mathematics

MDPI

Article

# Beltrami Equations on Rossi Spheres

**Elisabetta Barletta** †, **Sorin Dragomir** *,† and **Francesco Esposito** †

Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, 85100 Potenza, Italy; elisabetta.barletta@unibas.it (E.B.); f.esposito@unibas.it (F.E.)
* Correspondence: sorin.dragomir@unibas.it; Tel.: +39-0971-205843
† These authors contributed equally to this work.

**Abstract:** Beltrami equations $\overline{L}_t(g) = \mu(\cdot\,,\,t)\,L_t(g)$ on $S^3$ (where $L_t$, $|t| < 1$, are the Rossi operators i.e., $L_t$ spans the globally nonembeddable CR structure $\mathcal{H}(t)$ on $S^3$ discovered by H. Rossi) are derived such that to describe quasiconformal mappings $f : S^3 \to N \subset \mathbb{C}^2$ from the Rossi sphere $\big(S^3\,,\,\mathcal{H}(t)\big)$. Using the Greiner–Kohn–Stein solution to the Lewy equation and the Bargmann representations of the Heisenberg group, we solve the Beltrami equations for Sobolev-type solutions $g_t$ such that $g_t - v \in W^{1,2}_F\big(S^3,\,\theta\big)$ with $v \in \mathrm{CR}^\infty\big(S^3, \mathcal{H}(0)\big)$.

**Keywords:** CR manifold; Tanaka–Webster connection; Fefferman metric; Lewy operator; Heisenberg group; quasiconformal map; Beltrami equation; Rossi sphere; Bargmann representation; Fourier transform

## 1. Introduction and Statement of Main Result

Let $M$ be a 3-dimensional nondegenerate CR manifold, equipped with the CR structure $\mathcal{H}$. The global CR embedding problem for $M$ is to find a nondegenerate real hypersurface $N \subset \mathbb{C}^2$ and a CR isomorphism of $(M,\,\mathcal{H})$ onto $(N,\,T_{1,0}(N))$, where

$$T_{1,0}(N) = \big[T(N) \otimes \mathbb{C}\big] \cap T^{1,0}(\mathbb{C}^2)$$

is the CR structure on $N$ induced by the complex structure on $\mathbb{C}^2$. H. Rossi has produced (cf. [1]) a 1-parameter family $\big\{\mathcal{H}(t)\big\}_{|t|<1}$ of strictly pseudoconvex CR structures on the sphere $S^3$ such that none of the CR manifolds $\big(S^3, \mathcal{H}(t)\big)$, $t \neq 0$ (the *Rossi spheres*) is globally embeddable (cf. also D.M. Burns [2]). One of the purposes of the present paper is to start studying a natural weakening of the global CR embedding problem, seeking for an at least $K$-quasiconformal mapping from $M$ onto $N$. The problem is specialized to

$$(M,\,\mathcal{H}) \in \big\{\big(S^3\,,\,\mathcal{H}(t)\big)\;:\;|t| < 1\big\}.$$

A quasiconformal mapping $f = \big(f^1\,,\,f^2\big) : S^3 \to N$ (in the sense of A. Koranyi and H.M. Reimann [3]) is in particular a contact transformation of positive dilation $\lambda(f) > 0$, and then a vector bundle morphism $\mu_f(t) = \mu\big(f,\,\mathcal{H}(t)\big) : \mathcal{H}(t) \to \mathcal{H}(t)$ (the *complex dilation* of $f$) may be built such that quasiconformality is characterized by the Beltrami equations

$$\overline{L}_t\big(f^j\big) = \mu\big(\cdot\,,\,t\big)\,L_t\big(f^j\big), \quad j \in \{1,2\}, \quad |t| < 1, \tag{1}$$

where the functions $\mu\big(\cdot\,,\,t\big) : S^3 \to \mathbb{C}$ are determined by

$$\mu_f(t)\,L_t = \mu\big(\cdot\,,\,t\big)\,L_t\,,$$

$$L_t = Z + t\,\overline{Z}\,, \quad Z = \overline{w}\,\frac{\partial}{\partial z} - \overline{z}\,\frac{\partial}{\partial w}\,.$$

check for updates

**Citation:** Barletta, E.; Dragomir, S.; Esposito, F. Beltrami Equations on Rossi Spheres. *Mathematics* **2022**, *10*, 371. https://doi.org/10.3390/math10030371

Academic Editor: Juan De Dios Pérez

Received: 6 January 2022
Accepted: 19 January 2022
Published: 25 January 2022

Building on an idea by C-Y. Hsiao and P-L. Yung (cf. [4]) we use the canonical CR isomorphism (induced by the Cayley map) $H : U = S^3 \setminus \{(0, -1)\} \approx \mathbb{H}_1$ to transform the Beltrami Equation (1) into

$$\overline{u}\, \overline{V}(f) = \frac{\lambda(\,\cdot\,, t) - t}{1 - t\,\lambda(\,\cdot\,, t)}\, u\, V(f), \tag{2}$$

$$u(\zeta, t) = \frac{1}{2}\, \frac{\left(|\zeta|^2 - i\,\tau + 1\right)^2}{|\zeta|^2 + i\,\tau + 1}, \quad (\zeta, \tau) \in \mathbb{H}_1,$$

$$\lambda(x, t) = \mu\big(H^{-1}(x), t\big), \quad x \in \mathbb{H}_1, \ \ |t| < 1,$$

where $V \equiv \dfrac{\partial}{\partial\zeta} + i\,\overline{\zeta}\,\dfrac{\partial}{\partial\tau}$ (so that $\overline{V}$ is the unsolvable Lewy operator). Our main result is as follows.

**Theorem 1.** *Let* $\{\mu(\,\cdot\,, t)\}_{|t|<1}$ *be a smooth* 1*-parameter family of measurable functions* $\mu(\,\cdot\,, t) : S^3 \to \mathbb{C}$ *of compact support*

$$\operatorname{Supp}\big[\mu(\,\cdot\,, t)\big] \subset S^3 \setminus \{(0, -1)\}, \quad |t| < 1,$$

*such that*

$$\big\|\mu(\,\cdot\,, t)\big\|_\infty = \operatorname{ess\,sup}_{p \in S^3} \big|\mu(p, t)\big| < \frac{1 - |t|\sqrt{2}}{\sqrt{2} + |t|}.$$

*Let* $v \in \mathrm{CR}^\infty\big(S^3\big)$ *be a CR function* [*i.e.,* $\overline{Z}(v) = 0$]. *Let us set*

$$\alpha(x, t) = \frac{\lambda(x, t) - t}{1 - t\,\lambda(x, t)} \left[\frac{u(x)}{|u(x)|}\right]^2, \quad x \in \mathbb{H}_1, \quad |t| < 1.$$

*If one of the following conditions holds,*

(i) $\alpha(\,\cdot\,, t) \in L^2_+\big(\mathbb{H}_1, \theta_0\big)$, $\alpha(\,\cdot\,, t)\, V\big(v \circ H^{-1}\big) \in L^2_+\big(\mathbb{H}_1, \theta_0\big)$,
(ii) $\alpha(\,\cdot\,, t) \in D_{-2}$, $\alpha(\,\cdot\,, t)\, V\big(v \circ H^{-1}\big) \in D_{-1}$,
(iii) $\alpha(\,\cdot\,, t) \in D_{-2} \cap L^2_-\big(\mathbb{H}_1, \theta_0\big)$, $\alpha(\,\cdot\,, t)\, V\big(v \circ H^{-1}\big) \in D_{-1} \cap L^2_-\big(\mathbb{H}_1, \theta_0\big)$,

*then the Beltrami Equation* (2) *has a unique solution* $f_t$ *such that* $f_t - v \circ H^{-1} \in W^{1,2}_E\big(\mathbb{H}_1, \theta_0\big)$. *Consequently* $g_t = f_t \circ H$ *is a solution to*

$$\overline{L}_t(g) = \mu(\,\cdot\,, t)\, L_t(g) \tag{3}$$

*such that* $g_t - v \in W^{1,2}_F(U, \theta)$.

Here the spaces $L^2_\pm\big(\mathbb{H}_1, \theta_0\big) \subset L^2\big(\mathbb{H}_1, \theta_0\big)$ are

$$L^2_-\big(\mathbb{H}_1, \theta_0\big) = \big\{f \in L^2\big(\mathbb{H}_1, \theta_0\big) \ : \ \hat{f}(\lambda) = 0 \ \text{ a.e. } \ \lambda > 0\big\},$$

$$L^2_+\big(\mathbb{H}_1, \theta_0\big) = L^2\big(\mathbb{H}_1, \theta_0\big) \ominus L^2_-\big(\mathbb{H}_1, \theta_0\big),$$

and $\hat{f}(\lambda)$ is the Fourier transform of $f$ at $\lambda \in \mathbb{R} \setminus \{0\}$. The meaning of the sets $\{D_j\}_{j \in \mathbb{Z}}$ will be explained in Section 3.

The paper is organized as follows.

Section 2.1 is devoted to pseudohermitian geometry on a Rossi sphere $\big(S^3, \mathcal{H}(t)\big)$. We show that Rossi's CR structures $\{\mathcal{H}(t) \, : \, |t| < 1\}$ have the same Levi distribution (i.e., the maximally complex distribution associated to the standard CR structure $\mathcal{H}(0) = T_{1,0}(S^3)$) and, therefore, the same contact forms. We compute the pseudohermitian geometric objects of interest (the Tanaka–Webster connection, Fefferman's metric, etc.) of a Rossi sphere endowed with the canonical contact form $\theta = \frac{i}{2}\big(z\, d\overline{z} + w\, d\overline{w} - \overline{z}\, dz - \overline{w}\, dw\big)$.

Section 2.2 discusses the Folland–Stein spaces

$$W_H^{1,2}(M,\theta),\quad W_E^{1,2}(U,\iota^*\theta),$$

on a strictly pseudoconvex CR manifold $(M, T_{1,0}(M))$, equipped with the positively oriented contact form $\theta$, and $E = \{E_a : 1 \le a \le 2n\}$ is a $G_\theta$-orthonormal (local) frame of the Levi distribution $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$, defined on the open set $\iota : U \subset M$. If $U$ is also the domain of a local coordinate neighborhood $\chi : U \to \mathbb{R}^{2n+1}$, then $X \equiv \{\chi_* E_a : 1 \le a \le 2n\}$ is a Hörmander system of vector fields on $\Omega = \chi(U)$ (e.g., in the sense of [5]) and $W_E^{1,2}(U,\theta)$ are essentially the Sobolev-type spaces $W_X^{1,2}(\Omega)$ (e.g., in [6,7]). Our Theorem 2 in this section accounts for the fact that solving (3) in $W_F^{1,2}(S^3,\theta)$ is the same as solving (2) in $W_E^{1,2}(\mathbb{H}_1,\theta_0)$.

Section 2.3 discusses the basic differential geometric facts on quasiconformal maps of 3-dimensional nondegenerate CR manifolds and gives a proof of a characterization of $K$-quasiconformality due to A. Koranyi and H.M. Reimann (cf. [3]) yet proved by them only for the Heisenberg group.

In Section 2.4, we derive the Beltrami equations, describing quasiconformal maps of the Rossi sphere $(S^3, \mathcal{H}(t))$ into a real hypersurface $N \subset \mathbb{C}^2$.

Section 3 collects the needed tools of harmonic analysis (e.g., the Bargmann representations of the Heisenberg group $\mathbb{H}_1$, the corresponding Fourier transform of $f \in \mathcal{S}(\mathbb{H}_1)$, and the orthogonal decomposition $L^2(\mathbb{H}_1,\theta_0) = \bigoplus_{k\in\mathbb{Z}} U^k$) and complex analysis (e.g., the solution to the inhomogeneous tangential Cauchy–Riemann equations $\overline{V}(f) = g$ on $\mathbb{H}_1$) and provides the proof to Theorem 1.

## 2. Rossi's Spheres

### *2.1. CR Structures, Levi Form, Tanaka–Webster Connection*

We review a few notations, conventions and basic results in Cauchy–Riemann and pseudohermitian geometry, by mainly following the monograph [8].

#### 2.1.1. CR Manifolds, Pseudohermitian Structures

Let $M$ be a 3-dimensional, orientable, $C^\infty$ manifold. A *CR structure* on $M$ is a complex line subbundle $\mathcal{H} \subset T(M) \otimes \mathbb{C}$ such that

$$\mathcal{H}_x \cap \overline{\mathcal{H}}_x = (0),\quad x \in M.$$

The *tangential Cauchy–Riemann operator* is the first order differential operator

$$\overline{\partial}_{\mathcal{H}} : C^1(M,\mathbb{C}) \to C(\overline{\mathcal{H}}^*),$$

$$(\overline{\partial}_{\mathcal{H}} v)\overline{W} = \overline{W}(v),\quad v \in C^1(M,\mathbb{C}),\ \ W \in \mathcal{H}.$$

A *CR function* on $M$ is a $C^1$ solution $v$ to the *tangential CR equations* $\overline{\partial}_{\mathcal{H}} v = 0$. Let $\mathrm{CR}^k(M,\mathcal{H})$ be the space of all CR functions of class $C^k$, $k \ge 1$.

Let $H(M) = \mathrm{Re}\{\mathcal{H} \oplus \overline{\mathcal{H}}\}$ be the *Levi distribution*. It carries the complex structure

$$J : H(M) \to H(M),\ \ J(Z + \overline{Z}) = i(Z - \overline{Z}),\ \ Z \in \mathcal{H},$$

(with $i = \sqrt{-1}$). The *conormal bundle* is the real line subbundle $H(M)^\perp \subset T^*(M)$ given by

$$H(M)_x^\perp = \{\omega \in T_x^*(M) \,:\, \mathrm{Ker}(\omega) \supset H_x\},\quad x \in M.$$

The conormal bundle is trivial (i.e., $H(M)^\perp \approx M \times \mathbb{R}$, a vector bundle isomorphism), and hence it admits globally defined nowhere zero $C^\infty$ sections $\theta$, each of which is referred to as a *pseudohermitian structure* on $M$. Let $\mathcal{P} = \mathcal{P}(M, \mathcal{H})$ be the set of all pseudohermitian structures on $M$. For every $\theta \in \mathcal{P}$, the *Levi form* $G_\theta$ is

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M).$$

The CR structure $\mathcal{H}$ is *nondegenerate* if the Levi form $G_\theta$ is nondegenerate (i.e., $G_\theta(X, Y) = 0$ for every $Y \in H(M)$ yields $X = 0$) for some $\theta \in \mathcal{P}$. If $\mathcal{H}$ is nondegenerate, then every $\theta \in \mathcal{P}$ is a contact form, i.e., $\Psi = \theta \wedge d\theta$ is a volume form on $M$, and $\mathcal{P}$ splits into two orientation classes $\mathcal{P}_\pm = \mathcal{P}_\pm(M, \mathcal{H})$. A contact form $\theta \in \mathcal{P}_+$ is *positively oriented* (the Levi form $G_\theta$ is positive definite). For every $\theta \in \mathcal{P}_+$, the *Webster metric* is the Riemannian metric determined by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X, Y \in H(M)$.

2.1.2. Tanaka–Webster Connection, Canonical Circle Bundle, Fefferman's Metric

The *Tanaka–Webster connection* of $(M, \theta)$ is the linear connection $\nabla$ on $M$ uniquely determined by the following axioms: (i) $H(M)$ is parallel with respect to $\nabla$ i.e., $\nabla_Y X \in H(M)$ for any $X \in H(M)$ and any $Y \in \mathfrak{X}(M)$, (ii) the complex structure $J$ along $H(M)$ and the Webster metric $g_\theta$ are parallel with respect to $\nabla$ i.e., $\nabla J = 0$ and $\nabla g_\theta = 0$, (iii) the torsion $T_\nabla$ is pure, i.e.,

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2i\, G_\theta(Z, \overline{W})\, T,$$

$$\tau \circ J + J \circ \tau = 0, \quad \tau(Y) \equiv T_\nabla(T, Y),$$

$$Z,\, W \in \mathcal{H}, \quad Y \in \mathfrak{X}(M).$$

$\tau$ is the *pseudohermitian torsion* of the Tanaka–Webster connection $\nabla$. By a result of S.M. Webster (cf., for example, [8]), $\tau$ is self-adjoint (i.e., $g_\theta(\tau X, Y) = g_\theta(X, \tau Y)$) and $\tau(\mathcal{H}) \subset \overline{\mathcal{H}}$ (in particular, $\tau$ is traceless, i.e., $\mathrm{trace}(\tau) = 0$).

For every $C^1$ vector field $X$ on $M$, the divergence of $X$ is determined by $\mathcal{L}_X \Psi = \mathrm{div}(X)\, \Psi$ where $\mathcal{L}_X$ denotes the Lie derivative at $X$. The divergence of a vector field is most easily calculated as the trace of the covariant derivative, with respect to the Tanaka–Webster connection $\nabla$. Indeed (by axiom (ii) above), $\nabla \Psi = 0$, and hence,

$$\mathrm{div}(X) = \mathrm{trace}\Big\{ Y \longmapsto \nabla_Y X \Big\}.$$

A complex valued $p$-form $\eta \in \Omega^p(M) = C^\infty\big(\Lambda^p T^*(M) \otimes \mathbb{C}\big)$ is a $(p, 0)$*-form* if $\overline{\mathcal{H}} \rfloor \eta = 0$. Let $\Lambda^{p,0}(M) \to M$ be the relevant vector bundle (so that $\Omega^{p,0}(M) = C^\infty\big(\Lambda^{p,0}(M)\big)$ is the space of all $(p, 0)$-forms on $M$). Then $K(M, \mathcal{H}) = \Lambda^{n+1,0}(M)$ is a complex line bundle (the *canonical bundle* over $M$). $\mathbb{R}_+ = \mathrm{GL}^+(1, \mathbb{R})$ (the multiplicative positive reals) acts freely on $K_0(M, \mathcal{H}) = K(M, \mathcal{H}) \setminus \{\text{zero section}\}$, thus organizing the quotient space $C(M, \mathcal{H}) = K_0(M, \mathcal{H})/\mathbb{R}_+$ as the total space of a principal circle bundle $S^1 \to C(M, \mathcal{H}) \stackrel{\pi}{\to} M$. If $\omega \in K(M, \mathcal{H})_x$ with $\omega \neq 0$ then $[\omega] \in C\big(M, \mathcal{H}\big)_x$ denotes the class of $\omega$ mod $\mathbb{R}_+$. Let us assume that $(M, \mathcal{H})$ is strictly pseudoconvex and let $\theta \in \mathcal{P}_+(M, \mathcal{H})$. Let $\{T_\alpha : 1 \leq \alpha \leq n\} \subset C^\infty\big(U, \mathcal{H}\big)$ be a local frame of $\mathcal{H}$, defined on the open subset $U \subset M$. Let $T \in \mathfrak{X}(M)$ be the Reeb vector field of $(M, \theta)$. Let $\{\theta^\alpha : 1 \leq \alpha \leq n\}$ be the complex 1-forms on $U$ determined by

$$\theta^\alpha\big(T_\beta\big) = \delta^\alpha_\beta, \quad \theta^\alpha\big(T_{\overline{\beta}}\big) = 0, \quad \theta^\alpha\big(T\big) = 0.$$

$\{\theta^\alpha : 1 \leq \alpha \leq n\}$ is an *admissible coframe*. Then

$$\omega = \lambda\,\big(\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n\big)_x$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$. A local trivialization chart of $C(M, \mathcal{H})$ is

$$\Phi : \pi^{-1}(U) \to U \times S^1, \quad \Phi([\omega]) = \Big(x, \frac{\lambda}{|\lambda|}\Big).$$

The *Fefferman metric* is the Lorentzian metric $F_\theta \in \mathrm{Lor}\big[C(M, \mathcal{H})\big]$ given by (cf. [8] pp. 128–129)

$$F_\theta = \pi^* \tilde{G}_\theta + 2\big(\pi^*\theta\big) \odot \sigma, \tag{4}$$

where

$$\sigma = \frac{1}{n+2}\Big\{d\mathbf{s} + \pi^*\Big(i\,\omega_\alpha{}^\alpha - \frac{i}{2}\,g^{\alpha\overline{\beta}}\,dg_{\alpha\overline{\beta}} - \frac{R}{4(n+1)}\,\theta\Big)\Big\} \tag{5}$$

a connection 1-form on the principal bundle $S^1 \to C(M, \mathcal{H}) \to M$ (the *Graham connection*, cf. [9]). As to the notation in (4) and (5), the (degenerate) $(0,2)$-tensor field $\tilde{G}_\theta$ extends the Levi form $G_\theta$ to the whole of $T(M)$ by requesting that $\tilde{G}_\theta(T, W) = 0$ for any $W \in \mathfrak{X}(M)$ (and $\tilde{G}_\theta = G_\theta$ on $H(M) \otimes H(M)$). Additionally, $\mathbf{s}$ is a local fiber coordinate on $C(M, \mathcal{H})$ [a detailed description of $\mathbf{s}$ for $(M, \mathcal{H}) = \big(S^3, \mathcal{H}(t)\big)$ (a Rossi sphere) is given in Section 2.1.5]. Moreover,

$$g_{\alpha\overline{\beta}} = G_\theta\big(T_\alpha, T_{\overline{\beta}}\big), \quad \big[g^{\alpha\overline{\beta}}\big] = \big[g_{\alpha\overline{\beta}}\big],$$

$$\nabla T_\alpha = \omega_\alpha{}^\beta\, T_\beta, \quad R = g^{\alpha\overline{\beta}}\, R_{\alpha\overline{\beta}},$$

and $R_{\alpha\overline{\beta}}$ is the pseudohermitian Ricci tensor (cf. [8], p. 50).

#### 2.1.3. Heisenberg Group, Rossi Spheres

Let $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$ be the Heisenberg group, with the group law

$$(z, t) \cdot (\zeta, \tau) = \big(z + \zeta,\ t + \tau + 2\,\mathrm{Im}(z\overline{\zeta})\big),$$

for any $z, \zeta \in \mathbb{C}$ and $t, \tau \in \mathbb{R}$. The complex vector field $V = \partial/\partial\zeta + i\,\overline{\zeta}\,\partial/\partial\tau$ spans the left invariant CR structure $\mathcal{H}_x = \mathbb{C}V_x$, with $x \in \mathbb{H}_1$. Here, $\overline{V}$ is the *Lewy operator* and the tangential CR equations on $\mathbb{H}_1$ are $\overline{V}(F) = 0$. For instance, if $F(\zeta, \tau) = |\zeta|^2 - i\,\tau$, then $F \in \mathrm{CR}^\infty\big(\mathbb{H}_1, \mathcal{H}\big)$.

Let $S^3 = \big\{(z, w) \in \mathbb{C}^2 \ :\ z\overline{z} + w\overline{w} = 1\big\}$ be the standard sphere. The CR structure

$$T_{1,0}(S^3) = \big[T(S^3) \otimes \mathbb{C}\big] \cap T^{1,0}(\mathbb{C}^2)$$

(the *canonical CR structure* on $S^3$) is the span of $T_1 = \overline{w}\,\partial/\partial z - \overline{z}\,\partial/\partial w$. Let $H(S^3)$ be the Levi distribution of the CR manifold $\big(S^3, T_{1,0}(S^3)\big)$. Let us set

$$L_t = T_1 + t\,T_{\overline{1}}, \quad |t| < 1, \tag{6}$$

$$\mathcal{H}(t)_x = \big\{\lambda\, L_{t,x} \ :\ \lambda \in \mathbb{C}\big\}, \quad x \in S^3.$$

Here, $T_{\overline{1}} = \overline{T_1}$. Then, we have the following:

(i) $\mathcal{H}(t)$ is a nondegenerate CR structure on $S^3$ [such that $\mathcal{H}(0) = T_{1,0}(S^3)$].

(ii) The Levi distributions of $\big(S^3, \mathcal{H}(t)\big)$ and $\big(S^3, T_{1,0}(S^3)\big)$ coincide, i.e.,

$$\mathrm{Re}\Big\{\mathcal{H}(t) \oplus \overline{\mathcal{H}(t)}\Big\} = H(S^3), \quad |t| < 1.$$

(iii) The CR manifolds $\big(S^3, \mathcal{H}(t)\big)$ have the same positively oriented contact forms, i.e.,

$$\mathcal{P}_+\big(S^3, \mathcal{H}(t)\big) = \mathcal{P}_+\big(S^3, T_{1,0}(S^3)\big).$$

To prove (i)–(iii), we need some preparation. Let us consider the (real valued) differential 1-form $\theta \in \Omega^1(S^3)$ given by

$$\theta = \mathbf{j}^* \left[ \frac{i}{2} \left( z\, d\overline{z} + w\, d\overline{w} - \overline{z}\, dz - \overline{w}\, dw \right) \right] \tag{7}$$

(with $\mathbf{j} : S^3 \subset \mathbb{C}^2$). Then, we have the following:

Step 1. *$\theta \in \mathcal{P}_+ \left[ S^3 ,\ T_{1,0}(S^3) \right]$, i.e., $\theta$ is a positively oriented contact form on $S^3$ with respect to the ordinary CR structure $T_{1,0}(S^3)$.*

**Proof.** For simplicity, we drop $\mathbf{j}$. Then

$$d\theta = i \left( dz \wedge d\overline{z} + dw \wedge d\overline{w} \right)$$

and the Levi form $G_\theta$ is

$$G_\theta \left( T_1 , T_{\overline{1}} \right) = -i \, (d\,\theta) \left( T_1 , T_{\overline{1}} \right) = \left( dz \wedge d\overline{z} \right) \left( T_1 , T_{\overline{1}} \right) + \left( dw \wedge d\overline{w} \right) \left( T_1 , T_{\overline{1}} \right) =$$

$$= \frac{1}{2} \left\{ \left| dz \ (T_1) \right|^2 + \left| dw \ (T_1) \right|^2 \right\} = \frac{1}{2} \left\{ |z|^2 + |w|^2 \right\} = \frac{1}{2} > 0.$$

□

$\theta$ is referred to as the *canonical contact form* on $S^3$. The *Reeb vector* field of $\left( S^3 , \theta \right)$ is the nowhere zero globally defined vector field $T \in \mathfrak{X}(S^3)$ determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$.

Step 2. *The Reeb vector field $T$ of $(S^3 , \theta)$ is given by*

$$T = i \left( z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w} - \overline{z} \frac{\partial}{\partial \overline{z}} - \overline{w} \frac{\partial}{\partial \overline{w}} \right).$$

An *adapted coframe* is a frame $\{\theta^1\}$ in $T_{1,0}(S^3)^*$ such that

$$\theta^1 \left( T_1 \right) = 1, \quad \theta^1 \left( T_{\overline{1}} \right) = 0, \quad \theta^1 \left( T \right) = 0.$$

Step 3. *$\theta^1 = w\, dz - z\, dw$ is an adapted coframe on $(S^3 , \theta)$.*

We may now complete the proof of (i)–(iii). The complex distribution $\mathcal{H}(t) \oplus \overline{\mathcal{H}(t)}$ is the span of $\{ L_t , \overline{L}_t \}$ and then [by (6)] the span of $\{ T_1 , T_{\overline{1}} \}$. Hence, the CR manifolds $\left\{ \left( S^3 , \mathcal{H}(t) \right) \right\}_{|t|<1}$ have the same Levi distribution (i.e., $H\left(S^3\right) = \mathrm{Re}\left\{ \mathcal{H}(t) \oplus \overline{\mathcal{H}(t)} \right\}$) and therefore, the same pseudohermitian structures (i.e., $\mathcal{P}\left(S^3 , T_{1,0}(S^3)\right) = \mathcal{P}\left(S^3 , \mathcal{H}(t)\right)$).

Step 4. *The Levi form of $\left( S^3 , \mathcal{H}(t) \right)$*

$$G^t_\theta \left( X , Y \right) = (d\theta) \left( X,\ J^t Y \right), \quad X, Y \in H(S^3),$$

$$J^t : H(S^3) \to H(S^3), \quad J^t \left( Z + \overline{Z} \right) = i \left( Z - \overline{Z} \right), \quad Z \in \mathcal{H}(t),$$

*is given by*

$$G^t_\theta \left( L_t , \overline{L}_t \right) = \frac{1 - t^2}{2} > 0. \tag{8}$$

**Proof.** Indeed,

$$G^t_\theta \left( L_t , \overline{L}_t \right) =$$

[as $J^t L_t = i\, L_t$ and $J^t \overline{L}_t = -i\, \overline{L}_t$]

$$= -i \, (d\theta) \left( L_t , \overline{L}_t \right) =$$

$$= \frac{1}{2}\left\{\left|dz\,(L_t)\right|^2 - \left|dz\,(\overline{L}_t)\right|^2 + \left|dw\,(L_t)\right|^2 - \left|d\overline{w}\,(L_t)\right|^2\right\} =$$

$$= \frac{1-t^2}{2}\left(|z|^2 + |w|^2\right) = \frac{1-t^2}{2} > 0$$

proving that $\theta \in \mathcal{P}_+\big(S^3\,,\,\mathcal{H}(t)\big)$ and then

$$\mathcal{P}_+\big(S^3\,,\,\mathcal{H}(t)\big) = \mathcal{P}_+\big(S^3\,,\,\mathcal{H}(0)\big).$$

□

2.1.4. Tanaka–Webster Connection of a Rossi Sphere

We shall need the following commutation table

$$[T,\,T_1] = -2i\,T_1\,,\quad [T,\,T_{\overline{1}}] = 2i\,T_{\overline{1}}\,,\quad [T_1\,,\,T_{\overline{1}}] = -i\,T, \tag{9}$$

$$[T,\,L_t] = -\frac{2i(1+t^2)}{1-t^2}\,L_t + \frac{4it}{1-t^2}\,\overline{L}_t\,,\quad [L_t\,,\,\overline{L}_t] = -i(1-t^2)\,T. \tag{10}$$

Let $\nabla^t$ be the Tanaka–Webster connection of $\big(S^3\,,\,\mathcal{H}(t),\,\theta\big)$, where $\theta$ is given by (7), and let $\omega_t$ be the connection 1-form associated to the frame $\{L_t\} \subset C^\infty\big(\mathcal{H}(t)\big)$, i.e.,

$$\nabla^t L_t = \omega_t \otimes L_t\,,\quad \omega_t = \Gamma^1_{11}(t)\,\theta^1_t + \Gamma^1_{\overline{1}1}(t)\,\theta^{\overline{1}}_t + \Gamma^1_{01}(t)\,\theta\,.$$

Here, we have set

$$\theta^1_t = \frac{1}{1-t^2}\left(\theta^1 - t\,\theta^{\overline{1}}\right),\quad \theta^{\overline{1}}_t = \overline{\theta^1_t}\,,$$

so that $\{\theta^1_t\}$ is an adapted coframe relative to the CR structure $\mathcal{H}(t)$, i.e.,

$$\theta^1_t(L_t) = 1,\quad \theta^1_t(\overline{L}_t) = 0,\quad \theta^1_t(T) = 0.$$

By a result in [8] (p. 33),

$$\Gamma^1_{11}(t) = g^{1\overline{1}}(t)\Big\{L_t(g_{1\overline{1}}(t)) - g^t_\theta(L_t\,,\,[L_t\,,\,\overline{L}_t])\Big\}, \tag{11}$$

$$\Gamma^1_{\overline{1}1}(t) = g^{\overline{1}1}(t)\,g^t_\theta([\overline{L}_t\,,\,L_t],\,\overline{L}_t), \tag{12}$$

$$\Gamma^1_{01}(t) = g^{1\overline{1}}(t)\,g^t_\theta([T,\,L_t],\,\overline{L}_t), \tag{13}$$

where

$$g^{1\overline{1}}(t) = \frac{1}{g_{1\overline{1}}(t)}\,,\quad g_{1\overline{1}}(t) = G_\theta(L_t\,,\,\overline{L}_t) = \frac{1-t^2}{2}\,,$$

and $g^t_\theta$ is the Webster metric of $\big(S^3\,,\,\mathcal{H}(t),\,\theta\big)$, i.e., $g^t_\theta(X,Y) = G^t_\theta(X,Y)$, $g^t_\theta(X,T) = 0$ and $g^t_\theta(T,T) = 1$ for any $X, Y \in H(S^3)$. Substitution from (9) and (10) into (11)–(13) gives

$$\Gamma^1_{11}(t) = \Gamma^1_{\overline{1}1}(t) = 0,\quad \Gamma^1_{01}(t) = -\frac{2i(1+t^2)}{1-t^2}\,, \tag{14}$$

$$\omega_t = \Gamma^1_{01}(t)\,\theta = -\frac{2i(1+t^2)}{1-t^2}\,\theta. \tag{15}$$

The pseudohermitian torsion $\tau_t$ of $\nabla^t$ is given by

$$\tau_t(L_t) = A^{\overline{1}}_1(t)\,\overline{L}_t\,,\quad A^{\overline{1}}_1(t) = -\frac{4\,i\,t}{1-t^2}\,.$$

Let $R^t = R^{\nabla^t}$ be the curvature tensor field of $\nabla^t$. With the convention in [8], p. 50, the only nonzero component of $R^t$, with respect to the frame $\{L_t\}$, is

$$R^t(L_t\,,\,\overline{L}_t)L_t = R_1{}^1{}_{1\overline{1}}(t)\,, \quad R_1{}^1{}_{1\overline{1}}(t) = 2(1+t^2). \tag{16}$$

In particular, the pseudohermitian scalar curvature of $\nabla^t$ is

$$R(t) = g^{1\overline{1}}\,R_1{}^1{}_{1\overline{1}}(t) = \frac{4(1+t^2)}{1-t^2}\,.$$

To prove (16), one starts from (cf. [8], p. 51)

$$R^t(X,\,Y)L_t = 2\,(d\omega_t)(X,Y)\,L_t\,.$$

In addition (by taking the exterior differential of (15))

$$d\omega_t = -\frac{2i(1+t^2)}{1-t^2}\,d\theta$$

and hence (as $G^t_\theta = -i\,d\theta$ on $\mathcal{H}(t)\otimes\overline{\mathcal{H}(t)}$)

$$R^t(L_t\,,\,\overline{L}_t)L_t = \frac{4(1+t^2)}{1-t^2}\,G^t_\theta(L_t\,,\,\overline{L}_t)\,L_t = 2(1+t^2)\,L_t\,.$$

#### 2.1.5. Fefferman's Metric of Rossi's Sphere

Let $C\big(S^3\,,\,T_{1,0}(S^3)\big)$ be the canonical circle bundle over $(S^3,\,T_{1,0}(S^3))$. Then

$$C\big(S^3\,,\,T_{1,0}(S^3)\big)_x = \Big\{\big[\lambda\,(\theta\wedge\theta^1)_x\big]\;:\;\lambda\in\mathbb{C}\setminus\{0\}\Big\},\quad x\in S^3\,.$$

Let $\eta\in\Omega^2(S^3) = C^\infty\big(\Lambda^2T^*(S^3)\otimes\mathbb{C}\big)$ be a type $(2,0)$-form relative to the CR structure $\mathcal{H}(t)$ i.e., $\overline{\mathcal{H}(t)}\,\rfloor\,\eta = 0$. Then

$$\eta = h\,\theta\wedge\theta^1_t = \frac{h}{1-t^2}\,\theta\wedge\big(\theta^1 - t\,\theta^{\overline{1}}\big)$$

for some $h\in C^\infty\big(S^3\,,\,\mathbb{C}\big)$. Hence, the canonical circle bundle over $\big(S^3\,,\,\mathcal{H}(t)\big)$ is

$$C\big(S^3\,,\,\mathcal{H}(t)\big)_x = \Big\{\Big[\lambda\big(\theta\wedge\theta^1 - t\,\theta\wedge\theta^{\overline{1}}\big)_x\Big]\;:\;\lambda\in\mathbb{C}\setminus\{0\}\Big\},\quad x\in S^3\,.$$

Let $\pi^t : C\big(S^3\,,\,\mathcal{H}(t)\big)\to S^3$ be the canonical projection. Fefferman's metric

$$F^t_\theta = F\big(\mathcal{H}(t),\,\theta\big)\in\mathrm{Lor}\big[C\big(S^3\,,\,\mathcal{H}(t)\big)\big]$$

is

$$F^t_\theta = (\pi^t)^*\,\widetilde{G^t_\theta} + 2\,\big((\pi^t)^*\theta\big)\odot\sigma_t\,,$$

$$\sigma_t = \frac{1}{3}\Big\{d\mathbf{s}_t + (\pi^t)^*\Big[i\,\omega_t - \frac{i}{2}\,g^{1\overline{1}}(t)\,dg_{1\overline{1}}(t) - \frac{1}{8}\,R(t)\,\theta\Big]\Big\},$$

$$\widetilde{G^t_\theta}(X,Y) = G^t_\theta(X,Y),\;\; \widetilde{G^t_\theta}(T,W) = 0,\;\; X,Y\in H(S^3),\; W\in\mathfrak{X}(S^3).$$

Additionally, $\mathbf{s}_t$ is a local fiber coordinate on $C\big(S^3\,,\,\mathcal{H}(t)\big)$ that we now describe in some detail. Let us consider the $C^\infty$ diffeomorphism

$$\Phi_t : C\big(S^3\,,\,\mathcal{H}(t)\big)\to S^3\times S^1\,,\quad \Phi_t([\omega]) = \Big(x,\;\frac{\alpha}{|\alpha|}\Big),$$

$$\omega = \alpha\big(\theta \wedge \theta^1 - t\,\theta \wedge \theta^{\bar 1}\big)_x, \quad x \in S^3, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Note that $\Phi_t\big([\omega]\big)$ is invariant under a transformation $\alpha' = b\,\alpha$ with $b \in \mathbb{R}_+$ hence $\Phi_t$ is a well defined $C^\infty$ diffeomorphism. For every $\varphi_0 \in \mathbb{R}$, we set

$$U(\varphi_0) = \{e^{i\varphi} \,:\, |\varphi - \varphi_0| < \pi\} \subset S^1,$$

$$\psi : (\varphi_0 - \pi,\, \varphi_0 + \pi) \to U(\varphi_0), \quad \psi(\varphi) = e^{i\varphi},$$
$$\arg : U(\varphi_0) \to (\varphi_0 - \pi,\, \varphi_0 + \pi), \quad \arg = \psi^{-1},$$
$$\mathcal{U}(\varphi_0) = \Phi_t^{-1}\big[S^3 \times U(\varphi_0)\big] \subset C\big(S^3, \mathcal{H}(t)\big),$$
$$\mathbf{s}_t = \mathbf{s}_{t,\varphi_0} : \mathcal{U}(\varphi_0) \to (\varphi_0 - \pi,\, \varphi_0 + \pi), \quad \mathbf{s}_t\big([\omega]\big) = \arg\left(\frac{\alpha}{|\alpha|}\right).$$

**Lemma 1.** *Fefferman's metric $F_\theta^t$ of Rossi's sphere $\big(S^3, \mathcal{H}(t), \theta\big)$ is given by*

$$F_\theta^t = (\pi^t)^*\,\widetilde{G_\theta^t} + \frac{2}{3}\big[(\pi^t)^*\theta\big] \odot d\mathbf{s}_t + \frac{1+t^2}{1-t^2}(\pi^t)^*(\theta \odot \theta), \tag{17}$$

$$\widetilde{G_\theta^t} = \frac{1+t^2}{1-t^2}\,\theta^1 \odot \theta^{\bar 1} - \frac{t}{1-t^2}\Big\{(\theta^1)^2 + (\theta^{\bar 1})^2\Big\}.$$

**Proof.** By (15) and (16)

$$\sigma_t = \frac{1}{3}\,d\mathbf{s}_t + \frac{1+t^2}{2(1-t^2)}\,\pi^*\theta$$

[the Graham connection 1-form of $\big(S^3, \mathcal{H}(t), \theta\big)$] yielding (17). The second statement in Lemma 1 follows from

$$\widetilde{G_\theta^t} = 2\,g_{1\bar 1}(t)\,\theta_t^1 \odot \theta_t^{\bar 1}.$$

□

2.1.6. Siegel Domain, Cayley Map

Let

$$\Omega = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 \,:\, \mathrm{Im}(\zeta_2) > |\zeta_1|^2\}$$

be the Siegel domain. We shall need the CR isomorphisms

$$\mathcal{C} : S^3 \setminus \{(0,-1)\} \to \partial\Omega, \quad \mathcal{C}(z,w) = \left(\frac{z}{1+w},\, i\,\frac{1-w}{1+w}\right), \tag{18}$$

$$\psi : \mathbb{H}_1 \to \partial\Omega, \quad \psi(\zeta,\tau) = \big(\zeta,\, \tau + i|\zeta|^2\big), \tag{19}$$
$$(z,w) \in S^3, \quad w \neq -1, \quad \zeta \in \mathbb{C}, \quad \tau \in \mathbb{R},$$
$$H : U = S^3 \setminus \{(0,-1)\} \to \mathbb{H}_1, \quad H = \psi^{-1} \circ \mathcal{C}.$$

Then (i) for every $(z,w) \in S^3 \setminus \{(0,-1)\}$

$$H(z,w) = \left(\frac{z}{1+w},\, \frac{2\,\mathrm{Im}(w)}{|1+w|^2}\right), \tag{20}$$

(ii) for every $|t| < 1$

$$H_*\,L_t = u^H\,V^H + t\,\overline{u^H}\,\overline{V^H}, \tag{21}$$

where $u \in C^\infty\big(\mathbb{H}_1, \mathbb{C}\big)$ is given by

$$u(\zeta, \tau) = \frac{1}{2} \frac{\big(|\zeta|^2 - i\,\tau + 1\big)^2}{|\zeta|^2 + i\tau + 1}, \quad (\zeta, \tau) \in \mathbb{H}_1 . \tag{22}$$

Here, $u^H = u \circ H$ and $V^H = V \circ H$. Formula (20) follows from (18) (the restriction of the Cayley map $\mathcal{C} : \mathbb{C}^2 \setminus \{w + 1 = 0\} \to \mathbb{C}^2$ to $S^3 \setminus \{(0, -1)\}$) and (19) (the canonical CR isomorphism of the Heisenberg group $\mathbb{H}_1$ onto the boundary of the Siegel domain $\Omega \subset \mathbb{C}^2$). Formula (21) follows from

$$(d_{(z,w)} H) L_{t,\,(z,w)} = \frac{1+\overline{w}}{(1+w)^2}\, V_{H(z,w)} + t\, \frac{1+w}{(1+\overline{w})^2}\, \overline{V}_{H(z,w)}$$

together with the observation that (22) yields $u(H(z,w)) = \dfrac{1+\overline{w}}{(1+w)^2}$.

Recall the CR function $F(\zeta, \tau) = |\zeta|^2 - i\,\tau$. If $Z = T_1$, then

$$H_* Z = \frac{1+\overline{w}}{(1+w)^2}\, V^H \tag{23}$$

so that

$$f(z, w) = \frac{1-w}{1+w}, \quad (z, w) \in U \subset S^3 ,$$

is a CR function, i.e., $f \in \mathrm{CR}^\infty\big(U, \mathcal{H}(0)\big)$. Indeed, $f = F \circ H$ and then (by (23)) $\overline{Z}(f) = 0$. As to the proof of (23), it follows from

$$H_*\, Z = \frac{1+\overline{w}+\rho}{(1+w)^2}\, V^H - \frac{i\,\overline{z}\,\rho}{(1+w)^2(1+\overline{w})} \left(\frac{\partial}{\partial \tau}\right)^H \tag{24}$$

where $\rho(z,w) = z\overline{z} + w\overline{w} - 1$.

### 2.2. Folland–Stein Spaces

Let $\big(M, \mathcal{H}\big)$ be a strictly pseudoconvex CR manifold, of CR dimension $n$, and let $\theta \in \mathcal{P}_+$ be a positively oriented contact form on $M$. Let $\Psi = \theta \wedge (d\theta)^n$ and let $L^2(M, \theta)$ consist of all measurable functions $u : M \to \mathbb{R}$ such that

$$\|u\|_{L^2} = \|u\|_{L^2(M,\theta)} = \left(\int_M |u|^2\, \Psi\right)^{1/2} < \infty.$$

One tacitly identifies functions coinciding almost everywhere. Let $L^2\big(H(M), \theta\big)$ consist of all sections $X : M \to H(M)$ such that $G_\theta(X, X)^{1/2} \in L^2\big(M, \theta\big)$, i.e.,

$$\|X\|_{L^2} = \|X\|_{L^2(H(M),\theta)} = \left(\int_M G_\theta(X, X)\ \Psi\right)^{1/2} < \infty\, .$$

A function $u \in L^2(M, \theta)$ is *weakly differentiable* along $H(M)$ if there is $X_u \in L^2\big(H(M), \theta\big)$ such that

$$\int_M G_\theta\big(X_u\,, Y\big)\, \Psi = - \int_M u\, \mathrm{div}(Y)\, \Psi$$

for any $Y \in C^\infty_0\big(H(M)\big)$. Such $X_u$ is uniquely determined, up to a set of measure zero. Let $\mathcal{D}\big(\nabla^H\big)$ consist of all weakly differentiable $u \in L^2(M, \theta)$ and let us consider the linear operator

$$\nabla^H : \mathcal{D}\big(\nabla^H\big) \subset L^2\big(M, \theta\big) \to L^2\big(H(M), \theta\big)$$

given by $\nabla^H u \equiv X_u$. Note that $C_0^\infty(M) \subset \mathcal{D}(\nabla^H)$ so that $\nabla^H$ is densely defined. The Sobolev-type space $W_H^{1,2}(M,\theta)$ is $\mathcal{D}(\nabla^H)$ equipped with the norm

$$\|u\|_{W_H^{1,2}} = \|u\|_{W_H^{1,2}(M,\theta)} = \left(\|u\|^2_{L^2(M,\theta)} + \|\nabla^H u\|_{L^2(H(M),\theta)}\right)^{1/2}.$$

Let $E \equiv \{E_a : 1 \le a \le 2n\} \subset C^\infty(U, H(M))$ be a local $G_\theta$-orthonormal frame (i.e., $G_\theta(E_a, E_b) = \delta_{ab}$) defined on the open set $U \subset M$. Let $i : U \to M$ be the inclusion. A function $u \in L^2(U, i^*\theta)$ is *weakly E-differentiable* if for every $a \in \{1, \cdots, 2n\}$, there is $v_a \in L^2(U, i^*\theta)$ such that

$$\int_U v_a\, \varphi\, \Psi = -\int_U u\{E_a(\varphi) + \varphi \operatorname{div}(E_a)\}\, \Psi \tag{25}$$

for any $\varphi \in C_0^\infty(U)$. Such $v_a$ is uniquely determined, up to a set of measure zero, and denoted by $E_a(u) := v_a$. The Folland–Stein space $W_E^{1,2}(U, i^*\theta)$ consists of all weakly $E$-differentiable functions $u \in L^2(U, i^*\theta)$ and is equipped with the norm

$$\|u\|_{W_E^{1,2}} = \|u\|_{W_E^{1,2}(U,i^*\theta)} = \left(\|u\|^2_{L^2(U,i^*\theta)} + \sum_{a=1}^{2n} \|E_a(u)\|^2_{L^2(U,i^*\theta)}\right)^{1/2}.$$

Then, we have the following:

(i) The restriction map $r_U : W_H^{1,2}(M,\theta) \to W_H^{1,2}(U, i^*\theta)$ is a bounded linear operator,
(ii) $W_H^{1,2}(U, i^*\theta) \approx W_E^{1,2}(U, i^*\theta)$ (an isomorphism of Banach spaces).

The proof of (i) is straightforward. To prove (ii), note first that

$$W_H^{1,2}(U, i^*\theta) = W_E^{1,2}(U, i^*\theta)$$

as vector spaces. Indeed if $u \in W_H^{1,2}(U, i^*\theta)$ then $\nabla^H u \in L^2(H(U), i^*\theta)$ is well defined and one may consider the functions

$$v_a := G_\theta(\nabla^H u, E_a), \quad 1 \le a \le 2n.$$

Then (by the Cauchy–Schwartz inequality)

$$\int_U |v_a|^2\, \Psi \le \int_U G_\theta(\nabla^H u, \nabla^H u)\, G_\theta(E_a, E_a)\, \Psi = \|\nabla^H u\|^2_{L^2} < \infty$$

so that $v_a \in L^2(U, i^*\theta)$. On the other hand (as $u$ is weakly differentiable along $H(U)$) for every $\varphi \in C_0^\infty(U)$

$$\int_U v_a\, \varphi\, \Psi = \int_U G_\theta(\nabla^H u, \varphi\, E_a)\, \Psi = -\int_U u \operatorname{div}(\varphi\, E_a)\, \Psi$$

so that $u \in W_E^{1,2}(U, i^*\theta)$. The opposite inclusion $W_H^{1,2}(U, i^*\theta) \supset W_E^{1,2}(U, i^*\theta)$ may be proved in the same manner. Next, let us observe that

$$\nabla^H u = \sum_{a=1}^{2n} E_a(u)\, E_a \tag{26}$$

for every $u \in W_E^{1,2}(U,\theta)$. Indeed, for every $X \in C_0^\infty(H(U))$

$$\int_U G_\theta(\nabla^H u - \sum_{a=1}^{2n} E_a(u)\, E_a, X)\, \Psi = -\int_U u \operatorname{div}(X)\, \Psi - \sum_{a=1}^{2n} \int_U E_a(u)\, \varphi_a\, \Psi =$$

(where we have set $\varphi_a := G_\theta\big(E_a\,,\,X\big) \in C_0^\infty(U)\big)$

$$= - \int_U u \, \operatorname{div}\big(X - \sum_{a=1}^{2n} \varphi_a \, E_a\big)\, \Psi = 0$$

so that $\nabla^H - \sum_{a=1}^{2n} E_a(u)\, E_a$ is orthogonal to $C_0^\infty\big(H(U)\big)$ [a dense subspace of $L^2\big(H(U),\, i^*\theta\big)$]. The identity (26) is proved. Finally, one may check that the identity $I$ of $W_E^{1,2}(U,\, i^*\theta)$ preserves the norms. Indeed (by (26)),

$$\|\nabla^H u\|_{L^2}^2 = \int_U G_\theta\big(\nabla^H u,\, \nabla^H u\big)\, \Psi = \sum_{a=1}^{2n} \int_U \big|E_a(u)\big|^2\, \Psi = \sum_{a=1}^{2n} \|E_a(u)\|_{L^2}^2\,.$$

It is customary to endow $\big(\mathbb{H}_1\,,\,V\big)$ with the *canonical contact form*

$$\theta_0 = d\tau + i\big(\zeta\, d\overline{\zeta} - \overline{\zeta}\, d\zeta\big).$$

Then $G_{\theta_0}\big(V,\, \overline{V}\big) = 1$. Additionally,

$$H^*\,\theta_0 = \lambda\big(H\,;\,\theta,\,\theta_0\big)\,\theta, \quad \lambda\big(H\,;\,\theta,\,\theta_0\big)(z,w) = \frac{2}{\big|1+w\big|^2}\,.$$

Let us set as customary $\zeta = \xi + i\,\eta$ and

$$X = \frac{\partial}{\partial \xi} + 2\,\eta\,\frac{\partial}{\partial \tau}\,, \quad Y = \frac{\partial}{\partial \eta} - 2\,\xi\,\frac{\partial}{\partial \tau}\,,$$

so that $V = \frac{1}{2}\big(X - i\,Y\big)$. Then

$$E \equiv \big\{E_1\,,\,E_2\big\}, \quad E_1 = \frac{1}{\sqrt{2}}\,X, \quad E_2 = \frac{1}{\sqrt{2}}\,Y,$$

is a (globally defined) $G_{\theta_0}$-orthonormal frame on $\mathbb{H}_1$. The CR isomorphism $H : U \approx \mathbb{H}_1$ induces $L^2\big(\mathbb{H}_1\,,\,\theta_0\big) \approx L^2\big(U\,,\,\theta\big)$ (a vector space isomorphism). Indeed, if $f \in L^2\big(\mathbb{H}_1\,,\,\theta_0\big)$ and $u = f \circ H$ then [by $H^*\Psi_0 = \lambda^2\,\Psi$ with $\lambda = \lambda\big(H\,;\,\theta,\,\theta_0\big)$ and $\Psi_0 = \theta_0 \wedge d\theta_0$, $\Psi = \theta \wedge d\theta$]

$$\int_U |u|^2\,\Psi = \int_U \Big(\frac{|u|}{\lambda}\Big)^2\, H^*\,\Psi_0 = \int_{\mathbb{H}_1} \Big(\frac{|f|}{\lambda \circ H^{-1}}\Big)^2\,\Psi_0 =$$

$$= \int_{\mathbb{H}_1} \frac{4\,\big|f(\zeta,\,\tau)\big|^2}{\big[\big(1+|\zeta|^2\big)^2 + \tau^2\big]^2}\,\Psi_0(\zeta,\,\tau) \le 4\,\|f\|_{L^2(\mathbb{H}_1\,,\,\theta_0)}^2 < \infty\,.$$

As $H$ is a CR isomorphism $U \approx \mathbb{H}_1$, Cauchy–Riemann analysis is the same on $U$ and $\mathbb{H}_1$. However, $H$ does not preserve the contact forms $\theta$ and $\theta_0$ so that $(U,\,\theta)$ and $(\mathbb{H}_1\,,\,\theta_0)$ have rather different pseudohermitian geometries. On the same line of thought, we prove the following.

**Theorem 2.** *The map $f \longmapsto f \circ H$ is an isomorphism*

$$W_E^{1,2}\big(\mathbb{H}_1\,,\,\theta_0\big) \approx W_F^{1,2}\big(U\,,\,\theta\big).$$

Here, $U = S^3 \setminus \{(0,\,-1)\}$ while $E \equiv \{E_1\,,\,E_2\} \subset C^\infty\big(H(\mathbb{H}_1)\big)$, respectively $F \equiv \{F_1\,,\,F_2\} \subset C^\infty\big(H(U)\big)$, are the canonical $G_{\theta_0}$-orthonormal, respectively $G_\theta$-orthonormal, frames

$$E_1 = \frac{1}{\sqrt{2}}\big(V + \overline{V}\big), \quad E_2 = \frac{i}{\sqrt{2}}\big(V - \overline{V}\big),$$

$$F_1 = Z + \overline{Z}, \quad F_2 = i\big(Z - \overline{Z}\big).$$

**Lemma 2.** $\operatorname{div}\big(F_a\big) = 0$.

**Proof.** It follows from the fact that the only nonvanishing Christoffel symbol of the Tanaka–Webster connection $\nabla$ of $\big(S^3, \theta\big)$ is $\Gamma^1_{01} = -2i$ (itself a consequence of (14) with $t = 0$). □

**Proof of Theorem 2.** Given $f \in W^{1,2}_E\big(\mathbb{H}_1, \theta_0\big)$, we need to show that for every $a \in \{1, 2\}$, there is $v_a \in L^2\big(U, \theta\big)$ such that (by (25) and Lemma 2)

$$\int_U v_a\, \varphi\, \Psi = -\int_U (f \circ H)\, F_a(\varphi)\, \Psi \tag{27}$$

for any $\varphi \in C^\infty_0(U)$. The candidate for $v_a$ is, of course, obtained by computing $F_a\big(f \circ H\big)$ when $f$ is smooth.

**Lemma 3.** *Let $f \in C^1\big(\mathbb{H}_1\big)$ and $u = f \circ H$. Then*

$$F_1(u) = i\,\rho\,\big(g - \overline{g}\big)\left(\frac{\partial f}{\partial \tau}\right)^H + \frac{1}{\sqrt{2}}\big(h + \overline{h}\big)\, E_1(f)^H + \frac{i}{\sqrt{2}}\big(h - \overline{h}\big)\, E_2(f)^H, \tag{28}$$

$$F_2(u) = \rho\,\big(g + \overline{g}\big)\left(\frac{\partial f}{\partial \tau}\right)^H - \frac{i}{\sqrt{2}}\big(h - \overline{h}\big)\, E_1(f)^H + \frac{1}{\sqrt{2}}\big(h + \overline{h}\big)\, E_2(f)^H, \tag{29}$$

*where*

$$g,\, h : \mathbb{C}^2 \setminus \{w + 1 = 0\} \to \mathbb{C}^2,$$

$$g(z,w) = \frac{z}{(1 + \overline{w})\,(1 + w)^2}, \quad h(z,w) = \frac{1 + w + \rho(z,w)}{(1 + \overline{w})^2}, \tag{30}$$

*and* $\rho(z,w) = |z|^2 + |w|^2 - 1$.

**Proof.** It follows from (24), and its complex conjugate. □

Let $v_a$ be the (restrictions to $U$ of the) right-hand sides of (28) and (29), respectively. By a change of the variable under the integral sign,

$$\int_U v_a\, \varphi\, \Psi = \int_{\mathbb{H}_1} \left(\frac{v_a\, \varphi}{\lambda^2}\right)^{H^{-1}} \Psi_0$$

for every $\varphi \in C^\infty_0(U)$. Throughout, $v^{H^{-1}} = v \circ H^{-1}$, and the inverse of $H : U \to \mathbb{H}_1$ is

$$H^{-1}(\zeta, \tau) = \left(\frac{2i\zeta}{\tau + i\big(|\zeta|^2 + 1\big)},\; -\frac{\tau + i\big(|\zeta|^2 - 1\big)}{\tau + i\big(|\zeta|^2 + 1\big)}\right). \tag{31}$$

Next (by the very definition of $v_1$),

$$\int_U v_1\, \varphi\, \Psi =$$

$$= \frac{1}{\sqrt{2}} \int_{\mathbb{H}_1} \left(\frac{\varphi}{\lambda^2}\right)^{H^{-1}} \left\{ \big(h + \overline{h}\big)^{H^{-1}} E_1(f) + i\big(h - \overline{h}\big)^{H^{-1}} E_2(f) \right\} \Psi_0 =$$

(as $f$ is $E$-differentiable)

$$= -\frac{1}{\sqrt{2}} \int_{\mathbb{H}_1} f \left\{ E_1\Big(G_+^{H^{-1}}\Big) + i\, E_2\Big(G_-^{H^{-1}}\Big) \right\} \Psi_0$$

where $G_\pm = \frac{\varphi}{\lambda^2}\,(h \pm \overline{h})$. Next $E_1(G_+^{H^{-1}}) + i\,E_2(G_-^{H^{-1}})$ may be computed from

$$G_+ + G_- = \frac{2\varphi h}{\lambda^2}, \quad G_+ - G_- = \frac{2\varphi \overline{h}}{\lambda^2},$$

so that

$$\int_U f\,\varphi\,\Psi = \tag{32}$$

$$= \int_{\mathbb{H}_1} f \left\{ V\left[ \left( \frac{\varphi \overline{h}}{\lambda^2} \right)^{H^{-1}} \right] + \overline{V}\left[ \left( \frac{\varphi h}{\lambda^2} \right)^{H^{-1}} \right] \right\} \Psi_0 .$$

Note that (by (30) and (31))

$$h \circ H^{-1} = \frac{1}{2}\,\frac{(1+\overline{F})^2}{1+F}. \tag{33}$$

On the other hand (by (23)),

$$(H^{-1})_*\,V = \left( \frac{1}{\overline{h}}\,Z \right)^{H^{-1}}. \tag{34}$$

As $F$ is a CR function, i.e., $\overline{V}(F) = 0$ and $V(F) = 2\overline{\zeta}$ (by (33) and (34) and their complex conjugates)

$$V\left[ \left( \frac{\varphi \overline{h}}{\lambda^2} \right)^{H^{-1}} \right] = \overline{h}^{H^{-1}}\,V\left[ \left( \frac{\varphi}{\lambda^2} \right)^{H^{-1}} \right] + 2\,\overline{\zeta}\left( \frac{\varphi}{\lambda^2} \right)^{H^{-1}} \frac{1+F}{1+\overline{F}},$$

$$\overline{h}^{H^{-1}}\,V\left[ \left( \frac{\varphi}{\lambda^2} \right)^{H^{-1}} \right] = Z\left( \frac{\varphi}{\lambda^2} \right)^{H^{-1}},$$

and substitution into (32) followed by a change of variable under the integral sign gives

$$\int_U v_1\,\varphi\,\Psi = -\int_U u\,F_1(\varphi)\,\Psi + \tag{35}$$

$$-\int_U u\,\varphi \left\{ \lambda^2 F_1\left( \frac{1}{\lambda^2} \right) + 2\left( \frac{1+\overline{F}}{1+F}\,\zeta + \frac{1+F}{1+\overline{F}}\,\overline{\zeta} \right)^H \right\} \Psi .$$

Finally, by the identities

$$(1+F)^H = \frac{2}{1+w}, \quad \zeta^H = \frac{z}{1+w},$$

$$Z(\lambda) = \frac{\overline{z}\,\lambda}{1+w}, \quad \left( \frac{1}{\lambda^2} \right)^{H^{-1}} = \frac{4}{|1+F|^4}, \tag{36}$$

the last integral in (35) vanishes (yielding (27) with $a = 1$). □

The proof of the second relation in (27) is similar. Moreover, $v_a \in L^2(U, \theta)$ because

$$h\,E_a(f)^H \in L^2(U, \theta) \tag{37}$$

and $L^2(U, \theta)$ is a vector space. As to the proof of (37) (by a change of variable)

$$\int_U |h\,E_a(f)^H|^2\,\Psi = \int_{\mathbb{H}_1} \left( \frac{|h|^2}{\lambda^2} \right)^{H^{-1}} |E_a(f)|^2\,\Psi_0 =$$

(by (33) and the second identity in (36))

$$= \int_{\mathbb{H}_1} \frac{|E_a(f)|^2}{|1+F|^2} \Psi_0 \le \int_{\mathbb{H}_1} |E_a(f)|^2 \Psi_0 < \infty \,.$$

*2.3. Quasiconformal Maps*

Let $N$ be a (for now, abstract) strictly pseudoconvex 3-dimensional CR manifold. However, in the applications to come, $N$ will be a strictly pseudoconvex real hypersurface $N \subset \mathbb{C}^2$ endowed with the induced CR structure

$$T_{1,0}(N) = \left[T(N) \otimes \mathbb{C}\right] \cap T^{1,0}(\mathbb{C}^2).$$

**Definition 1.** *A $C^\infty$ diffeomorphism $f : S^3 \to N$ is a contact transformation if*

$$(d_x f) H(S^3)_x = H(N)_{f(x)} \,, \quad x \in S^3 \,.$$

Note that the notion of a contact transformation does not depend upon the particular CR structures one may set on $S^3$ and $N$, but only on their Levi distributions.

**Lemma 4.** *Let $\Theta \in \mathcal{P}_+(N)$ be a positively oriented contact form on N and let $f : S^3 \to N$ be a $C^\infty$ diffeomorphism. The following statements are equivalent:*

(i) *$f$ is a contact transformation of $(S^3\,,\, H(S^3))$ into $(N,\, H(N))$.*
(ii) *There is a $C^\infty$ function*

$$\lambda = \lambda(f) = \lambda\big(f;\, \theta,\, \Theta\big) : S^3 \to \mathbb{R} \setminus \{0\}$$

*such that $f^*\,\Theta = \lambda\,\theta$.*

**Proof.** (i) $\Longrightarrow$ (ii). There exist functions $\lambda\,,\, \lambda_1 \in C^\infty\big(S^3\,,\, \mathbb{C}\big)$ such that

$$f^*\Theta = \lambda\,\theta + \lambda_1\,\theta^1 + \lambda_{\bar 1}\,\theta^{\bar 1}$$

where $\lambda_{\bar 1} = \overline{\lambda_1}$. Then for any $x \in S^3$

$$\lambda_1(x) = \big((f^*\,\Theta)\, T_1\big)_x = \Theta_{f(x)}\big[(d_x f)\, T_{1,x}\big] = 0$$

because of

$$\big(d_x f\big)\, T_{1,x} \in \big(d_x f\big) H(S^3)_x \otimes_{\mathbb{R}} \mathbb{C} \subset H(N)_{f(x)} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Ker}\big(\Theta\big)_{f(x)} \otimes_{\mathbb{R}} \mathbb{C}.$$

Then $f^*\,\Theta = \lambda\,\theta$, and $\lambda$ is real valued. To show that $\lambda$ is nowhere vanishing, we argue by contradiction. Let us assume that $\lambda(x_0) = 0$ for some $x_0 \in S^3$. Then

$$0 = \lambda(x_0)\,\theta_{x_0} = \big(f^*\,\Theta\big)_{x_0} = \Theta_{f(x_0)} \circ (d_{x_0} f)$$

and hence for every $v \in T_{x_0}(S^3)$

$$(d_{x_0} f) v \in \mathrm{Ker}(\Theta)_{f(x_0)} = H(N)_{f(x_0)} \Longrightarrow (d_{x_0} f) T_{x_0}(S^3) \subset H(N)_{f(x_0)}\,,$$

a contradiction, because

$$\dim_{\mathbb{R}} T_{x_0}(S^3) = 3, \quad \dim_{\mathbb{R}} H(N)_{f(x_0)} = 2,$$

and $d_{x_0} f$ is a $\mathbb{R}$-linear isomorphism.

(ii) $\Longrightarrow$ (i) Let $v \in H(S^3)_x = \mathrm{Ker}(\theta)_x$. Then

$$0 = \lambda(x)\,\theta_x(v) = \big(f^*\Theta\big)_x(v) = \Theta_{f(x)}\big[(d_x f)v\big] \Longrightarrow$$
$$\Longrightarrow (d_x f)v \in \mathrm{Ker}(\Theta)_{f(x)} = H(N)_{f(x)}$$

and hence,

$$(d_x f)H(S^3)_x \subset H(N)_{f(x)}$$

for any $x \in S^3$. $\square$

It should be observed that in the proof of Lemma 4, use was made of the frames $\{T_1\} \subset C^\infty\big(T_{1,0}(S^3)\big)$ and $\{\theta^1,\ \theta^{\overline{1}},\ \theta\} \subset C^\infty\big(T^*(S^3)\otimes\mathbb{C}\big)$ and, therefore, of the canonical CR structure $T_{1,0}(S^3)$ of the sphere. Any other CR structure $\mathcal{H}$ with the same Levi distribution $H(S^3) = \mathrm{Re}\big\{\mathcal{H}\oplus\overline{\mathcal{H}}\big\}$ would have worked equally well.

**Definition 2.** *The function $\lambda = \lambda(f;\theta,\Theta)$ is called the dilation of $f$ with respect to the contact forms $\theta \in \mathcal{P}_+(S^3)$ and $\Theta \in \mathcal{P}_+(N)$.*

It obeys the following transformation law, with respect to a transformation of the given (positively oriented) pseudohermitian structures.

**Lemma 5.** *Let $f : S^3 \to N$ be a contact transformation of $\big(S^3, H(S^3)\big)$ into $(N, H(N))$. Let $\hat\theta = e^u\,\theta$ and $\hat\Theta = e^v\,\Theta$ with $u \in C^\infty\big(S^3,\mathbb{R}\big)$ and $v \in C^\infty\big(N,\mathbb{R}\big)$. Then*

$$\lambda\big(f;\hat\theta,\hat\Theta\big) = \exp\big(v\circ f - u\big)\,\lambda\big(f;\theta,\Theta\big). \tag{38}$$

*In particular,* $\mathrm{sign}\big[\lambda\big(f;\theta,\Theta\big)\big] \in \{\pm 1\}$ *is a CR invariant.*

**Proof.** Let us set $\lambda = \lambda\big(f;\theta,\Theta\big)$ and $\hat\lambda = \lambda\big(f;\hat\theta,\hat\Theta\big)$ for the sake of simplicity. Then

$$\big(f^*\hat\Theta\big)_x = \hat\Theta_{f(x)}\circ(d_x f) = e^{v(f(x))}\,\Theta_{f(x)}\circ(d_x f) =$$
$$= e^{v(f(x))}\,\big(f^*\Theta\big)_x = e^{v(f(x))}\,(\lambda\,\theta)_x$$

that is

$$f^*\,\hat\Theta = e^{v\circ f}\,\lambda\,\theta.$$

On the other hand

$$f^*\,\hat\Theta = \hat\lambda\,\hat\theta = \hat\lambda\,e^u\,\theta$$

yielding $\hat\lambda = e^{v\circ f - u}\,\lambda$. $\square$

To fix ideas, from now on, we shall work with contact transformations $f : S^3 \to N$ of positive dilation, i.e.,

$$\lambda\big(f;\theta,\Theta\big) > 0$$

with respect to some fixed contact form $\Theta \in \mathcal{P}_+(N)$. According to Lemma 5, this is a CR-invariant assumption.

Let $\mathcal{H}$ be an arbitrary CR structure on $S^3$ whose Levi distribution is $H(S^3)$, and let $K > 1$ be a constant.

**Definition 3.** *A contact transformation $f : S^3 \to N$ is called a K-quasiconformal mapping of the pseudohermitian manifold $(S^3, \mathcal{H}, \theta)$ into $(N, T_{1,0}(N), \Theta)$ if*

$$\frac{1}{K}\,G_{\theta,\mathcal{H}}(X, X) \le \frac{G^f_\Theta(f_*X, f_*Y)}{\lambda(f;\theta,\Theta)} \le K\,G_{\theta,\mathcal{H}}(X, X) \tag{39}$$

*for any* $X,\, Y \in H(S^3)$.

Here, $G_{\theta,\mathcal{H}}$ is the Levi form of $(S^3\,,\,\mathcal{H})$ and $G^f_\Theta = G_\Theta \circ f$. Additionally, $f_* X$ denotes the $C^\infty$ section in the pullback bundle $f^{-1}T(N) \to S^3$ given by

$$\big(f_* X\big)(x) = (d_x f) X_x \in T_{f(x)}(N) = \big(f^{-1} TN\big)_x\,, \quad x \in S^3\,.$$

The same symbol $f_*$ will denote the vector bundle morphism $f_* : T(S^3) \to f^{-1}T(N)$ (descending to a vector bundle morphism $f_* : H(S^3) \to f^{-1}H(N)$, because $f$ is a contact map) determined by the differential $df$. Let

$$J_N : H(N) \to H(N), \quad J_N\big(W + \overline{W}\big) = i\big(W - \overline{W}\big), \quad W \in T_{1,0}(N),$$

be the complex structure along the Levi distribution $H(N)$. Let us set

$$J_f : H(S^3) \to H(S^3),$$

$$J_{f,x} = (d_x f)^{-1} \circ J_{N,f(x)} \circ (d_x f), \quad x \in S^3\,.$$

Then $\big(J_f\big)^2 = -I$, and hence $J_f$ determines the CR structure

$$\mathcal{H}_f = \mathrm{Eigen}\big(J_f^{\mathbb{C}}\,;\, +i\big) \subset H(S^3) \otimes \mathbb{C}$$

whose Levi distribution is once again $H(S^3)$. Let $G_f$ be the Levi form of $\big(S^3\,,\,\mathcal{H}_f\big)$, i.e.,

$$G^f_\theta\big(X, Y\big) = (d\theta)\big(X,\, J_f\, Y\big), \quad X, Y \in H(S^3).$$

One has

$$G^f_\Theta\big(f_* X\,,\, f_* Y\big)_x = G_{\Theta, f(x)}\big((d_x f)X_x\,,\,(d_x f)Y_x\big) =$$
$$= \big(d\Theta\big)_{f(x)}\big((d_x f)X_x\,,\, J_{N,f(x)}(d_x f)Y_x\big) = (d\Theta)_{f(x)}\big((d_x f)X_x\,,\,(d_x f)\, J_{f,x}\, Y_x\big) =$$
$$= \big(f^*\, d\Theta\big)(X,\, J_f Y)_x = \big(d\, f^*\, \Theta\big)(X,\, J_f Y)_x = \big(d\,(\lambda\,\theta)\big)(X,\, J_f Y)_x =$$
$$= \big(d\lambda \wedge \theta + \lambda\, d\theta\big)(X,\, J_f Y\big)_x = \lambda(x)\,(d\theta)(X,\, J_f Y)_x$$

that is,

$$G^f_\Theta\big(f_* X\,,\, f_* Y\big) = \lambda\, G_f(X, Y) \tag{40}$$

for any $X, Y \in H(S^3)$. Consequently the $K$-quasiconformal requirement (39) may be rephrased as

$$\frac{1}{K}\, G_{\theta,\mathcal{H}}\big(X\,,\,X\big) \le G_f\big(X\,,\,X\big) \le K\, G_{\theta,\mathcal{H}}\big(X\,,\,X\big). \tag{41}$$

**Lemma 6.** *Let* $f : S^3 \to N$ *be a contact transformation of* $\big(S^3\,,\,H(S^3)\big)$ *into* $\big(N,\, H(N)\big)$*. Let* $\mathcal{H}$ *be a CR structure on* $S^3$ *such that* $H(S^3) = \mathrm{Re}\big\{\mathcal{H} \oplus \overline{\mathcal{H}}\big\}$ *and* $\theta \in \mathcal{P}_+\big(S^3\,,\,\mathcal{H}\big)$*. If* $x \in S^3$ *and* $w \in \overline{\mathcal{H}}_x$ *with* $w \neq 0$ *then*

$$(d_x f) w \notin T_{f(x)}(N)_{f(x)}$$

*that is*

$$T_{1,0}(N)_{f(x)} \cap (d_x f)\overline{\mathcal{H}}_x = (0). \tag{42}$$

**Proof.** We argue by contradiction, i.e., we assume that

$$(d_x f) w \in T_{1,0}(N)_{f(x)}$$

for some $w \in \overline{\mathcal{H}}_x \subset H(S^3)_x \otimes_{\mathbb{R}} \mathbb{C}$ with $w \neq 0$. Then (as $T_{1,0}(N)$ is nondegenerate and $\Theta$ is positively oriented)

$$0 < G_{\Theta, f(x)}\big((d_x f)w, \overline{(d_x f)w}\big) =$$
$$= (d\Theta)_{f(x)}\big((d_x f)w\,,\, J_{N,f(x)}\,\overline{(d_x f)w}\big) =$$

[as $\overline{(d_x f)w} \in T_{0,1}(N)_{f(x)}$ and $d_x f$ is real]

$$= -i\,(d\Theta)_{f(x)}\big((d_x f)w\,,\,(d_x f)\overline{w}\big) = -i\,\big(d\,f^*\,\Theta\big)_x(w, \overline{w}) =$$
$$= -i\,\big(d\,(\lambda\,\theta)\big)_x(w, \overline{w}) = i\,\big(d\lambda \wedge \theta + \lambda\,d\theta\big)_x(\overline{w}\,,\,w) =$$
$$= i\,\lambda(x)\,(d\theta)_x(\overline{w}\,,\,w) = -\lambda(x)\,(d\theta)_x(\overline{w}\,,\,J^{\mathcal{H}}_x\,w) =$$
$$= -\lambda(x)\,G_{\theta,\mathcal{H},x}(\overline{w}\,,\,w) < 0,$$

a contradiction. □

Here, we assumed that the canonical contact form (7) is positively oriented relative to $(S^3\,,\,\mathcal{H})$. Otherwise, one merely replaces $\theta$ by $-\theta$ to start with.

The contents of (42) are that, solely as a consequence of $f : S^3 \to N$ being a contact transformation of positive dilation $\lambda(f) > 0$,

$$(f_*\,\overline{\mathcal{H}}\,) \cap f^{-1}T_{1,0}(N) = (0)$$

for every CR structure $\mathcal{H}$ on $S^3$ whose Levi distribution is $H(S^3)$.

Let $f : S^3 \to N$ and $\mathcal{H}$ be as in Lemma 6. Next, let

$$\{L\} \subset C^\infty(U, \mathcal{H}), \quad \{T_1^N\} \subset C^\infty(V, T_{1,0}(N)),$$

be local frames in $\mathcal{H}$ and $T_{1,0}(N)$ respectively, defined on the open subsets $U \subset S^3$ and $V \subset N$ such that $U = f^{-1}(V)$. For every $x \in U$

$$(d_x f)\overline{L}_x = f_{\overline{1}}^{1}(x\,;\,\mathcal{H})\,T^N_{1,f(x)} + f_{\overline{1}}^{\overline{1}}(x\,;\,\mathcal{H})\,T^N_{\overline{1},f(x)}$$

for some functions

$$f_{\overline{1}}^{1}(\,\cdot\,;\,\mathcal{H}),\ \ f_{\overline{1}}^{\overline{1}}(\,\cdot\,;\,\mathcal{H}) \in C^\infty(U, \mathbb{C}).$$

The adopted notation emphasizes the dependence of the coefficients $f_{\overline{1}}^{1}$ and $f_{\overline{1}}^{\overline{1}}$ on the CR structure $\mathcal{H}$. Occasionally, if there is no danger of confusion, we drop $\mathcal{H}$ and write merely

$$f_{\overline{1}}^{1} = f_{\overline{1}}^{1}(\,\cdot\,;\,\mathcal{H})\,, \quad f_{\overline{1}}^{\overline{1}} = f_{\overline{1}}^{\overline{1}}(\,\cdot\,;\,\mathcal{H}).$$

**Lemma 7.** *One has*

$$f_{\overline{1}}^{\overline{1}}(x\,;\mathcal{H}) \neq 0 \tag{43}$$

*for any $x \in U$.*

**Proof.** We argue by contradiction, i.e., we assume that $f_{\overline{1}}^{\overline{1}}(x_0) = 0$ for some $x_0 \in U$. Then

$$(d_{x_0} f)\overline{L}_{x_0} = f_{\overline{1}}^{1}(x_0)\,T^N_{1,f(x_0)} \in T_{1,0}(N)_{f(x_0)}$$

and $\overline{L}_{x_0} \neq 0$, in contradiction with Lemma 6. □

We adopt the temporary notation

$$\hat{\mathcal{H}}_f = \big\{Z \in H(S^3) \otimes \mathbb{C}\ :\ f_* Z \in f^{-1}T_{1,0}(N)\big\}. \tag{44}$$

Then

$$\hat{\mathcal{H}}_f \cap \overline{\mathcal{H}} = (0)$$

for any CR structure $\mathcal{H}$ on $S^3$ as in Lemma 6.

**Lemma 8.** *Let $f : S^3 \to N$ be a contact transformation of positive dilation $\lambda(f) > 0$. For every CR structure $\mathcal{H}$ on $S^3$ such that*

$$H(S^3) = \mathrm{Re}\{\mathcal{H} \oplus \overline{\mathcal{H}}\}, \quad \theta \in \mathcal{P}_+(S^3, \mathcal{H}), \tag{45}$$

*there is a field*

$$\mu = \mu(f, \mathcal{H}) : \mathcal{H} \to \mathcal{H}$$

*of $\mathbb{C}$-anti-linear maps such that*

$$\hat{\mathcal{H}}_f = \{Z - \overline{\mu Z} \ : \ Z \in \mathcal{H}\}. \tag{46}$$

**Proof.** Let us start with $W \in \hat{\mathcal{H}}_f$ represented as

$$W = A^1 L + B^{\overline{1}} \overline{L}$$

with respect to the local frame $\{L, \overline{L}\}$ of $H(S^3) \otimes \mathbb{C}$. Then

$$f^{-1} T_{1,0}(N) \ni f_* W = \left(A^1 f_1^1 + B^{\overline{1}} f_{\overline{1}}^1\right) T_1^N + \left(A^1 f_1^{\overline{1}} + B^{\overline{1}} f_{\overline{1}}^{\overline{1}}\right) T_{\overline{1}}^N$$

yielding

$$B^{\overline{1}} = -\frac{f_1^{\overline{1}}}{f_{\overline{1}}^{\overline{1}}} A^1 .$$

Therefore

$$W = A^1 \left( L - \frac{f_1^{\overline{1}}}{f_{\overline{1}}^{\overline{1}}} \overline{L} \right),$$

i.e., $\hat{\mathcal{H}}_f$ is (locally, on $U$) the span of $\{L - (f_1^{\overline{1}}/f_{\overline{1}}^{\overline{1}})\, \overline{L}\}$. □

Let $x \in S^3$ be an arbitrary point and let us choose local frames $\{L\}$ and $\{T_1^N\}$ of the CR structures $\mathcal{H}$ and $T_{1,0}(N)$, defined on open neighborhoods of the points $x$ and $f(x)$

$$x \in U \subset S^3, \quad f(x) \in V \subset N, \quad U = f^{-1}(V).$$

Our rather pedantic approach to the construction of $\mu_x$ (see below) is devised to emphasize that the resulting $\mu$ is globally defined. Indeed we set by definition

$$\mu_x : \mathcal{H}_x \to \mathcal{H}_x , \quad \mu_x L_x = \frac{f_{\overline{1}}^1(x; \mathcal{H})}{f_1^1(x; \mathcal{H})} L_x , \tag{47}$$

followed by the $\mathbb{C}$-anti-linear extension to the whole of $\mathcal{H}_x$. The definition of $\mu_x$ does not depend upon the choice of local frames about $x$ and $f(x)$. Indeed, let us consider local frames

$$\{L'\} \subset C^\infty(U', \mathcal{H}), \quad \{T_1'^N\} \subset C^\infty(V', T_{1,0}(N)),$$

$$x \in U' \subset S^3, \quad f(x) \in V' \subset N, \quad U' = f^{-1}(V').$$

Then

$$L' = u_1^1 L \quad \text{on} \quad U \cap U', \quad T_1'^N = v_1^1 T_1^N \quad \text{on} \quad V \cap V',$$

for some $C^\infty$ functions $u_1^1 : U \cap U' \to \mathbb{C}$ and $v_1^1 : V \cap V' \to \mathbb{C}$. A comparison of the representations

$$f_* L = f_1^1 T_1^N + f_1^{\overline{1}} T_{\overline{1}}^N , \quad f_* L' = f'^1_1 T'^N_1 + f'^{\overline{1}}_1 T'^N_{\overline{1}} ,$$

yields

$$f'^1_1 v_1^1 = u_1^1 f_1^1 , \quad f'^{\overline{1}}_1 v_{\overline{1}}^{\overline{1}} = u_1^1 f_{\overline{1}}^{\overline{1}} , \tag{48}$$

where $v_{\overline{1}}^{\overline{1}} = \overline{v_1^1}$. Let $\mathcal{H}_A$ denote the portion of $\mathcal{H}$ over the open set $A \subset S^3$. If

$$\mu : \mathcal{H}_U \to \mathcal{H}_U , \quad \mu' : \mathcal{H}_{U'} \to \mathcal{H}_{U'} ,$$

$$\mu L = \frac{f_{\overline{1}}^1}{f_1^1} L, \quad \mu' L' = \frac{f'^1_{\overline{1}}}{f'^1_1} L',$$

what one needs to check is that $\mu|_{U \cap U'} = \mu'|_{U \cap U'}$. This is but a standard calculation relying on (48).

Summing up, we built a family of vector bundle morphisms

$$\mu(f, \mathcal{H}) : \mathcal{H} \to \mathcal{H} \tag{49}$$

associated to the contact transformation $f : S^3 \to N$ with $\lambda(f) > 0$, such that $\hat{\mathcal{H}}_f$ is represented by (46). Let $\mathrm{CR}\big[H(S^3)\big]$ be the set of all CR structures $\mathcal{H}$ on $S^3$ obeying to (45). The family of morphisms (49) is then parametrized by $\mathcal{H} \in \mathrm{CR}\big[H(S^3)\big]$.

**Definition 4.** *Each $\mu(f, \mathcal{H})$ is referred to as the complex dilation of $f$ with respect to the CR structure $\mathcal{H}$.*

We previously mentioned that $\hat{\mathcal{H}}_f$ is but a temporary name for the bundle on the right-hand side of (44). Indeed, one has

**Lemma 9.** $\hat{\mathcal{H}}_f = \mathcal{H}_f$.

**Proof.** If $J_N^f = J_N \circ f$, then

$$\mathcal{H}_f = \mathrm{Eigen}\big[(J_f)^{\mathbb{C}} ; +i\big] = \big\{ Z \in H(S^3) \otimes \mathbb{C} \, : \, J_f Z = i\, Z \big\} =$$

$$= \big\{ Z \in H(S^3) \otimes \mathbb{C} \, : \, J_N^f (f_* Z) = i\, f_* Z \big\} =$$

$$= \big\{ Z \in H(S^3) \otimes \mathbb{C} \, : \, f_* Z \in f^{-1} \mathrm{Eigen}\big[(J_N)^{\mathbb{C}} ; +i\big] \big\} =$$

$$= \big\{ Z \in H(S^3) \otimes \mathbb{C} \, : \, f_* Z \in f^{-1} T_{1,0}(N) \big\} = \hat{\mathcal{H}}_f .$$

□

By a result of H. Rossi (cf. [1]), the CR manifold $\big(S^3 , \mathcal{H}(t)\big)$ is not globally embeddable in $\mathbb{C}^2$, for any $0 < |t| < 1$. Hence, for every nondegenerate CR hypersurface $N \subset \mathbb{C}^2$, there is no CR isomorphism $f : \big(S^3 , \mathcal{H}(t)\big) \to N$, except of course for $t = 0$ (when one may consider $N = S^3$ and $f = 1_{S^3}$). We propose the following weaker version of the global CR embedding problem.

**Problem 1.** *Given a strictly pseudoconvex CR manifold $M$ of CR dimension $n$, find* (i) *a real hypersurface $N \subset \mathbb{C}^{n+1}$ whose induced CR structure $T_{1,0}(N)$ is strictly pseudoconvex,* (ii) *a constant $K > 1$, and* iii) *a $K$-quasiconformal map $f : M \to N$.*

Our treatment of the question in Problem 1 is confined to H. Rossi's nonembeddable examples $\left(S^3, \mathcal{H}(t)\right)$. Precisely, we shall discuss the following.

**Problem 2.** *Find* (i) *a function* $K : (-1, 1) \to (1, +\infty)$, (ii) *a family* $\{N_t\}_{0<|t|<1}$ *of nondegenerate real hypersurfaces* $N_t \subset \mathbb{C}^2$, *and* (iii) *a family* $\{f_t\}_{0<|t|<1}$ *of* $K(t)$*-quasiconformal maps* $f_t : \left(S^3, \mathcal{H}(t)\right) \to N_t$.

*2.4. Beltrami's Equation*

Let $N \subset \mathbb{C}^2$ be a nondegenerate real hypersurface, and let $f = \left(f^1, f^2\right) : S^3 \to N$ be a contact transformation of $\left(S^3, H(S^3)\right)$ into $\left(N, H(N)\right)$ with $\lambda(f) > 0$. By Lemmas 8 and 9

$$L_t - \overline{\mu_f(t)\, L_t} \in \mathcal{H}_f\,, \quad |t| < 1,$$

where we have set

$$\mu_f(t) = \mu\big[f,\ \mathcal{H}(t)\big] : \mathcal{H}(t) \to \mathcal{H}(t).$$

Hence (by the very definition of $\hat{\mathcal{H}}_f$)

$$f^{-1}T_{1,0}(N) \ni f_*\left[L_t - \overline{\mu_f(t)\, L_t}\right] = f_*\left[L_t - \mu_1^{\overline{1}}(t)\, \overline{L}_t\right] \tag{50}$$

where the functions $\mu_1^{\overline{1}}(t) : S^3 \to \mathbb{C}$ are given by

$$\mu_f(t)\, L_t = \mu_{\overline{1}}^{1}(t)\, L_t\,, \quad \mu_{\overline{1}}^{1}(t) = \frac{f_{\overline{1}}^1(t)}{f_1^1(t)}\,, \quad \mu_1^{\overline{1}}(t) = \overline{\mu_{\overline{1}}^{1}(t)}\,,$$

$$f_B^A(t) = f_B^A\big[\,\cdot\,;\ \mathcal{H}(t)\big], \quad A, B \in \big\{1,\ \overline{1}\big\}.$$

**Lemma 10.** *Let* $f = \left(f^1, f^2\right) : S^3 \to N$ *be a contact transformation of* $\left(S^3, H(S^3)\right)$ *into* $\left(N, H(N)\right)$ *with* $\lambda(f) > 0$. *The components* $f^j : S^3 \to \mathbb{C}$ *satisfy Beltrami's equations*

$$\overline{L}_t\big(f^j\big) = \mu_{\overline{1}}^{1}(t)\, L_t\big(f^j\big), \quad j \in \{1,2\}, \quad |t| < 1. \tag{51}$$

**Proof.** One has (by (50))

$$f^{-1}T^{1,0}(\mathbb{C}^2) \supset f^{-1}T_{1,0}(N) \ni L_t\big(f^j\big)\, \frac{\partial}{\partial \zeta^j} + L_t\big(\overline{f^j}\big)\, \frac{\partial}{\partial \overline{\zeta}^j} +$$

$$-\mu_1^{\overline{1}}(t)\left\{\overline{L}_t\big(f^j\big)\, \frac{\partial}{\partial \zeta^j} + \overline{L}_t\big(\overline{f^j}\big)\, \frac{\partial}{\partial \overline{\zeta}^j}\right\}$$

so that

$$L_t\big(\overline{f^j}\big) - \mu_1^{\overline{1}}(t)\, \overline{L}_t\big(\overline{f^j}\big) = 0$$

or (by taking complex conjugates)

$$\overline{L}_t\big(f^j\big) = \mu_{\overline{1}}^{1}(t)\, L_t\big(f^j\big)$$

which is (51). □

**Lemma 11.** *Let* $\mu = \mu\big[f,\ T_{1,0}(S^3)\big] : T_{1,0}(S^3) \to T_{1,0}(S^3)$ *be the complex dilation of* $f : S^3 \to N$ *relative to the canonical CR structure* $\mathcal{H}(0) = T_{1,0}(S^3)$. *If* $\mu\, T_1 = \mu_{\overline{1}}^1\, T_1$ *then*

$$\mu_{\overline{1}}^{1}(t) = \frac{\mu_{\overline{1}}^{1} + t}{1 + t\, \mu_{\overline{1}}^{1}} \tag{52}$$

*for every $|t| < 1$. In particular, the coefficients of the complex dilation $\mu_f(t)$ depend smoothly on the parameter $t$.*

**Proof.** As $\overline{L}_0 = T_{\overline{1}}$

$$(d_x f) T_{\overline{1},x} = f^1_{\overline{1}}(x, 0)\, T^N_{1,f(x)} + f^{\overline{1}}_{\overline{1}}(x, 0)\, T^N_{\overline{1},f(x)}$$

hence

$$\begin{aligned}(d_x f)\overline{L}_{t,x} = (d_x f) T_{1,x} + t\,(d_x f) T_{\overline{1},x} = \\ = f^1_1(x,0)\, T^N_{1,f(x)} + f^{\overline{1}}_1(x,0)\, T^N_{\overline{1},f(x)} + \\ + t\big[f^1_{\overline{1}}(x,0)\, T^N_{1,f(x)} + f^{\overline{1}}_{\overline{1}}(x,0)\, T^N_{\overline{1},f(x)}\big] = \\ = \big[f^1_1(x,0) + t\, f^1_{\overline{1}}(x,0)\big]\, T^N_{1,f(x)} + \big[f^{\overline{1}}_1(x,0) + t\, f^{\overline{1}}_{\overline{1}}(x,0)\big]\, T^N_{\overline{1},f(x)}\end{aligned}$$

yielding

$$f^1_{\overline{1}}(x,t) = f^1_1(x,0) + t\, f^1_{\overline{1}}(x,0), \quad f^{\overline{1}}_{\overline{1}}(x,t) = f^{\overline{1}}_1(x,0) + t\, f^{\overline{1}}_{\overline{1}}(x,0). \tag{53}$$

Let us set

$$f^A_B(x) = f^A_B(x, 0), \quad A, B \in \{1, \overline{1}\}.$$

According to the definition (47), the coefficients of the complex dilation $\mu = \mu_f(0)$ are given by

$$\mu\, T_1 = \mu^1_{\overline{1}}(\,\cdot\,, 0)\, T_1, \quad \mu^1_{\overline{1}}(\,\cdot\,, 0) = \frac{f^1_{\overline{1}}}{f^1_1}.$$

Next, let us set $t = 0$ into (51) to obtain

$$T_{\overline{1}}(f^j) = \mu^1_{\overline{1}}(\,\cdot\,, 0)\, T_1(f^j). \tag{54}$$

Let us set

$$\mu^A_B(x) = \mu^A_B(x, 0), \quad A, B \in \{1, \overline{1}\}.$$

Then

$$\mu^1_{\overline{1}}(x,t) = \frac{f^1_{\overline{1}}(x,t)}{f^1_1(x,t)} =$$

[by (53) and (54)]

$$= \frac{f^1_{\overline{1}}(x,0) + t\, f^1_1(x,0)}{f^1_1(x,0) + t\, f^1_{\overline{1}}(x,0)} = \frac{\mu^1_{\overline{1}}(x) + t}{1 + t\, \mu^1_{\overline{1}}(x)}.$$

□

**Corollary 1.** *The components $f^j$ of a contact transformation $f : S^3 \to N \subset \mathbb{C}^2$ under the assumptions of Lemma 10 satisfy the Beltrami equation*

$$\overline{L}_t(f^j) = \frac{\mu^1_{\overline{1}} + t}{1 + t\, \mu^1_{\overline{1}}}\, L_t(f^j)$$

*for any $j \in \{1, 2\}$.*

Let $(N, T_{1,0}(N))$ be a nondegenerate 3-dimensional CR manifold and let $\Theta \in \mathcal{P}_+(N)$ be a positively oriented contact form. Let $f : S^3 \to N$ be a contact transformation with $\lambda(f) = \lambda(f; \theta, \Theta) > 0$. Let $\mathcal{H} \in \mathrm{CR}[H(S^3)]$ and let $\mu_f = \mu(f, \mathcal{H}) : \mathcal{H} \to \mathcal{H}$ be the complex dilation of $f$.

**Definition 5.** *The pointwise norm of* $\mu_f$ *is the function* $\|\mu_f\| : S^3 \to [0, +\infty)$ *defined by*

$$\|\mu_f\|(x) = \left[ \sup_{0 \neq Z \in \mathcal{H}_x} \frac{G_{\theta,x}\big(\mu_{f,x} Z\,,\, \overline{\mu_{f,x} Z}\big)}{G_{\theta,x}\big(Z, \overline{Z}\big)} \right]^{1/2}, \quad x \in S^3.$$

We shall need the following

**Theorem 3.** *Let* $\mathcal{H} \in \mathrm{CR}\big[H(S^3)\big]$ *and let* $f : S^3 \to N$ *be a contact transformation with* $\lambda(f) > 0$*. The following statements are equivalent:*

(i) *There is* $K > 1$ *such that* $f$ *is K-quasiconformal.*

(ii) *There is* $K > 1$ *such that*

$$\|\mu_f\| \leq \frac{K-1}{K+1}\,. \tag{55}$$

Theorem 3 is stated in [3], p. 61, with $\big(S^3, \mathcal{H}\big)$ replaced by an arbitrary strictly pseudoconvex manifold $M$, yet the proof is confined to the case where $M = N = \mathbb{H}_n$ (the Heisenberg group). We give (by following the ideas in [3], pp. 63–65) a proof of the statement as it applies to Rossi's spheres, and refer to Theorem 3 as the *Koranyi–Reimann characterization theorem*.

**Proof of Theorem 3.** Let $x_0 \in S^3$ and let us choose an open neighborhood $V \subset N$ of $f(x_0)$ and local orthonormal frames

$$\{Z_1\} \subset C^\infty\big(U, \mathcal{H}\big), \quad \{T_1^N\} \subset C^\infty\big(V,\, T_{1,0}(N)\big), \quad U = f^{-1}(V),$$

$$G_\theta\big(Z_1\,, Z_{\overline{1}}\big) = 1, \quad G_\Theta\big(T_1^N\,, T_{\overline{1}}^N\big) = 1.$$

Next, let us set

$$E_1 = Z_1 + Z_{\overline{1}}\,, \quad E_2 = J E_1 = i\,\big(Z_1 - Z_{\overline{1}}\big),$$

$$E_1^N = T_1^N + T_{\overline{1}}^N\,, \quad E_2^N = J^N E_1^N = i\big(T_1^N - T_{\overline{1}}^N\big)\,,$$

so that $\big\{E_a \,:\, a \in \{1,2\}\big\}$ and $\big\{E_a^N \,:\, a \in \{1,2\}\big\}$ are respectively local frames of $H(S^3)$ and $H(N)$. Then

$$f_* E_b = F_b^a\, \big(E_a^N\big)^f$$

for some $C^\infty$ functions $F_b^a : U \to \mathbb{R}$ such that $\det\big[F_b^a(x)\big] \neq 0$ for any $x \in U$. Let us consider

$$g : U \to \mathrm{GL}(2, \mathbb{R}), \quad g = \frac{1}{\sqrt{\lambda(f)}} \begin{bmatrix} F_1^1 & F_2^1 \\ F_1^2 & F_2^2 \end{bmatrix}.$$

We shall need the symplectic group

$$\mathrm{Sp}\big(2, \mathbb{R}\big) = \big\{a \in \mathrm{GL}(2, \mathbb{R}) \,:\, a^\tau J_0\, a = J_0\big\}\,, \quad J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

**Lemma 12.** $g$ *is* $\mathrm{Sp}\big(2, \mathbb{R}\big)$*-valued.*

**Proof.** One has

$$(d\theta)\big(E_1\,, E_2\big) = -2\,i\,(d\theta)\big(Z_1\,, Z_{\overline{1}}\big) = 2\,G_\theta\big(Z_1\,, Z_{\overline{1}}\big) = 2$$

and similarly

$$(d\Theta)\big(E_1^N\,, E_2^N\big) = 2.$$

Then [by $f^*\Theta = \lambda(f)\,\theta$ and $\mathrm{Ker}(\theta) = H(S^3)$]

$$(d\Theta)^f\big(f_* E_1\,,\, f_* E_2\big) = \big(d\,f^*\,\Theta\big)\big(E_1\,,\,E_2\big) = \lambda(f)\,(d\theta)(E_1\,,\,E_2) = 2\,\lambda(f).$$

On the other hand

$$(d\Theta)^f\big(f_* E_1\,,\, f_* E_2\big) = \big(F_1^1\,F_2^2 - F_1^2\,F_2^1\big)(d\Theta)(E_1^N\,,\,E_2^N)\circ f =$$

$$= 2\big(F_1^1\,F_2^2 - F_1^2\,F_2^1\big) = 2\,\lambda(f)\ \det(g).$$

It follows that $\det(g) = 1$. □

Let us set

$$K = \mathrm{Sp}(2,\,\mathbb{R}) \cap \mathrm{O}(2), \quad A^+ = \left\{ \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \;:\; s \geq 0 \right\},$$

$$j : \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}\setminus\{0\} \to \mathrm{GL}(2,\mathbb{R}), \quad j(x + i\,y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

**Lemma 13.** $K = \mathrm{O}(2) \cap j\big[\mathrm{U}(1)\big]$.

The proof is straightforward and therefore omitted. Here $\mathrm{U}(n) = \big\{a \in \mathrm{GL}(n,\mathbb{C}) \;:\; \overline{a}^{\,\tau}\,a\; = I_n\big\}$ so that $\mathrm{U}(1) = \{a \in \mathbb{C} \;:\; |a| = 1\}$. We shall need the *Cartan decomposition* of $\mathrm{Sp}(2,\,\mathbb{R})$

$$\mathrm{Sp}(2,\mathbb{R}) = K\,A^+\,K.$$

By Lemma 12, there exist functions $k,\ k' : U \to K$ and $a : U \to A^+$ such that

$$g = k\;a\;k'$$

on $U$, i.e., there exist $x,\ y,\ u,\ v : U \to \mathbb{R}$ and $s : U \to [0,+\infty)$ such that

$$x^2 + y^2 = 1, \quad u^2 + v^2 = 1,$$

$$g = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}\begin{pmatrix} u & v \\ -v & u \end{pmatrix} = \tag{56}$$

$$= \begin{pmatrix} xue^s - yve^{-s} & xve^s + yue^{-s} \\ \\ -yue^s - xve^{-s} & -yve^s + xue^{-s} \end{pmatrix}$$

on $U$. Next

$$Z_1 = \frac{1}{2}\big(E_1 - i\,E_2\big), \quad Z_{\overline{1}} = \frac{1}{2}\big(E_1 + i\,E_2\big),$$

$$T_1^N = \frac{1}{2}\big(E_1^N - i\,E_2^N\big), \quad T_{\overline{1}}^N = \frac{1}{2}\big(E_1^N + i\,E_2^N\big),$$

$$f_*\,Z_1 = f_1^1\,\big(T_1^N\big)^f + f_1^{\overline{1}}\,\big(T_{\overline{1}}^N\big)^f, \quad f_*\,Z_{\overline{1}} = f_{\overline{1}}^1\big(T_1^N\big)^f + f_{\overline{1}}^{\overline{1}}\,\big(T_{\overline{1}}^N\big)^f,$$

yield

$$f_* Z_1 = \frac{1}{2}\big[F_1^1 + F_2^2 + i\big(F_1^2 - F_2^1\big)\big]\big(T_1^N\big)^f +$$

$$+\frac{1}{2}\big[F_1^1 - F_2^2 - i\big(F_1^2 + F_2^1\big)\big]\big(T_{\overline{1}}^N\big)^f$$

and if $g = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix}$ then

$$f_1^1 = \frac{1}{2}\left[F_1^1 + F_2^2 + i\left(F_1^2 - F_2^1\right)\right] = \tag{57}$$

$$= \frac{1}{2}\sqrt{\lambda(f)}\,\left(g_1^1 + g_2^2\right) + \frac{i}{2}\sqrt{\lambda(f)}\,\left(g_1^2 - g_2^1\right),$$

$$f_1^{\overline{1}} = \frac{1}{2}\left[F_1^1 - F_2^2 - i\left(F_1^2 + F_2^1\right)\right] = \tag{58}$$

$$= \frac{1}{2}\sqrt{\lambda(f)}\,\left(g_1^1 - g_2^2\right) - \frac{i}{2}\sqrt{\lambda(f)}\,\left(g_1^2 + g_2^1\right).$$

Let us substitute from (56) into (57) and (58) to obtain

$$f_1^1 = \sqrt{\lambda(f)}\,(xu - yv - iyu - ixv)\,\frac{e^s + e^{-s}}{2} = \tag{59}$$

$$= \sqrt{\lambda(f)}\,(x - iy)\,(u - iv)\,\cosh s,$$

$$f_1^{\overline{1}} = \sqrt{\lambda(f)}\,(xu + yv + iyu - ixv)\,\frac{e^s - e^{-s}}{2} = \tag{60}$$

$$= \sqrt{\lambda(f)}\,(x + iy)\,(u - iv)\,\sinh s.$$

**Lemma 14.** $\|\mu_f\| = \tanh s$.

**Proof.** We start from $\mu_{\overline{1}}^1 f_1^1 = f_{\overline{1}}^1$. Then (by (59) and (60) and their complex conjugates),

$$\mu_{\overline{1}}^1\,(x - iy)\,(u - iv)\,\cosh s = (x - iy)\,(u + iv)\,\sinh s$$

or

$$\mu_{\overline{1}}^1 = (u + iv)^2\,\tanh s. \tag{61}$$

Next, for every $x \in U$ and every $W \in \mathcal{H}_x \setminus \{0_x\}$ one has $W = h\,Z_{1,x}$ for some $h \in \mathbb{C} \setminus \{0\}$ and then

$$\frac{G_{\theta\,x}\left(\mu_{f,x}W\,,\,\overline{\mu_{f,x}W}\right)}{G_{\theta,x}\left(W\,,\,\overline{W}\right)} =$$

$$= \frac{(u - iv)^2(u + iv)^2(\tanh s)^2\overline{h}h}{h\overline{h}} = \left[\tanh s(x)\right]^2$$

and hence, $\|\mu_f\|(x) = \tanh s(x)$. □

At this point, we may attack the final part of the proof of Theorem 3. We start from $\|\mu_f\| = \tanh s$ (cf. Lemma 14) so that

$$\frac{1 + \|\mu_f\|}{1 - \|\mu_f\|} = \frac{1 + \tanh s}{1 - \tanh s} = \frac{e^s}{e^{-s}} = e^{2s}\,.$$

Recall that both $k$ and $k'$ are O(2)-valued. Then for any $x \in U$ and any $\mathbf{x} \in \mathbb{R}^2$

$$\sup_{|\mathbf{x}|=1}\left|g(x)\,\mathbf{x}\right| = \sup_{|\mathbf{x}|=1}\left|k(x)\,a(x)\,k'(x)\,\mathbf{x}\right| =$$

$$= \sup_{|\mathbf{x}|=1}\left|a(x)\,k'(x)\,\mathbf{x}\right| = \sup_{|\mathbf{y}|=1}\left|a(x)\mathbf{y}\right| = e^s$$

where $\mathbf{y} = k'(x)\mathbf{x}$. Similarly
$$\inf_{|\mathbf{x}|=1} |g(x)\mathbf{x}| = e^{-s}.$$

It follows that
$$\frac{1 + \|\mu_f\|(x)}{1 - \|\mu_f\|(x)} = \frac{\sup_{|\mathbf{x}|=1} |g(x)\mathbf{x}|}{\inf_{|\mathbf{x}|=1} |g(x)\mathbf{x}|}. \tag{62}$$

**Proof of (i) $\Longrightarrow$ (ii).** If $f$ is $K$-quasiconformal for some $K > 1$, then for every $x \in S^3$ and every $X \in H(S^3)_x$
$$\frac{\lambda(f)_x}{K} G_{\theta, x}(X, X) \le G_{\Theta, f(x)}\big((d_x f)X, (d_x f)X\big) \le \lambda(f)_x K G_{\theta, x}(X, X)$$
or
$$\frac{1}{K} |\mathbf{x}|^2 \le |g(x)\mathbf{x}|^2 \le K|\mathbf{x}|^2$$
where
$$\mathbf{x} = (x^1, x^2) \in \mathbb{R}^2, \quad X = x^1 E_{1,x} + x^2 E_{2,x} \in H(S^3)_x.$$

Consequently,
$$\frac{1}{\sqrt{K}} \le \inf_{|\mathbf{x}|=1} |g(x)\mathbf{x}|, \quad \sup_{|\mathbf{x}|=1} |g(x)\mathbf{x}| \le \sqrt{K},$$
so that [by (62)]
$$\frac{1 + \|\mu_f\|}{1 - \|\mu_f\|} \le K$$
or
$$\|\mu_f\| \le \frac{K-1}{K+1}.$$

□

**Proof of (ii) $\Longrightarrow$ (i).** If there is $K > 1$ such that (55) holds, then
$$\frac{e^{s(x)}}{e^{-s(x)}} = \frac{\sup_{|\mathbf{x}|=1} |g(x)\mathbf{x}|}{\inf_{|\mathbf{x}|=1} |g(x)\mathbf{x}|} = \frac{1 + \|\mu_f\|(x)}{1 - \|\mu_f\|(x)} \le K$$
so that $e^{s(x)} \le K e^{-s(x)}$. Let $x^1 E_{1,x} + x^2 E_{2,x} \in H(S^3)_x$ be a unit vector and let us set $\mathbf{x} = (x^1, x^2)$. Then
$$e^{-s(x)} \le |g(x)\mathbf{x}|^2 \le e^{s(x)} \le K e^{-s(x)} \le K.$$

Similarly
$$\frac{1}{K} \le |g(x)\mathbf{x}|^2.$$

□

□

*2.5. Quasiconformality to the Standard Sphere*

An interesting particular case of the CR embedding problem was considered by E. Barletta and S. Dragomir (cf. [10]) who asked which strictly pseudoconvex CR manifolds $M$, of CR dimension $n$, can be globally embedded as the standard sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with the ordinary CR structure $T_{1,0}(S^{2n+1})$ induced by the complex structure of $\mathbb{C}^{n+1}$. Their findings are that the Pontrjagin forms of the Fefferman metric $F_\theta \in \mathrm{Lor}[C(M)]$ are CR invariants of $M$ (and when a certain Pontrjagin form $P$ vanishes (i.e., $P = 0$), the corresponding transgression class $[T(P)]$ is a CR invariant, as well) and among those CR invariants, one pinpoints obstructions to the posed question (i.e., whether $M$ and $S^{2n+1}$ are CR equivalent).

A weaker version of E. Barletta and S. Dragomir's problem (cf. *op. cit.*) consistent with the formulation of our Problem 1, is to ask which strictly pseudoconvex CR manifolds $M$, of CR dimension $n$, are $K$-quasiconformally equivalent to the standard sphere $S^{2n+1}$. As with Problem 1, the question can be asked—and it is especially meaningful to ask—when $M$ fails to be globally embeddable. In the spirit of the present paper, we confine the question to the case of 3-dimensional (i.e., $n = 1$) CR manifolds and then to the particular case of Rossi's spheres

$$(M, \mathcal{H}) \in \Big\{ \big(S^3\,,\, \mathcal{H}(t)\big) \;:\; |t| < 1 \Big\}.$$

**Problem 3.** *Find a function $K : (-1, 1) \to (1, +\infty)$ and a family $\{f_t\}_{|t|<1}$ of $K(t)$-quasiconformal maps $f_t : S^3 \to S^3$ of the Rossi sphere $\big(S^3\,,\, \mathcal{H}(t)\big)$ onto the standard sphere $\big(S^3\,,\, T_{1,0}(S^3)\big)$.*

Of course $f_0 = 1_{S^3} : S^3 \to S^3$ is a CR equivalence of $\big(S^3\,,\, \mathcal{H}(0)\big)$ and itself $\big(S^3\,,\, T_{1,0}(S^3)\big)$. Yet given a constant $K > 1$ and a value of the parameter $0 < |t| < 1$, the identity mapping $1_{S^3}$ is not $K$-quasiconformal in general, and the pair $(K,\, t)$ is subject to constraints.

**Theorem 4.** *Let $K > 1$ and $0 < |t| < 1$ such that $f = 1_{S^3}$ is a $K$-quasiconformal map of $\big(S^3\,,\, \mathcal{H}(t)\big)$ onto $\big(S^3\,,\, T_{1,0}(S^3)\big)$. Then*

$$K \geq \frac{1+|t|}{1-|t|}\,. \tag{63}$$

**Proof.** Note that $f = 1_{S^3}$ is a contact transformation $f : S^3 \to S^3$ with $\lambda(f) = \lambda(f;\, \theta,\, \theta) = 1$. Let us consider the (globally defined) frames of $H(S^3)$

$$E_1 = T_1 + T_{\overline{1}}\,, \quad E_2 = i\big(T_1 - T_{\overline{1}}\big),$$

$$E_1^t = L_t + \overline{L}_t = (1+t)E_1\,, \quad E_2^t = i\big(L_t - \overline{L}_t\big) = (1-t)E_2\,.$$

The complex structures $J^t : H(S^3) \to H(S^3)$ and $J = J^0 : H(S^3) \to H(S^3)$ (determined by the CR structures $\mathcal{H}(t)$ and $T_{1,0}(S^3)$) are related by

$$J^t E_1 = \frac{1}{1+t}\, J^t E_1^t = \frac{1}{1+t}\, E_2^t = \frac{1-t}{1+t}\, E_2 = \frac{1-t}{1+t}\, J E_1$$

and similarly,

$$J^t E_2 = \frac{1+t}{1-t}\, J E_2\,.$$

Recall that

$$(d\theta)(E_1\,,\, E_2) = 2\, G_\theta\big(T_1\,,\, T_{\overline{1}}\big) = 1.$$

Then, for any $X = X^1 E_1 + X^2 E_2 \in H(S^3)$

$$G_\theta^t\big(X\,,\, X\big) = (d\theta)\big(X,\, J^t X\big) = \frac{1-t}{1+t}\,\big(X^1\big)^2 + \frac{1+t}{1-t}\,\big(X^2\big)^2$$

so that the Levi forms of the pseudohermitian manifolds $\big(S^3,\, \mathcal{H}(t),\, \theta\big)$ and $\big(S^3,\, T_{1,0}(S^3),\, \theta\big)$ are related by

$$G_\theta^t(X,\, X) = \frac{1}{1-t^2}\, G_\theta(X,\, X) + \frac{t}{1-t^2}\Big[(t-2)\big(X^1\big)^2 + (t+2)\big(X^2\big)^2\Big] =$$

$$= \frac{1-t}{1+t}\, G_\theta(X,\, X) + \frac{4t}{1-t^2}\,\big(X^2\big)^2 = \frac{1+t}{1-t}\, G_\theta(X,\, X) - \frac{4t}{1-t^2}\,\big(X^1\big)^2\,.$$

To establish the lower bound (63) on $K$, we distinguish two cases, as (I) $t > 0$ or (II) $t < 0$. In the first case the $K$-quasiconformality of $f = 1_{S^3}$

$$\frac{1}{K}\, G_\theta(X,X) \le G_\theta^t(X,X) \le K\, G_\theta(X,X) \tag{64}$$

for $X = E_2$ yields

$$K\, G_\theta(E_2\,,\,E_2) \ge G_\theta^t(E_2\,,\,E_2) = \frac{1+t}{1-t}\, G_\theta(E_2\,,\,E_2)$$

hence,

$$K \ge \frac{1+t}{1-t}\,.$$

In the second case, let us set $X = E_1$ in (64) so that

$$K\, G_\theta(E_1\,,\,E_1) \ge G_\theta^t(E_1\,,\,E_1) = \frac{1-t}{1+t}\, G_\theta(E_1\,,\,E_1)$$

hence

$$K \ge \frac{1-t}{1+t}\,.$$

□

The bound (63) is consistent with A. Korányi and H. Reimann's theorem. Indeed, if $\mu_f(t) : \mathcal{H}(t) \to \mathcal{H}(t)$ is the complex dilation of $f = 1_{S^3}$, then $\|\mu_f\| = |t|$ and (63) is a corollary of (55) in Theorem 3.

*2.6. Fefferman's Metrics*

Let $\big(N,\, T_{1,0}(N)\big)$ be a nondegenerate 3-dimensional CR manifold, and let $\Theta \in \mathcal{P}_+(N)$. Let $\{T_1^N\}$ be a local frame of $T_{1,0}(N)$, defined on the open set $V \subset N$. Let $T^N \in \mathfrak{X}(N)$ be the Reeb vector field of $(N, \Theta)$. Let $\{\Theta^1\}$ be the corresponding adapted coframe, i.e., $\Theta^1\big(T_1^N\big) = 1$, $\Theta^1\big(T_{\overline{1}}^N\big) = 0$ and $\Theta^1\big(T^N\big) = 0$. Let $f : S^3 \to N$ be a contact transformation with $\lambda(f) = \lambda(f\,;\,\theta,\,\Theta) > 0$. Here, $\theta$ is the canonical pseudohermitian structure on $S^3$ (given by (7)). Let $f^* : \Omega^p(N) \to \Omega^p(S^3)$ be the pullback by $f$ of differential $p$-forms on $N$, $p \in \{1, 2, 3\}$. Then,

$$f^*\,\Theta = \lambda(f)\,\theta, \quad f^*\,\Theta^1 = f_1^1\,\theta^1 + f_{\overline{1}}^1\,\theta^{\overline{1}} + f_0^1\,\theta.$$

Next, we consider the canonical circle bundles

$$\begin{array}{ccccccccc} S^1 \to & C\big(S^3\,,\,\mathcal{H}(t)\big) & & S^1 \to & C\big(S^3\,,\,\mathcal{H}_f\big) & & S^1 \to & C\big(N\,,\,T_{1,0}(N)\big) \\ & \downarrow \pi^t & & & \downarrow \pi_f & & & \downarrow \pi^N \\ & S^3 & & & S^3 & & & N \end{array}$$

(so that $\pi^0 = \pi$ (cf. our Section 2.1.5)) and the Fefferman metrics

$$F_\theta^t = F\big(\mathcal{H}(t),\,\theta\big) \in \mathrm{Lor}\big[C\big(S^3\,,\,\mathcal{H}(t)\big)\big],$$

$$F_f = F\big(\mathcal{H}_f\,,\,\theta\big) \in \mathrm{Lor}\big[C\big(S^3\,,\,\mathcal{H}_f\big)\big],$$

$$F_\Theta \in \mathrm{Lor}\big[C\big(N,\,T_{1,0}(M)\big)\big],$$

(so that $F^0_\theta = F_\theta$). We also write briefly $C(N) = C\big(N, T_{1,0}(M)\big)$. The principal bundle $S^1 \to C\big(S^3, \mathcal{H}(t)\big) \to S^3$ is described in Section 2. In addition, every $c \in C(N)_p$ with $p \in V$ may be represented as

$$c = \Big[\Lambda\left(\Theta \wedge \Theta^1\right)_p\Big], \quad \Lambda \in \mathbb{C} \setminus \{0\}.$$

To describe $S^1 \to C\big(S^3, \mathcal{H}_f\big) \to S^3$, we recall that, given a frame $\{Z_1\} \subset C^\infty(U, \mathcal{H})$, the CR structure $\mathcal{H}_f$ is the span of

$$L_f = Z_1 - \mu^{\overline{1}}_1 Z_{\overline{1}} \in C^\infty\big(\mathcal{H}_f\big), \quad \mu^{\overline{1}}_1 = \frac{f^{\overline{1}}_1}{f^{\overline{1}}_{\overline{1}}},$$

$$f^A_B = f^A_B\big(\,\cdot\,, \mathcal{H}\big), \quad \mathcal{H} \in \mathrm{CR}\big[H(S^3)\big], \quad A, B \in \{1, \overline{1}\}.$$

Let $\{\theta^1\}$ and $\{\theta^1_f\}$ be the adapted coframes determined by

$$\theta^1\big(Z_1\big) = 1, \quad \theta^1\big(Z_{\overline{1}}\big) = 0, \quad \theta^1(T) = 0,$$

$$\theta^1_f\big(L_f\big) = 1, \quad \theta^1_f\big(\overline{L}_f\big) = 0, \quad \theta^1_f(T) = 0.$$

Then

$$\theta^1_f = \frac{1}{1 - \big|\mu^1_{\overline{1}}\big|^2}\left(\theta^1 + \mu^1_{\overline{1}}\theta^{\overline{1}}\right)$$

and every $c \in C\big(S^3, \mathcal{H}_f\big)_x$ may be (locally) represented as

$$c = \left[\alpha\left(\theta \wedge \theta^1_f\right)_x\right] = \left[\alpha\left(\theta \wedge \theta^1 + \mu^1_{\overline{1}}\theta^{\overline{1}}\right)_x\right], \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

Every $(2,0)$-form on $N$ is locally represented as $\Omega = \Lambda\,\theta \wedge \Theta^1$ for some $\Lambda \in C^\infty\big(V, \mathbb{C}\big)$. Then [by $f^1_{\overline{1}} = \mu^1_{\overline{1}} f^1_1$]

$$f^*\Omega = f^1_1\,\Lambda^f\,\lambda(f)\,\theta \wedge \theta^1_f$$

where $\Lambda^f = \Lambda \circ f$.

**Proposition 1.** *Let $f : S^3 \to N$ be a contact transformation with $\lambda(f) > 0$ and let $\mathcal{H} \in \mathrm{CR}\big[H(S^3)\big]$ be a CR structure on $S^3$ whose Levi distribution is $H(S^3)$. The pullback $f^* : \Omega^2(N) \to \Omega^2(S^3)$ induces a $C^\infty$ diffeomorphism*

$$C(f) : C(N) \to C\big(S^3, \mathcal{H}_f\big),$$

$$C(f)(C) = \left[f^1_1(x)\,\Lambda\left(\theta \wedge \theta^1 + \mu^1_{\overline{1}}\theta \wedge \theta^{\overline{1}}\right)_x\right], \tag{65}$$

*for every $C \in C(N)_{f(x)}$ locally represented as*

$$C = \Big[\Lambda\left(\Theta \wedge \Theta^1\right)_x\Big], \quad \Lambda \in \mathbb{C}, \quad x \in U.$$

**Proof.** Let $y \in N$ and let $V \subset N$ be an open neighborhood of $y$, the domain of a (local) frame $\{T^N_1\} \subset C^\infty\big(V, T_{1,0}(N)\big)$. Let $C \in C(N)_y$ and let us set $x = f^{-1}(y) \in U = f^{-1}(V)$. Then $C = \big[\Lambda\,(\Theta \wedge \Theta^1)_y\big]$ for some $\Lambda \in \mathbb{C} \setminus \{0\}$ and we set

$$C(f)(C) = \Big[\Lambda\left\{f^*\left(\Theta \wedge \Theta^1\right)\right\}_x\Big]$$

thus yielding (65). The definition of $C(f)(C)$ does not depend upon the choice of local frame $\{T^N_1\}$ about $y = f(x)$. □

The investigation of the metric properties of $C(f)$ [in particular, the calculation of $C(f)^* F_f - F^t_\theta$ for $\mathcal{H} \in \{\mathcal{H}(t) : |t| < 1\}$] is an open problem.

## 3. Sobolev Solutions to Beltrami's Equation

The purpose of this section is to address the problem of solving the Beltrami equations

$$\overline{L}_t(g) = \mu(\,\cdot\,, t)\, L_t(g)\,, \quad |t| < 1, \tag{66}$$

under appropriate assumptions on a given family of functions $\mu(\,\cdot\,, t) : S^3 \to \mathbb{C}$, $|t| < 1$. To solve (66), we follow the approach by A. Koranyi and H. Reimann (cf. [3], pp. 69–74). There, one looks for weak solutions, in a Folland–Stein space, to the Beltrami equation

$$\overline{V}(f) = \mu\, V(f), \quad V \equiv \frac{\partial}{\partial \zeta} + i\,\overline{\zeta}\,\frac{\partial}{\partial \tau}\,,$$

on the Heisenberg group $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$, for a given function $\mu : \mathbb{H}_1 \to \mathbb{C}$ such that $\|\mu\|_\infty < 1$. Our problem (66) is formulated on the sphere $S^3$, rather than the Heisenberg group $\mathbb{H}_1$. Of course the sphere minus a point and the Heisenberg group may be identified by the Cayley transform, and we profit from certain ideas by C-Y. Hsiao and P-L. Yung (cf. [4]) to transpose (66) on $\mathbb{H}_1$. Equation (66) may also be written as

$$\overline{Z}(g) = \frac{\mu(\,\cdot\,, t) - t}{1 - t\,\mu(\,\cdot\,, t)}\, Z(g) \tag{67}$$

where $Z = T_1 = \overline{w}\,\partial/\partial z - \overline{z}\,\partial/\partial w$. By a change of dependent variable $f = g \circ H^{-1}$ or $g = f \circ H$, Equation (67) goes over to

$$\overline{u}\,\overline{V}(f) = \frac{\lambda(\,\cdot\,, t) - t}{1 - t\,\lambda(\,\cdot\,, t)}\, u\, V(f), \tag{68}$$

$$\lambda(x, t) = \mu\big(H^{-1}(x), t\big), \quad x \in \mathbb{H}_1\,, \quad |t| < 1.$$

Here, $H = \psi^{-1} \circ \mathcal{C} : S^3 \setminus \{(0, -1)\} \to \mathbb{H}_1$ is the $C^\infty$ diffeomorphism in Section 2.1.6. Equation (68) is central to the present section, and it is our purpose to solve it by an iterative argument relying on Banach's fixed-point theorem.

Let $\mathcal{S}(\mathbb{H}_1)$ be the Schwartz class, consisting of all functions $\varphi \in C^\infty(\mathbb{R}^3)$ such that $p_{\alpha,\beta}(\varphi) < \infty$ for any $\alpha, \beta \in \mathbb{Z}^3_+$. Here $\{p_{\alpha,\beta} : \alpha, \beta \in \mathbb{Z}^3_+\}$ is the separating family of semi-norms on $C^\infty(\mathbb{R}^3)$ given by

$$p_{\alpha,\beta}(\varphi) = \sup_{\mathbf{x} \in \mathbb{R}^3} \big|\mathbf{x}^\alpha\, D^\beta \varphi(\mathbf{x})\big|.$$

If $g \in \mathcal{S}(\mathbb{H}_1)$, then a necessary condition for solving

$$\overline{V}(f) = g \tag{69}$$

(the inhomogeneous tangential Cauchy–Riemann equation on $\mathbb{H}_1$) is that $g * \overline{S} = 0$ (i.e., $g$ must be orthogonal to the kernel of $V$) where

$$\overline{S}(\zeta, \tau) = \frac{1}{\pi^2\big(|\zeta|^2 + i\,\tau\big)^2}\,,$$

$$(g * \overline{S})(x) = \int_{\mathbb{H}_1} \overline{S}(y^{-1}x)\, g(y)\, dy.$$

The canonical solution to (69) (i.e., the solution orthogonal to the kernel of $\overline{V}$) is $f = g * k$, where

$$k(\zeta, \tau) = \frac{1}{\pi^2} \frac{\overline{\zeta}}{(\tau + i\,|\zeta|^2)(\tau - i\,|\zeta|^2)}\,.$$

Cf. P.C. Greiner, J.J. Kohn and E.M. Stein [7]. Let us set

$$b(\zeta, \tau) = (Vk)(\zeta, \tau) = \frac{2i}{\pi^2} \frac{\overline{\zeta}^2}{(\tau + i\,|\zeta|^2)(\tau - i\,|\zeta|^2)}$$

so that

$$f = g * k \Longrightarrow V(f) = g * V(k) = g * b = \overline{V}(f) * b.$$

The kernel $b(\zeta, \tau)$ is homogeneous [with respect to the parabolic dilations $\delta_s(\zeta, \tau) = (s\,\zeta, s^2\,\tau)$ on the Heisenberg group $\mathbb{H}_1$ (with $s > 0$)] of degree $-4$ and

$$\int_{\Sigma^2} b(\zeta, \tau)\; d\sigma = 0.$$

Here $\Sigma^2 = \Sigma^2(1)$, and

$$\Sigma^2(r) = \{x \in \mathbb{H}_1 \;:\; |x| = r\},$$

$$|x| = \left(|z|^4 + t^2\right)^{1/4}, \quad x = (\zeta, \tau) \in \mathbb{H}_1\,,$$

is the Heisenberg sphere of radius $r > 0$. Therefore (by a result of A. Korányi and S. Vági [11]) for every $1 < p < \infty$ the convolution operator $B(g) = g * b$ extends from $\mathcal{S}(\mathbb{H}_1)$ to a bounded operator on $L^p(\mathbb{H}_1\,, \theta_0)$. Additionally, $k(\zeta, \tau)$ is homogeneous of degree $-3$ so that the convolution operator $K(g) = g * k$ extends from $\mathcal{S}(\mathbb{H}_1)$ to a bounded operator

$$K : L^p(\mathbb{H}_1\,, \theta_0) \to L^q(\mathbb{H}_1\,, \theta_0), \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{4}, \quad 1 < p < q < \infty.$$

Cf. G.B. Folland and E.M. Stein [12]. Let $W^{1,2}_E(\mathbb{H}_1\,, \theta_0)$ be the Folland–Stein space of all $L^2$ functions on $\mathbb{H}_1$ admitting weak $E$-derivatives. Let $h \in C^\infty(\mathbb{H}_1)$ such that $\overline{V}(h) = 0$. Let us look for a solution $f \in L^2(\mathbb{H}_1\,, \theta_0)$ to the Beltrami Equation (68) such that $f - h \in W^{1,2}_E(\mathbb{H}_1\,, \theta_0)$. To this end, we set

$$g = \overline{V}(f - h).$$

Note that if $f$ were $C^1$, then we would have $\overline{V}(f) = g$ and

$$V(f) = V(f - h) + V(h) = B(g) + V(h).$$

At this point, we substitute $\overline{V}(f)$ and $V(f)$ into Equation (68), respectively, by $g$ and $B(g) + V(h)$ so as to obtain

$$\overline{u}\,g = \frac{\lambda(\,\cdot\,, t) - t}{1 - t\,\lambda(\,\cdot\,, t)}\, u\big[B(g) + V(h)\big]. \tag{70}$$

Solving for $g$ in (70) is equivalent to seeking a fixed point of

$$F_t(g) = \frac{\lambda(\,\cdot\,, t) - t}{1 - t\,\lambda(\,\cdot\,, t)}\, [B(g) + V(h)] \left(\frac{u}{|u|}\right)^2.$$

Let us set

$$\alpha(x,t) = \frac{\lambda(x,\,t) - t}{1 - t\,\lambda(x,\,t)}\left[\frac{u(x)}{|u(x)|}\right]^2, \quad x \in \mathbb{H}_1\,, \quad |t| < 1,$$

and consider the recurrent sequence

$$g_0 = 0, \quad g_{\nu+1} = F_t(g_\nu), \quad \nu \geq 0.$$

Then

$$g_{\nu+1} = \sum_{k=0}^{\nu}\left[\alpha(\,\cdot\,,t)\,B\right]^k\left(\alpha(\,\cdot\,,t)\,V(h)\right), \quad \nu \geq 0.$$

Then a formal solution to (70) is

$$g = \sum_{\nu=0}^{\infty}\left[\alpha(\,\cdot\,,t)\,B\right]^\nu\left(\alpha(\,\cdot\,,t)\,V(h)\right). \tag{71}$$

The series (71) converges in $L^2(\mathbb{H}_1\,,\,\theta_0)$ provided that

$$F_t : L^2(\mathbb{H}_1\,,\,\theta_0) \to L^2(\mathbb{H}_1\,,\,\theta_0), \quad |t| < 1,$$

are contractions. From now on, we assume that $\{\mu(\,\cdot\,,t)\}_{|t|<1}$ is a smooth 1-parameter family of measurable functions $\mu(\,\cdot\,,t) : S^3 \to \mathbb{C}$ of compact support

$$\mathrm{Supp}[\mu(\,\cdot\,,t)] \subset S^3 \setminus \{(0,\,-1)\}, \quad |t| < 1,$$

such that

$$\|\mu(\,\cdot\,,t)\|_\infty = \mathrm{ess\,sup}_{p\in S^3}|\mu(p,\,t)| < \frac{1-|t|}{1+|t|}\,.$$

The choice of the upper bound on the essential supremum of $|\mu(\,\cdot\,,t)|$ will be explained in a moment. As a consequence of our choice, the function $\lambda(\,\cdot\,,t)$ has compact support $\mathrm{Supp}[\lambda(\,\cdot\,,t)] \subset \mathbb{H}_1$ and

$$\|\lambda(\,\cdot\,,t)\|_\infty < \frac{1-|t|}{1+|t|}\,. \tag{72}$$

Then $\alpha(\,\cdot\,,t)$ has compact support $\mathrm{Supp}\alpha(\,\cdot\,,t) \subset \mathbb{H}_1$ and

$$\|\alpha(\,\cdot\,,t)\|_\infty = \inf\{C > 0 \,:\, |\alpha(x,t)| \leq C \text{ a.e. } x \in \mathbb{H}_1\} =$$

$$= \inf\left\{C > 0 \,:\, \left|\frac{\lambda(x,\,t) - t}{1 - t\,\lambda(x,\,t)}\right| \leq C \text{ a.e. } x \in \mathbb{H}_1\right\}$$

and hence (72) yields

**Lemma 15.**

$$\|\alpha(\,\cdot\,,t)\|_\infty < 1 \tag{73}$$

*for every* $|t| < 1$.

**Proof.** To prove (73), we ought to choose $0 < C_0 < 1$ such that

$$\left|\frac{\lambda(x,\,t) - t}{1 - t\,\lambda(x,\,t)}\right| \leq C_0 \text{ a.e. } x \in \mathbb{H}_1\,.$$

Yet
$$\left|\frac{\lambda(x,\,t)-t}{1-t\,\lambda(x,\,t)}\right| \le \frac{|\lambda(x,\,t)|+|t|}{1-|t|\,|\lambda(x,\,t)|}$$
so it suffices to choose $0 < C_0 < 1$ such that
$$\frac{|\lambda(x,\,t)|+|t|}{1-|t|\,|\lambda(x,\,t)|} \le C_0 \quad \text{a.e. } x \in \mathbb{H}_1$$
or
$$|\lambda(x,\,t)| \le \frac{C_0-|t|}{1+|t|\,C_0} \quad \text{a.e. } x \in \mathbb{H}_1\,.$$
Therefore, one ought to choose $0 < C_0 < 1$ such that
$$\|\lambda(\,\cdot\,,\,t)\|_\infty \le \frac{C_0-|t|}{1+|t|\,C_0} \Longleftrightarrow C_0 \ge \frac{\|\lambda(\,\cdot\,,\,t)\|_\infty+|t|}{1-|t|\,\|\lambda(\,\cdot\,,\,t)\|_\infty}$$
which is possible only provided that
$$\frac{\|\lambda(\,\cdot\,,\,t)\|_\infty+|t|}{1-|t|\,\|\lambda(\,\cdot\,,\,t)\|_\infty} < 1$$
or equivalently
$$\|\lambda(\,\cdot\,,\,t)\|_\infty < \frac{1-|t|}{1+|t|}$$
which is (72). □

As $\alpha(\,\cdot\,,\,t) \in L^2(\mathbb{H}_1\,,\,\theta_0)$ and $\mathrm{Supp}\big[\alpha(\,\cdot\,,\,t)\big]$ is compact, it must be that $\alpha(\,\cdot\,,\,t)\,V(h) \in L^2(\mathbb{H}_1\,,\,\theta_0)$. Then $F_t$ is a map of $L^2(\mathbb{H}_1\,,\,\theta_0)$ into $L^2(\mathbb{H}_1\,,\,\theta_0)$ and for any $g,\,v \in L^2(\mathbb{H}_1\,,\,\theta_0)$
$$\|F_t(g+v)-F_t(g)\|_{L^p(\mathbb{H}_1)} = \|\alpha(\,\cdot\,,\,t)\,B(v)\|_{L^p(\mathbb{H}_1)} \le \|\alpha(\,\cdot\,,\,t)\,B\|\,\|v\|_{L^p(\mathbb{H}_1)}$$
so that $F_t$ is a contraction provided that
$$\|\alpha(\,\cdot\,,\,t)\,B\| < 1. \tag{74}$$

If this is the case, the series (71) converges in $L^2(\mathbb{H}_1\,,\,\theta_0)$. Moreover, if the sum $g$ of the series (71) satisfies the integrability condition
$$g * \overline{S} = 0 \tag{75}$$
then solving for $f$ in $\overline{V}(f-h) = g$ gives the solution $f$ to the Beltrami Equation (68)
$$f = g*k+h, \quad f-h \in W^{1,2}_E(\mathbb{H}_1\,,\,\theta_0).$$

The property $f-h \in W^{1,2}_E(\mathbb{H}_1\,,\,\theta_0)$ of the solution describes its holomorphic behavior at $\infty$. The operator norm in (75) is
$$\|\alpha(\,\cdot\,,\,t)B\| = \sup\left\{\frac{\|\alpha(\,\cdot\,,\,t)\,B(g)\|_{L^2(\mathbb{H}_1\,,\,\theta_0)}}{\|g\|_{L^2(\mathbb{H}_1\,,\,\theta_0)}} \;:\; \begin{array}{c} g \in L^2(\mathbb{H}_1\,,\,\theta_0), \\ g \neq 0 \end{array}\right\}.$$

To compute the operator norm (and prove (75)), we need to represent $B$ as a multiplier on the Fourier transform. For every $\lambda \in \mathbb{R}\setminus\{0\}$ we consider the space
$$\mathfrak{H}_\lambda = L^2H(\mathbb{C},\,\gamma_\lambda) = \mathcal{O}(\mathbb{C}) \cap L^2(\mathbb{C},\,\gamma_\lambda)$$

of all holomorphic functions $\phi : \mathbb{C} \to \mathbb{C}$ such that

$$\|\phi\|_{\gamma_\lambda} = \left( \frac{|\lambda|}{\pi} \int_{\mathbb{C}} |\phi(z)|^2 \, \gamma_\lambda(z) \, d\, m(z) \right)^{1/2} < \infty \,,$$

$$\gamma_\lambda(z) = \exp\big( - |\lambda| \, |z|^2 \big),$$

where $m$ is the Lebesgue measure on $\mathbb{R}^2$. Then $\mathfrak{H}_\lambda$ is a Hilbert space with the scalar product

$$\langle \phi \,, \psi \rangle_{\mathfrak{H}_\lambda} = \frac{|\lambda|}{\pi} \int_{\mathbb{C}} \phi(z) \, \overline{\psi(z)} \, \gamma_\lambda(z) \, d\, m(z).$$

The *Bargmann representation* of the Heisenberg group $\mathbb{H}_1$ is the unitary representation of $\mathbb{H}_1$ on $\mathfrak{H}_\lambda$ given by

$$T_\lambda : \mathbb{H}_1 \to \mathrm{End}_{\mathbb{C}}(\mathfrak{H}_\lambda),$$

$$\big[T_\lambda(\zeta,\, \tau)\phi\big](z) = \begin{cases} \exp\left( -\dfrac{\lambda}{2} \, (i\tau + |\zeta|^2) - \lambda \, \overline{\zeta} \, z \right) \phi(z + \zeta) & \text{if } \lambda > 0, \\[2ex] \exp\left( -\dfrac{\lambda}{2} \, (i\tau - |\zeta|^2) + \lambda \, \overline{\zeta} \, z \right) \phi(z + \zeta) & \text{if } \lambda < 0, \end{cases}$$

$$(\zeta \,, \tau) \in \mathbb{H}_1 \,, \quad \phi \in \mathfrak{H}_\lambda \,, \quad z \in \mathbb{C}.$$

**Lemma 16.** $T_\lambda\big(\zeta,\, \tau\big) = T_{-\lambda}\big(\overline{\zeta} \,, -\tau\big)$.

Let $\mathfrak{h}_1$ be the Lie algebra of $\mathbb{H}_1$. The same symbol $T_\lambda$ will denote the induced representation of the Lie algebra $\mathfrak{h}_1$ on $\mathfrak{H}_\lambda$

$$T_\lambda : \mathfrak{h}_1 \to \mathrm{End}_{\mathbb{C}}(\mathfrak{H}_\lambda), \quad T_\lambda(A) = (d_0 T_\lambda) A_0 \,, \quad A \in \mathfrak{h}_1 \,.$$

The Lewy operator $\overline{V} = \partial/\partial\overline{\zeta} - i\,\zeta\,\partial/\partial\tau$ and the Reeb vector field $\partial/\partial\tau$ are known to be left invariant. Hence, $\mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C}$ is the span of $\{V,\, \overline{V} \,, \partial/\partial\tau\}$.

**Lemma 17.**

(i) *If* $\lambda > 0$ *then*

$$T_\lambda(V) = \frac{\partial}{\partial z} \,, \quad T_\lambda(\overline{V}) = -\lambda \, z, \quad T_\lambda\left( \frac{\partial}{\partial\tau} \right) = -\frac{i\,\lambda}{\zeta} \,.$$

(ii) *If* $\lambda < 0$ *then*

$$T_\lambda(V) = \lambda \, z, \quad T_\lambda(\overline{V}) = \frac{\partial}{\partial z} \,, \quad T_\lambda\left( \frac{\partial}{\partial\tau} \right) = -\frac{i\,\lambda}{\zeta} \,.$$

(iii) $T_\lambda : \mathfrak{h}_1 \to \mathrm{End}_{\mathbb{C}}\big(\mathfrak{H}_\lambda\big)$ *is a unitary representation.*

The *Fourier transform* at $\lambda$ of a function $f \in \mathcal{S}\big(\mathbb{H}_1\big)$ is the operator

$$T_\lambda(f) : \mathfrak{H}_\lambda \to \mathfrak{H}_\lambda \,,$$

$$\big[T_\lambda(f)\phi\big](z) = \int_{\mathbb{H}_1} f(\zeta,\, \tau) \, \big[T_\lambda(\zeta,\, \tau)\phi\big](z) \, d\,\xi \, d\,\eta \, d\tau \,,$$

$$\phi \in \mathfrak{H}_\lambda \,, \quad z \in \mathbb{C}.$$

Here $\zeta = \xi + i\,\eta$ are the real and imaginary parts of $\zeta$. We recall that a bounded linear operator $A : \mathfrak{H}_\lambda \to \mathfrak{H}_\lambda$ is an operator of *trace class* if

$$\|A\|_1 = \mathrm{Tr}\,|A| := \sum_{n=1}^{\infty} \big\langle (A^* A)^{1/2} \phi_n \,, \phi_n \big\rangle_\lambda < \infty$$

for some complete orthonormal system $\{\phi_n \ : \ n \geq 1\} \subset \mathfrak{H}_\lambda$ (and thus for all). If this is the case, then the *trace* of $A$

$$\mathrm{Tr}\, A := \sum_{\nu=1}^{\infty} \langle A\phi_n \, , \, \phi_n \rangle_\lambda$$

is an absolutely convergent series, and its sum is independent of the choice of a complete orthonormal system in $\mathfrak{H}_\lambda$.

**Lemma 18.** *The Fourier transform $T_\lambda(f) : \mathfrak{H}_\lambda \to \mathfrak{H}_\lambda$ of every $f \in \mathcal{S}(\mathbb{H}_1)$ is an operator of trace class.*

The norm of $T_\lambda(f)$ (the *trace norm*) is defined by

$$\left\| T_\lambda(f) \right\|^2 = \mathrm{Tr} \left\{ T_\lambda^*(f) \, T_\lambda(f) \right\}$$

where $T_\lambda^*(f) = T_\lambda(f)^*$ (the adjoint of $T_\lambda(f)$).

**Lemma 19.** *Let $f \in \mathcal{S}(\mathbb{H}_1)$.*

(i) *The inversion formula for the Fourier transform is*

$$f(\zeta, \tau) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \mathrm{Tr} \left\{ T_\lambda^*(\zeta, \tau) \, T_\lambda(f) \right\} |\lambda| \, d\lambda \, . \tag{76}$$

(ii) *The Plancherel formula for the Fourier transform is*

$$\|f\|^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}} \left\| T_\lambda(f) \right\|^2 |\lambda| \, d\lambda \tag{77}$$

*where $\|f\|$ is the $L^2$ norm of $f$.*

Cf. J. Faraut [13]. On the basis of the formulas (76) and (77), the Fourier transform $T_\lambda(f)$ may be extended from functions of Schwartz class $f \in \mathcal{S}(\mathbb{H}_1)$ to square integrable functions $f \in L^2(\mathbb{H}_1 \, , \, \theta_0)$.

**Lemma 20.** *The Fourier transform of the convolution product*

$$(f * g)(y) = \int_{\mathbb{H}_1} f(x) \, g(x^{-1}y) \, d\, x, \quad f, g \in \mathcal{S}(\mathbb{H}_1),$$

*is given by*

$$T_\lambda(f * g) = T_\lambda(f) \, T_\lambda(g).$$

**Lemma 21.** *The system $\{\phi_n \ : \ n \geq 1\} \subset \mathfrak{H}_\lambda$ given by*

$$\phi_n(z) = \sqrt{\frac{|\lambda|^n}{n!}} \, z^n \, , \quad z \in \mathbb{C}, \quad n \in \mathbb{N},$$

*is a complete orthonormal system in $\mathfrak{H}_\lambda$.*

Let

$$t^\lambda_{n,m}(\zeta, \tau) = \langle T_\lambda(\zeta, \tau)\phi_m \, , \, \phi_n \rangle_\lambda$$

be the Fourier coefficients of the operator $T_\lambda(\zeta, \tau)$ with respect to $\{\phi_n\}_{n \geq 1}$. This is an infinite matrix given by the following.

**Lemma 22.**

(i) *If $\lambda > 0$ and $m \geq n$, then*

$$t^{\lambda}_{n,m}(\zeta, \tau) =$$

$$= \sqrt{\frac{n!}{m!}}\,\big(\sqrt{\lambda}\,\zeta\big)^{m-n} \exp\Big(-\frac{i\lambda\tau}{2}\Big) \exp\Big(-\frac{\lambda|\zeta|^2}{2}\Big)\, L^{m-n}_{n}\big(\lambda|\zeta|^2\big).$$

(ii) *If $\lambda > 0$ and $m < n$, then*

$$t^{\lambda}_{n,m}(\zeta, \tau) = \overline{t^{\lambda}_{m,n}(-\zeta, -\tau)} =$$

$$= \sqrt{\frac{m!}{n!}}\,\big(-\sqrt{\lambda}\,\overline{\zeta}\big)^{n-m} \exp\Big(-\frac{i\lambda\tau}{2}\Big) \exp\Big(-\frac{\lambda|\zeta|^2}{2}\Big)\, L^{n-m}_{m}\big(\lambda|\zeta|^2\big).$$

(iii) *If $\lambda < 0$, then*

$$t^{\lambda}_{n,m}(\zeta, \tau) = \overline{t^{|\lambda|}_{n,m}(\zeta, \tau)}.$$

Cf. A. Korányi and H.M. Reimann [3], pp. 70–71. Here,

$$L^{\alpha}_{n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)!}\,\frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)}\, x^k, \quad \alpha > -1, \quad x \geq 0,$$

are the Laguerre polynomials. From now on, the Fourier transform of a function $f \in \mathcal{S}(\mathbb{H}_1)$ will be represented as an infinite matrix

$$\hat{f}(\lambda) = \big[\hat{f}^{\lambda}_{m,n}\big]_{m,n\in\mathbb{N}}, \quad \hat{f}^{\lambda}_{m,n} = \big\langle T_{\lambda}(f)\phi_n\,, \phi_m \big\rangle_{\gamma_\lambda},$$

so that

$$T_{\lambda}(f)\phi_m = \sum_{n=1}^{\infty} \hat{f}^{\lambda}_{n,m}\,\phi_n\,.$$

**Lemma 23.** *The Fourier transform $\hat{b}(\lambda) = \big[\hat{b}^{\lambda}_{m,n}\big]$ of*

$$b(\zeta, \tau) = Vk(\zeta, \tau) = \frac{2i}{\pi^2}\,\frac{\overline{\zeta}^2}{\big(\tau + i|\zeta|^2\big)\big(\tau - i|\zeta|^2\big)}$$

*is given by*

$$\hat{b}^{\lambda}_{m,n} = \begin{cases} -\delta_{m+2,\,n}\sqrt{\dfrac{m+1}{m}} & \text{if } \lambda > 0, \\[2ex] \delta_{m-2,\,n}\sqrt{\dfrac{m-2}{m-1}} & \text{if } \lambda < 0. \end{cases} \tag{78}$$

Let us consider the subspaces $L^2_{\pm}(\mathbb{H}_1\,, \theta_0) \subset L^2(\mathbb{H}_1\,, \theta_0)$ defined by

$$L^2_{-}(\mathbb{H}_1\,, \theta_0) = \big\{f \in L^2(\mathbb{H}_1\,, \theta_0) \;:\; \hat{f}(\lambda) = 0 \text{ a.e. } \lambda > 0\big\},$$

$$L^2_{+}(\mathbb{H}_1\,, \theta_0) = L^2(\mathbb{H}_1\,, \theta_0) \ominus L^2_{-}(\mathbb{H}_1\,, \theta_0).$$

**Lemma 24.** *$B\, L^2_{\pm}(\mathbb{H}_1\,, \theta_0) \subset L^2_{\pm}(\mathbb{H}_1\,, \theta_0)$.*

Next, for every $k \in \mathbb{Z}$ let us set

$$U^k = \big\{f \in L^2(\mathbb{H}_1\,, \theta_0) \;:\; f\big(\zeta\, e^{i\varphi}, \tau\big) = e^{i\,k\,\varphi} f(\zeta, \tau)\big\}.$$

**Lemma 25.** $L^2(\mathbb{H}_1\,,\,\theta_0) = \bigoplus_{k\in\mathbb{Z}} U^k$.

**Lemma 26.**

(i) *If* $f \in U^k$ *then*

$$m - n \neq +k \Longrightarrow \hat{f}^{\lambda}_{m,n} = 0 \ \text{ for a.e. } \ \lambda > 0,$$
$$m - n \neq -k \Longrightarrow \hat{f}^{\lambda}_{m,n} = 0 \ \text{ for a.e. } \ \lambda < 0.$$

(ii) $B\,U^k \subset U^{k+2}$.

Next let us consider the complete orthogonal sum

$$D_j = \widehat{\bigoplus}_{k\le j} U^k\,.$$

**Lemma 27.**

(i) *The complete orthogonal sums* $\{D_j\}_{j\in\mathbb{Z}}$ *satisfy the following multiplication law*

$$f \in D_j \ \text{ and } \ \alpha(\,\cdot\,,\,t) \in D_m \cap L^\infty(\mathbb{H}_1\,,\,\theta_0) \Longrightarrow$$
$$\Longrightarrow f \cdot \alpha(\,\cdot\,,\,t) \in D_{j+m}\,.$$

(ii) $L^2_\pm(\mathbb{H}_1\,,\,\theta_0)$ *are multiplication invariant, i.e.,*

$$f \in L^2_\pm(\mathbb{H}_1\,,\,\theta_0) \ \text{ and } \ \alpha(\,\cdot\,,\,t) \in L^2_\pm(\mathbb{H}_1\,,\,\theta_0) \cap L^\infty(\mathbb{H}_1\,,\,\theta_0) \Longrightarrow$$
$$\Longrightarrow f \cdot \alpha(\,\cdot\,,\,t) \in L^2_\pm(\mathbb{H}_1\,,\,\theta_0).$$

**Theorem 5.** *Let* $h \in \mathrm{CR}^\infty(\mathbb{H}_1)$ *be a CR function* [*i.e.,* $\overline{V}(h) = 0$] *and let us assume that* $\alpha(\,\cdot\,,\,t) \in L^\infty(\mathbb{H}_1)$. *Let us assume that one of the following conditions is satisfied*

(1) $\alpha(\,\cdot\,,\,t)$, $\alpha(\,\cdot\,,\,t)\,V(h) \in L^2_+(\mathbb{H}_1\,,\,\theta_0)$ *and* $\|\alpha(\,\cdot\,,\,t)\|_\infty < \dfrac{1}{\sqrt{2}}$.

(2) $\alpha(\,\cdot\,,\,t) \in D_{-2}$, $\alpha(\,\cdot\,,\,t)\,V(h) \in D_{-1}$ *and* $\|\alpha(\,\cdot\,,\,t)\|_\infty < \dfrac{1}{\sqrt{2}}$.

(3) $\alpha(\,\cdot\,,\,t) \in D_{-2} \cap L^2_-(\mathbb{H}_1\,,\,\theta_0)$, $\alpha(\,\cdot\,,\,t)\,V(h) \in D_{-1} \cap L^2_-(\mathbb{H}_1\,,\,\theta_0)$ *and* $\|\alpha(\,\cdot\,,\,t)\|_\infty < 1$.
*Then the Beltrami equation*

$$\overline{u}\,\overline{V}(f) = \frac{\lambda(\,\cdot\,,\,t) - t}{1 - t\,\lambda(\,\cdot\,,\,t)}\,u\,V(f)$$

*has a unique solution* $f_t$ *such that* $f_t - h \in W^{1,2}_E(\mathbb{H}_1\,,\,\theta_0)$.

**Proof.** We ought to show that the series

$$g = \sum_{\nu=0}^{\infty} \left[\alpha(\,\cdot\,,\,t)\,B\right]^\nu \big(\alpha(\,\cdot\,,\,t)\,V(h)\big)$$

converges in $L^2(\mathbb{H}_1\,,\,\theta_0)$, and its sum $g$ satisfies the integrability condition $g * \overline{S} = 0$. For every $f \in L^2(\mathbb{H}_1\,,\,\theta_0)$, its Fourier transform at $\lambda$ is

$$T_\lambda(f)\,\phi_n = \sum_{m=1}^{\infty} \hat{f}^{\lambda}_{m,n}\,\phi_m$$

and hence, its trace norm is

$$\left\|T_\lambda(f)\right\|^2 = \operatorname{Tr}\left\{T^*_\lambda(f)\,T_\lambda(f)\right\} = \sum_{n=1}^{\infty}\left\langle T^*_\lambda(f)\,T_\lambda(f)\,\phi_n\,,\,\phi_n\right\rangle_{\gamma_\lambda} =$$

$$= \sum_{n=1}^{\infty}\left\|T_\lambda(f)\,\phi_n\right\|^2 = \sum_{n,m=1}^{\infty}|\hat{f}^\lambda_{m,n}|^2.$$

Then

$$T_\lambda\big(Bf\big)\,\phi_n = T_\lambda\big(f*b\big) = T_\lambda(f)\,T_\lambda(b)\,\phi_n = \sum_{m,k=1}^{\infty}\hat{b}^\lambda_{m,n}\,\hat{f}^\lambda_{k,m}\,\phi_k$$

i.e.,

$$\widehat{(Bf)}^{\lambda}_{k,n} = \sum_m \hat{b}^\lambda_{m,n}\,\hat{f}^\lambda_{k,m} =$$

[by (78) in Lemma 23]

$$= \sum_m \hat{f}^\lambda_{k,m}\begin{cases} -\delta_{m+2,n}\sqrt{\dfrac{m+1}{m}} & \text{if } \lambda>0 \\[2ex] \delta_{m-2,n}\sqrt{\dfrac{m-2}{m-1}} & \text{if } \lambda<0 \end{cases} = \begin{cases} -\sqrt{\dfrac{n-1}{n-2}}\,\hat{f}^\lambda_{k,n-2} & \text{if } \lambda>0 \\[2ex] \sqrt{\dfrac{n}{n+1}}\,\hat{f}^\lambda_{k,n+2} & \text{if } \lambda<0 \end{cases}$$

so that

$$\left\|T_\lambda(Bf)\right\|^2 = \sum_{n,m}\begin{cases} \dfrac{n-1}{n-2}\left|\hat{f}^\lambda_{m,n-2}\right|^2 & \text{if } \lambda>0 \\[2ex] \dfrac{n}{n+1}\left|\hat{f}^\lambda_{m,n+2}\right|^2 & \text{if } \lambda<0 \end{cases} = \sum_{k,m}\begin{cases} \dfrac{k+1}{k}\left|\hat{f}^\lambda_{m,k}\right|^2 & \text{if } \lambda>0 \\[2ex] \dfrac{k-2}{k-1}\left|\hat{f}^\lambda_{m,k}\right|^2 & \text{if } \lambda<0 \end{cases}$$

and hence [by $(k-2)/(k-1)<1$ and $(k+1)/k<2$]

$$\left\|T_\lambda(Bf)\right\| \le \sqrt{2}\left\|T_\lambda(f)\right\|$$

and then

$$\left\|Bf\right\| \le \sqrt{2}\left\|f\right\|, \quad f\in L^2(\mathbb{H}_1\,,\,\theta_0). \tag{79}$$

Similarly, if $f\in L^2_-\left(\mathbb{H}_1\,,\,\theta_0\right)$ then

$$\left\|Bf\right\|^2 = \frac{1}{4\pi^2}\int_{\mathbb{R}}\left\|T_\lambda\big(Bf\big)\right\|\,|\lambda|\,d\lambda =$$

[as $\hat{f}(\lambda)=0$ for a.e. $\lambda>0$]

$$= \frac{1}{4\pi^2}\int_{-\infty}^{0}\sum_{n,m}\frac{n-2}{n-1}\left|\hat{f}^\lambda_{m,n}\right|^2|\lambda|\,d\lambda$$

yielding

$$\left\|Bf\right\| \le \left\|f\right\|, \quad f\in L^2_-(\mathbb{H}_1\,,\,\theta_0). \tag{80}$$

Let us examine now the three assumptions in Theorem 5. By (79) and $\left\|\alpha\big(\,,\,t\big)\right\|_\infty < \dfrac{1}{\sqrt{2}}$, it follows that the operator norm of

$$\alpha\big(\,\cdot\,,\,t\big)\,B : L^2(\mathbb{H}_1\,,\,\theta_0) \to L^2(\mathbb{H}_1\,,\theta_0)$$

is $\|\alpha(\,\cdot\,,\,t)\,B\| < 1$, and hence

$$\sum_{\nu=0}^{\infty} \big[\alpha(\,\cdot\,,\,t)\,B\big]^{\nu}\big[\alpha(\,\cdot\,,\,t)\,Vh\big]$$

converges to some $g_t \in L^2(\mathbb{H}_1\,,\,\theta_0)$.

(1) As $\alpha(\,\cdot\,,\,t) \in L^2_+(\mathbb{H}_1\,,\,\theta_0)$ and $\alpha(\,\cdot\,,\,t)\,Vh \in L^2_+(\mathbb{H}_1\,,\,\theta_0)$ and [by Lemma 24]

$$B\big[L^2_+(\mathbb{H}_1\,,\,\theta_0)\big] \subset L^2_+(\mathbb{H}_1\,,\,\theta_0)$$

it follows that $g_t \in L^2_+(\mathbb{H}_1\,,\,\theta_0)$.

(2) As

$$D_k = \widehat{\bigoplus}_{\ell \le k} U^{\ell}\,, \quad D_{k+2} = \widehat{\bigoplus}_{\ell \le k+2} U^{\ell} = \widehat{\bigoplus}_{\ell \le k} U^{\ell+2}\,,$$
$$B\big(U^{\ell}\big) \subset U^{\ell+2} \quad \text{[by (ii) in Lemma 26]}$$

one has

$$B\big(D_k\big) \subset D_{k+2}\,. \tag{81}$$

It should be observed that (81) is independent of any of the assumptions in Theorem 5. If $\alpha(\,\cdot\,,\,t) \in D_{-2} = \widehat{\bigoplus}_{\ell \le -2} U^{\ell}$, then (by (81)

$$\alpha(\,\cdot\,,\,t)\,B\,D_k \subset \alpha(\,\cdot\,,\,t) \cdot D_{k+2} \subset$$

(by (i) in Lemma 27 with $j = k+2$ and $m = -2$)

$$\subset D_k$$

so that $D_k$ is invariant by $\alpha(\,\cdot\,,\,t)\,B$. Moreover $\alpha(\,\cdot\,,\,t)\,Vh \in D_{-1}$ yields $g_t \in D_{-1}$.

(3) If $\alpha(\,\cdot\,,\,t) \in D_{-2} \cap L^2_-(\mathbb{H}_1\,,\,\theta_0)$ then $\alpha(\,\cdot\,,\,t) \in D_{-2}$ was already shown to imply $\alpha(\,\cdot\,,\,t)\,B\,D_k \subset D_k$ (here useful for $k = -1$). On the other hand (by $B\,L^2_-(\mathbb{H}_1\,,\,\theta_0) \subset L^2_-(\mathbb{H}_1\,,\,\theta_0)$),

$$\alpha(\,\cdot\,,\,t)\,B\,\big\{D_{-1} \cap L^2_-(\mathbb{H}_1\,,\,\theta_0)\big\} \subset D_{-1} \cap \big\{\alpha(\,\cdot\,,\,t) \cdot L^2_-(\mathbb{H}_1\,,\,\theta_0)\big\} \subset$$

(by $\alpha(\,\cdot\,,\,t) \in L^2_-(\mathbb{H}_1\,,\,\theta_0)$)

$$\subset D_{-1} \cap L^2_-(\mathbb{H}_1\,,\,\theta_0)$$

yielding $g_t \in D_{-1} \cap L^2_-(\mathbb{H}_1\,,\,\theta_0)$. Summing up, under the assumptions (1)–(3) in Theorem 5, the function $g_t$ belongs to one of the spaces

$$L^2_+(\mathbb{H}_1\,,\,\theta_0), \quad D_{-1}\,, \quad D_{-1} \cap L^2_-(\mathbb{H}_1\,,\,\theta_0).$$

The proof of Theorem 5 may be completed by applying the following lemma:

**Lemma 28.** *Let $S \in \big\{L^2_+(\mathbb{H}_1\,,\,\theta_0),\, D_{-1}\,,\, D_{-1} \cap L^2_-(\mathbb{H}_1\,,\,\theta_0)\big\}$. For every $g \in S$, one has $\hat{g}(\lambda)\,\hat{\overline{S}}(\lambda) = 0$. Equivalently, each $g \in S$ satisfies the integrability condition $g * \overline{S} = 0$.*

□

## 4. Conclusions and Open Problems

Sobolev-type solutions to the Beltrami equation

$$\overline{V}(f) = \mu\,V(f) \tag{82}$$

on the Heisenberg group $\mathbb{H}_1$ were first produced by A. Korányi and H.M. Reimann [3], relying on work by P.C. Greiner, J.J. Kohn and E.M. Stein [7], on the solution to the Lewy equation $\overline{V}(f) = g$. We consider the Beltrami equations associated to the non-embeddable

CR structures $\mathcal{H}(t)$, $|t| < 1$, on $S^3$ as discovered by H. Rossi [1], and transplant said equations on $\mathbb{H}_1$ by using the CR diffeomorphism $H : U = S^3 \setminus \{(0, -1)\} \approx \mathbb{H}_1$ (associated with the Cayley map). This gives a 1-parameter family of first order PDEs (with variable coefficients) on $\mathbb{H}_1$, similar to Korányi and Reimann's Beltrami Equation (82), which may be simultaneously treated by an outgrowth of Korányi and Reimann's techniques (borrowed from [7] for the part of complex analysis, and from J. Faraut [13] for the part of harmonic analysis). It is an open problem whether the same CR diffeomorphism $H$ may be used to transplant Fourier calculus from $\mathbb{H}_1$ to the open set $U \subset S^3$ (and whether the resulting tools are effective in a direct study of Equations (3)). We expect the resulting local harmonic analysis on $S^3$ to be similar to that proposed by R.S. Strichartz [14]. Cf. also [15].

The success in [10] to discover obstructions to CR equivalence of a strictly pseudoconvex real hypersurface $M \subset \mathbb{C}^{n+1}$ to the sphere $S^{2n+1}$ (such as the first Pontrjagin form of the Fefferman metric) prompts the question of whether (other) characteristic forms of $F^t_\theta$ [the Fefferman metric of a Rossi sphere $(S^3, \mathcal{H}(t))$] may be identified as obstructions to the existence of a $K$-quasiconformal map $f : (S^3, \mathcal{H}(t)) \to (S^3, \mathcal{H}(0))$. Our discussion of Fefferman's metric in Sections 2.1.5 and 2.6 is only tentative, and a deeper study is relegated to further work.

**Author Contributions:** Conceptualization, E.B., S.D. and F.E.; writing—original draft preparation, E.B., S.D. and F.E.; writing—review and editing, E.B., S.D. and F.E. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Acknowledgments:** Sorin Dragomir acknowledges support from P.R.I.N. 2015, Italy. Francesco Esposito is grateful for support from the joint Doctoral School of *Università degli Studi della Basilicata* (Potenza) and *Università del Salento* (Lecce) over the period 2018–2021.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Rossi, H. Attaching analytic spaces to an analytic space along a pseudoconcave boundary. In Proceedings of the Conference on Complex Analysis, Minneapolis, MA, USA, 16–21 March 1964; Springer: Berlin, Germany, 1965; pp. 242–256.
2. Burns, D.M., Jr. Global behavior of some tangential Cauchy-Riemann equations. In *Partial Differential Equations and Geometry*; Dekker: New York, NY, USA, 1979; pp. 51–56.
3. Korányi, A.; Reimann, H.M. Quasiconformal mappings on CR manifolds. In *Conference in Honor of Edoardo Vesentini*; Springer: Berlin, Germany, 1988; pp. 59–75.
4. Hsiao, C.-Y.; Yung, P.-L. The tangential Cauchy-Riemann complex on the Heisenberg group via conformal invariance. *Bull. Inst. Math. Acad. Sin.* **2013**, *8*, 359–375.
5. Nagel, A.; Stein, E.M.; Wainger, S. Balls and metrics defined by vector fields I: Basic properties. *Acta Math.* **1985**, *155*, 103–147. [CrossRef]
6. Franchi, B. Weighted Sobolev-Poincaré inequalities and pointwise estimates for a class of degenerate elliptic operators. *Trans. Amer. Math. Soc.* **1991**, *327*, 125–158.
7. Greiner, P.C.; Kohn, J.J.; Stein, E.M. Necessary and sufficient conditions for the solvability of the Lewy equation. *Proc. Nat. Acad. Sci. USA* **1975**, *72*, 3287–3289. [CrossRef] [PubMed]
8. Dragomir, S.; Tomassini, G. *Differential Geometry and Analysis on CR Manifolds*; Progress in Mathematics; Birkhäuser: Basel, Switzerland, 2006; Volume 246.
9. Graham, C.R. On Sparling's characterization of Fefferman metrics. *Am. J. Math.* **1987**, *109*, 853–874. [CrossRef]
10. Barletta, E.; Dragomir, S. New CR invariants and their application to the CR equivalence problem. *Ann. Scuola Norm. Sup. Pisa* **1997**, *XXIV*, 193–203.
11. Korányi, A.; Vági, S. Singular integrals on homogeneous spaces and some problems of classical analysis. *Ann. Scuola Norm. Sup. Pisa* **1971**, *25*, 575–648.
12. Folland, G.B.; Stein, E.M. Estimates for the $\overline{\partial}_b$-complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.* **1974**, *27*, 429–522. [CrossRef]
13. Faraut, J. Analyse harmonique et fonctions spéciales. In *Deux Courses d'Analyse Harmonique: École d'été d'Analyse Harmonique de Tunis*; Birkhäuser: Basel, Switzerland, 1987.
14. Strichartz, R.S. Local harmonic analysis on spheres. *J. Funct. Anal.* **1988**, *77*, 403–433. [CrossRef]
15. Sherman, T.O. Fourier analysis on the sphere. *Trans. Am. Math. Soc.* **1975**, *209*, 1–31. [CrossRef]