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The rectifiability of the entropy defect measure for Burgers equation

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A R T I C L E   I N F O

A B S T R A C T

We consider bounded weak solutions to the Burgers equation for which every entropy dissipation is representable by a measure and we prove that all these measures are concentrated on the graphs of countably many Lipschitz curves. The main tool is the Lagrangian representation, which is an extension of the method of characteristics to the non-smooth setting.
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1. Introduction

We consider the Cauchy problem for the Burgers equation:

$$
\begin{cases}
u_t + \left(\dfrac{u^2}{2}\right)_x = 0, & (t,x) \in [0,T] \times \mathbb{R} \\
u(0,\cdot) = u_0,
\end{cases}
\tag{1}
$$

with $T > 0$ and $u_0 \in L^\infty(\mathbb{R})$. It is known since the early stages [20] of the theory of scalar conservation laws that the Cauchy problem (1) is well-posed in the setting of bounded entropy solutions, namely weak solutions with the following additional constraint: for every smooth convex entropy $\eta : \mathbb{R} \to \mathbb{R}$ and relative flux $q : \mathbb{R} \to \mathbb{R}$ defined up to a constant by $q'(v) = \eta'(v)v$, it holds

$$\mu_\eta := \partial_t \eta(u) + \partial_x q(u) \leq 0$$

in the sense of distributions. The celebrated Oleinik one-sided Lipschitz estimate [29] implies that bounded entropy solutions to (1) belong to $\mathrm{BV}_{\mathrm{loc}}((0,T] \times \mathbb{R})$ and the structure of the solution is well described by means of the theory of BV functions.

In this paper we investigate the structure of more general weak solutions introduced in the following definition:

Definition 1.1. We say that u is a bounded weak solutions to (1) with *finite entropy production* if $u \in C^0([0,T]; L^1(\mathbb{R})) \cap L^\infty([0,T] \times \mathbb{R})$ and for every convex entropy η and corresponding flux q

$$\mu_\eta := \partial_t \eta(u) + \partial_x q(u) \in \mathscr{M}([0,T] \times \mathbb{R}),$$

where $\mathscr{M}$ denotes the set of finite Radon measures.

This class of solutions arises in several situations; we report here the main motivations and we refer to [24] for a more detailed presentation. Weak solutions arise in the study of large deviations for stochastic approximations of entropy solutions (see [35] for the case of the totally asymmetric simple exclusion process). In [6,26] such large deviations estimates are related to a Γ-limit problem for a family of energy functionals, whose asymptotic domain is the class of weak solutions with finite entropy production. Moreover there is a strong analogy between the solutions introduced above and the weak solutions of the eikonal equation

$$|\nabla \phi| = 1 \tag{2}$$

in the plane arising as domain of the Γ-limit as $\varepsilon \to 0$ of the Aviles-Giga functionals:

$$\mathrm{AG}_\varepsilon(\phi) := \int \left(\frac{\varepsilon}{2} |\nabla^2 \phi|^2 + \frac{1}{2\varepsilon} (|\nabla \phi|^2 - 1)^2 \right).$$

The same analogy holds for a very related model about thin ferro-magnetic films studied in [32,33], see also [31]. At a formal level the link between conservation laws and the eikonal equation is the following (see for example [10]): setting $m = \nabla^\perp \phi$ it holds div $m = 0$. Moreover we can impose the constraint $|m| = 1$ introducing a phase $\theta \in \mathbb{S}^1$ so that $m = (\cos \theta, \sin \theta)$ and the eikonal equation reduces to the $\mathbb{S}^1$-valued scalar conservation law

$$\partial_{x_1} \cos\theta + \partial_{x_2} \sin\theta = 0.$$

This analogy has been pushed much further when the notion of entropy has been transferred from the theory of conservation laws to the eikonal equation [18,2,15] and this allowed to provide a kinetic formulation for (2), see for example [19]. The kinetic formulation for scalar conservation laws has been introduced in [25] in the context of entropy solutions; in our setting it reads as follows (see for example [11]): given a bounded weak solution to (1) with finite entropy production u there exists a Radon measure $\mu \in \mathscr{M}([0,T] \times \mathbb{R} \times \mathbb{R})$ such that

$$\partial_t \chi + v \partial_x \chi = -\partial_v \mu, \qquad \text{where} \qquad \chi(t,x,v) := \begin{cases} 1 & \text{if } 0 < v \le u(t,x), \\ -1 & \text{if } u(t,x) \le v < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The kinetic measure μ encodes all the entropy production measures μ_η by means of the following formula:

$$\langle \mu_\eta, \phi \rangle = \int\limits_{[0,T]\times\mathbb{R}\times\mathbb{R}} \eta''(v)\phi(t,x)d\mu. \tag{4}$$

In particular we will consider the measure

$$\nu := (p_{t,x})_\sharp |\mu| = \bigvee_{\|\eta''\|_{L^\infty} \le 1} \mu_\eta, \tag{5}$$

where $p_{t,x} : [0,T]\times\mathbb{R}^2 \to [0,T]\times\mathbb{R}$ is the standard projection on the first two components and we denoted by $\vee$ the supremum operator in the set of measures. The equality in (5) follows from (4) (see for example [24]).

In contrast with the case of entropy solutions, bounded weak solutions with finite entropy production are not locally BV: they belong to $B^{1/3,3}_{\infty,\text{loc}}$ [17] and this regularity is optimal [14]. The same result has been established recently in the case of the eikonal equation [16]. Nevertheless these solutions share several fine properties with functions of bounded variation: in [22,11] it has been proved that the set

$$J := \left\{ (t,x) \in [0,T]\times\mathbb{R} : \limsup_{r\to 0} \frac{\nu(B_r(x))}{r} > 0 \right\} \tag{6}$$

is $\mathscr{H}^1$-rectifiable and it admits strong traces on both sides. In particular for every entropy η it holds

$$\mu_\eta \llcorner J = \big((\eta(u^+) - \eta(u^-))\mathbf{n}_t + (q(u^+) - q(u^-))\mathbf{n}_x) \big) \mathscr{H}^1 \llcorner J, \tag{7}$$

where $\mathbf{n} = (\mathbf{n}_t, \mathbf{n}_x)$ is the normal to J and $u^{\pm}$ are the traces. Moreover every point in J^c is a point of vanishing mean oscillation of u. Weak solutions with bounded variation enjoy the following additional properties:

(a) $\mathscr{H}^1$-a.e. point in J^c is a Lebesgue point;
(b) μ_η is concentrated on J for every convex η.

It is conjectured that properties (a) and (b) hold for any bounded weak solution with finite entropy production. The same results and open questions about the fine properties of solutions of scalar conservation laws appear in the context of the eikonal equation (see [4,10]). In all the mentioned contributions the rectifiability of the jump set and the vanishing mean oscillation property are obtained through the analysis of the blow-ups of the solutions. This analysis does not detect the a priori non trivial measure $\mu_{\eta \llcorner} J^c$, which is more diffuse than $\mathscr{H}^1$ by definition of J, and it does not distinguish between points with vanishing mean oscillation and Lebesgue points.

Only recently, it has been proved in [24] that the set of non-Lebesgue points of u has Hausdorff dimension at most 1, obtaining a partial result towards (a) (see also [28] for the same result for more general conservation laws and [34] for the case of entropy solutions, where every point in J^c is actually a continuity point of u).

Property (b) has been proved for entropy solutions of general conservation laws in one space dimensions [13,7], for continuous entropy solutions in several space dimensions [34,5] and for continuous weak solutions to general conservation laws in one space dimension [9]. In both settings of conservation laws and of the eikonal equation, the proof of this concentration property in the class of bounded weak solutions with finite entropy production is considered a fundamental step to prove the Γ-convergence of the families of functionals discussed above, see [6,23].

The main result of this work establishes that this property holds in the case of Burgers equation:

Theorem 1. *Let u be a bounded weak solution to* (1) *with finite entropy production. Then there exist countably many Lipschitz curves $\gamma_i : [0, T] \to \mathbb{R}$ such that ν (and therefore every entropy dissipation measure μ_η) is concentrated on*

$$J' := \bigcup_{i \in \mathbb{N}} \{(t, x) \in [0, T] \times \mathbb{R} : x = \gamma_i(t)\}.$$

We observe that by definition of J it follows that $\nu_{\llcorner} J^c$ does not charge any set with finite $\mathscr{H}^1$-measure, and therefore any 1-rectifiable set; in particular, in the statement of Theorem 1, we can replace the set J' with the set J defined in (6) and recover the representation (7) for $\mu_\eta = \mu_{\eta \llcorner} J$.

The fine properties of solutions to (1) and (2) that we discussed and the regularizing effect for conservation laws rely on the characteristics structure that these equations own. The main tool to get Theorem 1 is the Lagrangian representation, which is an

extension of the method of characteristics to deal with nonsmooth solutions. It has been introduced first for entropy solutions to general scalar conservation laws in [5], relying on the transport-collapse scheme by Brenier [8]. Then this notion has been extended to cover the case of bounded weak solutions with finite entropy production in [28], building on the kinetic formulation introduced in [11]: in the particular case of the Burgers equation it takes the form provided in Definition 2.1. We would like to point out that the notion of Lagrangian representation is inspired by the superposition principle for nonnegative measure-valued solutions to the continuity equation (see [1]). In particular it shares with it the main feature that the evolution of the solution u is described as the result of the evolutions of the single particles along the characteristics. Finally we mention that the notion of Lagrangian representation is available for solutions with finite entropy production to the eikonal equation too. In the forthcoming paper [27] the general strategy introduced here is exploited to obtain the analogous of Property (b) in the model introduced by Rivière and Serfaty in [32].

As a consequence of Theorem 1 and (7) we obtain the following result:

Theorem 2. *Let u be a bounded weak solution to the Burgers equation with finite entropy production. Then it holds*

$$|(p_{t,x})_\sharp \mu| = (p_{t,x})_\sharp |\mu|.$$

In particular, denoting by $\bar{\eta}(u) = u^2/2$, if $\mu_{\bar{\eta}} \leq 0$ then $\mu \leq 0$, namely if u dissipates the quadratic entropy $\bar{\eta}$, then u is the entropy solution to (1).

The second part of this statement is known even under milder assumptions: in [30] the result has been proved for all bounded weak solutions to (1) and in [12] it has been extended to weak solutions in L^4_{loc}. Both proofs rely on the link between entropy solutions to (1) and viscosity solutions to the Hamilton-Jacobi equation

$$v_t + \frac{v_x^2}{2} = 0.$$

The interest in looking for alternative proofs is motivated by the problem of extending this result to systems of conservation laws, where the link with the Hamilton-Jacobi equation is not available: see [21] for a recent result in this direction.

We finally observe that both Theorem 1 and Theorem 2 can be proved with the same strategy and minor modifications in the case of bounded weak solutions with finite entropy production to conservation laws with uniformly convex fluxes.

2. Lagrangian representation

In this section we present the notion of Lagrangian representation introduced in [28] and we discuss some of its properties. Since we consider bounded solutions, we can assume without loss of generality that u takes values in $[0, 1]$.

For every function $f : \mathbb{R} \to [0,1]$ we denote its hypograph and its epigraph by

$$E_f := \{(x,v) \in \mathbb{R} \times [0,1] : v \le f(x)\} \qquad \text{and} \qquad E_f^c := \{(x,v) \in \mathbb{R} \times [0,1] : v \ge f(x)\}$$

respectively. Moreover we denote by

$$\Gamma := \{\gamma = (\gamma_x, \gamma_v) \in BV([0,T]; \mathbb{R} \times [0,1]) : \gamma_x \text{ is Lipschitz}\}.$$

It will be useful to consider the standard decomposition of the measure $Df \in \mathscr{M}(\mathbb{R})$, where $f \in \mathrm{BV}(\mathbb{R}, \mathbb{R})$ (see for example [3]). We will adopt the following notation:

$$Df = D^{\mathrm{ac}}f + D^c f + D^j f = \tilde{D}f + D^j f, \tag{8}$$

where $D^{\mathrm{ac}}f$, $D^c f$ and $D^j f$ denote the absolutely continuous part, the Cantor part and the atomic part of Df respectively; we refer to $\tilde{D}f$ as the diffuse part of Df.

Definition 2.1. Let u be a weak solution to (1) with finite entropy production. We say that the Radon measure $\omega_h \in \mathscr{M}(\Gamma)$ is a *Lagrangian representation* of the hypograph of u if the following conditions hold:

(1) for every $t \in [0,T)$ it holds

$$(e_t)_\sharp \omega_h = \mathscr{L}^2 \llcorner E_{u(t)}, \tag{9}$$

where e_t denotes the evaluation map defined by

$$\begin{aligned} e_t : \Gamma &\to \mathbb{R}. \\ \gamma &\mapsto \gamma(t) \end{aligned} \tag{10}$$

(2) the measure ω_h is concentrated on the set of curves $\gamma \in \Gamma$ such that

$$\dot{\gamma}_x(t) = \gamma_v(t) \quad \text{for a.e. } t \in [0,T); \tag{11}$$

(3) it holds the integral bound

$$\int_\Gamma \mathrm{TotVar}_{[0,T)}\gamma_v d\omega_h(\gamma) < \infty. \tag{12}$$

Similarly we say that $\omega_e \in \mathscr{M}(\Gamma)$ is a *Lagrangian representation* of the epigraph of u if Conditions (2) and (3) hold and (1) is replaced by

$$(e_t)_\sharp \omega_e = \mathscr{L}^2 \llcorner E_{u(t)}^c \qquad \text{for every } t \in [0,T]. \tag{13}$$

A useful property of these representations is that the kinetic measure μ defined in (3) (and therefore any entropy dissipation measure μ_η) can be decomposed along the characteristics.

Given $\gamma \in \Gamma$ we consider

$$\mu_\gamma = (\mathrm{I}, \gamma)_\sharp \tilde{D}_t \gamma_v + \mathscr{H}^1 \llcorner E_\gamma^+ - \mathscr{H}^1 \llcorner E_\gamma^- \in \mathscr{M}([0, T] \times \mathbb{R} \times [0, 1]),$$

where

$$E_\gamma^+ := \{(t, x, v) : \gamma_x(t) = x, \gamma_v(t-) < \gamma_v(t+), v \in (\gamma_v(t-), \gamma_v(t+))\},$$
$$E_\gamma^- := \{(t, x, v) : \gamma_x(t) = x, \gamma_v(t+) < \gamma_v(t-), v \in (\gamma_v(t+), \gamma_v(t-))\},$$

$\mathrm{I} : [0, T] \to [0, T]$ denotes the identity and $\tilde{D}_t \gamma_v$ denotes the diffuse part of the measure $D_t \gamma_v$.

Proposition 3. *Let u be a weak solution to* (1) *with finite entropy production. Then there exist ω_h, ω_e Lagrangian representations of the hypograph and of the epigraph of u respectively enjoying the additional properties:*

$$\int_\Gamma \mu_\gamma d\omega_h(\gamma) = \mu = -\int_\Gamma \mu_\gamma d\omega_e(\gamma), \tag{14}$$

$$\int_\Gamma |\mu_\gamma| d\omega_h(\gamma) = |\mu| = \int_\Gamma |\mu_\gamma| d\omega_e(\gamma). \tag{15}$$

Eq. (14) asserts that μ can be decomposed along characteristics and Eq. (15) says that it can be done minimizing

$$\int_\Gamma \mathrm{TotVar}_{[0,T)} \gamma^2 d\omega_h(\gamma) \qquad \text{and} \qquad \int_\Gamma \mathrm{TotVar}_{[0,T)} \gamma^2 d\omega_e(\gamma).$$

Moreover it follows from (14) and (15) that we can separately represent the negative and the positive parts of μ:

$$\int_\Gamma \mu_\gamma^- d\omega_h(\gamma) = \mu^- = \int_\Gamma \mu_\gamma^+ d\omega_e(\gamma) \qquad \text{and} \qquad \int_\Gamma \mu_\gamma^+ d\omega_h(\gamma) = \mu^+ = \int_\Gamma \mu_\gamma^- d\omega_e(\gamma). \tag{16}$$

The assertion of Proposition 3 regarding ω_h is proved in [28] for more general conservation laws and it is straightforward to adapt the same argument to get the existence of an ω_e as in the statement. Alternatively, in order to prove the part concerning ω_e, we can consider a Lagrangian representation $\tilde{\omega}_h$ of the hypograph of $\tilde{u} = 1 - u$, which is a weak solution with finite entropy production to

$$\tilde{u}_t + g(\tilde{u})_x = 0, \qquad \text{with} \qquad g(z) = -\frac{(z-1)^2}{2}.$$

Let $T : \Gamma \to \Gamma$ be defined by

$$T(\gamma)(t) = (\gamma_x(t), 1 - \gamma_v(t)).$$

Then it is straightforward to check that the measure $\omega_e := T_\sharp \tilde{\omega}_h$ satisfies the requirements in Proposition 3.

The following lemma is an application of Tonelli theorem:

Lemma 4. *For ω_h-a.e. $\gamma \in \Gamma$ it holds that for $\mathscr{L}^1$-a.e. $t \in [0,T]$*

(1) $(t, \gamma_x(t))$ is a Lebesgue point of u;
(2) $\gamma_v(t) < u(t, \gamma_x(t))$.

We denote by Γ_h the set of curves $\gamma \in \Gamma$ such that the two properties above hold. Similarly for ω_e-a.e. $\gamma \in \Gamma$ it holds that for $\mathscr{L}^1$-a.e. $t \in [0,T]$

(1) $(t, \gamma_x(t))$ is a Lebesgue point of u;
(2) $\gamma_v(t) > u(t, \gamma_x(t))$

and we denote the set of these curves by Γ_e.

Proof. Let us prove the properties about Γ_h. We denote by $S \subset [0,T] \times \mathbb{R}$ the set of non-Lebesgue points of u and for every $t \in [0,T]$ we set

$$e_t^x : \Gamma \to \mathbb{R}.$$
$$\gamma \mapsto \gamma_x(t)$$

By (9) it follows that for every $t \in [0,T]$ it holds $(e_t^x)_\sharp \omega_h \le \mathscr{L}^1$. Since $\mathscr{L}^2(S) = 0$ it holds

$$\begin{aligned}
0 &= \left(\mathscr{L}^1 \otimes (e_t^x)_\sharp \omega_h \right)(S) \\
&= \int_0^T \omega_h(\{\gamma \in \Gamma : (t, \gamma_x(t)) \in S\}) dt \\
&= \int_\Gamma \mathscr{L}^1(\{t \in [0,T] : (t, \gamma_x(t)) \in S\}) d\omega_h(\gamma),
\end{aligned} \tag{17}$$

where the last equality follows by Tonelli theorem. Similarly by (9) it follows that for every $t \in [0,T]$

$$\omega_h(\{\gamma \in \Gamma : \gamma_v(t) \geq u(t, \gamma_x(t))\}) = (e_t)_\sharp \omega_h(\{(x, v) \in \mathbb{R} \times [0, 1] : v \geq u(t, x)\}) = 0, \quad (18)$$

where e_t is defined in (10). Therefore by Tonelli theorem it holds

$$
\begin{aligned}
0 &= \int_0^T \omega_h(\{\gamma \in \Gamma : \gamma_v(t) \geq u(t, \gamma_x(t))\}) dt \\
&= \int_\Gamma \mathscr{L}^1(\{t \in [0, T] : \gamma_v(t) \geq u(t, \gamma_x(t))\}) d\omega_h(\gamma).
\end{aligned}
\quad (19)
$$

By (17) and (19) it follows that ω_h is concentrated on Γ_h. The case of Γ_e is analogous. $\quad\square$

The following result is a well-known property of traces on Lipschitz curves:

Lemma 5. *Assume that $\bar{\gamma}_x : (t^-, t^+) \subset [0, T] \to \mathbb{R}$ is a Lipschitz curve and that for $\mathscr{L}^1$-a.e. $t \in (t^-, t^+)$ the point $(t, \bar{\gamma}_x(t))$ is a Lebesgue point of $u \in L^\infty([0, T] \times \mathbb{R})$. Then*

$$\lim_{\delta \to 0} \int_{t^-}^{t^+} \frac{1}{\delta} \int_{\gamma_x(t)}^{\gamma_x(t)+\delta} |u(t, x) - u(t, \bar{\gamma}_x(t))| dx\, dt = 0. \quad (20)$$

Proof. For $t \in (t^-, t^+)$ we set

$$v_\delta(t) := \frac{1}{\delta} \int_0^\delta |u(t, \bar{\gamma}_x(t) + y) - u(t, \bar{\gamma}_x(t))| dy,$$

so that (20) is equivalent to v_δ to converge to 0 in $L^1(t^-, t^+)$. For every $t \in [t^- + \delta, t^+ - \delta]$ we have

$$
\begin{aligned}
&|u(t, \bar{\gamma}_x(t) + y) - u(t, \bar{\gamma}(t))| \\
&\leq \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \left(|u(t, \bar{\gamma}_x(t) + y) - u(t', \bar{\gamma}_x(t'))| + |u(t', \bar{\gamma}_x(t')) - u(t, \bar{\gamma}_x(t))| \right) dt'.
\end{aligned}
\quad (21)
$$

Since $|v_\delta| \leq 1$, integrating (21) with respect to t we get

$$
\begin{aligned}
\int_{t^-}^{t^+} v_\delta(t) dt &\leq 2\delta + \int_{t^-+\delta}^{t^+-\delta} \frac{1}{2\delta^2} \int_0^\delta \int_{t-\delta}^{t+\delta} |u(t, \bar{\gamma}_x(t) + y) - u(t', \bar{\gamma}_x(t'))| dt'\, dy\, dt \\
&\quad + \int_{t^-+\delta}^{t^+-\delta} \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} |u(t', \bar{\gamma}_x(t')) - u(t, \bar{\gamma}_x(t))| dt'\, dt.
\end{aligned}
\quad (22)
$$

By Lebesgue differentiation theorem, $\frac{1}{2\delta}\int_{t-\delta}^{t+\delta}|u(t',\bar{\gamma}_x(t'))-u(t,\bar{\gamma}(t))|dt' \to 0$ as $\delta \to 0$ for $\mathscr{L}^1$-a.e. $t \in (t^-, t^+)$ therefore the last term in (22) converges to zero by dominated convergence theorem. By applying Tonelli theorem to the second term in the right-hand side of (22) we get

$$
\int\limits_{t^-+\delta}^{t^+-\delta}\frac{1}{2\delta^2}\int\limits_{0}^{\delta}\int\limits_{t'-\delta}^{t'+\delta}|u(t,\bar{\gamma}_x(t)+y)-u(t',\bar{\gamma}_x(t'))|dtdydt' \le
$$

$$
\int\limits_{t^-+\delta}^{t^+-\delta}\frac{1}{2\delta^2}\int\limits_{B_{(2+L)\delta}(t',\bar{\gamma}_x(t'))}|u(z)-u(t',\bar{\gamma}_x(t'))|dzdt',
$$
(23)

where L is the Lipschitz constant of $\bar{\gamma}_x$ and $B_{(2+L)\delta}(t',\bar{\gamma}_x(t'))$ denotes the ball in $\mathbb{R}^2$ of radius $(2+L)\delta$ and center $(t',\bar{\gamma}_x(t'))$. Since by assumption $\mathscr{L}^1$-a.e. $t \in (t^-, t^+)$ is a Lebesgue point of u, the right-hand side of (23) converges to 0 by the dominated convergence theorem and this concludes the proof. $\quad\square$

The following proposition formalizes the intuition that a curve lying in the hypograph of a function cannot cross from above a curve lying in the epigraph of the same function.

Proposition 6. *Let $\bar{\gamma} \in \Gamma_h, G \subset \Gamma_e$ and $\bar{t} > 0$ be such that for every $\gamma \in G$ it holds $\bar{\gamma}_x(\bar{t}) < \gamma_x(\bar{t})$. Then for every $t \in [\bar{t}, T]$ it holds*

$$
\omega_e(\{\gamma \in G : \gamma_x(t) < \bar{\gamma}_x(t)\}) = 0.
$$

Proof. Given $\delta > 0$ we define $\phi_\delta : \mathbb{R} \to \mathbb{R}$ by

$$
\phi_\delta(x) = \begin{cases} 1 & \text{if } x \le 0, \\ \frac{1}{\delta}(\delta - x) & \text{if } x \in (0, \delta), \\ 0 & \text{if } x \ge \delta, \end{cases}
$$

and for every $t \in [\bar{t}, T]$ we consider

$$
\Psi_\delta(t) = \int\limits_{G} \phi_\delta(\gamma_x(t) - \bar{\gamma}_x(t))d\omega_e(\gamma).
$$

Clearly for every $\delta > 0$ and every $t \in [\bar{t}, T]$ it holds

$$
\omega_e(\{\gamma^2 \in G : \gamma_x^2(t) < \gamma_x^1(t)\}) \le \Psi_\delta(t)
$$
(24)

and by assumption

$$\Psi_\delta(\bar{t}) \leq \omega_e(\gamma \in G : \gamma_x(\bar{t}) \in [\bar{\gamma}_x(\bar{t}), \bar{\gamma}_x(\bar{t}) + \delta]) \leq \delta.$$

Since ω_e is concentrated on curves satisfying (11), for every $t \in [\bar{t}, T]$

$$\Psi_\delta(t) = \Psi_\delta(\bar{t}) + \int_{\bar{t}}^{t} \frac{1}{\delta} \int_{G(\delta, t')} (\bar{\gamma}_v(t') - \gamma_v(t')) d\omega_e(\gamma) dt'$$

$$\leq \delta + \int_{\bar{t}}^{t} \frac{1}{\delta} \int_{G(\delta, t')} (\bar{\gamma}_v(t') - \gamma_v(t'))^+ d\omega_e(\gamma) dt',$$

where $G(\delta, t') := \{\gamma \in G : \gamma_x(t') \in [\bar{\gamma}_x(t'), \bar{\gamma}_x(t') + \delta]\}$. Since $\bar{\gamma} \in \Gamma_h$ for $\mathscr{L}^1$-a.e. $t' \in (t^-, t^+)$ it holds $\bar{\gamma}_v(t') < u(t', \bar{\gamma}_x(t'))$. Moreover for ω_e-a.e. $\gamma \in \Gamma$ and every $t \in [0, T]$ it holds $\gamma_v(t) \in [0, 1]$, therefore

$$\Psi_\delta(t) \leq \delta + \int_{\bar{t}}^{t} \frac{1}{\delta} \int_{G(\delta, t')} (u(t', \bar{\gamma}_x(t')) - \gamma_v(t'))^+ d\omega_e(\gamma) dt'$$

$$\leq \delta + \int_{\bar{t}}^{t} \frac{1}{\delta} \left(\omega_e(\{\gamma \in G(\delta, t') : \gamma_v(t') < u(t', \bar{\gamma}_x(t'))\}) \right) dt'$$

$$\leq \delta + \int_{\bar{t}}^{t} \frac{1}{\delta} \int_{0}^{\delta} |u(t', \gamma_x(t')) - u(t', \gamma_x(t') + y)| dy dt',$$

where the last inequality follows by (13). The claim follows by Lemma 5 and (24) by letting $\delta \to 0$. $\square$

For every $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}$ we denote by

$$G_{\bar{t}, \bar{x}}^l := \{\gamma \in \Gamma_h : \gamma_x(\bar{t}) < \bar{x}\}, \qquad G_{\bar{t}, \bar{x}}^r := \{\gamma \in \Gamma_e : \gamma_x(\bar{t}) > \bar{x}\}. \qquad (25)$$

Corollary 7. *Let $(\bar{t}, \bar{x}) \in [0, T) \times \mathbb{R}$ and consider $G_{\bar{t}, \bar{x}}^l, G_{\bar{t}, \bar{x}}^r$ as above. Then there exists a Lipschitz function $f_{\bar{t}, \bar{x}} : [\bar{t}, T] \to \mathbb{R}$ such that for every $t \in [\bar{t}, T]$ it holds*

$$\omega_h(\{\gamma \in G_{\bar{t}, \bar{x}}^l : \gamma_x(t) > f_{\bar{t}, \bar{x}}(t)\}) = 0 \qquad and \qquad \omega_e(\{\gamma \in G_{\bar{t}, \bar{x}}^r : \gamma_x(t) < f_{\bar{t}, \bar{x}}(t)\}) = 0. \qquad (26)$$

Proof. If $G_{\bar{t}, \bar{x}}^l = \emptyset$ we set $f_{\bar{t}, \bar{x}}(t) = \bar{x}$ and (26) follows by (11) since for ω_e-a.e. $\gamma \in \Gamma$, it holds $\dot{\gamma}_x(t) \geq 0$ for $\mathscr{L}^1$-a.e. $t \in (0, T)$. If $G_{\bar{t}, \bar{x}}^l \neq \emptyset$, then we set

$$f_{\bar{t}, \bar{x}} = \sup_{\gamma \in G_{\bar{t}, \bar{x}}^l} \gamma_x. \qquad (27)$$

The first condition in (26) is trivially satisfied and by Proposition 6 for every $\bar\gamma \in G^l_{\bar t,\bar x}$ it holds

$$\omega_e(\{\gamma \in G^r_{\bar t,\bar x} : \gamma_x(t) < \bar\gamma_x(t)\}) = 0. \qquad (28)$$

Let $\{t_i\}_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap (\bar t, T)$ and for every $i, n \in \mathbb{N}$ let $\gamma^{i,n} \in G^l_{\bar t,\bar x}$ be such that $\gamma^{i,n}_x(t_j) \geq f_{\bar t,\bar x}(t_i) - \frac{1}{n}$. For every $\gamma \in G^l_{\bar t,\bar x}$ the x-component γ_x is 1-Lipschitz, hence $f_{\bar t,\bar x} = \sup_{i,n\in\mathbb{N}} \gamma^{i,n}_x$. Therefore it follows by (28) that for every $t \in [\bar t, T]$ it holds

$$\omega_e(\{\gamma \in G^r_{\bar t,\bar x} : \gamma_x(t) < f_{\bar t,\bar x}(t)\}) = \omega_e\left(\bigcup_{i,n\in\mathbb{N}}\left\{\gamma \in G^r_{\bar t,\bar x} : \gamma_x(t) < \gamma^{i,n}_x(t)\right\}\right) = 0. \quad (29)$$

$\square$

Thanks to Corollary 7 we can identify the candidate rectifiable set J^- on which $(p_{t,x})_\sharp \mu^-$ is concentrated: let $(\bar t_i)_{i\in\mathbb{N}}$ be an enumeration of $[0, T] \cap \mathbb{Q}$ and $(\bar x_j)_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$. Then we set

$$J^- := \bigcup_{i,j\in\mathbb{N}} C_{f_{\bar t_i, \bar x_j}}, \qquad \text{where} \qquad C_{f_{\bar t,\bar x}} := \{(t, x) \in [\bar t, T] \times \mathbb{R} : x = f_{\bar t,\bar x}(t)\}. \qquad (30)$$

3. Concentration of the entropy dissipation

In this section we prove Theorem 1 and Theorem 2. In the next lemma we couple the two representations ω_h, ω_e taking into account that they represent the same measure μ^- as in (16). We denote by $X = [0, T] \times \mathbb{R} \times [0, 1]$ and consider the measures $\omega_h \otimes \mu^-_\gamma$ and $\omega_e \otimes \mu^+_\gamma$ defined on the set $\Gamma \times X$ by

$$\omega_h \otimes \mu^-_\gamma(G \times E) = \int_G \mu^-_\gamma(E) d\omega_h(\gamma) \qquad \text{and} \qquad \omega_e \otimes \mu^+_\gamma(G \times E) = \int_G \mu^+_\gamma(E) d\omega_e(\gamma),$$

for every measurable $E \subset X, G \subset \Gamma$.

Lemma 8. *Denote by $p_1, p_2 : (\Gamma \times X)^2 \to \Gamma \times X$ the standard projections. Then there exists a plan $\pi^- \in \mathscr{M}((\Gamma \times X)^2)$ with marginals*

$$\begin{aligned}(p_1)_\sharp \pi^- &= \omega_h \otimes \mu^-_\gamma, \\ (p_2)_\sharp \pi^- &= \omega_e \otimes \mu^+_\gamma,\end{aligned} \qquad (31)$$

concentrated on the set

$$\mathcal{G} := \{((\gamma, t, x, v), (\gamma', t', x', v')) \in (\Gamma \times X)^2 : t = t', \gamma_x(t) = x = x' = \gamma'_x(t'), v = v',$$
$$v \in [\gamma_v(t+), \gamma_v(t-)] \cap [\gamma'_v(t-), \gamma'_v(t+)]\}.$$

Proof. First we observe that by definition, $\omega_h \otimes \mu_\gamma^-$ is concentrated on the set

$$\mathcal{G}_h^- := \{(\gamma, t, x, v) \in \Gamma \times X : \gamma_x(t) = x, v \in [\gamma_v(t+), \gamma_v(t-)]\}$$

and $\omega_e \otimes \mu_\gamma^+$ is concentrated on the set

$$\mathcal{G}_e^+ := \{(\gamma, t, x, v) \in \Gamma \times X : \gamma_x(t) = x, v \in [\gamma_v(t-), \gamma_v(t+)]\}.$$

Denoting by $p_X : \Gamma \times X \to X$ the standard projection it follows from (16) that

$$(p_X)_\sharp(\omega_h \otimes \mu_\gamma^-) = \mu^- = (p_X)_\sharp(\omega_e \otimes \mu_\gamma^+).$$

By the disintegration theorem (see for example [3]) there exist two measurable families of probability measures $(\mu_{t,x,v}^{-,h})_{(t,x,v)\in X}, (\mu_{t,x,v}^{+,e})_{(t,x,v)\in X} \in \mathcal{P}(\Gamma \times X)$ such that

$$\omega_h \otimes \mu_\gamma^- = \int_X \mu_{t,x,v}^{-,h} d\mu^- \qquad \text{and} \qquad \omega_e \otimes \mu_\gamma^+ = \int_X \mu_{t,x,v}^{+,e} d\mu^- \tag{32}$$

and for μ^--a.e. (t,x,v) the measures $\mu_{t,x,v}^{-,h}$ and $\mu_{t,x,v}^{+,e}$ are concentrated on the set

$$p_X^{-1}(\{t,x,v\}) = \{(\gamma, t', x', v') \in \Gamma \times X : t' = t, x' = x, v' = v\}.$$

Moreover, since $\omega_h \otimes \mu_\gamma^-$ is concentrated on the set $\mathcal{G}_h^-$ and $\omega_e \otimes \mu_\gamma^+$ is concentrated on the set $\mathcal{G}_e^+$, we have that for μ^--a.e. (t,x,v) the measure $\mu_{t,x,v}^{-,h}$ is concentrated on $p_X^{-1}(\{t,x,v\}) \cap \mathcal{G}_h^-$ and $\mu_{t,x,v}^{+,e}$ is concentrated on $p_X^{-1}(\{t,x,v\}) \cap \mathcal{G}_e^+$. We eventually set

$$\pi^- := \int_X \left(\mu_{t,x,v}^{-,h} \otimes \mu_{t,x,v}^{+,e}\right) d\mu^-.$$

From (32) it directly follows (31) and by the above discussion for μ^--a.e. $(t,x,v) \in X$ the measure $\mu_{t,x,v}^{-,h} \otimes \mu_{t,x,v}^{+,e}$ is concentrated on $(p_X^{-1}(\{t,x,v\}) \cap \mathcal{G}_h^-) \times (p_X^{-1}(\{t,x,v\}) \cap \mathcal{G}_e^+)$, therefore π^- is concentrated on

$$\bigcup_{(t,x,v)\in X} (p_X^{-1}(\{t,x,v\}) \cap \mathcal{G}_h^-) \times (p_X^{-1}(\{t,x,v\}) \cap \mathcal{G}_e^+) = \mathcal{G}$$

and this concludes the proof. □

The following elementary lemma is about functions of bounded variation of one variable: we refer to [3] for the theory of BV functions.

Lemma 9. *Let $v : (a,b) \to \mathbb{R}$ be a* BV *function and denote by D^-v the negative part of the measure Dv. Then for $\tilde{D}^-v$-a.e. $\bar{x} \in (a,b)$ there exists $\delta > 0$ such that*

$$\bar{v}(x) > \bar{v}(\bar{x}) \quad \forall x \in (\bar{x} - \delta, \bar{x}) \qquad \text{and} \qquad \bar{v}(x) < \bar{v}(\bar{x}) \quad \forall x \in (\bar{x}, \bar{x} + \delta).$$

The proof of Theorem 1 is obtained considering separately the positive and the negative parts of μ; in the following theorem we deal with μ^-.

Theorem 10. *The measure $(p_{t,x})_\sharp \mu^-$ is concentrated on the set J^-, defined in* (30).

Proof. Step 1. For every $(\bar{t}, \bar{x}) \in [0, T) \times \mathbb{R}$ we consider the measure

$$\pi_{\bar{t}, \bar{x}}^- := \pi^- \llcorner \big(G_{\bar{t}, \bar{x}}^l \times [\bar{t}, T] \times \mathbb{R} \times [0, 1]\big) \times (G_{\bar{t}, \bar{x}}^r \times [\bar{t}, T] \times \mathbb{R} \times [0, 1])\big),$$

where $G_{\bar{t}, \bar{x}}^l, G_{\bar{t}, \bar{x}}^r$ are defined in (25) and we set

$$p_{t,x}^1 : (\Gamma \times [0, T] \times \mathbb{R} \times [0, 1])^2 \to [0, T] \times \mathbb{R}.$$
$$(\gamma, t, x, v, \gamma', t', x', v') \mapsto (t, x)$$

We prove that the measure $(p_{t,x}^1)_\sharp \pi_{\bar{t}, \bar{x}}^-$ is concentrated on the graph of $f_{\bar{t}, \bar{x}}$, namely $C_{f_{\bar{t}, \bar{x}}}$.
First we observe that

$$\begin{aligned} (p_{t,x}^1)_\sharp \pi_{\bar{t}, \bar{x}}^- &\leq (p_{t,x}^1)_\sharp \big[\pi^- \llcorner \big((G_{\bar{t}, \bar{x}}^l \times [\bar{t}, T] \times \mathbb{R} \times [0, 1]) \times (\Gamma \times X) \big) \big] \\ &= \omega_h \llcorner G_{\bar{t}, \bar{x}}^l \otimes \mu_\gamma^- \llcorner ([\bar{t}, T] \times \mathbb{R} \times [0, 1]). \end{aligned} \tag{33}$$

Since for ω_h-a.e. $\gamma \in G_{\bar{t}, \bar{x}}^l$ it holds $(t, \gamma_x(t)) \notin \Omega_{\bar{t}, \bar{x}}^+$ for every $t \in [\bar{t}, T]$, then for ω_h-a.e. $\gamma \in G_{\bar{t}, \bar{x}}^l$ it holds $\mu_\gamma^-(\Omega_{\bar{t}, \bar{x}}^+ \times [0, 1]) = 0$ and therefore it follows by (33) that

$$(p_{t,x}^1)_\sharp \pi_{\bar{t}, \bar{x}}^- (\Omega_{\bar{t}, \bar{x}}^+) = 0. \tag{34}$$

In the same way we get

$$(p_{t,x}^2)_\sharp \pi_{\bar{t}, \bar{x}}^- (\Omega_{\bar{t}, \bar{x}}^-) = 0, \tag{35}$$

where

$$p_{t,x}^2 : (\Gamma \times [0, T] \times \mathbb{R} \times [0, 1])^2 \to [0, T] \times \mathbb{R}$$
$$(\gamma, t, x, v, \gamma', t', x', v') \mapsto (t', x').$$

Finally, since π^- is concentrated on $\mathcal{G}$, then

$$(p_{t,x}^1 \otimes p_{t,x}^2)_\sharp \pi^- \in \mathscr{M}(([0, T] \times \mathbb{R})^2)$$

is concentrated on the graph of the identity on $[0, T] \times \mathbb{R}$ and in particular $(p_{t,x}^1)_\sharp \pi_{\bar{t}, \bar{x}}^- = (p_{t,x}^2)_\sharp \pi_{\bar{t}, \bar{x}}^-$. Therefore it follows from (34) and (35) that $(p_{t,x}^1)_\sharp \pi_{\bar{t}, \bar{x}}^-$ is concentrated on

$$([0, T] \times \mathbb{R}) \setminus (\Omega_{\bar{t}, \bar{x}}^+ \cup \Omega_{\bar{t}, \bar{x}}^-) = C_{f_{\bar{t}, \bar{x}}}.$$

Step 2. We prove that for π^--a.e. pair $(\gamma, t, x, v, \gamma', t', x', v') \in (\gamma \times X)^2$ there exists $\delta > 0$ such that for every $s \in [t - \delta, t)$ it holds $\gamma_x(s) < \gamma'_x(s)$.

By applying Lemma 9 to the BV function $\gamma_v : (\bar{t}, T) \to [0, 1]$ we get that for $(\tilde{D}\gamma_v)^-$-a.e. $t \in (\bar{t}, T)$ there exists $\delta > 0$ such that for every $s \in (t - \delta, t)$ it holds $\gamma_v(s) < \gamma_v(t)$. Moreover for every t such that γ_v has a negative jump and for every $v \in [\gamma_v(t+), \gamma_v(t-))$ there exists $\delta > 0$ such that for every $s \in (t - \delta, t)$ it holds $\gamma_v(s) < v$. Since for every $\gamma \in \Gamma$ the measure μ_γ^- has no atoms and the set

$$E_\gamma := \{(t, x, v) \in X : \gamma_v(t-) = v > \gamma_v(t+)\}$$

is countable, then $\mu_\gamma^-(E_\gamma) = 0$ and in particular

$$\omega_h \otimes \mu_\gamma^- \left(\{(\gamma, t, x, v) \in \Gamma \times X : \gamma_v(t-) = v > \gamma_v(t+)\} \right) = 0.$$

Therefore it follows by the argument above that for $\omega_h \otimes \mu_\gamma^-$-a.e. (γ, t, x, v) there exists $\delta_h > 0$ such that for every $s \in (t - \delta_h, t)$ it holds

$$\gamma_v(s) > v \qquad \text{and hence} \qquad \gamma_x(s) < \gamma_x(t) - v(t - s)$$

by (11). In the same way we obtain that for $\omega_e \otimes \mu_\gamma^+$-a.e. (γ', t', x', v') there exists $\delta_e > 0$ such that for every $s \in (t' - \delta_e, t')$ it holds

$$\gamma_x(s) > \gamma_x(t') - v'(t' - s).$$

Finally, since π^- is concentrated on $\mathcal{G}$, then for π^--a.e. $(\gamma, t, x, v, \gamma', t, x', v')$ we set $\delta = \min(\delta_h, \delta_e) > 0$, so that for every $s \in (t - \delta, t)$ it holds

$$\gamma_x(s) < \gamma_x(t) - v(t - s) = x - v(t - s) = x' - v'(t' - s)\gamma'_x(t') - v'(t' - s) < \gamma'_x(s)$$

and this concludes Step 2.

By Step 2 it follows that for π^--a.e. $(\gamma, t, x, v, \gamma', t, x', v')$ there exists $(\bar{t}_i, \bar{x}_j) \in ([0, T] \cap \mathbb{Q}) \times \mathbb{Q}$ such that

$$(\gamma, t, x, v, \gamma', t, x', v') \in (G^l_{\bar{t}_i, \bar{x}_j} \times [\bar{t}_i, T] \times \mathbb{R} \times [0, 1]) \times (G^r_{\bar{t}_i, \bar{x}_j} \times [\bar{t}_i, T] \times \mathbb{R} \times [0, 1])$$

so that π^- is concentrated on

$$\bigcup_{i,j \in \mathbb{N}} (G^l_{\bar{t}_i, \bar{x}_j} \times [\bar{t}_i, T] \times \mathbb{R} \times [0, 1]) \times (G^r_{\bar{t}_i, \bar{x}_j} \times [\bar{t}_i, T] \times \mathbb{R} \times [0, 1]).$$

Therefore by Step 1 it follows that $(p^1_{t,x})_\sharp \pi^-$ is concentrated on

$$\bigcup_{i,j \in \mathbb{N}} C_{f_{\bar{t}_i, \bar{x}_j}} = J^-.$$

We finally observe that from (31) and (16) it follows that

$$(p^1_{t,x})_\sharp \pi^- = (p_{t,x})_\sharp ((p_1)_\sharp \pi^-) = (p_{t,x})_\sharp (\omega_h \otimes \mu^-_\gamma) = (p_{t,x})_\sharp \mu^-$$

and therefore the proof is completed. □

The analogous statement for the positive part of μ follows from the observation that reversing the direction of time turns entropy solutions into anti-entropic solutions.

Proposition 11. *The measure $(p_{t,x})_\sharp \mu^+$ is concentrated on a 1-rectifiable set J^+.*

Proof. We consider the two reflections $R : [0, T] \times \mathbb{R} \to [0, T] \times \mathbb{R}$ and $\bar{R} : X \to X$ defined by

$$R(t, x) = (T - t, -x) \qquad \text{and} \qquad \bar{R}(t, x, v) = (T - t, -x, v).$$

Observe that $R^{-1} = R$ and $\bar{R}^{-1} = \bar{R}$. From the kinetic formulation for u:

$$\chi(t, x, v) := \begin{cases} 1 & \text{if } u(t, x) > v \\ 0 & \text{otherwise} \end{cases}, \qquad \partial_t \chi + v \partial_x \chi = -\partial_v \mu$$

we deduce that $\tilde{\chi} := \chi \circ \bar{R}$ satisfies

$$\partial_t \tilde{\chi} + v \partial_x \tilde{\chi} = -\partial_v \tilde{\mu}, \qquad \text{with} \qquad \tilde{\mu} = -\bar{R}_\sharp \mu \tag{36}$$

and it is the kinetic formulation of

$$\tilde{u}(t, x) = \int_0^1 \tilde{\chi}(t, x, v) dv = (u \circ R)(t, x).$$

In particular $\tilde{u}$ is a weak solution with finite entropy production to the Burgers equation (1) and by Theorem 10 the measure $(\pi_{t,x})_\sharp \tilde{\mu}^-$ is concentrated on a 1-rectifiable set $\tilde{J}^-$. From (36) it follows that $\mu^+ = \bar{R}_\sharp \tilde{\mu}^-$ and therefore that $(p_{t,x})_\sharp \mu^+$ is concentrated on $J^+ := R(\tilde{J}^-)$, which is obviously 1-rectifiable. □

Theorem 1 immediately follows from Theorem 10 and Proposition 11 setting

$$J' = J^- \cup J^+.$$

We conclude by proving Theorem 2, which essentially follows from the observation that every negative shock of u is entropic and every positive shock is anti-entropic.

Proof. By (4) it follows that

$$|(p_{t,x})_\sharp \mu| = \mu_{\bar\eta} \vee \mu_{-\bar\eta},$$

where $\bar\eta(v) = v^2/2$. Therefore in order to prove the theorem, it is sufficient to check that for ν-a.e. $(t,x) \in [0,T] \times \mathbb{R}$ the supremum in (5) is attained at $\bar\eta$ or at $-\bar\eta$. Let us denote by $\tilde{J}^+ \subset J$ the set of points for which $u^- < u^+$ and by $\tilde{J}^- \subset J$ the set of points for which $u^+ < u^-$. In view of Theorem 1 and (7) it holds $\nu(([0,T] \times \mathbb{R}) \setminus (\tilde{J}^+ \cup \tilde{J}^-)) = 0$. We claim that

$$\nu \llcorner \tilde{J}^+ = \mu_{\bar\eta} \qquad \text{and} \qquad \nu \llcorner \tilde{J}^- = \mu_{-\bar\eta}. \tag{37}$$

We consider the second equality: by the Rankine-Hugoniot conditions the normal to J in (7) is determined for $\mathcal{H}^1$-a.e. $(t,x) \in J$ by

$$\mathbf{n} = (\mathbf{n}_t, \mathbf{n}_x) = \frac{1}{\sqrt{1+\lambda^2}}(\lambda, -1), \qquad \text{where} \qquad \lambda = \frac{u^- + u^+}{2}.$$

In particular the density in (7) takes the form

$$\frac{1}{\sqrt{1+\lambda^2}}\left(q(u^-) - q(u^+) - \lambda(\eta(u^-) - \eta(u^+))\right)$$

$$= \frac{1}{\sqrt{1+\lambda^2}} \int\limits_{u^+}^{u^-} (q'(v) - \lambda\eta'(v))dv$$

$$= \frac{1}{\sqrt{1+\lambda^2}} \int\limits_{u^+}^{u^-} (v - \lambda)\eta'(v)dv$$

$$= \frac{(v-\lambda)^2}{2\sqrt{1+\lambda^2}}\bigg|_{u^+}^{u^-} - \frac{1}{\sqrt{1+\lambda^2}} \int\limits_{u^+}^{u^-} \frac{(v-\lambda)^2}{2}\eta''(v)dv,$$

which is maximized by $\eta = -\bar\eta$ in the set $\{\eta \in C^2(\mathbb{R}) : \|\eta''\|_{C^0} \leq 1\}$. Being the first inequality of (37) completely analogous, this concludes the proof. $\square$

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