Dolomites Research Notes on Approximation

Special Issue *Multivariate Approximation: Theory and Application, MATA2020*, Volume 14 · 2021 · Pages 26–38

The numerical solution of Cauchy singular integral equations with additional fixed singularities

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Abstract

In this paper we propose a quadrature method for the numerical solution of Cauchy singular integral equations with additional fixed singularities. The unknown function is approximated by a weighted polynomial which is the solution of a finite dimensional equation obtained discretizing the involved integral operators by means of a Gauss-Jacobi quadrature rule. Stability and convergence results for the proposed procedure are proved. Moreover, we prove that the linear systems one has to solve, in order to determine the unknown coefficients of the approximate solutions, are well conditioned. The efficiency of the proposed method is shown through some numerical examples.

1 Introduction

This paper deals with the numerical solution of the following Cauchy singular integral equation (CSIE) with additional terms having fixed singularities

$$a\,u(\tau)+\frac{b}{\pi}\int_{-1}^{1}\frac{u(t)}{t-\tau}dt+\int_{-1}^{1}k(t,\tau)u(t)dt+\int_{-1}^{1}h(t,\tau)u(t)dt=g(\tau),\quad |\tau|<1, \tag{1}$$

where $u(\tau)$ is the unknown function, $h(t,\tau)$ and $g(\tau)$ are given sufficiently smooth functions on $[-1,1]\times[-1,1]$ and $[-1,1]$, respectively, a and b are given real constants such that $a^2+b^2=1$, and $k(t,\tau)$ is a known kernel assuming one of the forms

$$k(t,\tau)=\frac{1}{1+t}\bar{k}\left(\frac{1-\tau}{1+t}\right) \tag{2}$$

or

$$k(t,\tau)=\frac{1}{1-t}\bar{k}\left(\frac{1+\tau}{1-t}\right), \tag{3}$$

for some given function $\bar{k}$ on $[0,+\infty)$. The first integral is understood in the Cauchy principal value sense.
Note that the kernels $k(t,\tau)$ in (2) and (3) have fixed singularities at the points $(-1,1)$ and $(1,-1)$, respectively, and differ from the Mellin convolution type kernels that become singular if t and τ tend to the same point simultaneously.

Setting

$$(Du)(\tau)=a\,u(\tau)+\frac{b}{\pi}\int_{-1}^{1}\frac{u(t)}{t-\tau}dt, \tag{4}$$

$$(Ku)(\tau)=\int_{-1}^{1}k(t,\tau)u(t)dt, \tag{5}$$

and

$$(Hu)(\tau)=\int_{-1}^{1}h(t,\tau)u(t)dt, \tag{6}$$

we can rewrite the equation (1) as follows

$$(D+K+H)u=g. \tag{7}$$

The solution u is searched in the following form

$$u(t)=v^{\alpha,\beta}(t)f(t), \tag{8}$$

where f is a smooth function and $v^{\alpha,\beta}(t)=(1-t)^{\alpha}(1+t)^{\beta}$ is a Jacobi weight. The exponents $-1<\alpha,\beta<1$ depend on the coefficients a and b of the operator D as follows

$$\alpha=M-\frac{1}{2\pi i}\log\left(\frac{a+ib}{a-ib}\right),\quad \beta=N+\frac{1}{2\pi i}\log\left(\frac{a+ib}{a-ib}\right),$$

where M and N are integers chosen so that the index $\chi=-(\alpha+\beta)=-(M+N)=0$, i.e. $\beta=-\alpha$.

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Since several mathematical problems in physics and engineering can be reduced to the solution of Cauchy type integral equations, the development of numerical methods for approximating their solution has been receiving an increasing interest in recent years. Many papers are available in the literature on this topic especially in the case when only the compact perturbation Hu appears in addition to the singular Cauchy operator and no other singular integral operator is involved (see, for instance, [1, 17, 16, 20, 24, 22, 8, 2, 4, 3] and the references therein). For the complete equation, discretization schemes based on polynomial approximation have been more recently proposed in [18, 11, 14, 15, 12, 13], where the case of kernels having additional fixed singularities of Mellin type

$$k(t,\tau)=\frac{1}{1-t}\bar{k}\left(\frac{1-\tau}{1-t}\right)$$

and/or

$$k(t,\tau)=\frac{1}{1+t}\bar{k}\left(\frac{1+\tau}{1+t}\right),$$

is treated. In particular, in [14, 15, 12, 13] the authors considered some special choices of the function $\bar{k}$. Moreover, the stability of the proposed collocation methods is proved in weighted L^2 spaces w.r.t. Chebyshev weights and collocation is performed with respect to Chebyshev nodes.

To our knowledge, integral equations of type (1) with singular kernels of the form (2) or (3) have not been studied until now. They are of interest, for instance, when one has to solve a boundary CSIE defined on a simple open wedge or on a polygonal contour in the plane. Our aim is to propose a discrete collocation method for the approximation of the solution of equation (1), proving its stability and convergence in more general weighted L^2 spaces, suitably connected with the original problem (1). The application of this procedure to the numerical solution of the above mentioned boundary CSIE will be the object of a further investigation.

The method follows the scheme of a collocation-quadrature method proposed in [1] in the case where the kernels are smooth and/or weakly singular. The unknown solution u of (7) is approximated by a suitable weighted polynomial u_n which is the solution of a finite dimensional equation. Such equation is deduced from (7) by using first an appropriate Gauss-Jacobi quadrature rule in order to discretize the integral operators H and K, and then a suitable Lagrange projector applied to both sides of the equation having H and K replaced by their respective approximations. Hence, the numerical solution u_n is determined by solving a linear system obtained by collocating such equation at suitable nodes.

Due to the fixed singularity of the kernel $k(t,\tau)$ at the point $(-1,1)$ or $(1,-1)$ (according to whether it takes the form (2) or (3), respectively), the Gauss-Jacobi formula for the approximation of $(Ku)(\tau)$ diverges when τ approaches 1 or -1. For this reason, in many papers (see, for example, [9, 10, 5, 6, 7, 21]) dealing with the numerical solution of integral equations with fixed singularities of Mellin type, a slight modification of the quadrature formula has been employed in order to achieve stability and convergence of the proposed numerical methods. Neverthless, in virtue of the choice of the collocation points, here we are able to achieve stability and convergence results simply applying the classical Gauss-Jacobi rule, also providing an estimate of the error in weighted L^2 norm. We also pay special attention to the study of the conditioning of the involved linear systems. In particular, we are able to prove that the sequence of their condition numbers converges to the condition number of the operator $D+K+H$.

The plan of the paper is as follows. In Section 2 we give some notation while in Section 3 we state some preliminary results dealing with the mapping properties of the operators D, K and H. Section 4 contains the description of the numerical method and the main results regarding stability, convergence and well-conditioning of the involved linear systems. The proofs of the main results are given in Section 5. Finally, in Section 6 we present some numerical tests showing the performance of the method and the reliability of the theoretical results.

2 Notation and basic facts

In the sequel $\mathcal{C}$ denotes a positive constant which may assume different values in different formulas. We will write $\mathcal{C}(a,b,\dots)$ to say that $\mathcal{C}$ depends on the parameters $a,b,\dots$ and $\mathcal{C}\neq\mathcal{C}(a,b,\dots)$ to say that $\mathcal{C}$ is independent of the parameters $a,b,\dots$. Moreover, if $A,B\geq 0$, the symbol $A\sim B$ means that there exists a constant $0<\mathcal{C}\neq\mathcal{C}(A,B)$ such that $\mathcal{C}^{-1}B\leq A\leq \mathcal{C}B$.

Let us introduce the function spaces where we are going to study equation (7).
With $v^{\gamma,\delta}(t)=(1-t)^{\gamma}(1+t)^{\delta}$, $\gamma,\delta>-1$, we denote by $L^2_{v^{\gamma,\delta}}$, the weighted space of all real-valued measurable functions F on $[-1,1]$ such that

$$\|F\|_{L^2_{v^{\gamma,\delta}}}:=\|F\|_{v^{\gamma,\delta}}:=\left(\int_{-1}^{1}|F(t)|^2v^{\gamma,\delta}(t)dt\right)^{\frac{1}{2}}<+\infty.$$

Let $\{p_n^{\gamma,\delta}\}_n$ be the system of the orthonormal polynomials w.r.t. the scalar product

$$<u,v>_{v^{\gamma,\delta}}:=\int_{-1}^{1}u(t)v(t)v^{\gamma,\delta}(t)dt \qquad (9)$$

with positive leading coefficients. The system $\{\tilde{p}_n^{\gamma,\delta}\}_n:=\{v^{-\gamma,-\delta}p_n^{-\gamma,-\delta}\}_n$ is orthonormal w.r.t. the scalar product (9), too. Using the above orthonormal systems, for a real number $s\geq 0$, we define the following subspaces of $L^2_{v^{\gamma,\delta}}$

$$L^2_{v^{\gamma,\delta},s}:=\left\{u\in L^2_{v^{\gamma,\delta}}\ :\ \|u\|_{v^{\gamma,\delta},s}:=\left(\sum_{i=0}^{\infty}(1+i)^{2s}|<u,p_i^{\gamma,\delta}>_{v^{\gamma,\delta}}|^2\right)^{\frac{1}{2}}<+\infty\right\}$$

and

$$\tilde{L}^2_{v^{\gamma,\delta},s} = \left\{ u \in L^2_{v^{\gamma,\delta}} \ : \ \|u\|_{v^{\gamma,\delta},s,\sim} := \left(\sum_{i=0}^{\infty} (1+i)^{2s} | < u, \tilde{p}_i^{\gamma,\delta} >_{v^{\gamma,\delta}} |^2 \right)^{\frac{1}{2}} < +\infty \right\}.$$

In $L^2_{v^{\gamma,\delta},s}$, the error of best approximation by means of polynomials of degree at most n ($P \in \mathbb{P}_n$) is defined as follows

$$E_n(u)_{L^2_{v^{\gamma,\delta},s}} = \inf_{P\in\mathbb{P}_n} \|u - P\|_{v^{\gamma,\delta},s}.$$

In the sequel, since $L^2_{v^{\gamma,\delta},0} = \tilde{L}^2_{v^{\gamma,\delta},0} = L^2_{v^{\gamma,\delta}}$, we will write $E_n(u)_{L^2_{v^{\gamma,\delta},0}} = E_n(u)_{L^2_{v^{\gamma,\delta}}}$. We recall that, for all $s > 0$, the following equivalence holds true [25, (3.13)]

$$\|u\|_{v^{\gamma,\delta},s} \sim \left(\sum_{i=0}^{\infty} (1+i)^{2s-1} E_i^2(u)_{L^2_{v^{\gamma,\delta}}} \right)^{\frac{1}{2}}. \tag{10}$$

For a continuous function u, we denote by $L_n^{\gamma,\delta} u$ the Lagrange polynomial interpolating u at the zeros

$$-1 < t^{\gamma,\delta}_{n,1} < t^{\gamma,\delta}_{n,2} < \ldots < t^{\gamma,\delta}_{n,n} < 1$$

of $p_n^{\gamma,\delta}$. We use the following representation for $L_n^{\gamma,\delta} u$

$$L_n^{\gamma,\delta}(u,t) := (L_n^{\gamma,\delta} u)(t) = \sum_{i=1}^{n} \psi_i^{\gamma,\delta}(t) \sqrt{\lambda^{\gamma,\delta}_{n,i}}\, u(t^{\gamma,\delta}_{n,i}),$$

where

$$\psi_i^{\gamma,\delta}(t) = \frac{l^{\gamma,\delta}_{n,i}(t)}{\sqrt{\lambda^{\gamma,\delta}_{n,i}}}, \quad l^{\gamma,\delta}_{n,i}(t) = \frac{p_n^{\gamma,\delta}(t)}{[p_n^{\gamma,\delta}]'(t^{\gamma,\delta}_{n,i})(t - t^{\gamma,\delta}_{n,i})}, \tag{11}$$

with $\lambda^{\gamma,\delta}_{n,i}$, $i = 1, \ldots, n$, the Christoffel numbers related to the weight $v^{\gamma,\delta}$.
We recall that, if $s > \frac{1}{2}$, $L_n^{\gamma,\delta}$ can be defined in $L^2_{v^{\gamma,\delta},s}$ (see [1, Theorem 2.5]). More precisely, given a function $u \in L^2_{v^{\gamma,\delta},s}$, with $s > \frac{1}{2}$, in the equivalent class of $L^2_{v^{\gamma,\delta},s}$ containing u there exists a representative u_0 which is locally continuous on $[-1,1]$ (see [1, Remark 2.6]). Then $L_n^{\gamma,\delta} u$ is defined as $L_n^{\gamma,\delta} u_0$.
Moreover, if $u \in L^2_{v^{\gamma,\delta},s}$, $s > \frac{1}{2}$, then for $0 \le r \le s$, we have

$$\|u - L_n^{\gamma,\delta} u\|_{v^{\gamma,\delta},r} \le \frac{\mathcal{C}}{n^{s-r}} \|u\|_{v^{\gamma,\delta},s} \tag{12}$$

and, consequently,

$$\|L_n^{\gamma,\delta} u\|_{v^{\gamma,\delta},r} \le \mathcal{C} \|u\|_{v^{\gamma,\delta},s}, \tag{13}$$

where $\mathcal{C} \ne \mathcal{C}(n,u)$. Note that (12) can be found in [20, Th. 3.4] for the case $0 < r \le s$, while it can be deduced from [20, Th. 3.1] and [20, eq. (3.13)] by easy computations.

3 Mapping properties of the operators D, K and H

In this section we are going to establish sufficient conditions for the existence and uniqueness of the solution of integral equation (7). To this end, we assume that, for some $s > 0$ and $-1 < \alpha < 1$, the given functions appearing in (1) have the following properties. We start from the case where the singular kernel $k(t,\tau)$ has the form (2). In this case we suppose that the kernel $k(t,\tau)$ satisfies

$$\|k(t,\cdot)\|_{v^{-\alpha,\alpha},s} < +\infty, \qquad \forall\, t \in (-1,1], \tag{14}$$

$$\|k(\cdot,\tau)\|_{v^{\alpha,-\alpha}} \le \mathcal{C}(1-\tau)^{-\frac{1+\alpha}{2}}, \qquad \mathcal{C} \ne \mathcal{C}(\tau), \tag{15}$$

$$\|k(\cdot,\tau)\|_{v^{\alpha,-\alpha},s} \le \mathcal{C}(1-\tau)^{-\frac{s+1+\alpha}{2}}, \qquad \mathcal{C} \ne \mathcal{C}(\tau). \tag{16}$$

for $-1 < \alpha < 0$. If the function $k(t,\tau)$ has the form (3), we assume that, for $0 < \alpha < 1$,

$$\|k(t,\cdot)\|_{v^{-\alpha,\alpha},s} < +\infty, \qquad \forall\, t \in [-1,1), \tag{17}$$

$$\|k(\cdot,\tau)\|_{v^{\alpha,-\alpha}} \le \mathcal{C}(1+\tau)^{-\frac{1-\alpha}{2}}, \qquad \mathcal{C} \ne \mathcal{C}(\tau) \tag{18}$$

$$\|k(\cdot,\tau)\|_{v^{\alpha,-\alpha},s} \le \mathcal{C}(1+\tau)^{-\frac{s+1-\alpha}{2}}, \qquad \mathcal{C} \ne \mathcal{C}(\tau). \tag{19}$$

Moreover, for the smoother kernel $h(t,\tau)$ we suppose that

$$\sup_{|t|\le 1}\|h(t,\cdot)\|_{v^{-\alpha,\alpha},s} < +\infty, \tag{20}$$

$$\sup_{|\tau|\le 1}\|h(\cdot,\tau)\|_{v^{\alpha,-\alpha},s} < +\infty. \tag{21}$$

The following theorem establishes the invertibility of the operator D defined in (4) in the couple of spaces $(\tilde{L}^2_{v^{-\alpha,\alpha},s}, L^2_{v^{-\alpha,\alpha},s})$.

Theorem 3.1. *Let $-1<\alpha<1$. For all $s\ge 0$, $D:\tilde{L}^2_{v^{-\alpha,\alpha},s}\to L^2_{v^{-\alpha,\alpha},s}$ is a continuous and invertible operator. Its adjoint operator $\widehat{D}: L^2_{v^{-\alpha,\alpha},s}\to \tilde{L}^2_{v^{-\alpha,\alpha},s}$ is bounded and is defined as*

$$(\widehat{D}g)(\tau) = a g(\tau) - \frac{b}{\pi} v^{\alpha,-\alpha}(\tau)\int_{-1}^{1}\frac{g(t)}{t-\tau}v^{-\alpha,\alpha}(t)dt.$$

Moreover $\widehat{D}$ is the inverse operator of D and the following relations

$$\|D\|_{\tilde{L}^2_{v^{-\alpha,\alpha},s}\to L^2_{v^{-\alpha,\alpha},s}} = \|\widehat{D}\|_{L^2_{v^{-\alpha,\alpha},s}\to \tilde{L}^2_{v^{-\alpha,\alpha},s}} = 1, \tag{22}$$

$$D v^{\alpha,-\alpha} p_n^{\alpha,-\alpha} = p_n^{-\alpha,\alpha}, \tag{23}$$

and

$$\widehat{D} p_n^{-\alpha,\alpha} = v^{\alpha,-\alpha} p_n^{\alpha,-\alpha}$$

hold true.

The mapping properties of the operator K defined in (5) are stated in the following result.

Theorem 3.2. *Let $-1<\alpha<1$. If, for some $s>0$, the kernel $k(t,\tau)$ satisfies*

$$\left(\int_{-1}^{1}\|k(t,\cdot)\|^2_{v^{-\alpha,\alpha},s}\, v^{\alpha,-\alpha}(t)dt\right)^{\frac{1}{2}} < +\infty, \tag{24}$$

then the operator $K: L^2_{v^{-\alpha,\alpha}}\to L^2_{v^{-\alpha,\alpha},r}$ is continuous for all $0\le r\le s$ and compact for all $0\le r<s$. Moreover, if (24) holds true with $s=0$, $K: L^2_{v^{-\alpha,\alpha}}\to L^2_{v^{-\alpha,\alpha}}$ is a compact operator.

Concerning the operator H given in (6), we state the following theorem.

Theorem 3.3. *Let $-1<\alpha<1$. If, for some $s>0$, the kernel $h(t,\tau)$ satisfies (20), then the operator $H: L^2_{v^{-\alpha,\alpha}}\to L^2_{v^{-\alpha,\alpha},r}$ is continuous for all $0\le r\le s$ and compact for all $0\le r<s$.*

Using the mapping properties of the integral operators K and H stated above, from the Fredholm Alternative one can easily deduce the following result concerning the existence and uniqueness of the solution of equation (7).

Theorem 3.4. *Under the assumptions of theorems 3.2 and 3.3, if $\mathrm{Ker}(D+K+H)=\{0\}$ in $\tilde{L}^2_{v^{-\alpha,\alpha},r}$ for some $0\le r<s$, then the equation (7) admits a unique solution u in $\tilde{L}^2_{v^{-\alpha,\alpha},r}$ for each right-hand side $g\in L^2_{v^{-\alpha,\alpha},r}$.*

4 The Method

Our aim is to propose a quadrature method for the numerical solution of the integral equation (7). We suppose that, for some $s>\frac{1}{2}$, conditions (14)-(21) are fulfilled and that the right-hand side function g satisfies

$$g\in L^2_{v^{-\alpha,\alpha},s}. \tag{25}$$

The numerical method consists in approximating the unknown solution u of (7) in the form (8) with $\beta=-\alpha$ by the weighted polynomial u_n, belonging to the subset

$$\mathcal{P}_{n-1} := \{v^{\alpha,-\alpha}p_{n-1} \ : \ p_{n-1}\in\mathbb{P}_{n-1}\}$$

of $L^2_{v^{-\alpha,\alpha}}$, which satisfies the finite dimensional equation

$$L_n^{-\alpha,\alpha}(D+K_n+H_n)u_n = L_n^{-\alpha,\alpha}g, \tag{26}$$

where $K_n u_n$ and $H_n u_n$ are suitable approximations of $K u_n$ and $H u_n$, respectively, obtained by applying a Gauss-Jacobi quadrature formula. More precisely, $K_n u_n$ and $H_n u_n$ are defined as follows

$$(K_n u_n)(\tau) = \sum_{j=1}^{n}\lambda_{n,j}^{\alpha,-\alpha} k(t_{n,j}^{\alpha,-\alpha},\tau) v^{-\alpha,\alpha}(t_{n,j}^{\alpha,-\alpha}) u_n(t_{n,j}^{\alpha,-\alpha}),$$

$$(H_n u_n)(\tau) = \sum_{j=1}^{n}\lambda_{n,j}^{\alpha,-\alpha} h(t_{n,j}^{\alpha,-\alpha},\tau) v^{-\alpha,\alpha}(t_{n,j}^{\alpha,-\alpha}) u_n(t_{n,j}^{\alpha,-\alpha}),$$

where $t_{n,j}^{\alpha,-\alpha}$, $j=1,\dots,n$, are the zeros of $p_n^{\alpha,-\alpha}$ and $\lambda_{n,j}^{\alpha,-\alpha}$, $j=1,\dots,n$, are the Christoffel numbers corresponding to the weight $v^{\alpha,-\alpha}$.
Taking into account the property (23) of the operator D, (26) can also be written as

$$(D+L_n^{-\alpha,\alpha}(K_n+H_n))u_n = L_n^{-\alpha,\alpha}g. \tag{27}$$

We point out that equation (27) makes sense assuming the conditions (14)-(21) and (25) satisfied for $s>\frac{1}{2}$ (see Section 2).

The following theorem establishes the stability and the convergence of the above numerical method under suitable assumptions.

Theorem 4.1. *Let $s>\frac{1}{2}$, $-1<\alpha<0$ (resp. $0<\alpha<1$), and $\eta=\min\{s,-2\alpha\}$ (resp. $\eta=\min\{s,2\alpha\}$). Let us assume that the kernel $k(t,\tau)$ given in (2) (resp. (3)) satisfies (14)-(16) (resp. (17)-(19)) and (24), and also that (20)-(21) and (25) are fulfilled. Then, if $\mathrm{Ker}(D+K+H)=\{0\}$ in $\tilde{L}^2_{v^{-\alpha,\alpha},r}$ for some $0\le r<s$ and $\eta>r$, the inverses of the operators*

$$D+L_n^{-\alpha,\alpha}(K_n+H_n):\left(\mathcal{P}_{n-1},\|\cdot\|_{v^{-\alpha,\alpha},r,\sim}\right)\to\left(\mathbb{P}_{n-1},\|\cdot\|_{v^{-\alpha,\alpha},r}\right)$$

exist and are uniformly bounded for all sufficiently large n.
Moreover, if u denotes the unique solution of (7) and u_n is the unique solution of (27), the following estimate

$$\|u_n-u\|_{v^{-\alpha,\alpha},r,\sim}=\mathcal{O}\left(\frac{1}{n^{\eta-r}}\right) \tag{28}$$

holds true.

At this point, we can obtain the solution u_n of (27) by solving a system of linear equations equivalent to (27). In order to get such linear system, we represent u_n in the basis $\{v^{\alpha,-\alpha}\psi_i^{\alpha,-\alpha}\}_{i=1}^n$ of $\mathcal{P}_{n-1}$ and the right hand side $g_n:=L_n^{-\alpha,\alpha}g$ in the basis $\{\psi_i^{-\alpha,\alpha}\}_{i=1}^n$ of $\mathbb{P}_{n-1}$. We recall that the definition of the polynomials $\psi_i^{\alpha,-\alpha}$ and $\psi_i^{-\alpha,\alpha}$ is given in (11) and we also remark that the two bases are both orthonormal with respect to the scalar product $<\cdot,\cdot>_{v^{-\alpha,\alpha}}$.
Then, we write

$$u_n(\tau)=v^{\alpha,-\alpha}(\tau)\sum_{i=1}^n a_i\psi_i^{\alpha,-\alpha}(\tau)=:v^{\alpha,-\alpha}(\tau)f_n(\tau),\quad a_i=\sqrt{\lambda_{n,i}^{\alpha,-\alpha}}f_n(t_{n,i}^{\alpha,-\alpha}), \tag{29}$$

and

$$g_n(\tau)=\sum_{i=1}^n b_i\psi_i^{-\alpha,\alpha}(\tau),\quad b_i=\sqrt{\lambda_{n,i}^{-\alpha,\alpha}}g(t_{n,i}^{-\alpha,\alpha}).$$

Collocating the equation (27) at the zeros $t_{n,i}^{-\alpha,\alpha}$, $i=1,\dots,n$, of $p_n^{-\alpha,\alpha}$ and taking into account that, by (11) and (23), we have (see [26, pag. 448])

$$(Du_n)(t_{n,i}^{-\alpha,\alpha})=\frac{b}{\pi}\sum_{j=1}^n\sqrt{\lambda_{n,j}^{\alpha,-\alpha}}\frac{a_j}{t_{n,j}^{\alpha,-\alpha}-t_{n,i}^{-\alpha,\alpha}},$$

we get the linear system

$$\sqrt{\lambda_{n,i}^{-\alpha,\alpha}}\sum_{j=1}^n\sqrt{\lambda_{n,j}^{\alpha,-\alpha}}\left[\frac{b}{\pi(t_{n,j}^{\alpha,-\alpha}-t_{n,i}^{-\alpha,\alpha})}+k(t_{n,j}^{\alpha,-\alpha},t_{n,i}^{-\alpha,\alpha})+h(t_{n,j}^{\alpha,-\alpha},t_{n,i}^{-\alpha,\alpha})\right]a_j=b_i,\qquad i=1,\dots,n. \tag{30}$$

Consequently, the array $\mathbf{a}=(a_1,\dots,a_n)^T$ is solution of (30) if and only if u_n defined in (29) is solution of (27). Moreover, the matrix M_n of the coefficients of the linear system (30), that is the matrix of the isomorphism $(D+L_n^{-\alpha,\alpha}(K_n+H_n)):\mathcal{P}_{n-1}\to\mathbb{P}_{n-1}$ with respect to the pair of bases $\{v^{\alpha,-\alpha}\psi_i^{\alpha,-\alpha}\}_{i=1}^n$ and $\{\psi_i^{-\alpha,\alpha}\}_{i=1}^n$, satisfies the following property.

Theorem 4.2. *Assuming that the hypotheses of Theorem 4.1 are fulfilled for $r=0$, the condition number $\mathrm{cond}(M_n)$ of the matrix M_n w.r.t. the spectral norm satisfies the equality*

$$\lim_n \mathrm{cond}(M_n)=\mathrm{cond}(D+K+H), \tag{31}$$

being

$$\mathrm{cond}(D+K+H)=\|D+K+H\|_{L^2_{v^{-\alpha,\alpha}}\to L^2_{v^{-\alpha,\alpha}}}\cdot\|(D+K+H)^{-1}\|_{L^2_{v^{-\alpha,\alpha}}\to L^2_{v^{-\alpha,\alpha}}}.$$

5 Proofs

Proof of Theorem 3.1. The proof easily follows from [1, p. 204], taking into account that for any $u\in L^2_{v^{-\alpha,\alpha}}$

$$Du=Av^{-\alpha,\alpha}u,\quad \widehat{D}u=v^{\alpha,-\alpha}\widehat{A}u,$$

where

$$(Af)(\tau)=a\,v^{\alpha,-\alpha}(\tau)f(\tau)+\frac{b}{\pi}\int_{-1}^1\frac{f(t)}{t-\tau}v^{\alpha,-\alpha}(t)dt$$

and

$$(\widehat{A}f)(\tau)=a\,v^{-\alpha,\alpha}(\tau)f(\tau)-\frac{b}{\pi}\int_{-1}^1\frac{f(t)}{t-\tau}v^{-\alpha,\alpha}(t)dt.$$

□

Proof of Theorem 3.2. If we assume that (24) holds true for $s=0$, it is well known that $K: L^2_{v^{-\alpha,\alpha}} \to L^2_{v^{-\alpha,\alpha}}$ is a compact operator (see, for instance, [1]). In the case $s>0$, since

$$\begin{aligned} |<Ku, p_i^{-\alpha,\alpha}>_{v^{-\alpha,\alpha}}|^2 &= \left|\int_{-1}^{1}\left(\int_{-1}^{1} k(t,\tau)u(t)dt\right)p_i^{-\alpha,\alpha}(\tau)v^{-\alpha,\alpha}(\tau)d\tau\right|^2 \\ &\le \|u\|^2_{v^{-\alpha,\alpha}} \int_{-1}^{1} |<k(t,\cdot), p_i^{-\alpha,\alpha}>_{v^{-\alpha,\alpha}}|^2 v^{\alpha,-\alpha}(t)dt, \end{aligned}$$

for $0\le r\le s$, we get

$$\begin{aligned} \|Ku\|^2_{v^{-\alpha,\alpha},r} &= \sum_{i=0}^{\infty}(1+i)^{2r}|<Ku, p_i^{-\alpha,\alpha}>_{v^{-\alpha,\alpha}}|^2 \\ &\le \|u\|^2_{v^{-\alpha,\alpha}} \int_{-1}^{1}\left(\sum_{i=0}^{\infty}(1+i)^{2r}|<k(t,\cdot), p_i^{-\alpha,\alpha}>_{v^{-\alpha,\alpha}}|^2\right)v^{\alpha,-\alpha}(t)dt \\ &= \|u\|^2_{v^{-\alpha,\alpha}} \int_{-1}^{1} \|k(t,\cdot)\|^2_{v^{-\alpha,\alpha},r} v^{\alpha,-\alpha}(t)dt \\ &\le \mathcal{C}\|u\|^2_{v^{-\alpha,\alpha}}. \end{aligned}$$

This proves that $K: L^2_{v^{-\alpha,\alpha}} \to L^2_{v^{-\alpha,\alpha},r}$ is bounded. Moreover, it is compact for any $0\le r<s$ in virtue of the compact embedding of $L^2_{v^{-\alpha,\alpha},s}$ in $L^2_{v^{-\alpha,\alpha},r}$ for any $r<s$ (see [1, Conclusion 2.3]). □

Proof of Theorem 3.3. The proof can be obtained by proceeding as in the proof of Theorem 3.2, by replacing the kernel k with the kernel h and using assumption (20). □

In order to prove Theorem 4.1, we need the following results.

Lemma 5.1. *Let $-1<\alpha<1$. If, for some $s>\frac{1}{2}$, the kernel $h(t,\tau)$ satisfies (20) and (21) then, for all $u_n\in\mathcal{P}_{n-1}$ we have*

$$\|(H-L_n^{-\alpha,\alpha}H_n)u_n\|_{v^{-\alpha,\alpha},r} \le \frac{\mathcal{C}}{n^{s-r}}\|u_n\|_{v^{-\alpha,\alpha}}, \quad 0\le r<s, \tag{32}$$

where $\mathcal{C}\ne\mathcal{C}(n,u_n)$.

Proof. We have

$$\begin{aligned} \|(H-L_n^{-\alpha,\alpha}H_n)u_n\|_{v^{-\alpha,\alpha},r} &\le \|(H-L_n^{-\alpha,\alpha}H)u_n\|_{v^{-\alpha,\alpha},r} + \|L_n^{-\alpha,\alpha}(H-H_n)u_n\|_{v^{-\alpha,\alpha},r} \\ &=: A+B. \end{aligned} \tag{33}$$

Since, by Theorem 3.3, $Hu_n\in L^2_{v^{-\alpha,\alpha},s}$, using (12) we get

$$A\le \frac{\mathcal{C}}{n^{s-r}}\|Hu_n\|_{v^{-\alpha,\alpha},s} \le \frac{\mathcal{C}}{n^{s-r}}\|u_n\|_{v^{-\alpha,\alpha}}. \tag{34}$$

Concerning B, for $r=0$, we have

$$B = \|L_n^{-\alpha,\alpha}(H-H_n)u_n\|_{v^{-\alpha,\alpha}} \tag{35}$$

and for $r>0$, recalling (10), we get

$$\begin{aligned} B &\le \mathcal{C}\left(\sum_{i=0}^{n-1}(1+i)^{2r-1}E_i^2\left(L_n^{-\alpha,\alpha}(H-H_n)u_n\right)_{L^2_{v^{-\alpha,\alpha}}}\right)^{\frac{1}{2}} \\ &\le \mathcal{C}\left(\sum_{i=0}^{n-1}(1+i)^{2r-1}\|L_n^{-\alpha,\alpha}(H-H_n)u_n\|^2_{v^{-\alpha,\alpha}}\right)^{\frac{1}{2}} \\ &\le \mathcal{C}\|L_n^{-\alpha,\alpha}(H-H_n)u_n\|_{v^{-\alpha,\alpha}}\left(\sum_{i=0}^{n-1}(1+i)^{2r-1}\right)^{\frac{1}{2}} \\ &\le \mathcal{C}n^r\|L_n^{-\alpha,\alpha}(H-H_n)u_n\|_{v^{-\alpha,\alpha}}. \end{aligned} \tag{36}$$

Then, it remains to estimate

$$\|L_n^{-\alpha,\alpha}(H-H_n)u_n\|_{v^{-\alpha,\alpha}} = \left(\int_{-1}^{1}|L_n^{-\alpha,\alpha}(H-H_n)u_n(\tau)|^2 v^{-\alpha,\alpha}(\tau)d\tau\right)^{\frac{1}{2}}.$$

Using the Gaussian rule based on the zeros of $p_n^{-\alpha,\alpha}$, we get

$$\|L_n^{-\alpha,\alpha}(H-H_n)u_n\|_{v^{-\alpha,\alpha}} = \left(\sum_{j=1}^{n}\lambda_{n,j}^{-\alpha,\alpha}\left[(H-H_n)u_n(t_{n,j}^{-\alpha,\alpha})\right]^2\right)^{\frac{1}{2}}.$$

Now, writing

$$(H_n u_n)(\tau) = \int_{-1}^{1} L_n^{\alpha,-\alpha}(h(\cdot,\tau),t)f_n(t)v^{\alpha,-\alpha}(t)dt$$

and using Hölder's inequality and (12) with $r=0$, we get

$$\begin{aligned} |(H-H_n)u_n(\tau)| &\le \int_{-1}^{1}|h(t,\tau)-L_n^{\alpha,-\alpha}(h(\cdot,\tau),t)||f_n(t)|v^{\alpha,-\alpha}(t)dt \\ &\le \|f_n\|_{v^{\alpha,-\alpha}}\|h(\cdot,\tau)-L_n^{\alpha,-\alpha}(h(\cdot,\tau))\|_{v^{\alpha,-\alpha}} \\ &\le \frac{\mathcal{C}}{n^s}\|u_n\|_{v^{-\alpha,\alpha}}\|h(\cdot,\tau)\|_{v^{\alpha,-\alpha},s}. \end{aligned} \tag{37}$$

Then, under the assumption (21), we obtain

$$\begin{aligned} \|L_n^{-\alpha,\alpha}(H-H_n)u_n\|_{v^{-\alpha,\alpha}} &\le \frac{\mathcal{C}}{n^s}\|u_n\|_{v^{-\alpha,\alpha}}\left(\sum_{j=1}^{n}\lambda_{n,j}^{-\alpha,\alpha}\|h(\cdot,t_{n,j}^{-\alpha,\alpha})\|^2_{v^{\alpha,-\alpha},s}\right)^{\frac{1}{2}} \\ &\le \frac{\mathcal{C}}{n^s}\|u_n\|_{v^{-\alpha,\alpha}}, \end{aligned}$$

with $\mathcal{C}\neq\mathcal{C}(n,u_n)$. Replacing the above estimate into (35) and (36), we get

$$B \le \frac{\mathcal{C}}{n^{s-r}}\|u_n\|_{v^{-\alpha,\alpha}}, \quad 0\le r<s. \tag{38}$$

Finally, combining (34) and (38) with (33), the thesis follows. □

Lemma 5.2. *Let $s>\frac{1}{2}$ and $-1<\alpha<0$ (resp. $0<\alpha<1$). If the kernel $k(t,\tau)$ given in (2) (resp. (3)) satisfies (14)-(16) (resp. (17)-(19)), then, for all $u_n\in\mathcal{P}_{n-1}$ we have*

$$\|(K-L_n^{-\alpha,\alpha}K_n)u_n\|_{v^{-\alpha,\alpha},r} \le \frac{\mathcal{C}}{n^{\eta-r}}\|u_n\|_{v^{-\alpha,\alpha}}, \quad 0\le r<\eta, \tag{39}$$

where $\eta=\min\{s,-2\alpha\}$ (resp. $\eta=\min\{s,2\alpha\}$) and $\mathcal{C}\neq\mathcal{C}(n,u_n)$.

Proof. We prove the lemma in the case where $k(t,\tau)$ has the form (2). The other case can be treated analogously. Repeating the same steps of the proof of Lemma 5.1 with K and K_n in place of H and H_n, respectively, we just need to prove that

$$\begin{aligned} \|L_n^{-\alpha,\alpha}(K-K_n)u_n\|_{v^{-\alpha,\alpha}} &= \left(\int_{-1}^{1}\left[L_n^{-\alpha,\alpha}(K-K_n)u_n(\tau)\right]^2 v^{-\alpha,\alpha}(\tau)d\tau\right)^{\frac{1}{2}} \\ &= \left(\sum_{j=1}^{n}\lambda_{n,j}^{-\alpha,\alpha}\left[(K-K_n)u_n(t_{n,j}^{-\alpha,\alpha})\right]^2\right)^{\frac{1}{2}} \\ &\le \frac{\mathcal{C}}{n^\eta}\|u_n\|_{v^{-\alpha,\alpha}}. \end{aligned} \tag{40}$$

Proceeding as done for the estimate (37) and using the assumption (16), for any $\tau\in[-1,1)$ we get

$$\begin{aligned} |(K-K_n)u_n(\tau)| &\le \frac{\mathcal{C}}{n^s}\|u_n\|_{v^{-\alpha,\alpha}}\|k(\cdot,\tau)\|_{v^{\alpha,-\alpha},s} \\ &\le \frac{\mathcal{C}}{n^s}\|u_n\|_{v^{-\alpha,\alpha}}(1-\tau)^{-\frac{s+1+\alpha}{2}}. \end{aligned} \tag{41}$$

Consequently, we obtain

$$\begin{aligned} \|L_n^{-\alpha,\alpha}(K-K_n)u_n\|_{v^{-\alpha,\alpha}} &\le \frac{\mathcal{C}}{n^s}\|u_n\|_{v^{-\alpha,\alpha}}\left(\sum_{j=1}^{n}\lambda_{n,j}^{-\alpha,\alpha}(1-t_{n,j}^{-\alpha,\alpha})^{-s-1-\alpha}\right)^{\frac{1}{2}} \\ &\le \frac{\mathcal{C}}{n^s}\|u_n\|_{v^{-\alpha,\alpha}}\left(\sum_{j=1}^{n-1}\lambda_{n,j}^{-\alpha,\alpha}(1-t_{n,j}^{-\alpha,\alpha})^{-s-1-\alpha}+\lambda_{n,n}^{-\alpha,\alpha}(1-t_{n,n}^{-\alpha,\alpha})^{-s-1-\alpha}\right)^{\frac{1}{2}} \\ &=: \frac{\mathcal{C}}{n^s}\|u_n\|_{v^{-\alpha,\alpha}}(A_1+A_2)^{\frac{1}{2}}. \end{aligned} \tag{42}$$

Let us to estimate the terms in the brackets. We recall that the zeros of $p_n^{-\alpha,\alpha}$ are labeled in increasing order, i.e.

$$-1 =: t_{n,0}^{-\alpha,\alpha} < t_{n,1}^{-\alpha,\alpha} < \ldots < t_{n,n}^{-\alpha,\alpha} < t_{n,n+1}^{-\alpha,\alpha} := 1.$$

About the first term, by using the generalized Markov-Stieltjes inequalities (see [28]) we have

$$\sum_{j=1}^{n-1} \lambda_{n,j}^{-\alpha,\alpha}(1-t_{n,j}^{-\alpha,\alpha})^{-s-1-\alpha} \quad \leq \quad \int_{-1}^{t_{n,n}^{-\alpha,\alpha}} (1-\tau)^{-s-1-2\alpha}(1+\tau)^{\alpha} d\tau.$$

If $\alpha \leq -\frac{s}{2}$, one immediately has

$$A_1 \leq \mathcal{C}, \qquad \mathcal{C} \neq \mathcal{C}(n),$$

while, if $\alpha > -\frac{s}{2}$, we can write

$$\begin{aligned} A_1 &\leq \left\{ \int_{-1}^{0} + \int_{0}^{t_{n,n}^{-\alpha,\alpha}} \right\} (1-\tau)^{-s-1-2\alpha}(1+\tau)^{\alpha} d\tau \\ &\leq \mathcal{C} + \mathcal{C}(1-t_{n,n}^{-\alpha,\alpha})^{-s-2\alpha} \leq \mathcal{C} n^{2s+4\alpha}, \end{aligned}$$

being $1-t_{n,n}^{-\alpha,\alpha} \sim n^{-2}$ (see [27, Theorem 3]). Taking also into account the relation [29, p. 353. eq. (15.3.10)],

$$\lambda_{n,j}^{-\alpha,\alpha} \sim \Delta t_{n,j}^{-\alpha,\alpha} v^{-\alpha,\alpha}(t_{n,j}^{-\alpha,\alpha}), \quad \Delta t_{n,j}^{-\alpha,\alpha} = t_{n,j+1}^{-\alpha,\alpha} - t_{n,j}^{-\alpha,\alpha}, \tag{43}$$

for A_2 we have

$$\begin{aligned} A_2 &\leq \mathcal{C}\Delta t_{n,n}^{-\alpha,\alpha}(1-t_{n,n}^{-\alpha,\alpha})^{-s-1-2\alpha}(1+t_{n,n}^{-\alpha,\alpha})^{\alpha} \\ &\leq \mathcal{C} n^{2s+2\alpha} \Delta t_{n,n}^{-\alpha,\alpha}(1-t_{n,n}^{-\alpha,\alpha})^{-1-\alpha} \\ &\leq \mathcal{C} n^{2s+2\alpha} \int_{t_{n,n}^{-\alpha,\alpha}}^{1} (1-\tau)^{-1-\alpha} d\tau \\ &\leq \mathcal{C} n^{2s+2\alpha}(1-t_{n,n}^{-\alpha,\alpha})^{-\alpha} \\ &\leq \mathcal{C} n^{2s+4\alpha}. \end{aligned}$$

Let us observe that for $\alpha \leq -\frac{s}{2}$, the previous inequality reduces to

$$A_2 \leq \mathcal{C}, \qquad \mathcal{C} \neq \mathcal{C}(n).$$

Hence, we can conclude that the term $\|L_n^{-\alpha,\alpha}(K-K_n)u_n\|_{v^{-\alpha,\alpha}}$ can be estimated as follows

$$\|L_n^{-\alpha,\alpha}(K-K_n)u_n\|_{v^{-\alpha,\alpha}} \leq \begin{cases} \dfrac{\mathcal{C}}{n^s}\|u_n\|_{v^{-\alpha,\alpha}} & \alpha \leq -\frac{s}{2} \\ \\ \dfrac{\mathcal{C}}{n^{-2\alpha}}\|u_n\|_{v^{-\alpha,\alpha}} & \alpha > -\frac{s}{2} \end{cases}.$$

□

Proof of Theorem 4.1. Recalling that the operator $D+K+H : \tilde{L}^2_{v^{-\alpha,\alpha},r} \to L^2_{v^{-\alpha,\alpha},r}$ is invertible under our assumptions, for $u_n \in \mathcal{P}_{n-1}$ we have

$$\|u_n\|_{v^{-\alpha,\alpha},r,\sim} = \|(D+K+H)^{-1}(D+K+H)u_n\|_{v^{-\alpha,\alpha},r,\sim}. \tag{44}$$

Since we can write

$$(D+K+H)u_n = (D+L_n^{-\alpha,\alpha}(K_n+H_n))u_n + (K-L_n^{-\alpha,\alpha}K_n)u_n + (H-L_n^{-\alpha,\alpha}H_n)u_n,$$

from (44), (32) and (39) we get

$$\|u_n\|_{v^{-\alpha,\alpha},r,\sim} \quad \leq \quad \|(D+K+H)^{-1}\|_{L^2_{v^{-\alpha,\alpha},r}\to\tilde{L}^2_{v^{-\alpha,\alpha},r}} \times \left(\|(D+L_n^{-\alpha,\alpha}(K_n+H_n))u_n\|_{v^{-\alpha,\alpha},r} + \varepsilon_n\|u_n\|_{v^{-\alpha,\alpha},r,\sim}\right),$$

with $\varepsilon_n = \mathcal{O}(n^{r-\eta})$ and then

$$\|(D+L_n^{-\alpha,\alpha}(K_n+H_n))u_n\|_{v^{-\alpha,\alpha},r} \geq \left(\|(D+K+H)^{-1}\|^{-1}_{L^2_{v^{-\alpha,\alpha},r}\to\tilde{L}^2_{v^{-\alpha,\alpha},r}} - \varepsilon_n\right)\|u_n\|_{v^{-\alpha,\alpha},r,\sim}.$$

From the last inequality, since

$$D : \left(\mathcal{P}_{n-1}, \|\cdot\|_{v^{-\alpha,\alpha},r,\sim}\right) \to \left(\mathbb{P}_{n-1}, \|\cdot\|_{v^{-\alpha,\alpha},r}\right)$$

is an invertible operator (see Theorem 3.1) and, by (23),

$$\left(D+L_n^{-\alpha,\alpha}(\bar{K}_n+H_n)\right)(\mathcal{P}_{n-1}) \subseteq \mathbb{P}_{n-1},$$

we can deduce that, for n large enough (say $n \geq n_0$), the continuous operators

$$D + L_n^{-\alpha,\alpha}(K_n + H_n) : \left(\mathcal{P}_{n-1}, \|\cdot\|_{v^{-\alpha,\alpha},r,\sim}\right) \to \left(\mathbb{P}_{n-1}, \|\cdot\|_{v^{-\alpha,\alpha},r}\right)$$

are invertible and their inverses are uniformly bounded (see, for instance, [19, p. 214]. Hence, for $n \geq n_0$, (27) has a unique solution $u_n \in \mathcal{P}_{n-1}$ and

$$\|u_n\|_{v^{-\alpha,\alpha},r,\sim} \leq \mathcal{C}\|L_n^{-\alpha,\alpha} g\|_{v^{-\alpha,\alpha},r} \leq \mathcal{C}\|g\|_{v^{-\alpha,\alpha},s}, \tag{45}$$

according to (13). Therefore, the estimate

$$\begin{aligned}\|u-u_n\|_{v^{-\alpha,\alpha},r,\sim} &\leq \|(D+K+H)^{-1}\|_{L^2_{v^{-\alpha,\alpha},r}\to \tilde{L}^2_{v^{-\alpha,\alpha},r}} \times \|(g-L_n^{-\alpha,\alpha}g)-(K-L_n^{-\alpha,\alpha}K_n)u_n-(H-L_n^{-\alpha,\alpha}H_n)u_n\|_{v^{-\alpha,\alpha},r} \\ &\leq \|(D+K+H)^{-1}\|_{L^2_{v^{-\alpha,\alpha},r}\to \tilde{L}^2_{v^{-\alpha,\alpha},r}} \times \left[\|(g-L_n^{-\alpha,\alpha}g)\|_{v^{-\alpha,\alpha},r}+\|(K-L_n^{-\alpha,\alpha}K_n)u_n\|_{v^{-\alpha,\alpha},r}+\|(H-L_n^{-\alpha,\alpha}H_n)u_n\|_{v^{-\alpha,\alpha},r}\right]\end{aligned}$$

combined with (12), (39), (32) and (45) implies (28). □

Before proving Theorem 4.2 we state the following useful lemma that can be demonstrated by proceeding analogously to the proof of [23, Theorem 2.3] (see, also, [20]).

Lemma 5.3. *Let $v^{\gamma,\delta}$, $\gamma,\delta > -1$, be a Jacobi weight and $T : L^2_{v^{\gamma},\delta} \to L^2_{v^{\gamma},\delta}$ be a bounded linear operator. Assume that $X_n \subset L^2_{v^{\gamma},\delta}$ is a finite dimensional dense subspace of the space $L^2_{v^{\gamma},\delta}$ and that there exists a projector $\Pi_n : L^2_{v^{\gamma},\delta} \to X_n$ such that*

$$\sup_n \|\Pi_n\|_{L^2_{v^{\gamma},\delta}\to L^2_{v^{\gamma},\delta}} < \infty.$$

Then, one has that

$$\lim_n \left\|T|_{X_n}\right\|_{L^2_{v^{\gamma},\delta}\to L^2_{v^{\gamma},\delta}} = \|T\|_{L^2_{v^{\gamma},\delta}\to L^2_{v^{\gamma},\delta}}.$$

Proof of Theorem 4.2. In what follows, for simplicity of notation, we shall omit to write the subscript $L^2_{v^{-\alpha,\alpha}} \to L^2_{v^{-\alpha,\alpha}}$ in the symbol $\|\cdot\|_{L^2_{v^{-\alpha,\alpha}}\to L^2_{v^{-\alpha,\alpha}}}$ used for the norm of an operator acting from the space $L^2_{v^{-\alpha,\alpha}}$ into itself. Moreover, we will briefly denote the operator $D+K+H$ by B and the operator $D+L_n^{-\alpha,\alpha}(K_n+H_n)$ by B_n. We begin by observing that M_n is the matrix of the isomorphism

$$B_n|_{\mathcal{P}_{n-1}} : (\mathcal{P}_{n-1}, \|\cdot\|_{v^{-\alpha,\alpha}}) \to (\mathbb{P}_{n-1}, \|\cdot\|_{v^{-\alpha,\alpha}})$$

with respect to the pair of bases $\{v^{\alpha,-\alpha}\psi_i^{\alpha,-\alpha}\}_{i=1}^n$ and $\{\psi_i^{-\alpha,\alpha}\}_{i=1}^n$ which are both orthonormal in the space $L^2_{v^{-\alpha,\alpha}}$ endowed with the scalar product $<\cdot,\cdot>_{v^{-\alpha,\alpha}}$. Consequently, it can be easily seen that for the spectral norm of the matrices M_n and M_n^{-1} one has

$$\|M_n\| = \left\|B_n|_{\mathcal{P}_{n-1}}\right\|$$

as well as

$$\|M_n^{-1}\| = \left\|\left(B_n|_{\mathcal{P}_{n-1}}\right)^{-1}\right\|.$$

Then, our aim becomes to show that

$$\lim_n \left\|B_n|_{\mathcal{P}_{n-1}}\right\| = \|B\| \tag{46}$$

and

$$\lim_n \left\|\left(B_n|_{\mathcal{P}_{n-1}}\right)^{-1}\right\| = \left\|B^{-1}\right\|. \tag{47}$$

Let us consider the linear operator $\Pi_n : L^2_{v^{-\alpha,\alpha}} \to \mathcal{P}_{n-1}$ defined as follows

$$\Pi_n f = v^{\alpha,-\alpha} S_n^{\alpha,-\alpha}(v^{-\alpha,\alpha} f), \qquad f \in L^2_{v^{-\alpha,\alpha}},$$

with $S_n^{\alpha,-\alpha}$ the Fourier operator given by

$$S_n^{\alpha,-\alpha} u = \sum_{i=0}^{n-1} < u, p_i^{\alpha,-\alpha} >_{v^{\alpha,-\alpha}} p_i^{\alpha,-\alpha}.$$

It is a projector of $L^2_{v^{-\alpha,\alpha}}$ onto the subspace $\mathcal{P}_{n-1}$ satisfying the assumptions of Lemma 5.3. Then, we can deduce that

$$\lim_n \left\|B|_{\mathcal{P}_{n-1}}\right\| = \|B\|. \tag{48}$$

Furthermore, from (32) and (39) we can deduce

$$\lim_n \left\|B_n|_{\mathcal{P}_{n-1}} - B|_{\mathcal{P}_{n-1}}\right\| = 0$$

and, consequently,

$$\lim_n \left|\left\|B_n|_{\mathcal{P}_{n-1}}\right\| - \left\|B|_{\mathcal{P}_{n-1}}\right\|\right| = 0. \tag{49}$$

Combining (48) and (49) we get (46).
In order to prove (47), first we consider the set of Fourier operators $S_n^{-\alpha,\alpha} : L^2_{v^{-\alpha,\alpha}} \to \mathbb{P}_{n-1}$ defined as

$$S_n^{-\alpha,\alpha} u = \sum_{i=0}^{n-1} < u, p_i^{-\alpha,\alpha} >_{v^{-\alpha,\alpha}} p_i^{-\alpha,\alpha}$$

which is a uniformly bounded sequence of projectors onto the dense subspace $\mathbb{P}_{n-1}$ of $L^2_{v^{-\alpha,\alpha}}$. Consequently, in virtue of Lemma 5.3, we get

$$\lim_n \left\| B^{-1}|_{\mathbb{P}_{n-1}} \right\| = \|B^{-1}\|. \tag{50}$$

It remains to show the equality

$$\lim_n \left\| \left(B_n|_{\mathcal{P}_{n-1}}\right)^{-1} - B^{-1}|_{\mathbb{P}_{n-1}} \right\| = 0 \tag{51}$$

which, combined with (50), allows us to deduce (47).
Recalling Theorem 4.1, for any fixed polynomial $p_{n-1} \in \mathbb{P}_{n-1}$ there exists a unique function $u_n \in \mathcal{P}_{n-1}$ such that $B_n u_n = p_{n-1}$. Then, under our assumptions, we can write

$$\begin{aligned} B_n^{-1} p_{n-1} - B^{-1} p_{n-1} &= B^{-1} B u_n - B^{-1} p_{n-1} \\ &= B^{-1} \left(B u_n - p_{n-1}\right) \\ &= B^{-1} \left(B u_n - B_n u_n\right) \\ &= B^{-1} \left(B|_{\mathcal{P}_{n-1}} - B_n|_{\mathcal{P}_{n-1}}\right) u_n \end{aligned}$$

from which, taking into account (32) and (39), it follows that

$$\begin{aligned} \|B_n^{-1} p_{n-1} - B^{-1} p_{n-1}\|_{v^{-\alpha,\alpha}} &\leq \|B^{-1}\| \, \varepsilon_n \, \|u_n\|_{v^{-\alpha,\alpha}} \\ &\leq \|B^{-1}\| \, \varepsilon_n \left\| \left(B_n|_{\mathcal{P}_{n-1}}\right)^{-1} \right\| \|p_{n-1}\|_{v^{-\alpha,\alpha}} \end{aligned}$$

with $\varepsilon_n \to 0$ as $n \to \infty$. But Theorem 4.1 assures that, for n large enough (say $n \geq n_0$),

$$\left\| \left(B_n|_{\mathcal{P}_{n-1}}\right)^{-1} \right\| \leq \mathcal{C}, \qquad \mathcal{C} \neq \mathcal{C}(n),$$

therefore we can conclude that if $n \geq n_0$, for any $p_{n-1} \in \mathbb{P}_{n-1}$, $p_{n-1} \neq 0$,

$$\frac{\left\| B_n^{-1} p_{n-1} - B^{-1} p_{n-1} \right\|_{v^{-\alpha,\alpha}}}{\|p_{n-1}\|_{v^{-\alpha,\alpha}}} \leq \mathcal{C} \, \varepsilon_n. \tag{52}$$

From (52), (51) immediately follows and the proof is complete. □

6 Numerical examples

In this section we will show the numerical results obtained by applying the proposed method to some test integral equations of type (1). For each example, chosen a suitable value of the parameter α, the constant coefficients a and b appearing in (1) will be given by $a = \cos(\pi\alpha)$ and $b = -\sin(\pi\alpha)$. Since the solution u is unknown, we will consider as exact the approximating one u_{1024}.

In the tables that follow we will report the pointwise weighted absolute errors

$$e_n(\tau) = \left| u_n(\tau) - u_{1024}(\tau) \right| v^{-\alpha,\alpha}(\tau),$$

the L^2 weighted errors

$$err_n = \|u_n - u_{1024}\|_{v^{-\alpha,\alpha}} = \left(\sum_{k=1}^{1024} \lambda_{1024,k}^{\alpha,-\alpha} \left| f_n\left(t_{1024,k}^{\alpha,-\alpha}\right) - f_{1024}\left(t_{1024,k}^{\alpha,-\alpha}\right) \right|^2 \right)^{\frac{1}{2}},$$

the corresponding estimated orders of convergence

$$eoc_n = \frac{\log(err_n/err_{2n})}{\log 2},$$

and the condition numbers cond(M_n) in the spectral norm of the matrices M_n associated with the linear systems (30). As one can see, the numerical results confirm the stability and the convergence of the proposed method. Actually, the numerical convergence order appears to be higher than the theoretical one. More precisely, the numerical evidence allows us to presume that this order is about 2η (see the values of eoc_n reported in the tables). Moreover, in accordance with (31), the sequence $\{\text{cond}(M_n)\}_n$ is convergent as n goes to infinity.

Finally, we remark that, in correspondence of the chosen weight $v^{-\alpha,\alpha}$, all the assumptions on $k(t,\tau)$, $h(t,\tau)$ and $g(\tau)$ are fulfilled for every $s > 1/2$.

Table 1: Example 6.1

n	$e_n(-0.99)$	$e_n(-0.45)$	$e_n(0.3)$	$e_n(0.8)$	$e_n(0.99)$
8	2.31e-02	8.66e-03	3.94e-03	1.78e-03	2.05e-03
16	1.62e-03	3.06e-04	8.49e-05	1.18e-04	2.67e-03
32	6.43e-05	1.04e-05	1.43e-06	2.60e-05	5.24e-05
64	4.13e-06	1.68e-06	1.34e-06	1.47e-06	3.62e-05
128	2.45e-07	3.12e-08	7.23e-08	3.99e-08	1.65e-06
256	1.97e-08	2.40e-09	4.99e-10	3.42e-09	4.40e-08
512	2.60e-11	8.92e-11	2.09e-11	5.25e-10	7.07e-09

Table 2: Example 6.1

n	err_n	eoc_n	cond(M_n)
8	1.97e-02	2.50	**9.2**69840292397463
16	3.49e-03	2.51	**9.27**5792936415662
32	6.11e-04	2.55	**9.2761**21867983536
64	1.04e-04	2.58	**9.27613**7709516037
128	1.73e-05	2.61	**9.2761384**34721977
256	2.82e-06	2.72	**9.27613846**7436798
512	4.29e-07		**9.276138468**912372

Example 6.1. Let us assume that the known functions in the equation (1) are the following

$$k(t,\tau)=\frac{1}{\pi}(1+t)\frac{\sqrt{(1+t)(1-\tau)}}{(1+t)^3+(1-\tau)^3},\quad h(t,\tau)=\sin(t\tau),\quad g(\tau)=e^\tau+\tau^2$$

and that the parameter α defining the Jacobi weights $v^{\alpha,-\alpha}$ and $v^{-\alpha,\alpha}$ is $-\frac{2}{3}$. Consequently, by (28) the convergence order is at least $\eta=\frac{4}{3}$. In tables 1 and 2 we show the obtained results.

Example 6.2. In this example we consider an integral equation of type (1) with

$$k(t,\tau)=\frac{1}{2\pi}\frac{1+\tau}{(1-t)^2+(1+\tau)^2},\quad h(t,\tau)=\frac{(t^2+\tau^2)}{1+e^\tau},\quad g(\tau)=\tau^5.$$

Choosing $\alpha=0.9$ the value of η in (28) is 1.8. We report in tables 3 and 4 the obtained numerical results.

Table 3: Example 6.2

n	$e_n(-0.9)$	$e_n(-0.2)$	$e_n(0.5)$	$e_n(0.7)$	$e_n(0.8)$
8	1.40e-04	1.10e-04	1.35e-04	1.84e-06	1.95e-04
16	2.65e-06	1.60e-06	4.41e-06	1.45e-07	4.90e-06
32	5.16e-07	6.84e-08	1.79e-07	3.70e-08	3.15e-07
64	2.31e-09	2.61e-09	3.99e-09	3.03e-09	3.41e-11
128	2.29e-10	1.02e-10	1.37e-10	1.71e-10	1.06e-10
256	1.31e-11	3.10e-12	3.76e-12	1.01e-11	2.46e-11
512	1.60e-12	7.32e-13	7.19e-13	7.23e-13	5.14e-12

Example 6.3. Let

$$k(t,\tau)=\frac{1}{5}\frac{(1-\tau)}{(1+t)^2}e^{-\frac{1-\tau}{1+t}},\quad h(t,\tau)=e^{t+\tau},\quad g(\tau)=\log(\tau^3+3)$$

be the kernels and the right-hand side in equation (1) and let $\alpha=-4/5$ ($\eta=1.6$). By applying the numerical method proposed in Section 4, the results shown in tables 5 and 6 were obtained.

Acknowledgements The authors are partially supported by INdAM-GNCS 2020 project "Approssimazione multivariata ed equazioni funzionali per la modellistica numerica" and by University of Basilicata (local funds). This research has been accomplished within the RITA "Research ITalian network on Approximation".

Table 4: Example 6.2

n	err_n	eoc_n	cond(M_n)
8	7.07e-04	3.36	**1.1290**55966832389e+01
16	6.85e-05	3.60	**1.129088**076447376e+01
32	5.65e-06	3.65	**1.1290887**27287702e+01
64	4.48e-07	3.67	**1.129088740**182178e+01
128	3.50e-08	3.69	**1.1290887404**44359e+01
256	2.71e-09	3.74	**1.1290887404**49769e+01
512	2.02e-10		**1.129088740450**248e+01

Table 5: Example 6.3

n	$e_n(-0.99)$	$e_n(-0.8)$	$e_n(-0.4)$	$e_n(0.3)$	$e_n(0.9)$
8	7.87e-03	1.03e-03	6.43e-05	8.41e-05	7.59e-05
16	9.57e-06	7.54e-07	3.13e-07	3.82e-07	7.61e-06
32	2.11e-07	3.84e-08	1.08e-08	1.58e-08	1.16e-07
64	5.85e-09	2.57e-10	7.15e-10	7.89e-10	1.92e-10
128	2.53e-10	2.39e-11	1.94e-11	2.94e-11	4.02e-11
256	2.61e-12	1.90e-12	1.92e-12	4.86e-14	4.20e-12
512	3.20e-12	6.28e-13	1.24e-12	9.69e-14	1.37e-13

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Table 6: Example 6.3

n	err_n	eoc_n	cond(M_n)
8	3.29e-04	3.80	**1.279**878322557983e+01
16	2.36e-05	3.13	**1.279922**823833182e+01
32	2.68e-06	3.31	**1.2799227**14909900e+01
64	2.70e-07	3.36	**1.279922723**345081e+01
128	2.61e-08	3.39	**1.2799227235**53726e+01
256	2.49e-09	3.44	**1.279922723559**195e+01
512	2.29e-10		**1.279922723559**774e+01

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