procedure that determines a leader class among those on the *SEC*. The set of robots in this class is denoted by R_l, and they will assume the special color GOLD (see Algorithm 2). Note that, due to asynchrony, some robots in R_l might become GOLD while others are not active yet; still, the identification of R_l is not affected as long as the robots do not move. Therefore:

Lemma 3: Within finite time from the start of the algorithm, the robots agree on the leaders, and the leaders are colored GOLD.

Algorithm 2 SETUP()

Requires: Γ is the current configuration.

```
If All robots' colors are GOLD and/or OFF Then
    $R_l$ := IDENTIFYLEADERS($\Gamma$);
    $q = |R_l|$;
    If All robots in $R_l$ are GOLD Then
        Return
    Else
        If I am in $R_l$ and I am not GOLD Then
            Become GOLD.
Else
    $q$ = Number of robots having color in {GOLD, GOLD-IN,
    GOLD-OUT };
    Return
```

Throughout the lifetime of the system, the leaders can transition only through color GOLD-OUT or GOLD-IN before becoming GOLD again. Thus, once the initial set-up has been completed, the set R_l of leader is always uniquely identifiable by those colors.

C. COLORS

Before describing and analyzing the core steps of the algorithm, we describe the mechanism to decide the colors used by the robots. As already mentioned, the GOLD color is used exclusively by the leader class.

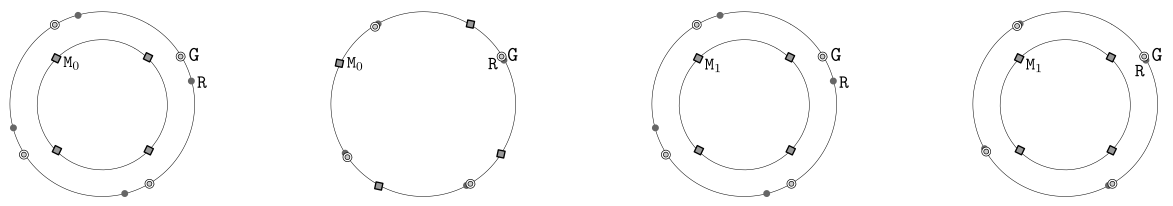
Our algorithm, as will be evident later, always preserves during its execution the symmetricity classes $\mathcal{Q}$ of the initial configuration $\overline{\Gamma}$. For each pattern S_i, it maps the symmetricity classes $\mathcal{Q}$ to those $\mathcal{Q}_i$ of S_i (specifying in this way which class of S_i should be formed by which classes of robots). Let $\tau_i : \mathcal{Q} \rightarrow \mathcal{Q}_i$ denote such a mapping; notice that this mapping is not necessarily injective, as $\alpha(S_i)$ is possibly smaller than $\alpha(\overline{\Gamma})$. To take into account repetitions and contractions in the sequence, as well as non-rigid movements and asynchrony of the robots, colors are assigned to classes carefully during the whole process.

The algorithm uses a set $\Lambda(\mathcal{Q}, \mathcal{S})$ of coloring functions $\lambda_i : \mathcal{Q} \rightarrow C$, $0 \leq i < \alpha(\overline{\Gamma})$, where λ_i specifies which color the robots of a class should assume when forming S_i. Thus, since $\alpha(S_i)$ is possibly smaller than $\alpha(\overline{\Gamma})$, λ_i indirectly (i.e., in conjunction with τ_i) specifies which set of colors will be associated to each symmetricity class of S_i once the robots form S_i. Let $\hat{S_i}$ denote the pattern S_i colored according to λ_i, and let $q_c(S_i)$ denote its chromatic symmetricity.

The set $\Lambda(\mathcal{Q}, \mathcal{S})$, called *overall coloring*, is chosen so to have the following properties:

Coloring Properties:

1) If r' and r'' belong to the same symmetricity class in $\overline{\Gamma}$, then $\forall i$, $\lambda_i(r') = \lambda_i(r'')$. That is, robots that initially belong to the same symmetricity class, will never be assigned different colors.
2) The number of *chromatic* symmetricity classes of each colored pattern is the same and equal to the number



(a) The robots formed S_0: in this coloring function, the only class that changes color is the *mutant* one (squared robots). Here, they get color MUTANT$_0$.
(b) The robots formed S_1, where two chromatic classes are contracted.
(c) The robots formed S_2, which is a repetition of S_0: the *mutant* class gets color MUTANT$_1$.
(d) The robots formed S_3, where two chromatic classes are contracted.

FIGURE 4. An example of coloring function (Section IV-C) with one *mutant* class (the squared robots).

of classes of the initial configuration; i.e., $\forall i$, $q_c(S_i) = \alpha(\overline{\Gamma})$.

3) For $i \neq j$, $\hat{S}_i \not\triangleq \hat{S}_j$.
4) $\forall i, j$, $\lambda_i^{-1}(\texttt{GOLD}) = \lambda_j^{-1}(\texttt{GOLD})$. That is, a GOLD robot is always assigned that color in every pattern of the sequence.
5) $\forall i$, the class colored GOLD according to λ_i is on the *SEC* of S_i.
6) For $i \neq j$, $\hat{S}_i\neg\texttt{GOLD} \neq \hat{S}_j\neg\texttt{GOLD}$, where $\hat{S}_*\neg\texttt{GOLD}$ denotes the colored pattern $\hat{S}_*$ from which the color GOLD has been removed; if the class colored GOLD had no other colors in $\hat{S}_*$, then $\hat{S}_*\neg\texttt{GOLD}$ denotes the colored pattern $\hat{S}_*$ from which that class has been removed.

Any overall coloring with such properties, if deterministically identifiable from $\mathcal{Q}$ and S, can be employed by our algorithm. Note that there are many such colorings. Consider, for example, the overall coloring $\Lambda^*(\mathcal{Q}, \mathcal{S})$ defined by the following process: (i) Assign a distinct color to each class of $\mathcal{Q}$, with GOLD assigned to the leader class, and MUTANT$_0$ to an arbitrarily chosen class, called *mutant* class; (ii) every class maintains its originally assigned color in all patterns, except for the mutant class which assumes a new color MUTANT$_i$ at each repetition of the same pattern (see the example depicted in Figure 4). This overall coloring clearly satisfies the Coloring Properties, and is deterministically identifiable from $\mathcal{Q}$ and S.

In the following, we denote by $\Lambda = \Lambda(\mathcal{Q}, \mathcal{S})$ the global coloring employed by the algorithm. As already mentioned, the leader class determined in the SETUP step is assigned color GOLD by Λ. We will denote by GETCOLORS(S_i) the routine that returns the coloring function λ_i for S_i according to Λ; finally, we will say that a robot is λ_i-colored, or simply *colored*, when colored according to λ_i.

In addition to the colors employed by Λ, the algorithm uses some special service colors: WHITE-S, WHITE-R, WHITE-F, BRONZE, SILVER, PLATINUM, GOLD-IN, GOLD-OUT, BLACK, as well as OFF which is the initial color of all robots.

D. PATTERN IDENTIFICATION

This step of the algorithm is used to identify the pattern of the choreography that the robots should form next or are currently forming (refer to Algorithm 3). To aid in this process, two tools are used by the algorithm.

Algorithm 3 PATTERNIDENTIFICATION()

```
Requires: The current configuration.

$R_l$ := set of robots colored GOLD, GOLD-OUT, and GOLD-IN;
If Color of all robots but those in $R_l$ is OFF Then
    $P := S_0$;
    SIGNAL(P);
Else
    If $\exists \lambda_i \in \Lambda | \mathcal{R} \setminus R_l$ are colored according to $\lambda_i \setminus$ {GOLD, GOLD-OUT, GOLD-IN } Then
        $P := S_{(i+1) \bmod m}$;
        SIGNAL(P);
    Else
        /* A pattern is already under formation */
        $rad$ := Radius of $SEC(\mathcal{R})$;
        $c$ := Center of $SEC(\mathcal{R})$;
        $rad'$ := Radius of smallest non-degenerated populated circle centered in $c$;
        $P := S_i, 0 \leq i \leq m-1$, such that $f(S_i) \cdot rad' = rad$;
Return
```

The first tool is an injective function $f : \mathcal{S} \mapsto \mathbb{R}$ that maps each element $S_i \in \mathcal{S}$ into a real number $f(S_i)$. The only requirement on $f(S_i)$ is the following: let $\mathcal{C}$ be the smallest populated circle in Γ, let $S_i(\mathcal{C})$ be the pattern S_i scaled so that its smallest populated circle has the same radius as $\mathcal{C}$, let $SEC_i(\mathcal{C})$ be the SEC of $S_i(\mathcal{C})$, and let $SEC(\mathcal{C})$ be the largest of all $SEC_i(\mathcal{C})$; then, each $f(S_i)$ must be greater that the radius of $SEC(\mathcal{C})$.

This mapping is used to inform the non-leader robots of the pattern of the sequence, say S_i, they are currently forming or should form. This is done by having the leader robots positioning themselves so that the smallest non-degenerated

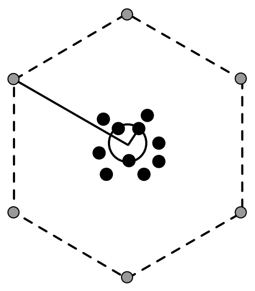


FIGURE 5. The configuration RATIO(x), where the gray dots represent the leader robots. The small circle is $\mathcal{C}$ (refer to Section IV-D).

populated circle $\mathcal{C}$ in Γ has a radius that is $1/f(S_i)$ times the distance between the leader robots and the center of $\mathcal{C}$ (refer to the example depicted in Figure 5).

Let Ratio($f(S_i)$) denote the set of configurations with this property. Since function $f(\cdot)$ is injective, once a configuration in Ratio($f(S_i)$) has been created, all robots can uniquely agree on the pattern S_i that is being currently formed. In particular, observe that, in a configuration of type Ratio(x), any robot can compute the exact value of x as a ratio of two distances measured in its own coordinate system. Thus, even if the robots do not agree on the unit distance, they can agree on the value of this ratio x. This means that, once the leaders have positioned themselves appropriately, all robots understand which pattern should be formed or is already being formed, regardless of their movements provided $\mathcal{C}$ is not changed. Our protocol ensures that $\mathcal{C}$ will never be changed. This general idea is based on one used in [9], where, however, the movements were rigid and the model $\mathcal{S}$Sync, while in this paper the movements are non-rigid and the model $\mathcal{A}$Sync.

To position themselves in such a Ratio configuration, the leaders must know where to go; i.e., they need to know which pattern should be done next. Furthermore, they should continue to have this information, despite their obliviousness, if they are stopped before reaching their destination, due to non-rigidity. In the algorithm, the leaders obtain this information by just observing the set of the non-leader robots; this is achieved through the second tool of the algorithm: the coloring function Λ. In fact, one of the Coloring Properties of Λ is that for $i \neq j$, $\hat{S}_i \neg\texttt{GOLD} \neq \hat{S}_j \neg\texttt{GOLD}$; that is, each pattern S_i, once colored according to Λ, is different from all other patterns colored according to Λ *even if* the leaders are removed from the patterns. In other words, just by looking at the non-leader robots, the leaders know what pattern has just been formed and hence where they should go so to communicate the identity of the next pattern. Note that, at the very beginning, after the SetUp, all robots are OFF while the leaders are GOLD. Also in this case, just by looking at the non-leader robots, the leaders know the pattern to be formed, S_0, and hence their destination.

Using these two tools, the PatternIdentification() step (described in Algorithm 3) determines whether (1) a new pattern has to be formed or (2) a pattern is already under formation; in both cases, it identifies the pattern, say $P \in \mathcal{S}$.

If P is a new pattern to be formed, using procedure Signal(P) reported in Algorithm 4, the leaders are first colored GOLD; then they start moving along their radiuses, to reach a configuration in Ratio($f(P)$)). Note that, during these movements, the leaders temporarily change color to GOLD-OUT, so that this transition phase can be clearly identified by all robots. Within finite time, a configuration in Ratio($f(P)$) is formed, signalling to the other robots that pattern $P \in \mathcal{S}$ has to be formed, and the leaders change color back to GOLD. Summarizing:

Algorithm 4 Signal(P)

```
$x :=$ Compute $f(P)$;
$c :=$ Center of $SEC(\mathcal{R} \setminus R_l)$;
$rad :=$ Radius of smallest non-degenerated populated circle centered in $c$;
$\overrightarrow{w} :=$ half-line from $c$ through my current position;
$p :=$ Point on $\overrightarrow{w}$, at distance $rad \cdot x$ from $c$;
If I am GOLD and I am not at $p$ Then
    Become GOLD-OUT;
    Move towards $p$.
Else
    If I am GOLD-OUT Then
        If I am not at $p$ Then
            Move towards $p$.
        Else
            Become GOLD.
If All $R_l$ are GOLD and at distance $rad \cdot x$ from $c$ Then
    Return
```

Lemma 4: Starting either from the initial configuration or from any S_i colored according to λ_i, in finite time the leaders reach, avoiding any collision, a configuration where they are at distance $f(S_{(i+1) mod\ m})$.

Proof: First of all, note that once a robot turns its color from OFF to some other color, it will never become OFF again. Hence, by Lemma 3, within finite time the leader class R_l is identified, and those robots are colored GOLD.

Let us assume now that, at time t, the robots are either in the initial configuration (i.e., all robots, but the leaders, are colored OFF); or they form pattern $S_i \in \mathcal{S}$, with all robots colored according to λ_i. As soon as a robot becomes active after time t, we distinguish the two possible cases:

1) At time t all robots are OFF (i.e., we are in the initial configuration, and the robots are about to form S_0): hence, Signal(P) is executed, which places the robots in a configuration in Ratio($f(S_0)$). In particular, the only robots that move are the leaders: they first compute $f(S_0)$; then, they change their color to GOLD-OUT. Once all leaders are colored GOLD-OUT, they all move radially at distance $rad \cdot f(S_0)$ from c, where c is the center of $SEC(\mathcal{R} \setminus R_l)$ and rad is the radius of the smallest non-degenerated populated circle centered in c. Once they reach such a point, by construction, Ratio($f(S_0)$) is achieved and the lemma follows.

2) At time t the robots form pattern $S_i \in \mathcal{S}$, with all robots colored according to λ_i: that is, the robots just completed the formation of S_i, and the next pattern to be formed is $S_{(i+1) mod\ m}$. As soon as a robot becomes active after time t, it executes SIGNAL(P).
Following the argument of the previous case, the leaders compute $f(S_{(i+1) mod\ m})$, change color to GOLD-OUT, and they all move radially at distance $rad \cdot f(S_0)$ from c. During these moves, no other robot is allowed to move; hence, within finite time all leaders will reach p. At this time, by construction, RATIO($f(S_{(i+1) mod\ m})$) is achieved and the lemma follows.

Finally, we note that both Algorithm 2 and 3 are collision free: in fact, the only robots that are allowed to move are the leaders, and they move outwards radially starting from the *SEC* of the current configuration. ■

After the leaders reach distance $f(\cdot)$, the non-leader robots will start forming the next pattern in the sequence within the circle $\mathcal{C}$, by performing the SEPARATE, ROTATE, and FORM steps of the algorithm. Finally, if PATTERNIDENTIFICATION() instead determines that pattern P is already under formation, the robots will continue with their execution of the algorithm.

E. SEPARATION

After PATTERNIDENTIFICATION(), the next step of the algorithm is SEPARATION() (Algorithm 5). The purpose of this step is to create a configuration where each of the $\alpha(\overline{\Gamma})$ classes of the initial configuration is on a distinct concentric circle. This step uses the control color SILVER and the service color WHITE-S.

Before describing it and discussing its behaviour and properties, some useful definitions of properties of a configuration during the execution of this step are in order.

Definition 5 (Ready-to-Separate): We say that a configuration Γ is *ready-to-separate* under coloring $\lambda_i \in \Lambda$ if the following two conditions hold:

1) the robots on *SEC* are GOLD and $\Gamma \in$RATIO($f(S_i)$);
2) the robots not on *SEC* are either all OFF or they form $\hat{S}_{i-1}\neg$GOLD.

Note that the condition that all robots not on *SEC* are OFF is satisfied only after the first execution of PATTERNIDENTIFICATION() (Algorithm 3).

Definition 6 (In-Separation): We say that a configuration Γ is *in-separation* under coloring $\lambda_i \in \Lambda$ if the following conditions hold:

1) the robots on *SEC* are GOLD and $\Gamma \in$ RATIO($f(S_i)$);
2) among the robots not on *SEC* :
   a) there is at least a SILVER robot; or
   b) there is at least a WHITE-S robot and $\sigma(\Gamma) \neq \alpha(\overline{\Gamma})$; or
   c) not all robots are WHITE-S and $\sigma(\Gamma) = \alpha(\overline{\Gamma})$.

Definition 7 (Separated): We say that a configuration Γ is *separated* under coloring $\lambda_i \in \Lambda$ if the following two conditions hold:

1) the robots on *SEC* are GOLD and $\Gamma \in$ RATIO($f(S_i)$);
2) the robots not on *SEC* are all WHITE-S, and $\sigma(\Gamma) = \alpha(\overline{\Gamma})$.

The SEPARATION(Γ) step transforms a ready-to-separate configuration Γ into a separated configuration Γ' where each of the $\alpha(\overline{\Gamma})$ classes of the initial configuration is on a distinct concentric circle. Let $Cir_1, \ldots, Cir_{\sigma(\Gamma)}$ denote the populated concentric circles of the ready-to-separate Γ, with Cir_1 being the smallest one. Informally, starting from $i = 1$, if Cir_i contains multiple (chromatic) classes, we move one class at a time, in an ordered fashion, to a slightly smaller concentric circle, until all classes originally on Cir_i have been separated on different circles. When a class is alone on a circle, it is then colored WHITE-S; the process is then applied to Cir_{i+1}.

In more detail, the separation is performed by iterating the strategy described below.

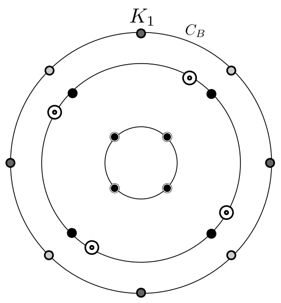
Since at the beginning of the separation process the configuration is not separated yet, there exists at least one circle that contains multiple classes: let C_B be the largest one. Since the robots agree on a common chirality, the classes on C_B can be totally ordered: let $K_1, \ldots, K_k$ be the distinct classes on C_B. Now, all the robots in K_1 become SILVER, and no robot can move as long as there is a non SILVER robot in K_1. In particular, this process is carried out as follows: let q be symmetricity of the initial configuration (i.e., the number of GOLD robots); if a robot $r \in K_1$ observes at least two classes on C_B and it sees that the number of SILVER robots on C_B is not q, then r colors itself SILVER. After a finite amount of time, all robots in K_1 become SILVER. Once this occurs, C_B contains an integral number k of classes, with $k \geq 2$, and exactly q robots on C_B are SILVER (those in K_1).

Let C_A be the largest populated circle, smaller than C_B, that does not contain any SILVER robot, and let *dis* be the distance between C_A and C_B. Now the algorithm forces all robots in K_1 to move inwards on the circle at distance $dis/2$ from C_B, call it C_T. Note that, during their movements, due to asynchrony and non-rigidity, these SILVER robots will create circles between C_B and C_A that contain only SILVER robots. Routine MOVETOCIR() (Algorithm 6) makes sure that all SILVER robots reach C_T. Once all SILVER robots reach C_T, C_T is the only circle that is populated with just SILVER robots: these robots can now uncolor themselves.

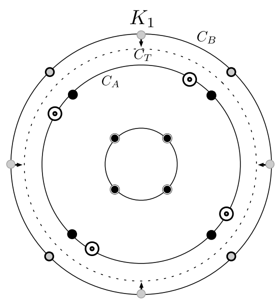
Now C_B contains one less class. These operations are repeated until C_B contains only one class; when this happens this class uncolors itself. The entire process is applied to the new largest circle that contains multiple classes, until the configuration is fully separated.

Lemma 8: Starting from a ready-to-separate configuration Γ under coloring λ, within finite time the robots reach a separated configuration Γ' under coloring λ, such that $\alpha(\Gamma') = \sigma(\Gamma') = \alpha(\overline{\Gamma})$.

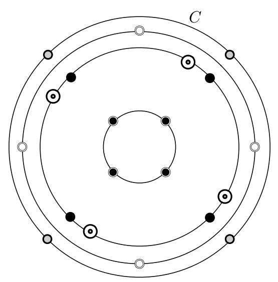
Proof: Since Γ is ready-to-separate, then, by definition, no robot is SILVER or WHITE-S. Any robot observing Γ in its *Look* operation, will correctly identify the largest circle C other than *SEC* (which contains only the GOLD robots);



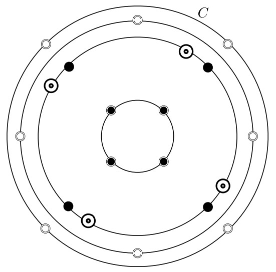
(a) A ready-to-separate configuration. Here, $q = 4$. The largest circle with colored robots, C_B, has more than one class: the robots in K_1 are the darker ones.



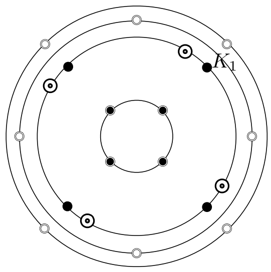
(b) Robots in K_1 become SILVER and move towards C_T, the circle half-way between C_A and C_B.



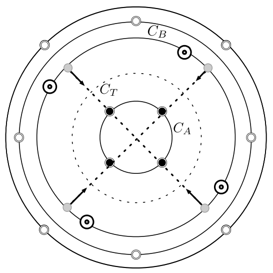
(c) All SILVER robots are on C_T: they now become WHITE-S. On C, the largest circle with colored robots, there is now only one class.



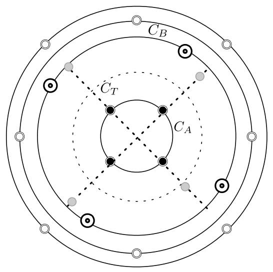
(d) All robots on C become WHITE-S.



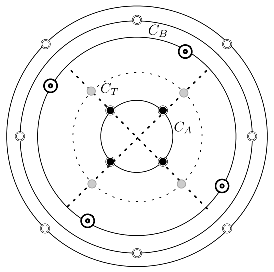
(e) The largest circle with colored robots has more than one class: let K_1 be the class colored black.



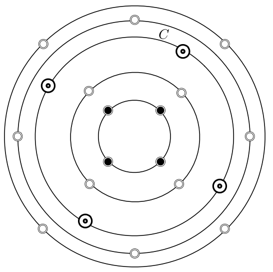
(f) The robots in K_1 become SILVER. They will move towards C_T.



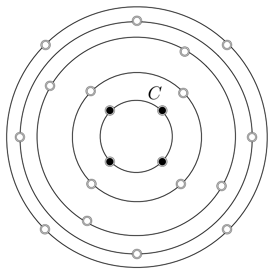
(g) The SILVER robots move towards C_T.



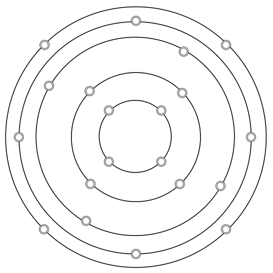
(h) All SILVER robots are on C_T.



(i) All SILVER robots become WHITE-S. Now, on C there is only one class that becomes WHITE-S.



(j) Again, on C there is only one class.



(k) All robots on C_B become WHITE-S, and the separation process is over.

FIGURE 6. An example for the separation step of SEPARATION(Γ) in Algorithm 5. The *SEC*, where all the GOLD robots are located, is not shown in the figure.

hence, within finite time, one or more of the robots on C will independently and asynchronously become WHITE-S or SILVER (depending on whether on C there is one or more than one class, respectively). As soon as the first of these robots changes color, the configuration becomes in-separation and appears as such to any robot performing its

Algorithm 5 SEPARATION(Γ)

```
Requires: The current configuration $\Gamma$.

If $\Gamma$ is ready-to-separate Then
    $C$ = largest populated circle other than the $SEC$;
    If $C$ contains only one class and I am in $C$ Then
        Become WHITE-S.
    Else
        $K_1, \dots, K_k$ = The classes on $C$;
        If I am in $K_1$ Then
            Become SILVER.
If $\Gamma$ is in-separation Then
    If I am SILVER Then
        $C$ = The largest circle that contains SILVER robots;
        If $C$ contains colored robots Then
            $C_B = C$;
        Else
            $C_B$ = Smallest populated circle, larger than $C$, where
            there are colored robots;
        $C_A$ = The largest populated circle, smaller than $C_B$, that
        does not contain SILVER robots; if no such a circle
        exists then $C_A$ = center of $SEC$;
        If $C$ contains $q$ robots and $C$ is halfway between $C_B$ and
        $C_A$ Then
            Become WHITE-S.
        Else
            If there are $q$ SILVER robots Then %I should move
            to destination if not there already%
                MOVETOCIR($C_A$, $C_B$).
    Else
        $C$ = largest populated circle where there are colored
        robots;
        If I am in $C$ and I am colored Then
            If $C$ contains only one class Then
                Become WHITE-S.
            Else
                If There are no SILVER or all SILVER robots are
                on $C$ Then
                    $K_1, \dots, K_k$ = The distinct classes on $C$;
                    If There are $< q$ SILVER robots and I am in
                    $K_1$ Then
                        Become SILVER.
```

Look operation. Notice that no robot can move during this time.

From this moment on, the algorithm proceeds as follows. Let C_B be the largest circle with no GOLD robots on it, containing colored robots; let $K_1, \dots, K_k$ be the colored classes of robots on C_B, and let t_0 be the first time a robot executes the SEPARATION(Γ) process.

1) If $k = 1$, according to the Algorithm, within finite time these robots will first become SILVER, and then WHITE-S, and no robot can move during this time.

Algorithm 6 MOVETOCIR(C_A, C_B)

```
$dis$ := Distance between $C_A$ and $C_B$;
$C_T$ = Circle of radius $dis/2$
If I am not on $C_T$ Then
    Move inwards radially to $C_T$.
```

2) If $k > 1$, let C_A be the largest circle smaller than C_B with colored robots on it, and C_T be the circle halfway between C_B and C_A.
   a) Within finite time, say at time $t_1 > t_0$, all robots in K_1 become SILVER, and no robot moves between t_0 and t_1.
   b) Within finite time, each SILVER robot will leave C_B, moving radially toward C_T (MOVETOCIR(C_A, C_B)); during this time no other robot is allowed to move, hence C_B, C_A and C_T stay invariant. Hence, within finite time, say at time $t_2 > t_1$, all SILVER robots will be on C_T. Note that at this time the number of chromatic classes is still $\alpha(\overline{\Gamma})$.
   c) Once all SILVER robots are on C_T (hence, there are exactly q robots on it), each SILVER robot becomes WHITE-S and, within finite time, say at time $t_3 > t_2$, all robots on C_T become WHITE-S.

   Thus, at time t_3, the number of classes on C_B decreased by one. By iterating the above process, within finite time, say at $t_4 > t_3$, between C_B and C_A there will be $k - 1$ circles each containing exactly one class, each colored WHITE-S (note that at time t_4 also C_B contains one class colored WHITE-S).

Finally, by iterating the above argument, the claim follows. ■

F. TARGETS

Before continuing the description and analysis of the algorithm, some discussion is now required on the mechanism used to decide where each robot should go to form the next pattern S_i.

As shown in the previous section, procedure SEPARATION() disassembles the previous configuration (or initially $\overline{\Gamma}$) so that each symmetricity class $Q \in \mathcal{Q}$ of $\overline{\Gamma}$ is in a distinct concentric circle. Once this is done, the algorithm will start the assembly of the next pattern.

In this process the algorithm uses the mapping $\tau_i : \mathcal{Q} \to \mathcal{Q}_i$, where $\mathcal{Q}_i$ denotes the set of symmetricity classes of S_i. It assigns to each class $Q \in \mathcal{Q}$ a unique symmetricity class $\tau_i(Q) \in \mathcal{Q}_i$ of S_i, called *target class*, following the global ordering of the symmetricity classes of S_i (this ordering is uniquely identified because of the common chirality). In other words, the mapping τ_i, called *targeting* function, specifies that the goal of the robots of Q should be to move to form $\tau_i(Q)$. Notice that this mapping, while surjective, is not necessarily injective since $\alpha(S_i)$ is possibly smaller than $\alpha(\overline{\Gamma})$.

Among the many possible targeting functions usable by the algorithm, we are going to employ the one described in the following. Let $\mathcal{Q} = Q_1, \dots, Q_a$ be the $a = \alpha(\Gamma)$ classes,

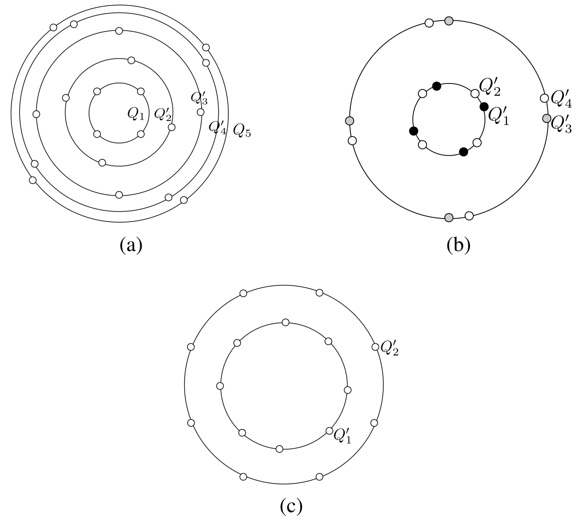


FIGURE 7. Computation of target classes. (a) The current configuration Γ, with $a = 5$. (b) The next pattern S_i to be formed, with $a' = 4$ (colors are used only to show to which class they belong). Here, $\phi_i = 4/4 = 1$; $k = \lceil j/1 \rceil$, for $1 \le j \le 4 * 1 = 4$. Hence, $\tau_i(Q_1) = 1$, $\tau_i(Q_2) = 2$, $\tau_i(Q_3) = 3$, $\tau_i(Q_4) = 4$. Here, the only *extra* class is Q_5, whose target is Q_4'. (c) A different next pattern S_i to be formed, where: $a' = 2$; $\phi_i = 8/4 = 2$; $k = \lceil j/2 \rceil$, for $1 \le j \le 2 * 2 = 4$. Hence, $\tau_i(Q_1) = 1$, $\tau_i(Q_2) = 1$, $\tau_i(Q_3) = 2$, $\tau_i(Q_4) = 2$. Also, here the only *extra* class is Q_5, whose target is Q_2'.

each on a distinct concentric circle, after SEPARATION(), where Q_j is in a smaller circle than Q_{j+1} $(1 \le j < a)$. Let $Q' = Q'_1, \ldots, Q'_{a'}$ be the (totally ordered) $a' = \alpha(S_i) \le a$ classes of S_i, where Q'_k is in the same circle as or in a smaller circle that Q'_{k+1} $(1 \le k \le a')$. Let $\phi_i = \frac{q(S_i)}{q(\Gamma)}$. Then $\tau_i(Q_j) = Q'_k$ where $k = \min\{a', \lceil \frac{j}{\phi_i} \rceil\}$.

In other words, ϕ_i classes in Γ are needed to form a single class in S_i. Hence, there are $e = a - a' \cdot \phi_i$ classes which are not necessary to form S_i; these *extra* classes will be all positions on $Q'_{a'}$ (refer to Figure 7).

G. ROTATION

After SEPARATION(), the next step of the algorithm is ROTATION(), reported in Algortihm 7: this step begins when the current configuration Γ is separated under some coloring $\lambda_i \in \Lambda$. By construction, after SEPARATION() the robots on each of the concentric circles Cir_i on Γ, with $1 \le i \le \sigma(\Gamma) = \alpha(\Gamma)$, belong to the same symmetricity class, and on each Cir_i there is exactly one symmetricity class. In the following, we will use the concepts of targets and targeting function introduced and defined in the previous section (Section IV-F).

The aim of this phase is to rotate each symmetricity class in Γ so that, if each Q_j, $1 \le j \le a$, is projected onto $\tau(Q_j)$, then S_i would be formed. More specifically, let us first introduce the following definition (refer to Figure 8):

Definition 9 (Properly Rotated): A *properly rotated* robot is defined as follows:

1) All robots on Q_1 are properly rotated.
2) Robots on Q_j, with $j \le a' \cdot \phi_i$ and $(j \bmod \phi_i) \neq 1$, are properly rotated if they form an angle of $\theta' = 360/q(S_i)$ degrees with respect to Q_{j-1}.
3) Given Q_j, with $j \le a' \cdot \phi_i$ and $(j \bmod \phi_i) = 1$, let $Q'_l = \tau(Q_j)$, and θ'' be the clockwise angle between Q'_1 and Q'_l. Robots in Q_j are *properly rotated* are all robots in $Q_r, r < j$, are properly rotated, and Q_j forms a clockwise angle of θ'' with respect to Q_1.
4) Given Q_j, with $j > a' \cdot \phi_i$, robots in Q_j are *properly rotated* if all robots in Q_r, $r < j$, are properly rotated, and Q_j form an angle of zero degrees with respect to $Q_{a' \cdot \phi_i}$.

We will refer to the angle θ used in 2) and 3) as the *proper rotation angle*. We will say that Q_j is properly rotated if all robots on Q_j are properly rotated.

This phase uses service colors `BRONZE` and `WHITE-R` to rotate all robots, one class at the time according to their total order so that, within finite time, all of them are properly rotated. In particular, by definition, all robots on Q_1 are properly rotated, so they just color themselves `WHITE-R`. Then, robots on Q_2 color themselves `BRONZE`, compute their proper rotational angle, and rotate so to become properly rotated; once their rotation process is over, they all color themselves `WHITE-R`. And so on, one concentric circle at a time, until also the robots on *SEC* (the `GOLD` robots) rotate, completing this phase.

Note that the existence of a `BRONZE` robot in some Cir_l implies that Cir_l is under the process of rotation. When the rotation is completed all robots will become either `WHITE-R` (the non leaders) or `GOLD` (the leaders). This process is denoted by ROTATION(Γ), and reported in Algorithm 7, where routine `ProperlyRotated()` and `ComputeProperRotationAngle()` are defined according to Definition 9.

In the following, we will also make use of the following definitions:

Definition 10 (In-Rotation): We say that a configuration Γ is *in-rotation* under coloring function $\lambda_i \in \Lambda$ if the following conditions hold:

1) the robots on *SEC* are `GOLD`, and $\Gamma \in$ RATIO$(f(S_i))$;
2) the robots are on $\sigma(\Gamma) = \alpha(\overline{\Gamma})$ concentric circles each containing q robots, the robots in Q_1 are all `WHITE-R`, and there are robots not `WHITE-R` (beside the leaders).

Definition 11 (Rotated): We say that a configuration Γ with coloring function λ is *rotated* if the following conditions hold:

1) the robots on *SEC* are `GOLD`, and $\Gamma \in$ RATIO$(f(S))$, for some $S_i \in \mathcal{S}$;
2) the robots not on *SEC* are all `WHITE-R`, and $q_c(\Gamma, \lambda) = \sigma(\Gamma)$.
3) all robots are properly rotated.

Lemma 12: Starting from a separated configuration Γ with coloring function λ, within finite time the robots reach a rotated configuration Γ' with coloring function λ' such that $q_c(\Gamma, \lambda) = q_c(\Gamma', \lambda')$, $\sigma(\Gamma) = \sigma(\Gamma')$.

Proof: By hypothesis, in Γ there are either `WHITE-S` or `GOLD` robots. By construction, all robots on Q_1 are properly rotated (these robots are on Cir_1^{Γ}); hence, by Algorithm 7,

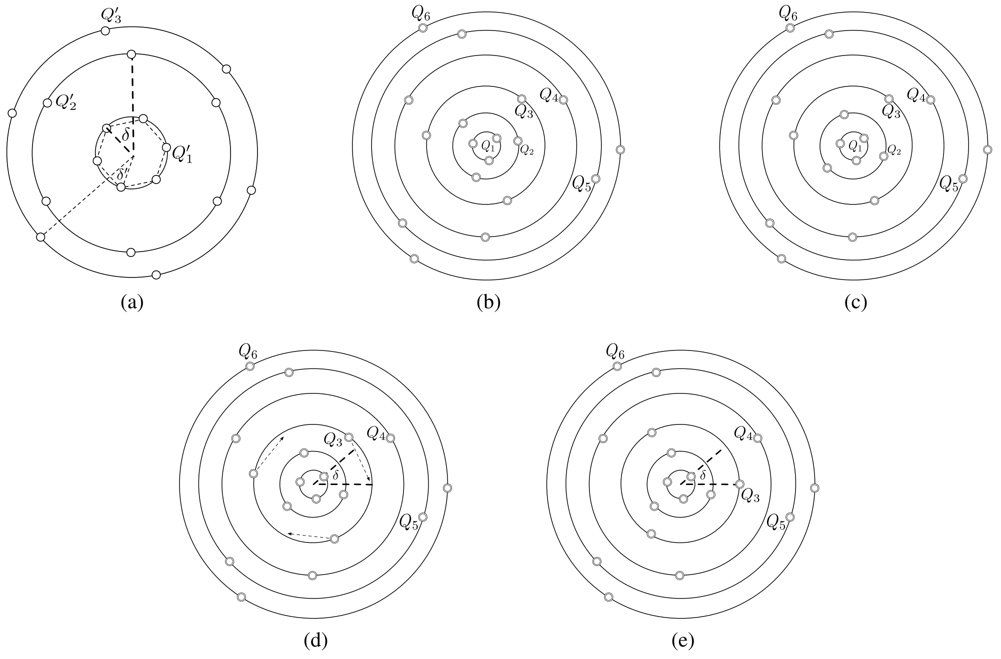


FIGURE 8. Example of properly rotated robots. (a) The next pattern S_i to be formed. Here, $q(S_i) = 6$. (b) The current separated configuration Γ, with $q(\Gamma) = 3$. Hence, $\phi_i = 2$, and $Q'_1 = \tau(Q_1)$, $Q'_1 = \tau(Q_2)$, $Q'_2 = \tau(Q_3)$, $Q'_2 = \tau(Q_4)$, $Q'_3 = \tau(Q_5)$, $Q'_3 = \tau(Q_6)$, and there are no extra robots. Q_1 is already properly rotated. Robots in Q_2 have to rotate of an angle of 360/6 degrees in order to be properly rotated. (c) Q_2 after rotation. (d) Now, let δ be the clockwise angle between Q'_1 and Q'_2 (see (a)): robots on Q_3, to become properly rotated, have to form a clockwise angle of δ with respect to Q_1. (e) Q_3 is now properly rotated; Q_4 has to rotate of δ with respect to Q_2 to be properly rotated. Same argument applies to Q_5 and Q_6, that have to rotate of an angle of δ' (see (a)) with respect to Q_1 and Q_2, respectively.

within finite time, say at time t, they all become WHITE-R, and no other robot is allowed to move until this time.

Now, let C_i, $1 < i \leq \sigma(\Gamma)$, be the smallest concentric circle that it is not properly rotated. Also, let $t_0 \geq t$ be the first time any robot executes the ROTATION(Γ) process.

1) Within finite time, say at time $t_1 > t_0$, all robots on C_i will become BRONZE, and no robot moves between time t_0 and t_1. Thus, at t_1, on C_i there are q BRONZE robots.
2) Now, the robots on C_i compute the proper rotation angle θ, according to Definition 9, and they rotate on C_i of an angle θ. Within finite time, say at $t_2 > t_1$, all robots on C_i complete their rotation.
3) Finally, within finite time, say at $t_3 > t_2$, all robots on C_i change their color: they become WHITE-R.

Finally, by Definition 9 and by induction on the number of concentric circles in Γ, the lemma follows. ■

H. FORMATION

In this last phase, the robots start from a separated and rotated configuration Γ; its outcome is to bring the robots in their correct final positions, so that the next pattern of the sequence, say S_i, is formed.

Let us first compute the positions of the *target circles* where the robots have to move to form S_i, call them $T_1, \ldots, T_{\sigma(S_i)}$: these are obtained by mapping the smallest concentric circle of S_i (i.e., the circle where the first class $Q'_1 = \tau_i(Q_1)$ resides), onto the smallest concentric circle of Γ (i.e., the one containing Q_1), and by scaling consequently all other concentric circles of S_i. Also, let k_l be the number of classes that have to end up on T_l in order to correctly form S_i.

Definition 13 (In-Formation): We say that a configuration Γ with coloring function λ is *in-formation* if the following conditions hold:

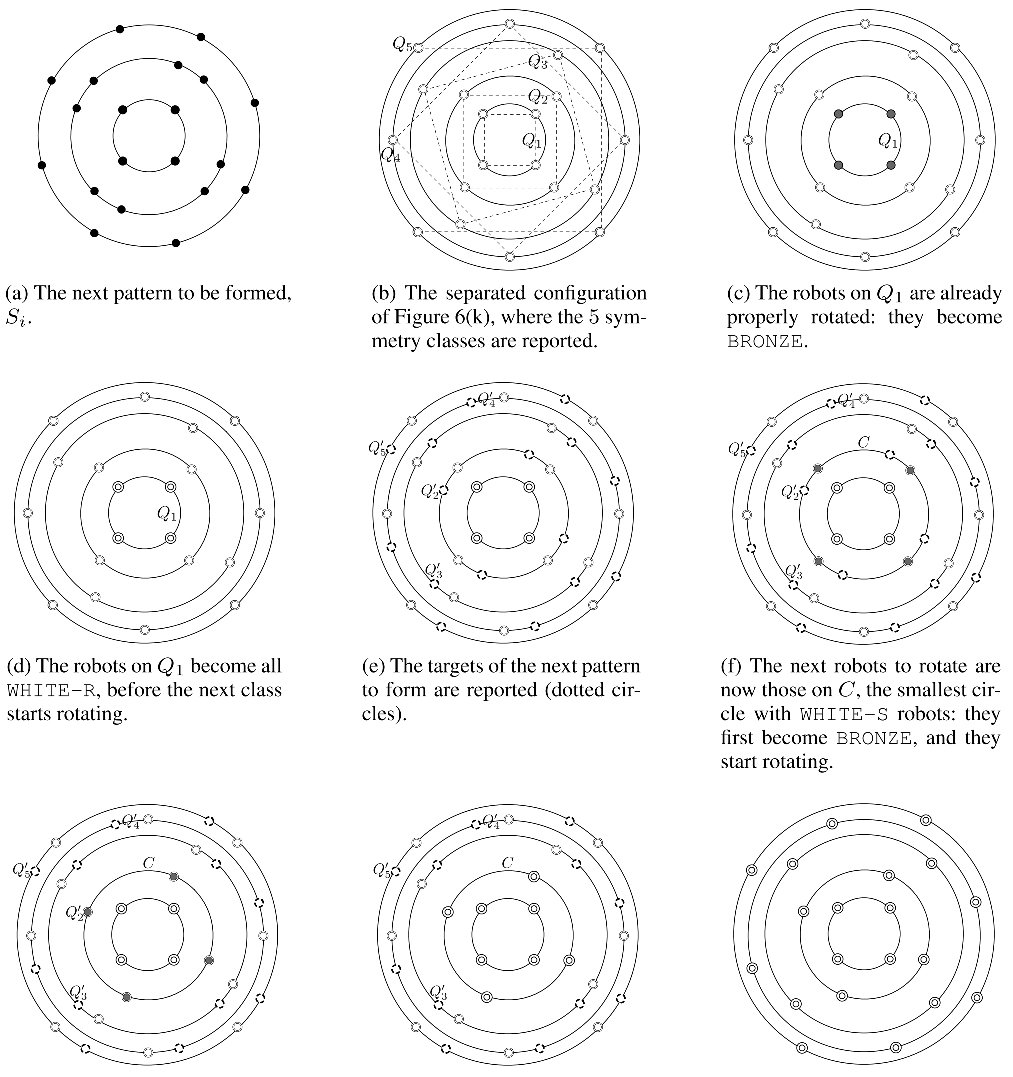
1) the robots on *SEC* are GOLD, and $\Gamma \in$ RATIO($f(S_i)$), for some $S_i \in \mathcal{S}$;
2) the robots not on *SEC* are either PLATINUM or WHITE-R or λ-colored; if a robot is λ-colored, it is on one of the target circles.

Definition 14 (Formed): We say that a configuration Γ with coloring function λ is *formed* if $\Gamma = S_i$, for some $S_i \in \mathcal{S}$, and $\lambda = \lambda_i$.

We say that a target T_l is *complete* with respect to S_i at time t if the positions of the robots on T_l form $Cir_l^{S_i}$, and they are correctly colored according to λ_i. By convention, we will say that T_0 is *complete*.

By construction, the robots on the smallest concentric circle in Γ, Cir_1, are already on T_1, i.e. they are on the correct positions to form S_i. The formation process iterates on all concentric circles in Γ, one at the time, from the smallest to the largest one.

In particular, let us assume that the first $j-1$ concentric circles in Γ are *complete* and that there are no robots below T_{j-1} that are not on their final targets; hence, T_j is the first target



(a) The next pattern to be formed, S_i.

(b) The separated configuration of Figure 6(k), where the 5 symmetry classes are reported.

(c) The robots on Q_1 are already properly rotated: they become BRONZE.

(d) The robots on Q_1 become all WHITE-R, before the next class starts rotating.

(e) The targets of the next pattern to form are reported (dotted circles).

(f) The next robots to rotate are now those on C, the smallest circle with WHITE-S robots: they first become BRONZE, and they start rotating.

(g) The robots on C are properly rotated.

(h) The robots on C can now turn WHITE-R.

(i) The rotation process has been completed for all classes.

FIGURE 9. An example for the rotation step of ROTATION(Γ) in Algorithm 7. The GOLD robots are not included in the figure.

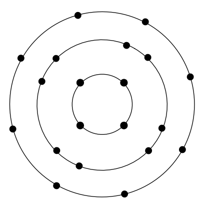
that is not complete. The algorithm, reported in Algorithm 8, distinguishes the only three possible cases:

1) Between T_j and T_{j-1} there are robots whose targets are not T_j: these robots must move above T_j. In this case, the class of robots whose target is not T_j and that is closest to T_j is first colored PLATINUM, and then moved to a new circle larger than T_j. When this occurs, the number of classes whose target is not T_j, between T_j and above T_{j-1}, decreases; this process is repeated until this case no longer applies.
2) Between T_j and T_{j-1} there are only classes whose target is T_j: in this case, these robots are moved, one class at the time, to T_j, using the special PLATINUM color.
3) Between T_j and T_{j-1} there are no robots: the class of robots that is on the smallest circle closest to T_j whose target is T_j is moved onto T_j, using the special PLATINUM color. After finite time, this class reaches T_j, and the number of classes on T_j increases.

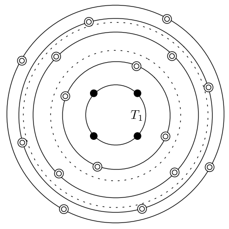
Thus, after finite time, all circles in Γ are correctly processed, and S_i correctly formed.

Lemma 15: Starting from a rotated configuration Γ with coloring function λ, within finite time the robots reach a formed configuration Γ′ with coloring function λ′ with respect to S, where S is the next pattern of the sequence to be formed, such that $q_c(\Gamma, \lambda) = q_c(\Gamma', \lambda')$.

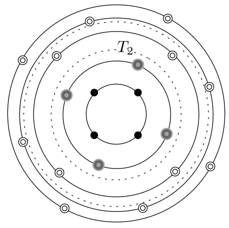
Proof: By definition, right after the rotation process in Γ there are no PLATINUM robots; in particular, all robots not on *SEC* are WHITE-R, and those on *SEC* are GOLD. By construction, the robots on the smallest concentric circle of Γ are already on their target, T_1: within finite time, say at time t_0, they have computed and assumed the color assigned by routine GETCOLORS(S). Starting from this moment,



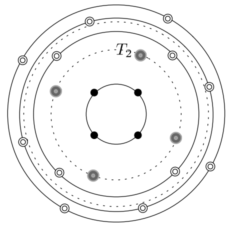
(a) The next pattern to be formed, S_i.



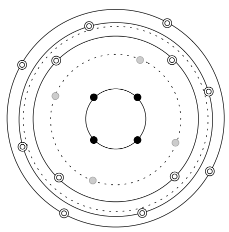
(b) Starting from the rotated configuration of Figure 9(i), the robots on the smallest concentric circle are already on their target circle T_1, hence they become colored. Dotted circles represents target circles.



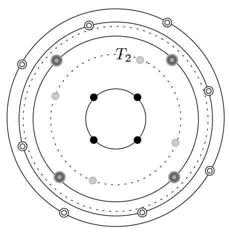
(c) The robots on the second smallest circle have to reach the second target circle T_2 ($up_2 = 1$); before moving, they all become PLATINUM. Let us assume that 2 classes of robots have as target T_2 (i.e., $k_2 = 2$).



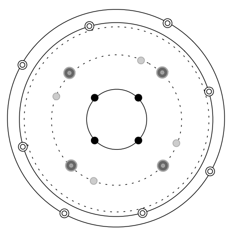
(d) Here, $0 < up_i = 1 \leq k_2 = 2$; hence, all PLATINUM robots move towards T_2.



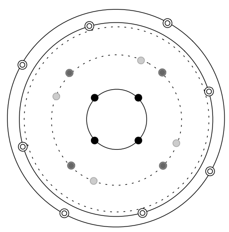
(e) Once all PLATINUM robots reach T_2, they can become colored.



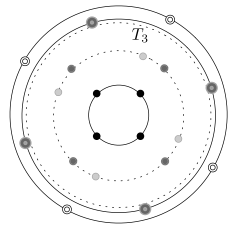
(f) Here, T_2 is not completed yet (2 classes of robots have as target T_2); also, $up_2 = 0$. Hence, the robots on the third smallest circle move towards T_2. Before moving, they all become PLATINUM.



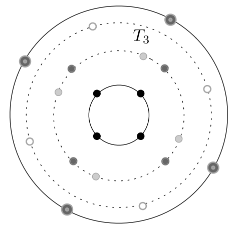
(g) All PLATINUM robots reached T_2, that is now completed.



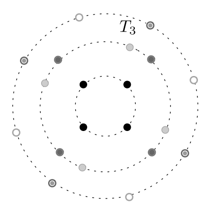
(h) The PLATINUM robots on T_2 can now become colored.



(i) Finally, T_3 is addressed: here, $up_3 = 0$, and $k_3 = 1$. Thus, the robots on the third smallest circle move towards T_3. Again, before moving, they all become PLATINUM.



(j) All PLATINUM robots reached their final destinations on T_3, that is now completed.



(k) The formation step has been completed for all but GOLD robots: they will be the last ones to reach the last target circle.

FIGURE 10. An example for the formation step of FORMATION(Γ) in Algorithm 8. The GOLD robots are not included in the figure.

let T_j, $j \geq 1$, be the smallest target circle which is not complete; i.e., some of the classes that have to be on T_j are still missing. Also, let us denote by up_j the number of classes between T_{j-1} and T_j, including T_j, whose target is not T_j (by convention, T_0 is complete). The algorithm now distinguishes three cases:

1) $up_j > 0$. In this case, we have that between T_{j-1} and T_j there are robots whose target is not T_j.
   a) By construction of routine MoveFromToDes() (Algorithm 9), within finite time the class of robots (whose target is not T_j) on the largest concentric circle between T_{j-1} and T_j, including T_j, say C_{from}, starts becoming PLATINUM. Hence, there is a time $t_1 > t_0$ where there are exactly q PLATINUM robots on C_{from}.
   b) At this stage, the goal of the algorithm is to move above T_j all robots on C_{from}. In particular, all these robots will be moved to an empty circle C_{des} computed as follows.
   Let C^* be the smallest circle populated by non PLATINUM robots between T_j and T_{j+1}, including T_{j+1} (note that, at time t_1, there are no PLATINUM

Algorithm 7 ROTATION(Γ)

```
If I am in $Q_1$ and I am not WHITE-R Then
    Become WHITE-R.
If $\Gamma$ is in-rotation Then
    If $\Gamma$ contains BRONZE robots Then
        $C$ := the circle that contains BRONZE robots;
        If I am in $C$ Then
            If I am WHITE-S Then
                Become BRONZE.
            If all robots in $C$ are BRONZE Then
                $\theta$ := ComputeProperRotationAngle();
                If Rotation of BRONZE robots has not been completed Then
                    If I am $\neg$ ProperlyRotated() Then
                        Rotate($\theta$).
                Else
                    Become WHITE-R.
    Else
        If $\Gamma$ contains WHITE-S robots Then
            $C$ := the smallest circle that contains WHITE-S robots;
            If I am in $C$ and I am WHITE-S Then
                Become BRONZE.
    If Rotation of robots on $SEC$ has not been completed Then
        If All robots not on $SEC$ are WHITE-R Then
            If There are GOLD robots on $SEC$ Then
                If If I am on $SEC$ and I am GOLD Then
                    Become WHITE-R.
            Else
                If I am on $SEC$ Then
                    $\theta$:=ComputeProperRotationAngle();
                    Rotate($\theta$).
    Else
        If There are robots on $SEC$ that are not GOLD Then
            If I am on $SEC$ and I am not GOLD Then
                Become GOLD.
```

robots; hence C^* is the smallest populated circle between T_j and T_{j+1}). Then, C_{des} is the circle halfway between T_j and C^*.
Starting at t_1, robots on C_{from} move radially to C_{des}, again by calling routine MOVEFROMTODES(). Within finite time, say at $t_2 > t_1$, all PLATINUM robots will be on C_{des}.

c) Once all q PLATINUM robots are on C_{des}, they all become WHITE-F at some time $t_3 > t_2$.

Note that, between time t_1 and t_3 no other robot is allowed neither to move, hence C_{des} stays invariant, nor to change color. Thus, at time t_3, the number of classes between T_{j-1} and T_j whose target is not on T_j (i.e., the value of up_j), decreases by one.

2) $up_j = 0$ and between T_j and T_{j-1} there are only classes whose target is T_j. In this case, all these robots, one class at the time, move to T_j. At this point, the proof

Algorithm 8 FORMATION(Γ)

Requires: The current configuration Γ is either *rotated* or *in-formation*. Also, let S be the next pattern of the sequence to be formed.

```
If There are WHITE-R robots on the smallest concentric circle of $\Gamma$ Then
    If I am on the smallest concentric circle of $\Gamma$, and I am WHITE-R Then
        $Color$ = Color assigned to robots on the smallest concentric circle by routine GETCOLORS($S$);
        Become $Color$.
Else
    $T_j$ = The smallest target circle which is not complete;
    $up_j$ = Number of classes between $T_{j-1}$ and $T_j$, including $T_j$, whose target is not $T_j$;
    Case $up_j$
      • $> 0$
        $C^*$ = The smallest circle populated by non PLATINUM robots between $T_j$ and $T_{j+1}$, including $T_{j+1}$;
        $C_{des}$ = Circle halfway between $T_j$ and $C^*$;
        $C_{from}$ = Largest circle between $T_{j-1}$ and $T_j$, including $T_j$, populated by classes whose target is not $T_j$;
        MOVEFROMTODES($C_{from}$, $C_{des}$, PLATINUM, WHITE-F).
      • $= 0$
        If There are populated circles between $T_{j-1}$ and $T_j$ Then
            $C_{from}$ = Largest populated circle between $T_{j-1}$ and $T_j$;
            $new\_color$ = Color assigned to robots on $C_{from}$ by routine GETCOLORS($S$)
            MOVEFROMTODES($C_{from}$, $T_j$, PLATINUM, $new\_color$).
        Else
            $C_{from}$ = Smallest circle larger than $T_j$, populated by classes whose target is $T_j$;
            If $C_{from}$ is $SEC$ or There is at least one GOLD-IN Then
                MOVEFROMTODES($C_{from}$, $T_j$, GOLD-IN, GOLD).
            Else
                $new\_color$ = Color assigned to robots on $C_{from}$ by routine GETCOLORS($S$)
                MOVEFROMTODES($C_{from}$, $T_j$, PLATINUM, $new\_color$).
```

is analogous to the previous case, where C_{des} is T_j, and the color assigned at the end of the movement is the one assigned by GETCOLORS(S): this is because the robots have reached their final destination to form S. Hence, within finite time, one more class with target T_j reaches T_j.

Algorithm 9 MoveFromToDes(*From*, *Des*, *MoveColor*, *FinalColor*)

```
If I am $MoveColor$ Then
    If There are $q$ $MoveColor$ robots and I am not on $Des$ Then
        Move radially to $Des$.
    If All $MoveColor$ robots are on $Des$ Then
        Become $FinalColor$.
Else
    If There are less than $q$ $MoveColor$ robots, and all of them are on $From$ Then
        If I am on $From$ Then Become $MoveColor$.
```

3) $up_j = 0$ and all robots whose target is T_j are on circles larger than T_j. Also in this case, all these robots, one class at the time, move onto T_j by calling routine MoveFromToDes(); the only difference with the previous case is that the robots are moving from a larger circle, T_{from}, to a smaller circle, T_j. Hence, within finite time, one more class with target T_j reaches T_j.

Hence, by induction, within finite time, the last class to move is that of the GOLD robots. Once again, this case is handled by routine MoveFromToDes(), where the robots use the color GOLD-IN instead of PLATINUM, and GOLD instead of the the color assigned by GetColors($\mathcal{S}$). Therefore, within finite time all targets become complete. Since, by definition, a complete target T_l coincides with $Cir_l^{\mathcal{S}}$, the theorem follows. ■

I. $\overline{\Gamma}$ SYMMETRIC AND $\mathcal{S}$ CENTRAL

Algorithm 1 assumes both $q(\overline{\Gamma}) > 1$ and $\mathcal{S}$ non-central. In the following, we will briefly outline how to extend it to cover also the case when $q(\overline{\Gamma}) > 1$ and $\mathcal{S}$ is central.

In this case, by Theorem 2, $\mathcal{S}$ is feasible if $q(S_i^-)$ is a multiple of $q(\overline{\Gamma})$, for all i. Thus, at the beginning, the first action to do is to move one class to the center c of SEC (note that, being $\overline{\Gamma}$ symmetric, there cannot be any robot at c at the beginning): this class will assume the special color BLACK.

In more detail, the SetUp() algorithm (Algorithm 2) is changed as follows: (I) the guard of the first **If** statement checks also for the presence of BLACK robots (*All robots' colors are* GOLD *and/or* OFF *and/or* BLACK); (II) instead of the first **Return** statement, routine FixCentral() (Algorithm 10) is called. This routine ensures that all robots in the class of $\overline{\Gamma}$ closest to c first change their color to BLACK, and then move to c.

At this point the execution runs following Algorithm 1, where BLACK robots never move; also, they are never considered by all the other robots (i.e., it is like they do not exists). In other words, Γ^- it is considered instead of Γ, and S_i^- instead of S_i, for all $S_i \in \mathcal{S}$.

By following the same techniques used in the previous algorithms, it is easy to see that within finite time Algorithm 10 moves one class to c: the first class on the smallest populated circle according to the total ordering of the classes. At this point, the BLACK robots will never move again, and the correctness of the overall solution (that covers both central and non-central sequences) follows from previous Lemmas 3–16.

Algorithm 10 FixCentral()

Requires: Γ is the current configuration.

```
If There are robots at $c$ and all BLACK are at $c$ Then Return
If No robot is at $c$ and less than $q$ robots are BLACK Then
    $K$ := First class of robots closest to the center $c$ of $SEC$;
    If I am in $K$ and I am not BLACK Then
        Become BLACK.
Else
    If I am BLACK and I am not at $c$ Then
        Move radially towards $c$.
```

J. ASYMMETRIC INITIAL CONFIGURATION

In case the initial configuration is *asymmetric*, that is $q(\overline{\Gamma}) = 1$, the entire protocol is much simpler. Note that, since $q(\overline{\Gamma}) = 1$, each class contains only one robot, and there is a total order of the classes; hence, the robots can agree on a unique leader and on a common orientation of SEC, i.e., there is a common chirality. In the algorithm, the main difference is that we keep invariant the initial SEC, instead of the smallest concentric populated circle of the starting configuration. Another difference is that the unique leader uses color RAINBOW instead of GOLD (and thus RAINBOW-OUT and RAINBOW-IN instead of GOLD-OUT, GOLD-IN) throughout the algorithm.

More precisely, there are only three phases: SetUp, PatternIdentification and Formation (Algorithm 11), described in the following.

Algorithm 11 AsymmetricPatternSequence ()

```
SetUp(); %Section IV-J(a)%
PatternIdentification(); %Algorithm 12%
If $\Gamma$ is in-formation Then
    Formation(); %Section IV-J(c)%
```

1) SETUP PHASE

In this phase, if no robot is colored RAINBOW, the first one in the total order whose removal does not destroy the SEC becomes RAINBOW.

2) PATTERNIDENTIFICATION PHASE

This phase, described in Algorithms 12 and 13, follows an idea similar to that described in Section IV-D. The main differences with Algorithms 3 and 4 is that rad' in Algorithm 12 is computed with respect to the $SEC \setminus \{r_R\}$.

3) FORMATION PHASE

In this phase, we have the following three steps:

Algorithm 12 PATTERNIDENTIFICATION()

```
Requires: The current configuration.

$r_R$ := robot colored {RAINBOW, RAINBOW-OUT, RAINBOW-IN };
If Color of robots in $\mathcal{R} \setminus \{r_R\}$ is OFF Then
    $P := S_0$;
    SIGNAL(P);
Else
    If $\exists \lambda_i \in \Lambda | \mathcal{R} \setminus \{r_R\}$ are colored according to $\lambda_i \setminus$ {RAINBOW, RAINBOW-OUT, RAINBOW-IN } Then
        $P := S_{(i+1) \mod m}$;
        SIGNAL(P);
    Else
        /* A new pattern is already under construction */
        $c$ := Center of $SEC(\mathcal{R})$;
        $rad$ := Segment between $c$ and the robot RAINBOW;
        $rad'$ := Radius of $SEC(\mathcal{R} \setminus \{r_R\})$;
        $P := S_i, 0 \leq i \leq m-1$, such that $f(S_i) \cdot rad' = rad$
Return
```

Algorithm 13 SIGNAL(P)

```
$x$ := Compute $f(P)$;
$c$ := Center of $SEC(\mathcal{R} \setminus \{r_R\})$;
$rad$ := Radius of $SEC(\mathcal{R} \setminus \{r_R\})$;
$\overrightarrow{w}$ := half-line from $c$ through my current position;
$p$ := Point on $\overrightarrow{w}$, at distance $rad \cdot x$ from $c$;
If I am RAINBOW and I am not at $p$ Then
    Become RAINBOW-OUT;
    Move towards $p$.
Else
    If I am RAINBOW-OUT Then
        If I am not at $p$ Then
            Move towards $p$.
        Else
            Become RAINBOW.
If $R_r$ is at distance $r \cdot x$ from $c$ Then Return
```

1) $SEC(\mathcal{R} \setminus \{r_R\})$ is mapped to $SEC(S)$: the first robot on $SEC(\mathcal{R} \setminus \{r_R\})$ is considered already in its final position, and thus changes color according to GETCOLORS(S). At this point, the pattern to be formed is scaled and rotated accordingly, and the final destination the robots have to reach in order to correctly form S can be computed.
Now, a one-to-one mapping is established between the other robots in Γ and the other destinations of the pattern to be formed for which there exists a sequential scheduling of the movements of the robots onto the assigned destination, such that the $SEC(\mathcal{R} \setminus \{r_R\})$ does not change during each movement. Note that such a scheduler trivially exists.

2) Each robot, excluding the one colored RAINBOW, one at the time (according to scheduling associated to the one-to-one mapping):
   a) Becomes PLATINUM;
   b) Moves to its destination;
   c) Gets the color assigned by GETCOLORS(S).

3) Finally, the only robot that still needs to be placed is the RAINBOW: it moves to the last available target, by using RAINBOW-IN color during its movement. Once it reaches its destination, it becomes RAINBOW.

The correctness of the case with $q = 1$ follows the same lines of previous Lemmas 3–15, thus we can conclude that:

Lemma 16: Starting from any asymmetric initial configuration $\overline{\Gamma}$ (i.e., $q(\overline{\Gamma}) = 1$), ASYMMETRICPATTERNSEQUENCE allows the robots to perform any feasible choreography in $\mathcal{F}(\overline{\Gamma})$.

K. FINAL ALGORITHM

The final algorithm can be easily obtained by merging the two cases for $q(\overline{\Gamma}) = 1$ and $q(\overline{\Gamma}) > 1$. The distinction between these two cases is trivially done at the starting of the execution. In any subsequent configuration, it can be done depending on whether there are GOLD or RAINBOW robots.

By previous Lemmas 3–16, we can conclude that:

Theorem 17: Starting from any initial configuration $\overline{\Gamma}$, the robots can perform any feasible choreography in $\mathcal{F}(\overline{\Gamma})$.

V. LIMITATIONS TO THE NUMBER OF COLORS

Since the focus of this paper is on the characterization of the set $\mathcal{F}(\overline{\Gamma})$ of sequences of patterns that can be formed by luminous robots starting from an arbitrary initial configuration $\overline{\Gamma}$, and the feasibility of constructing them, the number of colors used in the construction has not been a concern. It is however an important element worth discussing.

Our protocol employs a coloring function Λ, which assigns colors to the classes of $\overline{\Gamma}$ so that, once those classes are mapped by our algorithm to the classes of the pattern to be formed, the Coloring Properties (see Section IV-C) are guaranteed to hold.

From the point of view of characterization and feasibility, the choice of Λ is irrelevant. However, it has an impact on the number of colors used by the robots. For example, the function Λ^* described in Section IV-C, provides the following upperbound:

$$c(\overline{\Gamma}) \leq \alpha(\mathcal{S}) + \mu(\mathcal{S}) + O(1) \tag{1}$$

The immediate question that arises is how many colors are truly necessary to form any sequence $\mathcal{S} \in \mathcal{F}(\overline{\Gamma})$, starting from a configuration $\overline{\Gamma}$. Note that in the $\mathcal{OBLOT}$ model, where the robots have a single color, it is possible only to form sequences without repetitions and contractions [9]. Intuitively, this is because, if two robots move to occupy the same point performing a contraction, they will not be able to separate anymore and will thus not be able to form alternatively patterns with different number of vertices.

Moreover, the presence of repetitions would not allow the robots to "understand" which repetition of the same pattern they are currently forming without an external indication. Multiple colors are needed to deal with contractions, repetitions, as well as with asynchrony. We now establish bounds on how many colors are actually needed.

Theorem 18: A set of luminous robots starting from initial configuration $\overline{\Gamma}$, *cannot* perform a choreography $\mathcal{S}^\infty = \langle S_0, \dots, S_{m-1}\rangle^\infty$ with less than $\max\{\alpha(\mathcal{S})/\min_i\{\alpha(S_i)\}, \sqrt[\alpha(\mathcal{S})]{\mu(\mathcal{S})}\}$ colors. The lower bound holds even if the robots are fully-synchronous and rigid.

Proof: First of all, the robots need to distinguish between at least $a = \alpha(\mathcal{S})$ distinct classes, during the execution of any algorithm. Even if the initial configuration $\overline{\Gamma}$ is such that $\alpha(\overline{\Gamma}) > \alpha(\mathcal{S})$, $\overline{\Gamma}$ can be contracted to a configuration containing $a = \alpha(\mathcal{S})$ distinct classes, without any loss. However, for forming any pattern $S_i \in \mathcal{S}$ with $\alpha(S_i) < \alpha(\mathcal{S})$, after the contraction, the robots from distinct classes must be distinctly colored before they are contracted to a single class. Thus at least $\alpha(\mathcal{S})/\alpha(S_i)$ colors are needed to perform this contraction and thus the total number of colors $c(\overline{\Gamma})$ must satisfy

$$c(\overline{\Gamma}) \geq \alpha(\mathcal{S})/\min_i\{\alpha(S_i)\} \tag{2}$$

For two patterns S_i and S_j, such that $i \neq j$ and $S_i \equiv S_j$, the coloring λ_i and λ_j must be visibly distinguishable by the robots. With a distinct (ordered) classes and c colors, it is possible to have c^a visibly distinct colorings for patterns having a classes. This gives the following bound on the total number of colors

$$c(\overline{\Gamma})^{\alpha(\mathcal{S})} \geq \mu(\mathcal{S}) = \max_i\{\mu(S_i)\} \tag{3}$$

Combining the above two inequalities we obtain the following lower bound on the number of colors for forming sequence $\mathcal{S} \in F(\overline{\Gamma})$

$$c(\overline{\Gamma}) \geq \max\left\{\frac{\alpha(\mathcal{S})}{\min_i\{\alpha(S_i)\}}, \sqrt[\alpha(\mathcal{S})]{\mu(\mathcal{S})}\right\} \tag{4}$$

■

There is a large gap between the lower bound of (4) and the upper bound of (1). To narrow this gap, e.g. by identifying a more efficient coloring or by designing a different pattern formation protocol, is a very interesting and important open question.

VI. CONCLUDING REMARKS

In this paper we have characterized the set $\mathcal{F}(\overline{\Gamma})$ of sequences of patterns that can be formed by luminous robots starting from an initial configuration $\overline{\Gamma}$, showing that $\mathcal{F}(\overline{\Gamma})$ is much larger than the set of sequences of patterns formable without lights characterized in [9].

Our characterization is complete and constructive: We have provided a universal protocol that, for any initial configuration $\overline{\Gamma}$ and for any pattern sequence $\mathcal{S} \in \mathcal{F}(\overline{\Gamma})$, allows the robots to form $\mathcal{S}$ even if asynchronous and with non-rigid movements.

Note that we have assumed that $\alpha(\overline{\Gamma}) = \alpha(\mathcal{S})$. It is actually possible that $\alpha(\overline{\Gamma}) > \alpha(\mathcal{S})$; should this be the case, it suffices to apply a simple pre-processing phase that will reduce the number of classes to $\alpha(\mathcal{S})$. In more detail, all extra classes of $\overline{\Gamma}$ are moved to the center c of *SEC* and colored `BLACK`, following an approach similar to the one adopted in Section IV-I to handle central sequences. From now on: if the next pattern to be formed is central, the `BLACK` will occupy c; otherwise, after all other classes have occupied their final destinations (hence the pattern is formed up to the robots at c) each of them will choose as destination one of the points of any other class (e.g., the closest class to c), and move there. Once the pattern has been formed they will move back to the center to start the formation of the next pattern.

There are several interesting open questions. The results for formation of sequences of patterns without lights established in [9], and those with luminous robots established in this paper have been derived under the assumption that there is chirality; what happens when there is no chirality ? What happens when robots may fail? What kind of failures can be tolerated in forming patterns? Is it possible to have self-stabilizing algorithms when the robots may start from any arbitrary color?

As mentioned in the previous section, the determination of a better upper-bound on the number of colors, through a more efficient coloring or different pattern formation protocol, is an immediate and natural open problem.

ACKNOWLEDGMENT

Some of these results were presented at the 7th International Conference on Fun with Algorithms.

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