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A numerical method for the generalized Love integral equation in 2D

Luisa Fermo[a] · Maria Grazia Russo[b] · Giada Serafini[c]

Abstract

This paper deals with the numerical solution of the generalized Love integral equation defined on the square. The method is of Nyström type and is based on the approximation of the integral by a product cubature rule whose coefficients are approximated by a "dilation" scheme. The theoretical analysis of the presented method is discussed by proving its stability and convergence in weighted spaces equipped with the uniform norm. To support the theoretical estimates, some numerical tests are presented.

1 Introduction

Let us consider the following Fredholm integral equation of the second kind defined on the square $D = [-1, 1] \times [-1, 1]$

$$\mathbf{f}(\mathbf{y}) - \mu \int_D \mathbf{k}_\omega(\mathbf{x}, \mathbf{y})\mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x} = \mathbf{g}(\mathbf{y}), \qquad \mathbf{y} \in D, \tag{1}$$

where $\mu \in \mathbb{R}$, $\mathbf{f}$ is the unknown solution,

$$\mathbf{k}_\omega(\mathbf{x}, \mathbf{y}) = \frac{\omega^{-1}}{|\mathbf{x}-\mathbf{y}|^2 + \omega^{-2}} \tag{2}$$

is the kernel function depending on a real positive parameter ω, $\mathbf{g}$ is the known right-hand side, and $\mathbf{w}(\mathbf{x})$ is a bivariate Jacobi weight in the variable $\mathbf{x} = (x_1, x_2)$ defined as

$$\mathbf{w}(\mathbf{x}) = \prod_{i=1}^{2}(1-x_i)^{\alpha_i}(1+x_i)^{\beta_i}, \qquad \alpha_i, \beta_i > -1. \tag{3}$$

The main pathology of (1) is the presence of a "nearly" singular kernel that is a function which is "close" to be singular at $\mathbf{x} = \mathbf{y}$ when $\omega^{-1} \to 0$. In addition, the presence of the Jacobi weight $\mathbf{w}$ implies that the unknown function $\mathbf{f}$ could have algebraic singularities along the boundary of D.

Let us mention that in the case $\mathbf{g} \equiv \mathbf{w} \equiv 1$ and $\mu = \frac{1}{\pi^2}$, equation (1) is the bivariate Love integral equation, that is the corresponding extension in 2D of the classical univariate Love integral equation [6].

Very recently we have developed a numerical method in order to approximate the solution of (1) in the univariate case [4]. The approach is based on the discretization of the integral operator by means of a product quadrature rule whose coefficients are computed by using a "dilation" formula [2, 11]. Then, a Nyström method is given and very accurate results are obtained also in the case when ω is large which is the undisputed most interesting case.

In this paper, we want to extend the method presented in [4] to the bivariate case. Then, first, we approximate the integral by using a cubature formula given in [10]. In this way, we isolate the kernel, and consequently the pathology of the equation, in the coefficients of such a cubature rule. At this point, to face this pathology, we use a "dilation" cubature formula to approximate the coefficients [10]. Hence, a Nyström method is developed. It leads to a well-conditioned linear system whose size does not depend on the magnitude of the parameter ω. Its unique solution allows us to compute the Nyström interpolant which converges to the exact solution. The convergence and the stability of the method are proved in weighted spaces equipped with the uniform norm. In addition, the provided error estimate claims that the error of the method is essentially of the order of the best polynomial approximation of the unknown function, independently of the value of ω (the extra term $(\frac{d}{\omega})^{m-1}\log^2 m$ in the convergence estimate, vanishes exponentially).

We want to underline that the proposed mixed scheme, namely the product cubature rule combined with the dilation technique, can be also used to approximate other nearly singular integrals whose accurate approximation plays an important role in the boundary element method (BEM) which has wide applications [14]. These include evaluating the solution near the boundary in potential problems and calculating displacements and stresses near the boundary in elasticity problems, for example, displacement around open crack tips, contact problems, sensitivity problems, etc.

[a]Department of Mathematics and Computer Science, University of Cagliari, Via Ospedale 72, 09124 Cagliari, Italy. Member of the INdAM Research group GNCS and the "Research ITalian network on Approximation (RITA)", email: fermo@unica.it
[b]Department of Mathematics, Computer Science and Economics, University of Basilicata, Viale dell'Ateneo Lucano 10, 85100 Potenza, Italy. Member of the INdAM Research group GNCS and the "Research ITalian network on Approximation (RITA)", email: mariagrazia.russo@unibas.it
[c]Department of Mathematics, Computer Science and Economics, University of Basilicata, Viale dell'Ateneo Lucano 10, 85100 Potenza, Italy. Member of the "Research ITalian network on Approximation (RITA)", email: giada.serafini88@gmail.com

In more details, the paper is organized as follows. In Section 2 we fix the spaces in which we look for the solution and we review three different cubature schemes. Among them, we give a dilation cubature formula by providing new error estimates. In Section 3 we focus on the numerical solution of equation (1) by presenting at first a mixed cubature rule in Section 3.1 and then developing a Nyström method in Section 3.2. In Section 4 we give several numerical tests and in Section 5 we confine the proofs of our theoretical analysis.

2 Preliminaries

2.1 Functional spaces

Let us denote by

$$v^{\gamma,\delta}(x)=(1-x)^{\gamma}(1+x)^{\delta}, \quad x\in(-1,1)$$

a generic univariate Jacobi weight function with parameters $\gamma,\delta \geq 0$ and let us introduce the bivariate weight

$$\boldsymbol{\sigma}(\mathbf{x})=v^{\gamma_1,\delta_1}(x_1)v^{\gamma_2,\delta_2}(x_2), \qquad \mathbf{x}=(x_1,x_2)\in D. \tag{4}$$

Define the space of weighted continuous function as

$$C_{\sigma}=\left\{\mathbf{f}\in C(D\setminus\partial D): \lim_{\mathbf{x}\to\partial D}(\mathbf{f}\boldsymbol{\sigma})(\mathbf{x})=0\right\},$$

where ∂D denotes the boundary of the square D. We endow the space with the weighted uniform norm

$$\|\mathbf{f}\|_{C_{\sigma}}=\|\mathbf{f}\boldsymbol{\sigma}\|_{\infty}=\sup_{\mathbf{x}\in D}|(\mathbf{f}\boldsymbol{\sigma})(\mathbf{x})|.$$

For $r>0$, setting $\mathbf{f}^{(r)}_{x_i}=\dfrac{\partial^r}{\partial x_i^r}\mathbf{f}(x_1,x_2)$ and $\varphi(z)=\sqrt{1-z^2}$, let us also introduce the following Sobolev-type space

$$W^r_{\sigma}=\left\{\mathbf{f}\in C_{\sigma}: \|\mathbf{f}^{(r)}_{x_i}\varphi^r\boldsymbol{\sigma}\|_{\infty}<\infty,\ \forall i=1,2\right\}$$

equipped with the norm

$$\|\mathbf{f}\|_{W^r_{\sigma}}=\|\mathbf{f}\boldsymbol{\sigma}\|_{\infty}+\max_{i=1,2}\|\mathbf{f}^{(r)}_{x_i}\varphi^r\boldsymbol{\sigma}\|_{\infty}.$$

For our aims, it is also useful to define the error of best polynomial approximation in C_{σ} as

$$E_m(\mathbf{f})_{\sigma}=\inf_{\mathbf{P}\in\mathbb{P}_m}\|(\mathbf{f}-\mathbf{P})\boldsymbol{\sigma}\|_{\infty},$$

where $\mathbb{P}_m$ denotes the set of all algebraic polynomials of two variables of degree at most m in each variable.
It is known that for each $f\in W^r_{\sigma}$ we have [9]

$$E_m(\mathbf{f})_{\sigma}\leq\frac{\mathcal{C}}{m^r}\|\mathbf{f}\|_{W^r_{\sigma}}. \tag{5}$$

Here, $\mathcal{C}$ is a positive constant independent of m and $\mathbf{f}$. In the sequel, $\mathcal{C}$ will denote any positive constant having different meaning in different formulas. We will write $\mathcal{C}\neq\mathcal{C}(a,b,\ldots)$ to say that $\mathcal{C}$ is a positive constant independent of the parameters $a,b,\ldots$, and $\mathcal{C}=\mathcal{C}(a,b,\ldots)$ to say that $\mathcal{C}$ depends on $a,b,\ldots$. If $A,B>0$ are quantities depending on some parameters, we will write $A\sim B$, if there exists a constant $\mathcal{C}\neq\mathcal{C}(A,B)$ such that $\mathcal{C}^{-1}B\leq A\leq\mathcal{C}B$.

2.2 Cubature schemes

The aim of this section is to recall three different cubature formulae having the Jacobi weight $\mathbf{w}$ given in (3) as part of the integrand function. According to the introduced notation, such a weight can be also rewritten as

$$\mathbf{w}(\mathbf{x})=v^{\alpha_1,\beta_1}(x_1)v^{\alpha_2,\beta_2}(x_2), \qquad \alpha_i,\beta_i>-1, \quad i=1,2. \tag{6}$$

In details, we aim to approximate the following three integrals

$$\int_D \mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})\,d\mathbf{x}, \qquad \int_D \mathbf{k}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x}, \qquad \text{and} \qquad \int_D \mathbf{k}_{\omega}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x},$$

where $\mathbf{f}$, $\mathbf{k}$ and $\mathbf{k}_{\omega}$ are given functions, and the last one is a "nearly" singular kernel depending on a real positive parameter ω.
For the first two integrals we propose the standard Gaussian [9] and product [10] cubature rule, respectively. For the last one, we generalize what has already been done in [10] and [4], and present new theoretical results.

2.2.1 Gaussian cubature rule

In this subsection we want to recall the Gaussian cubature rule presented in [9] approximating the integrals of the form

$$\int_D \mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})\, d\mathbf{x} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{f}(x_1,x_2)v^{\alpha_1,\beta_1}(x_1)v^{\alpha_2,\beta_2}(x_2)dx_1dx_2,$$

where $\mathbf{f}$ is defined in D and $\mathbf{w}$ is as in (6). To this end, let us denote by $\{p_m(v^{\alpha_i,\beta_i},x_i)\}_m$ the sequences of orthonormal polynomials with respect to the weight v^{α_i,β_i} for $i=1,2$ and by

$$\xi_1^{\alpha_i,\beta_i} < \xi_2^{\alpha_i,\beta_i} < \cdots < \xi_m^{\alpha_i,\beta_i}, \quad i=1,2$$

the zeros of $p_m(v^{\alpha_i,\beta_i},x_i)$. Then, the Gaussian cubature rule reads as [9]

$$\int_D \mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})\, d\mathbf{x} = \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i^{\alpha_1,\beta_1}\lambda_j^{\alpha_2,\beta_2}\mathbf{f}(\xi_i^{\alpha_1,\beta_1},\xi_j^{\alpha_2,\beta_2}) + \mathcal{R}_m(\mathbf{f}), \tag{7}$$

where $\{\lambda_k^{\alpha_i,\beta_i}\}_{k=1}^m$ denote the Christoffel numbers with respect to the weight v^{α_i,β_i} and $\mathcal{R}_m$ is the remainder term.

Let us note that $\mathcal{R}_m(\mathbf{P})=0$ for any bivariate polynomial $\mathbf{P}\in\mathbb{P}_{2m-1}$. Next two propositions give an error estimate for $\mathcal{R}_m$.

Proposition 2.1. *[9] Let $\boldsymbol{\sigma}$ and $\mathbf{w}$ be the weights defined as in (4) and (6), respectively such that*

$$0 \le \gamma_i < \alpha_i + 1, \qquad 0 \le \delta_i < \beta_i + 1, \tag{8}$$

for each $i=1,2$. Then, for all $\mathbf{f}\in C_{\boldsymbol{\sigma}}$, we have

$$|\mathcal{R}_m(\mathbf{f})| \le \mathcal{C}E_{2m-1}(\mathbf{f})_{\boldsymbol{\sigma}},$$

where $\mathcal{C}\neq\mathcal{C}(m,\mathbf{f})$.

Proposition 2.2. *Let $\mathbf{f}$ be a bivariate function defined on D having $2m$ continuous partial derivatives with respect to both variables x_1 and x_2. Then*

$$|\mathcal{R}_m(\mathbf{f})| \le \max_{i=1,2}\|\mathbf{f}_{x_i}^{(2m)}\|_\infty \frac{1}{(2m)!}\prod_{i=1}^{2}\frac{1}{(\gamma_m(v^{\alpha_i,\beta_i}))^2},$$

where $\gamma_m(v^{\alpha_i,\beta_i})$ is the leading coefficient of $p_m(v^{\alpha_i,\beta_i})$ for each $i=1,2$.

2.2.2 Product cubature rule

Let us now consider the following integral

$$\int_D \mathbf{k}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x}$$

where $\mathbf{f}\in C_{\boldsymbol{\sigma}}$, $\mathbf{w}$ is defined in (6) and $\mathbf{k}$ is a known kernel function.

In order to approximate such an integral, in [10] the authors propose the following product cubature rule

$$\int_D \mathbf{k}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x} = \sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}(\mathbf{y})\mathbf{f}(\xi_i^{\alpha_1,\beta_1},\xi_j^{\alpha_2,\beta_2}) + \mathcal{E}_m(\mathbf{f},\mathbf{y}). \tag{9}$$

Here $\mathcal{E}_m(\mathbf{f},\mathbf{y})$ denotes the raminder term whereas A_{ij} are the coefficients given by

$$A_{ij}(\mathbf{y}) = \int_D \mathbf{k}(\mathbf{x},\mathbf{y})\boldsymbol{\ell}_{ij}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x} \qquad \text{with} \qquad \boldsymbol{\ell}_{ij}(\mathbf{x}) = \ell_i^{\alpha_1,\beta_1}(x_1)\ell_j^{\alpha_2,\beta_2}(x_2),$$

where

$$\ell_i^{\alpha_k,\beta_k}(x_k) = \frac{p_m(v^{\alpha_k,\beta_k},x_k)}{p_m'(v^{\alpha_k,\beta_k},\xi_i^{\alpha_k,\beta_k})(x_k-\xi_i^{\alpha_k,\beta_k})}$$

denotes the i-th fundamental Lagrange polynomial based on the zeros of $p_m(v^{\alpha_k,\beta_k},x_k)$, for each $k=1,2$.

The following theorem, proved in [10], guarantees the stability and the convergence of formula (9).

Theorem 2.3. *[10] Let $\boldsymbol{\sigma}$ and $\mathbf{w}$ be defined as in (4) and (6), respectively such that their parameters satisfy the following conditions*

$$\max\left\{0,\ \frac{\alpha_i}{2}-\frac{1}{4}\right\} < \gamma_i < \min\left\{\alpha_i+\frac{1}{2},\ \frac{\alpha_i}{2}+\frac{1}{4}\right\}, \qquad \max\left\{0,\ \frac{\beta_i}{2}-\frac{1}{4}\right\} < \delta_i < \min\left\{\beta_i+\frac{1}{2},\ \frac{\beta_i}{2}+\frac{1}{4}\right\},$$

for each $i=1,2$. Moreover, assume that

$$\sup_{\mathbf{y}\in D}\int_D \mathbf{k}^2(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x})\,d\mathbf{x} < \infty.$$

Then, the cubature scheme (9) is stable since

$$\sup_{\mathbf{y}\in \mathbf{D}}\left|\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}(\mathbf{y})\mathbf{f}(\xi_i^{\alpha_1,\beta_1},\xi_j^{\alpha_2,\beta_2})\right| \le \mathcal{C}\|\mathbf{f}\boldsymbol{\sigma}\|_\infty,$$

and the following error estimate holds true

$$\sup_{\mathbf{y}\in D}|\mathcal{E}_m(\mathbf{f},\mathbf{y})| \le \mathcal{C}E_{m-1}(\mathbf{f})_{\boldsymbol{\sigma}},$$

where in all cases $\mathcal{C}\neq\mathcal{C}(m,\mathbf{f})$.

2.2.3 A 2D-dilation formula

In this section we focus on the approximation of the integrals of the form

$$\mathcal{I}_\omega(\mathbf{f},\mathbf{y}) = \int_D \mathbf{k}_\omega(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x}, \tag{10}$$

where $\mathbf{f}$ is a given function, $\mathbf{w}$ is as in (6) and $\mathbf{k}_\omega(\mathbf{x},\mathbf{y})$ is a known "nearly singular" kernel which is close to be singular if $\omega^{-1} \to 0$. An example is the kernel function appearing in the Love equation (1). As already mentioned, such kind of integrals arise in several contexts as, for instance, in the boundary element methods. Consequently, for the their numerical approximation, different numerical formulas have been proposed over the years [5, 8, 12].

Here, our idea is to approximate the integral (10) by "generalizing" the dilation techniques proposed in [4, 10] providing new convergence and stability results.

Following [4, 10], we aim to dilate the domain of integration from the square D into the square $D_\omega = [-\omega, \omega] \times [-\omega, \omega]$. Thus, in (10) we make the following change of variables

$$\mathbf{x} = \frac{\boldsymbol{\eta}}{\omega}, \qquad \mathbf{y} = \frac{\boldsymbol{\theta}}{\omega}, \qquad \boldsymbol{\eta} = (\eta_1, \eta_2) \in D_\omega, \quad \boldsymbol{\theta} = (\theta_1, \theta_2) \in D_\omega.$$

Then, by partitioning the new domain D_ω into S^2 squares of area d^2 with d such that $S = \frac{2\omega}{d} \in \mathbb{N}$, i.e.

$$D_\omega = \bigcup_{i=1}^{S}\bigcup_{j=1}^{S}[-\omega+(i-1)d, -\omega+id] \times [-\omega+(j-1)d, -\omega+jd] =: \bigcup_{i=1}^{S}\bigcup_{j=1}^{S} D_{ij},$$

we get

$$\begin{aligned}\mathcal{I}_\omega(\mathbf{f},\mathbf{y}) &= \frac{1}{\omega^2}\int_{D_\omega} \mathbf{k}_\omega\left(\frac{\boldsymbol{\eta}}{\omega},\frac{\boldsymbol{\theta}}{\omega}\right)\mathbf{f}\left(\frac{\boldsymbol{\eta}}{\omega}\right)\mathbf{w}\left(\frac{\boldsymbol{\eta}}{\omega}\right)d\boldsymbol{\eta} =: \frac{1}{\omega^2}\int_{D_\omega} \boldsymbol{\kappa}(\boldsymbol{\eta},\boldsymbol{\theta})\mathbf{f}\left(\frac{\boldsymbol{\eta}}{\omega}\right)\mathbf{w}\left(\frac{\boldsymbol{\eta}}{\omega}\right)d\boldsymbol{\eta} \\ &= \frac{1}{\omega^2}\sum_{i=1}^{S}\sum_{j=1}^{S}\int_{D_{ij}} \boldsymbol{\kappa}(\boldsymbol{\eta},\omega\mathbf{y})\mathbf{f}\left(\frac{\boldsymbol{\eta}}{\omega}\right)\mathbf{w}\left(\frac{\boldsymbol{\eta}}{\omega}\right)d\boldsymbol{\eta}.\end{aligned} \tag{11}$$

Now, by using the invertible linear maps $\boldsymbol{\Psi}_{ij} : D_{ij} \to D$ defined as

$$\mathbf{x} = \boldsymbol{\Psi}_{ij}(\boldsymbol{\eta}) = \left(\frac{2}{d}(\eta_1+\omega)-(2i-1), \frac{2}{d}(\eta_2+\omega)-(2j-1)\right) =: (\Psi_i(\eta_1), \Psi_j(\eta_2)),$$

we can remap each integral into the unit square D. In fact, by making in (11) the following change of variables

$$\boldsymbol{\eta} = \boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x}) = \left(\left(\frac{x_1+1}{2}\right)d-\omega+(i-1)d, \left(\frac{x_2+1}{2}\right)d-\omega+(j-1)d\right) =: \left(\Psi_i^{-1}(x_1), \Psi_j^{-1}(x_2)\right) \tag{12}$$

we have

$$\mathcal{I}_\omega(\mathbf{f},\mathbf{y}) = \frac{d^2}{4\omega^2}\sum_{i=1}^{S}\sum_{j=1}^{S}\int_D \boldsymbol{\kappa}_{ij}(\mathbf{x},\omega\mathbf{y})\mathbf{f}_{ij}(\mathbf{x})\mathbf{u}_{ij}(\mathbf{x})d\mathbf{x}. \tag{13}$$

Here

$$\mathbf{u}_{ij}(\mathbf{x}) := u_i^{\alpha_1,\beta_1}(x_1)u_j^{\alpha_2,\beta_2}(x_2), \qquad \text{with} \qquad u_i^{\alpha_k,\beta_k}(x_k) := \begin{cases} v^{0,\beta_k}(x_k), & i=1 \\ v^{0,0}(x_k), & 2 \le i \le S-1 \\ v^{\alpha_k,0}(x_k), & i=S \end{cases} \quad \text{for} \quad k=1,2, \tag{14}$$

$\mathbf{f}_{ij}(\mathbf{x}) := \mathbf{f}\left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega}\right)$, and $\boldsymbol{\kappa}_{ij}$ is the new kernel function defined as

$$\boldsymbol{\kappa}_{ij}(\mathbf{x},\omega\mathbf{y}) := \mathbf{k}_\omega\left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega},\mathbf{y}\right)\mathbf{U}_{ij}(\mathbf{x}), \tag{15}$$

with

$$\mathbf{U}_{ij}(\mathbf{x}) := U_i(x_1)U_j(x_2) \qquad \text{being} \qquad U_i(x_k) := \begin{cases} \left(\frac{d}{2\omega}\right)^{\beta_k} v^{\alpha_k,0}\left(\frac{\Psi_i^{-1}(x_k)}{\omega}\right), & i=1 \\ v^{\alpha_k,\beta_k}\left(\frac{\Psi_i^{-1}(x_k)}{\omega}\right), & 2 \le i \le S-1 \\ \left(\frac{d}{2\omega}\right)^{\alpha_k} v^{0,\beta_k}\left(\frac{\Psi_i^{-1}(x_k)}{\omega}\right), & i=S \end{cases} \quad \text{for} \quad k=1,2. \tag{16}$$

By approximating each integral appearing in (13) by means of the n-point Gaussian cubature rule (7) with $\mathbf{u}_{ij}$ in place of $\mathbf{w}$ and $\boldsymbol{\kappa}_{ij}\mathbf{f}_{ij}$ instead of $\mathbf{f}$, we have the following "dilation" cubature formula

$$\Sigma_\omega^n(\mathbf{f}, \omega\mathbf{y}) = \frac{d^2}{4\omega^2} \sum_{i=1}^{S} \sum_{j=1}^{S} \sum_{h=1}^{n} \sum_{\nu=1}^{n} \lambda_{h,i}^{\alpha_1,\beta_1} \lambda_{\nu,j}^{\alpha_2,\beta_2} \boldsymbol{\kappa}_{ij} \left((\xi_{h,i}^{\alpha_1,\beta_1}, \xi_{\nu,j}^{\alpha_2,\beta_2}), \omega\mathbf{y} \right) \mathbf{f}_{ij}(\xi_{h,i}^{\alpha_1,\beta_1}, \xi_{\nu,j}^{\alpha_2,\beta_2}), \tag{17}$$

where $\lambda_{h,i}^{\alpha_k,\beta_k}$ is the h-th Christoffel coefficient with respect to the weight $u_i^{\alpha_k,\beta_k}$, $\xi_{h,i}^{\alpha_k,\beta_k}$ is the h-th node of $p_n(u_i^{\alpha_k,\beta_k})$. Moreover, we will denote by Λ_ω^n the remainder term, namely

$$\mathcal{I}_\omega(\mathbf{f}, \mathbf{y}) = \Sigma_\omega^n(\mathbf{f}, \omega\mathbf{y}) + \Lambda_\omega^n(\mathbf{f}, \omega\mathbf{y}). \tag{18}$$

The given rule is stable and convergent as the following theorem shows.

Theorem 2.4. *Let $\boldsymbol{\sigma}$ and $\mathbf{w}$ be the weights defined as in (4) and (6) respectively, such that their parameters satisfy*

$$0 \le \gamma_i < \min\{1, \alpha_i + 1\}, \qquad 0 \le \delta_i < \min\{1, \beta_i + 1\},$$

for each $i = 1, 2$ and assume that the kernel function $\mathbf{k}_\omega$ is such that $\max_{\mathbf{x},\mathbf{y}\in D} |\mathbf{k}_\omega(\mathbf{x}, \mathbf{y})| < \infty$.

Then, for each $\mathbf{f} \in C_{\boldsymbol{\sigma}}$, we have that the cubature rule (17) is stable, i.e.

$$\sup_{\mathbf{y}\in D} |\Sigma_\omega^n(\mathbf{f}, \omega\mathbf{y})| \le \mathcal{C}\|\mathbf{f}\boldsymbol{\sigma}\|_\infty, \quad \mathcal{C} \ne \mathcal{C}(n, \mathbf{f}). \tag{19}$$

Moreover, for any $\mathbf{f} \in W_{\boldsymbol{\sigma}}^r$, if

$$\max_{\mathbf{y}\in D} \left(\max_{i=1,2} \sup_{\mathbf{x}\in D} \left| \frac{\partial^r}{\partial x_i^r} \mathbf{k}_\omega(\mathbf{x}, \mathbf{y}) \varphi^r(x_i) \right| \right) < \infty, \tag{20}$$

we get

$$\sup_{\mathbf{y}\in D} |\Lambda_\omega^n(\mathbf{f}, \omega\mathbf{y})| \le \frac{\mathcal{C}}{n^r} \left(\frac{d}{\omega} \right)^r \|\mathbf{f}\|_{W_{\boldsymbol{\sigma}}^r}, \tag{21}$$

where $\mathcal{C} \ne \mathcal{C}(n, \mathbf{f}, \omega, d)$.

Next result gives an estimate for the remainder term Λ_ω^n in the case when $\mathbf{f}$ and $\mathbf{k}_\omega$ are analytical functions.

Corollary 2.5. *Let $\mathbf{f}(\mathbf{x})$ and $\mathbf{k}_\omega(\mathbf{x}, \mathbf{y})$ two continuous functions having $2n$ continuous partial derivatives with respect to each component x_i of the variable $\mathbf{x}$. Then*

$$\sup_{\mathbf{y}\in D} |\Lambda_\omega^n(\mathbf{f}, \omega\mathbf{y})| \le \frac{\mathcal{C}}{(2n)^{2n+\frac{1}{2}}} \left(\frac{d}{2\omega} \right)^{2n+2} e^{\frac{48n^2+1}{24n}} \left[\|\mathbf{f}\|_\infty + \max_{i=1,2} \|\mathbf{f}_{x_i}^{(2n)}\|_\infty \right],$$

with $\mathcal{C} \ne \mathcal{C}(n, \mathbf{f}, \omega, d)$.

3 The numerical method

The goal of this section is to propose a Nyström method for the bivariate Love integral equation (1). Setting

$$(\mathbf{K}_\omega \mathbf{f})(\mathbf{y}) = \mu \int_D \mathbf{k}_\omega(\mathbf{x}, \mathbf{y}) \mathbf{f}(\mathbf{x}) \mathbf{w}(\mathbf{x}) d\mathbf{x}, \tag{22}$$

with $\mu \in \mathbb{R}$ and $\mathbf{k}_\omega$ defined as in (2), equation (1) can be also rewritten as

$$(\mathbf{I} - \mathbf{K}_\omega)\mathbf{f} = \mathbf{g}, \tag{23}$$

where $\mathbf{I}$ is the identity bivariate operator. Next proposition shows the mapping properties of the operator $\mathbf{K}_\omega$.

Proposition 3.1. *Let $\boldsymbol{\sigma}$ and $\mathbf{w}$ be defined in (4) and (6), respectively such that the parameters γ_i, δ_i, α_i and β_i satisfy (8) for each $i = 1, 2$. Then $\mathbf{K}_\omega : C_{\boldsymbol{\sigma}} \to C_{\boldsymbol{\sigma}}$ is continuous, bounded and compact. Moreover, $\forall \mathbf{f} \in C_{\boldsymbol{\sigma}}$, it results $\mathbf{K}_\omega \mathbf{f} \in W_{\boldsymbol{\sigma}}^r$, for all $r \in \mathbb{N}$.*

According to the previous proposition, by virtue of the Fredholm Alternative theorem, under the assumption $Ker\{\mathbf{I} - \mathbf{K}_\omega\} = \{\mathbf{0}\}$ equation (23) has a unique solution for any fixed $\mathbf{g} \in C_{\boldsymbol{\sigma}}$. The next two subsections deal with the approximation of such a solution. Specifically, in the next one we introduce a suitable cubature formula which approximate the integral operator $\mathbf{K}_\omega$, whereas the second one contains our Nyström method.

3.1 A mixed cubature formula

Let us consider the operator (22) and let us approximate it by using the product rule (9) that is

$$(\mathbf{K}_\omega \mathbf{f})(\mathbf{y}) = \mu \sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}(\mathbf{y})\mathbf{f}(\xi_i^{\alpha_1,\beta_1}, \xi_j^{\alpha_2,\beta_2}) + \mathcal{E}_\omega^m(\mathbf{f},\mathbf{y}),$$

where we remind that $\xi_i^{\alpha_k,\beta_k}$ is the i-th node of $p_m(v^{\alpha_k,\beta_k})$ for each $k=1,2$.

At this point let us approximate the coefficients

$$A_{ij}(\mathbf{y}) = \int_D \mathbf{k}_\omega(\mathbf{x},\mathbf{y})\boldsymbol{\ell}_{ij}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x}$$

by using the n-point “dilation” cubature formula (17), i.e. by the following

$$A_{ij}^n(\mathbf{y}) = \frac{d^2}{4\omega^2}\sum_{i=1}^{S}\sum_{j=1}^{S}\sum_{p=1}^{n}\sum_{q=1}^{n}\lambda_{p,i}^{\alpha_1,\beta_1}\lambda_{q,j}^{\alpha_2,\beta_2}\boldsymbol{\kappa}_{ij}(\boldsymbol{\xi}_{pi,qj},\omega\mathbf{y})\boldsymbol{\ell}_{\mathbf{ij}}\left(\boldsymbol{\Psi}_{ij}^{-1}\left(\frac{\boldsymbol{\xi}_{pi,qj}}{\omega}\right)\right),$$

with $\lambda_{p,i}^{\alpha_k,\beta_k}$ the p-th Christoffel coefficient with respect to the weight $u_i^{\alpha_k,\beta_k}$ given in (14) and $\boldsymbol{\xi}_{pi,qj} = (\xi_{p,i}^{\alpha_1,\beta_1}, \xi_{q,j}^{\alpha_2,\beta_2})$ with $\xi_{p,i}^{\alpha_k,\beta_k}$ the p-th zero of $p_n(u_i^{\alpha_k,\beta_k})$.

In this way we get the following mixed cubature rule

$$(\mathbf{K}_\omega \mathbf{f})(\mathbf{y}) = \mu \sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}^n(\mathbf{y})\mathbf{f}(\xi_i^{\alpha_1,\beta_1}, \xi_j^{\alpha_2,\beta_2}) + \mathcal{E}_\omega^{n,m}(\mathbf{f},\mathbf{y}) =: \mathbf{K}_\omega^{n,m}(\mathbf{f},\mathbf{y}) + \mathcal{E}_\omega^{n,m}(\mathbf{f},\mathbf{y}), \tag{24}$$

where $\mathcal{E}_\omega^{n,m}$ is the remainder term.

Next theorem gives the conditions on the weights which ensure the convergence of the above formula by providing an error estimate in the case when $n \equiv m$.

Theorem 3.2. *Let $\boldsymbol{\sigma}$ and $\mathbf{w}$ be defined in (4) and (6), respectively with parameters such that*

$$\max\left\{0, \frac{\alpha_i}{2}+\frac{1}{4}\right\} < \gamma_i < \min\left\{1, \alpha_i+1, \frac{\alpha_i}{2}+\frac{5}{4}\right\}, \qquad \max\left\{0, \frac{\beta_i}{2}+\frac{1}{4}\right\} < \delta_i < \min\left\{1, \beta_i+1, \frac{\beta_i}{2}+\frac{5}{4}\right\}, \tag{25}$$

for each $i=1,2$. Then, if $\mathbf{f} \in C_\sigma$ the following error estimate holds true

$$|\mathcal{E}_\omega^{m,m}(\mathbf{f})| \le \mathcal{C}\left[E_m(\mathbf{f})_\sigma + \left(\frac{d}{\omega}\right)^{m-1}\log^2 m \|\mathbf{f}\boldsymbol{\sigma}\|_\infty\right],$$

where $\mathcal{C} \neq \mathcal{C}(m,\omega)$.

Let us remark that the previous theorem gives the error estimate for $n=m$. Nevertheless, from the practical point of view, we can apply our method with a fixed and low value of n (for instance n=20), since in virtue of Corollary 2.5 the coefficients of the mixed formula are approximated with an error which decreases exponentially.

Remark 1. Let us underline that the proposed cubature mixed scheme can be also applied to other kind of integral operators, namely, to integrals of the type (22) having a different “nearly” singular kernel. In this case, Theorem 3.2 is still true but the kernel function must satisfy the conditions given in Theorem 2.3 and Theorem 2.4 with $r = m-1$.

3.2 The Nyström method

In order to approximate the solution of (23) let us consider the operator equations

$$\left(\mathbf{I} - \mathbf{K}_\omega^{n,m}\right)\mathbf{f}^{n,m} = \mathbf{g}, \tag{26}$$

where $\mathbf{f}^{n,m}$ is unknown and $\mathbf{K}_\omega^{n,m}$ is the discrete operator arising by the mixed cubature formula introduced in (24).

We multiply both sides of equation (26) by the weight function $\boldsymbol{\sigma}$ and we collocate it on the pairs $\boldsymbol{\xi}_{ij} = (\xi_i^{\alpha_1,\beta_1}, \xi_j^{\alpha_2,\beta_2})$, $i,j = 1,\dots,m$.

In this way we have the following $m^2 \times m^2$ linear system

$$a_{ij} - \mu\boldsymbol{\sigma}(\boldsymbol{\xi}_{ij})\sum_{h=1}^{m}\sum_{\nu=1}^{m}\frac{A_{h,\nu}^n(\boldsymbol{\xi}_{ij})}{\boldsymbol{\sigma}(\boldsymbol{\xi}_{ij})}a_{h\nu} = (\mathbf{g}\boldsymbol{\sigma})(\boldsymbol{\xi}_{ij}), \qquad i,j=1,\dots,m, \tag{27}$$

where the unknowns

$$a_{ij} = (\mathbf{f}^{n,m}\boldsymbol{\sigma})(\boldsymbol{\xi}_{ij}), \qquad i,j=1,\dots,m,$$

allow us to construct the weighted bivariate Nyström interpolant

$$(\mathbf{f}^{n,m}\boldsymbol{\sigma})(\mathbf{y}) = \mu\boldsymbol{\sigma}(\mathbf{y})\sum_{h=1}^{m}\sum_{\nu=1}^{m}\frac{A_{h,\nu}^n(\mathbf{y})}{\boldsymbol{\sigma}(\boldsymbol{\xi}_{ij})}a_{h\nu}^* + (\mathbf{g}\boldsymbol{\sigma})(\mathbf{y}). \tag{28}$$

Next theorem states that the above described Nyström method is stable, convergent and the condition number of the system we solve does not depend on m.

Theorem 3.3. *Let $\boldsymbol{\sigma}$ and $\mathbf{w}$ be defined as in (4) and (6) respectively, with the parameters satisfying (25) and let us assume that $Ker\{\mathbf{I}-\mathbf{K}\}=\{0\}$ in $C_{\boldsymbol{\sigma}}$. Then for m sufficiently large, the operators $\left(\mathbf{I}-\mathbf{K}_{\omega}^{n,m}\right)^{-1}$ exist and are uniformly bounded and system (27) is well conditioned. Moreover, if $\mathbf{g}\in W_{\boldsymbol{\sigma}}^{r}$, $r\geq 1$, the following convergence estimate holds true*

$$\|[\mathbf{f}-\mathbf{f}^{m,m}]\boldsymbol{\sigma}\|_{\infty}\leq \mathcal{C}\left[\frac{1}{m^r}+\left(\frac{d}{\omega}\right)^{m-1}\log^2 m\right]\|\mathbf{f}\|_{W_{\boldsymbol{\sigma}}^r},\qquad \mathcal{C}\neq\mathcal{C}(m,\mathbf{f}). \tag{29}$$

Remark 2. Let us note that, as stated in estimate (29), the proposed global approximation method allows us to find the solution of equation (1) with an order of convergence essentially given by the order of the best polynomial approximation of the unknown function f since the extra term $(\frac{d}{\omega})^{m-1}\log^2 m$ vanishes exponentially. Consequently, the convergence order is independent of the magnitude of ω. However, the function f naturally depends on ω, being the solution of an equation in which such a parameter appears and therefore the constant $\mathcal{C}$ in (29) depends on ω. This is the only reason why for a large value of ω we need to increase the dimension of the system in order to get high precision (see the numerical results given in Section 4).

4 Numerical Tests

The aim of this section is to show the accuracy of our method by some numerical examples. For each considered test equation, we solve system (27) and we compute the weighted Nyström interpolant $(\mathbf{f}^{n,m}\boldsymbol{\sigma})(\mathbf{y})$ (28) in several points $\mathbf{y}$ of the square D with $n=20$ fixed, for different values of m and with $\omega=10$ or $\omega=10^2$.

All the numerical experiments were performed in double precision arithmetic on an IntelCore i7 system (4 cores), running the Mac-Os operating system and using Matlab R2018a.

Example 4.1. Let us consider the classical bivariate Love integral equation

$$\mathbf{f}(\mathbf{y})-\frac{1}{\pi^2}\int_D \mathbf{k}_{\omega}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})d\mathbf{x}=1,$$

in the space $C_{\boldsymbol{\sigma}}$ with $\boldsymbol{\sigma}(x)=\sqrt{(1-x^2)(1-y^2)}$. Table 1 contains the results we get for $\omega=10$ and $\omega=10^2$. As we can see, the convergence is very fast and for $m=64$ we get the machine precision if $\omega=10$ and an absolute error of the order 10^{-12} if $\omega=10^2$. Moreover, for different values of m, in Table 2 we report the condition numbers in infinity norm of the matrix of coefficient $\mathbb{A}_{m^2}$ of the linear system (27), showing that they are extremely well-conditioned.

ω	m	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.5,0.5)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.3,0.99)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0,0)$
10	8	$8.706379638969600e-01$	$1.478716528767691e-01$	$1.179654926512359e+00$
	16	$8.706406847444945e-01$	$1.478727042826599e-01$	$1.179642631846515e+00$
	32	$8.706406048629998e-01$	$1.478727063066576e-01$	$1.179642776897315e+00$
	64	$8.706406048626485e-01$	$1.478727063065337e-01$	$1.179642776903225e+00$
	128	$8.706406048626485e-01$	$1.478727063065337e-01$	$1.179642776903225e+00$
ω	m	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.9,0.7)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.1,0.6)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.5,0.2)$
10^2	8	$3.189258239649260e-01$	$8.195662167349387e-01$	$8.739905398475708e-01$
	16	$3.189266279099560e-01$	$8.195642531444833e-01$	$8.739905373874824e-01$
	32	$3.189263910313820e-01$	$8.195643111474382e-01$	$8.739907639414651e-01$
	64	$3.189263896191077e-01$	$8.195643136782380e-01$	$8.739907628529722e-01$
	128	$3.189263896171025e-01$	$8.195643136779556e-01$	$8.739907628538929e-01$

Table 1: Numerical results for Example 4.1

ω	m	$cond(\mathbb{A}_{m^2})$	ω	m	$cond(\mathbb{A}_{m^2})$
10	8	1.357259159118987e+00	10^2	8	1.033095026557234e+00
	16	1.441462903398752e+00		16	1.042950050720626e+00
	32	1.489800370704548e+00		32	1.052106842473793e+00
	64	1.509508024196380e+00		64	1.060421800715730e+00
	128	1.515495801392271e+00		128	1.067223416255047e+00

Table 2: Condition numbers for Example 4.1

Example 4.2. Let us test our method on the equation

$$\mathbf{f}(\mathbf{y})-\frac{1}{\pi^2}\int_D \mathbf{k}_{\omega}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})\sqrt{(1-\mathbf{x}^2)}\sqrt{1-\mathbf{y}^2}d\mathbf{x}=\log(10-\mathbf{x}-\mathbf{y}),$$

where the weight appearing inside the integral is of the type in (6) with $\alpha_i=\beta_i=1/2$, $i=1,2$. Let us consider such a equation in the space $C_{\boldsymbol{\sigma}}$ with $\boldsymbol{\sigma}$ as in (4) with $\gamma_i=\delta_i=1$, $i=1,2$, according to (25). In Tables 3 and 4 we report our numerical results. They show the fast convergence of our method and that the linear systems we solve is well conditioned for each fixed value of m.

ω	m	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(-0.5,-0.2)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0,0)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.9,-0.9)$
10	8	$1.914882727738103e+00$	$2.646981926729218e+00$	$8.604583371836652e-02$
	16	$1.914882654327322e+00$	$2.646985122795177e+00$	$8.604581564470247e-02$
	32	$1.914882643576744e+00$	$2.646985144586979e+00$	$8.604581566290904e-02$
	64	$1.914882643578620e+00$	$2.646985144594662e+00$	$8.604581566290784e-02$
	128	$1.914882643578620e+00$	$2.646985144594662e+00$	$8.604581566290784e-02$
ω	m	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(-0.9,-0.3)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.1,0)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.9,0.9)$
10^2	8	$4.226891240890577e-01$	$2.333974818974274e+00$	$7.642142716682045e-02$
	16	$4.226892784081589e-01$	$2.333973917132355e+00$	$7.642143745504938e-02$
	32	$4.226892750350204e-01$	$2.333973908025957e+00$	$7.642143721145409e-02$
	64	$4.226892751293564e-01$	$2.333973908025349e+00$	$7.642143721927090e-02$
	128	$4.226892751295805e-01$	$2.333973908023352e+00$	$7.642143721928943e-02$

Table 3: Numerical results for Example 4.2

ω	m	$cond(\mathbb{A}_{m^2})$	ω	m	$cond(\mathbb{A}_{m^2})$
10	8	$1.330159520557044e+00$	10^2	8	$1.032981756338642e+00$
	16	$1.422185660580896e+00$		16	$1.042132938492258e+00$
	32	$1.475093994785804e+00$		32	$1.051654415624680e+00$
	64	$1.497315163110278e+00$		64	$1.060128520091822e+00$
	128	$1.503729683343860e+00$		128	$1.066998666939352e+00$

Table 4: Condition numbers for Example 4.2

Example 4.3. Let us now apply our Nyström method on the equation

$$\mathbf{f}(\mathbf{y}) - \frac{1}{\pi^2}\int_D \mathbf{k}_\omega(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})\frac{1}{\sqrt{(1-\mathbf{x}^2)}\sqrt{1-\mathbf{y}^2}}d\mathbf{x} = |\mathbf{x}|^{\frac{9}{2}}\mathbf{y}^3,$$

where the weight appearing inside the integral is of the type in (6) with $\alpha_i = \beta_i = -1/2$, $i = 1,2$, in order to approximate its unique solution in the space $C_{\boldsymbol{\sigma}}$ where $\boldsymbol{\sigma}$ is as in (4) with $\gamma_i = \delta_i = 1/4$, $i = 1,2$. In this case, since the right-hand side $\mathbf{g} \in W^4_{\boldsymbol{\sigma}}$, according to Theorem 3.3, we expect an order of convergence of m^{-4}. Table 5 confirms our theoretical estimate and Table 6 shows the well-conditioning of our system. We underline that the presence of the singularities along the boundary of D in the kernel, does not make any influence on the rate of convergence of the method.

ω	m	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(-0.5,-0.2)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0,0)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.5,-0.9)$
10	8	$-6.014268164506133e-03$	$-3.515284717562412e-18$	$-4.481539403944196e-02$
	16	$-6.032447421683561e-03$	$7.030569435124824e-19$	$-4.497567218241906e-02$
	32	$-6.032420390247492e-03$	0	$-4.497474464346271e-02$
	64	$-6.032420285926134e-03$	$1.230349651146844e-18$	$-4.497474503492876e-02$
	128	$-6.032420283626174e-03$	$-9.491268737418512e-18$	$-4.497474502742697e-02$
ω	m	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(-0.9,-0.3)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.1,0)$	$(\mathbf{f}^{20,m}\boldsymbol{\sigma})(0.9,0.7)$
10^2	8	$-1.234047743548169e-02$	$8.766158391001223e-20$	$1.311258296739617e-01$
	16	$-1.229899599514160e-02$	$1.314923758650183e-19$	$1.311416050215052e-01$
	32	$-1.229606685111479e-02$	$1.424500738537699e-19$	$1.311213605521988e-01$
	64	$-1.229630417628584e-02$	$2.739424497187882e-20$	$1.311209550570823e-01$
	128	$-1.229630551950528e-02$	$-1.123164043847032e-19$	$1.311209577202236e-01$

Table 5: Numerical results for Example 4.3

ω	m	$cond(\mathbb{A}_{m^2})$	ω	m	$cond(\mathbb{A}_{m^2})$
10	8	$2.214479076630497e+00$	10^2	8	$1.462137835998365e+00$
	16	$2.707286451771667e+00$		16	$1.580015950348097e+00$
	32	$2.991900403334029e+00$		32	$2.239311336538747e+00$
	64	$3.167377814865466e+00$		64	$2.557976384124247e+00$
	128	$3.374806255262317e+00$		128	$2.831292261327492e+00$

Table 6: Condition numbers for Example 4.3

5 Proofs

Proof of Proposition 2.2. Let us introduce the bivariate Lagrange polynomial $\mathcal{L}_{2m}(\mathbf{f})$ of degree $2m-1$ in each variable x_i [9], interpolating the function $\mathbf{f}$ at the pairs (t_1^k, t_2^k) where $\{t_i^k\}_{k=1}^{2m}$ are the zeros of the polynomial $p_m(v^{\alpha_i,\beta_i})q_m(x_i)$ for $i=1,2$ with q_m a monic univariate polynomial of degree m in the variable x_i.

By virtue of the exactness of the Gaussian cubature rule for algebraic polynomials of degree $2m-1$ in each variable, we can state

$$\mathcal{R}_m(\mathbf{f}) = \mathcal{R}_m(\mathbf{f}-\mathcal{L}_{2m}(\mathbf{f})) = \int_D [\mathbf{f}(\mathbf{x}) - \mathcal{L}_{2m}(\mathbf{f},\mathbf{x})]\,\mathbf{w}(\mathbf{x})d\mathbf{x}.$$

Since we have,

$$\mathbf{f}(\mathbf{x}) - \mathcal{L}_{2m}(\mathbf{f},\mathbf{x}) = \frac{\partial^{2m}}{\partial x_i^{2m}}\mathbf{f}(\xi_1,\xi_2)\frac{1}{(2m)!}\prod_{i=1}^{2}\frac{1}{\gamma_m(v^{\alpha_i,\beta_i})}p_m(v^{\alpha_i,\beta_i},x_i)q_m(x_i),$$

where the point $(\xi_1,\xi_2)\in D$ depends on the variable x_i with respect to we make the derivative, we can deduce

$$\begin{aligned}|\mathcal{R}_m(\mathbf{f})| &\le \left|\frac{\partial^{2m}}{\partial x_i^{2m}}\mathbf{f}(\xi_1,\xi_2)\right|\frac{1}{(2m)!}\prod_{i=1}^{2}\frac{1}{\gamma_m(v^{\alpha_i,\beta_i})}\int_{-1}^{1}p_m(v^{\alpha_i,\beta_i})q_m(x_i)v^{\alpha_i,\beta_i}(x_i)\,dx_i\\ &\le \max_{i=1,2}\|\mathbf{f}_{x_i}^{(2m)}\|_\infty\frac{1}{(2m)!}\prod_{i=1}^{2}\frac{1}{(\gamma_m(v^{\alpha_i,\beta_i}))^2}.\end{aligned}$$

□

Proof of Theorem 2.4. First, we prove the stability of the formula. By definition (17)

$$\begin{aligned}|\Sigma_\omega^n(\mathbf{f},\omega\mathbf{y})| &= \frac{d^2}{4\omega^2}\left|\sum_{i=1}^{S}\sum_{j=1}^{S}\sum_{h=1}^{n}\sum_{\nu=1}^{n}\lambda_{h,i}^{\alpha_1,\beta_1}\lambda_{\nu,j}^{\alpha_2,\beta_2}\boldsymbol{\kappa}_{ij}\left((\xi_{h,i}^{\alpha_1,\beta_1},\xi_{\nu,j}^{\alpha_2,\beta_2}),\omega\mathbf{y}\right)\mathbf{f}_{ij}(\xi_{h,i}^{\alpha_1,\beta_1},\xi_{\nu,j}^{\alpha_2,\beta_2})\right|\\ &\le \frac{d^2}{4\omega^2}\|\mathbf{f}\boldsymbol{\sigma}\|_\infty\max_{i,j}\|\boldsymbol{\kappa}_{ij}(\omega\mathbf{y})\|_\infty\sum_{i=1}^{S}\sum_{j=1}^{S}\sum_{h=1}^{n}\frac{\lambda_{h,i}^{\alpha_1,\beta_1}}{\sigma(\xi_{h,i}^{\alpha_1,\beta_1})}\sum_{\nu=1}^{n}\frac{\lambda_{\nu,j}^{\alpha_2,\beta_2}}{\sigma(\xi_{\nu,j}^{\alpha_2,\beta_2})}\end{aligned}$$

from which taking into account that

$$\sum_{h=1}^{n}\frac{\lambda_{h,i}^{\alpha_1,\beta_1}}{\sigma(\xi_{h,i}^{\alpha_1,\beta_1})}\le\int_{-1}^{1}\frac{v_i^{\alpha_1,\beta_1}(x_1)}{v^{\gamma_1,\delta_1}(x_1)}dx_1 \quad\text{and}\quad \sum_{\nu=1}^{n}\frac{\lambda_{\nu,j}^{\alpha_2,\beta_2}}{\sigma(\xi_{\nu,j}^{\alpha_2,\beta_2})}\le\int_{-1}^{1}\frac{v_j^{\alpha_2,\beta_2}(x_2)}{v^{\gamma_2,\delta_2}(x_2)}dx_2$$

by the assumptions on the weights and on the kernel we get estimate (19).

Let us now prove (21). By applying Proposition 2.1 and taking into account the well-known estimate [7]

$$E_{2n-1}(\mathbf{h}_1\,\mathbf{h}_2)_\sigma \le \|\mathbf{h}_1\boldsymbol{\sigma}\|_\infty E_{[\frac{2n-1}{2}]}(\mathbf{h}_2) + 2\|\mathbf{h}_2\|_\infty E_{[\frac{2n-1}{2}]}(\mathbf{h}_1)_{\boldsymbol{\sigma}}, \qquad \forall \mathbf{h}_1\mathbf{h}_2\in C_{\boldsymbol{\sigma}},$$

where $[a]$ denotes the greatest integer smaller than or equal to $a>0$, we can write

$$|\Lambda_\omega^n(\mathbf{f},\omega y)| \le \mathcal{C}\sum_{i=1}^{S}\sum_{j=1}^{S}\left(\|\mathbf{f}_{ij}\boldsymbol{\sigma}\|_\infty E_{[\frac{2n-1}{2}]}(\boldsymbol{k}_{ij}) + \sup_{\mathbf{x}\in D}|\boldsymbol{\kappa}_{ij}(\mathbf{x},\omega\mathbf{y})|\; E_{[\frac{2n-1}{2}]}(\mathbf{f}_{ij})_{\boldsymbol{\sigma}}\right).$$

Then, by using (5) we get

$$|\Lambda_\omega^n(\mathbf{f},\omega\mathbf{y})| \le \frac{\mathcal{C}}{n^r}\sum_{i=1}^{S}\sum_{j=1}^{S}\|\mathbf{f}_{ij}\|_{W_\sigma^r}\left(\mathcal{N}_r(\boldsymbol{\kappa}_{ij},\mathbf{y}) + \sup_{\mathbf{x}\in D}|\boldsymbol{\kappa}_{ij}(\mathbf{x},\omega\mathbf{y})|\right),$$

with

$$\mathcal{N}_r(\boldsymbol{\kappa}_{ij},\omega\mathbf{y}) := \max_{k=1,2}\left\{\max_{\mathbf{x}\in D}\left|\left[\frac{\partial^r}{\partial x_k^r}\boldsymbol{\kappa}_{ij}(\mathbf{x},\omega\mathbf{y})\right]\varphi^r(x_k)\right|\right\}.$$

By definitions (15) and (16) we can deduce

$$\left|\left[\frac{\partial^\ell}{\partial x_k^\ell}\mathbf{U}_{ij}(\mathbf{x})\right]\varphi^\ell(x_k)\right| \le \mathcal{C}\left(\frac{d}{2\omega}\right)^\ell,$$

so that by applying assumptions (20) and taking into account (12), we get

$$\begin{aligned}\left|\left[\frac{\partial^r}{\partial x_k^r}\boldsymbol{\kappa}_{ij}(\mathbf{x},\omega\mathbf{y})\right]\varphi^r(x_k)\right| &\le \sum_{\ell=0}^{r}\binom{r}{\ell}\left|\frac{\partial^\ell}{\partial x_k^\ell}\boldsymbol{k}\left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega},\mathbf{y}\right)\varphi^\ell(x_k)\right|\left|\left[\frac{\partial^{r-\ell}}{\partial x_k^{r-\ell}}\mathbf{U}_{ij}(\mathbf{x})\right]\varphi^{r-\ell}(x_k)\right|\\ &\le \mathcal{C}\sum_{\ell=0}^{r}\binom{r}{\ell}\left(\frac{d}{2\omega}\right)^{r-\ell}\left(\frac{d}{2\omega}\right)^\ell\\ &= \mathcal{C}\left(\frac{d}{\omega}\right)^r.\end{aligned}$$

Therefore

$$\sup_{\mathbf{y}\in D} |\Lambda_\omega^n(\mathbf{f}, \omega\mathbf{y})| \le \frac{\mathcal{C}}{n^r} \left(\frac{d}{\omega}\right)^r \|\mathbf{f}\|_{W_\sigma^r}.$$

□

Proof of Corollary 2.5. By (18), taking into account (17) and by applying Proposition 2.2, we have

$$|\Lambda_\omega^n(\mathbf{f}, \omega\mathbf{y})| \le \frac{d^2}{4\omega^2} \sum_{i=1}^{S}\sum_{j=1}^{S} |\mathcal{R}_n(\mathbf{f}_{ij}\,\boldsymbol{\kappa}_{ij}, \omega\mathbf{y})| \le \frac{d^2}{4\omega^2} \frac{1}{(2n)!} \prod_{k=1}^{2} \frac{1}{(\gamma_n(v^{\alpha_k,\beta_k}))^2} \sum_{i=1}^{S}\sum_{j=1}^{S} \sup_{\mathbf{x}\in D} \max_{k=1,2} \left| \frac{\partial^{2n}}{\partial x_k^{2n}} [\mathbf{f}_{ij}(\mathbf{x})\boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y})] \right|.$$

By the Leibnitz rule we can write

$$\begin{aligned}
\left| \frac{\partial^{2n}}{\partial x_k^{2n}} [\mathbf{f}_{ij}(\mathbf{x})\boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y})] \right| &\le \sum_{r=0}^{2n} \binom{2n}{r} \left| \frac{\partial^{2n-r}}{\partial x_k^{2n-r}} \mathbf{f}\left(\frac{\Psi_{ij}^{-1}(\mathbf{x})}{\omega} \right) \right| \left| \frac{\partial^r}{\partial x_k^r} \boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y}) \right| \\
&\le \sum_{r=0}^{2n} \binom{2n}{r} \left(\frac{d}{2\omega}\right)^{2n-r} \|\mathbf{f}_{x_k}^{(2n-r)}\|_\infty \sup_{\mathbf{x}\in D} \left| \frac{\partial^r}{\partial x_k^r} \boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y}) \right|,
\end{aligned}$$

from which [3] we get

$$\left| \frac{\partial^{2n}}{\partial x_k^{2n}} [\mathbf{f}_{ij}(\mathbf{x})\boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y})] \right| \le \|\mathbf{f}\|_\infty \sum_{r=0}^{2n} \binom{2n}{r} \left(\frac{d}{4\omega}\right)^{2n-r} \left| \frac{\partial^r}{\partial x_k^r} \boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y}) \right| + \|\mathbf{f}_{x_k}^{(2n)}\|_\infty \sum_{r=0}^{2n} \binom{2n}{r} \left(\frac{d}{2\omega}\right)^{2n-r} 2^r \left| \frac{\partial^r}{\partial x_k^r} \boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y}) \right|. \tag{30}$$

Let us now estimate the partial derivative of the kernels $\boldsymbol{\kappa}_{ij}$. By their definition (15) and taking into account the form of the functions $\mathbf{U}_{ij}$ given in (16), we can write

$$\begin{aligned}
\left| \frac{\partial^r}{\partial x_k^r} \boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y}) \right| &\le \sum_{\ell=0}^{r} \binom{r}{\ell} \left| \frac{\partial^{r-\ell}}{\partial x_k^{r-\ell}} \left[\mathbf{k}_\omega \left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega}, \mathbf{y} \right) \right] \right| \left| \frac{\partial^\ell}{\partial x_k^\ell} \mathbf{U}_{ij}(\mathbf{x}) \right| \\
&\le \mathcal{C} \sum_{\ell=0}^{r} \binom{r}{\ell} \left(\frac{d}{2\omega}\right)^{r-\ell} \left(\frac{d}{2\omega}\right)^{\ell} \sup_{\mathbf{x}\in D} \left| \frac{\partial^{r-\ell}}{\partial x_k^{r-\ell}} \mathbf{k}_\omega \left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega}, \mathbf{y} \right) \right|.
\end{aligned}$$

Thus, being [3]

$$\sup_{\mathbf{x}\in D} \left| \frac{\partial^{r-\ell}}{\partial x_k^{r-\ell}} \mathbf{k}_\omega \left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega}, \mathbf{y} \right) \right| \le \mathcal{C} \left[\left(\frac{1}{2}\right)^{r-\ell} \sup_{\mathbf{x}\in D} \left| \mathbf{k}_\omega \left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega}, \mathbf{y} \right) \right| + 2^{2n-r+\ell} \sup_{\mathbf{x}\in D} \left| \frac{\partial^{2n}}{\partial x^{2n}} \mathbf{k}_\omega \left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega}, \mathbf{y} \right) \right| \right],$$

in virtue of the assumptions on the kernel $\boldsymbol{k}$, we have

$$\left| \frac{\partial^r}{\partial x_k^r} \boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y}, \omega) \right| \le \mathcal{C}\, 2^{2n} \left(\frac{3d}{4\omega}\right)^r.$$

Consequently, by replacing the above estimate in (30) we have

$$\left| \frac{\partial^{2n}}{\partial x_k^{2n}} [\mathbf{f}_{ij}(\mathbf{x})\boldsymbol{\kappa}_{ij}(\mathbf{x}, \omega\mathbf{y})] \right| \le \mathcal{C} \left(\frac{d}{\omega}\right)^{2n} \left[\|\mathbf{f}\|_\infty + \|\mathbf{f}_{x_k}^{(2n)}\|_\infty \right],$$

from which we deduce

$$|\Lambda_\omega^n(\mathbf{f}, \omega\mathbf{y})| \le \frac{d^2}{4\omega^2} \frac{1}{(2n)!} \prod_{k=1}^{2} \frac{1}{(\gamma_n(v^{\alpha_k,\beta_k}))^2} \left(\frac{d}{\omega}\right)^{2n} \left[\|\mathbf{f}\|_\infty + \|\mathbf{f}_{x_k}^{(2n)}\|_\infty \right]$$

and then taking into account that $\gamma_n(w) \sim 2^n$ [7]

$$|\Lambda_\omega^n(\mathbf{f}, \omega\mathbf{y})| \le \left(\frac{d}{2\omega}\right)^{2n+2} \frac{1}{(2n)!} \left[\|\mathbf{f}\|_\infty + \|\mathbf{f}_{x_k}^{(2n)}\|_\infty \right].$$

Therefore, by using the well-known Stirling formula

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{-\frac{1}{12n}} \le n! \le \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{-\frac{1}{12n+1}},$$

we get the thesis. □

Proof of Proposition 3.1 . The boundedness of the operator follows by (8) and by the fact that $\sup_{\mathbf{x}\in D} \|\mathbf{k}_\omega(\mathbf{x},\cdot)\boldsymbol{\sigma}\|_\infty < \infty$. In fact, by definition we can assert

$$|(\mathbf{K}_\omega \mathbf{f})(\mathbf{y})\boldsymbol{\sigma}(\mathbf{y})| \le |\mu|\,\|\mathbf{f}\boldsymbol{\sigma}\|_\infty \int_D |\mathbf{k}_\omega(\mathbf{x},\mathbf{y})\boldsymbol{\sigma}(\mathbf{y})| \frac{\mathbf{w}(\mathbf{x})}{\boldsymbol{\sigma}(\mathbf{x})} d\mathbf{x} \le \mathcal{C}\|\mathbf{f}\boldsymbol{\sigma}\|_\infty.$$

In order to prove its compactness, it is sufficient to prove that the operator $\mathbf{K}_\omega$ satisfies the following condition [13]

$$\lim_{m\to\infty} \sup_{\|\mathbf{f}\boldsymbol{\sigma}\|_\infty=1} E_m(\mathbf{K}_\omega \mathbf{f})_{\boldsymbol{\sigma}} = 0. \tag{31}$$

To this end, we note that for $r > 0$

$$\begin{aligned}\left\|\frac{\partial^r}{\partial y_i^r}(\mathbf{K}_\omega \mathbf{f})\varphi^r\boldsymbol{\sigma}\right\|_\infty &\le |\mu|\,\|\mathbf{f}\boldsymbol{\sigma}\|_\infty \int_D \left|\frac{\partial^r}{\partial y_i^r}\mathbf{k}_\omega(\mathbf{x},\mathbf{y})\varphi^r(y)\boldsymbol{\sigma}(y)\right| \frac{\mathbf{w}(\mathbf{x})}{\boldsymbol{\sigma}(\mathbf{x})}d\mathbf{x}\\ &\le |\mu|\,\|\mathbf{f}\boldsymbol{\sigma}\|_\infty \sup_{\mathbf{x}\in D}\left\|\frac{\partial^r}{\partial y_i^r}\mathbf{k}_\omega(\mathbf{x},\cdot)\varphi^r\boldsymbol{\sigma}\right\|_\infty \int_D \frac{\mathbf{w}(\mathbf{x})}{\boldsymbol{\sigma}(\mathbf{x})}d\mathbf{x}.\end{aligned}$$

Hence, $\mathbf{K}_\omega \mathbf{f} \in W_\sigma^r$ for each $\mathbf{f} \in C_{\boldsymbol{\sigma}}$, and by using (5) we deduce (31). □

Proof of Theorem 3.2. At first, let us mention that the kernel $\mathbf{k}_\omega$ satisfies the assumptions of Theorem 2.3 and 2.4 for any $r \ge 1$. Now, by (24) we can write

$$\begin{aligned}|\mathcal{E}_\omega^{m,m}(\mathbf{f},\mathbf{y})| &\le \left|\int_D k(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x} - \mu\sum_{i=1}^m\sum_{j=1}^m A_{ij}(\mathbf{y})\mathbf{f}(\xi_i^{\alpha_1,\beta_1},\xi_j^{\alpha_2,\beta_2})\right| + |\mu|\left|\sum_{i=1}^m\sum_{j=1}^m [A_{ij}(\mathbf{y}) - A_{ij}^m(\mathbf{y})]\mathbf{f}(\xi_i^{\alpha_1,\beta_1},\xi_j^{\alpha_2,\beta_2})\right|\\ &\le |\mathcal{E}_\omega^m(\mathbf{f},\mathbf{y})| + |\mu|\,\|\mathbf{f}\boldsymbol{\sigma}\|_\infty \sum_{h=1}^m\sum_{\nu=1}^m \frac{|\Lambda_\omega^m(\boldsymbol{\ell}_{i,j},\omega\mathbf{y})|}{\boldsymbol{\sigma}(\xi_i^{\alpha_1,\beta_1},\xi_j^{\alpha_2,\beta_2})}.\end{aligned}$$

Then, by applying (21) with $r = m-1$ we have

$$|\Lambda_\omega^m(\boldsymbol{\ell}_{i,j},\omega\mathbf{y})| \le \frac{\mathcal{C}}{m^{m-1}}\left(\frac{d}{\omega}\right)^{m-1}\|\boldsymbol{\ell}_{i,j}\|_{W_\sigma^{m-1}}.$$

Consequently, by using the weighted Bernstein inequality (see, for instance [7, p. 170]) we get

$$|\Lambda_\omega^m(\boldsymbol{\ell}_{i,j},\omega\mathbf{y})| \le \mathcal{C}\left(\frac{d}{\omega}\right)^{m-1}\|\boldsymbol{\ell}_{i,j}\boldsymbol{\sigma}\|_\infty,$$

from which, taking into account the bahaviour of the Lebesgue constants [7] we deduce

$$|\mathcal{E}_\omega^{m,m}(\mathbf{f},\mathbf{y})| \le |\mathcal{E}_m(\mathbf{f},\mathbf{y})| + \mathcal{C}\left(\frac{d}{\omega}\right)^{m-1}\log^2 m\,\|\mathbf{f}\boldsymbol{\sigma}\|_\infty.$$

Then the thesis is obtained by applying Theorem 2.3 to the first term. □

Proof of Theorem 3.3. By Theorem 3.2 we can deduce that $\|(\mathbf{K}_\omega - \mathbf{K}_\omega^{m,m})\mathbf{f}\boldsymbol{\sigma}\|$ tends to zero for any $\mathbf{f} \in C_{\boldsymbol{\sigma}}$. Moreover, by proceeding as done in the proof of Theorem 4.1 in [4] we can state that the operators $\{\mathbf{K}_\omega^{m,m}\}_m$ are collectively compact. Thus, in virtue of the principle of uniform boundedness, we can deduce that $\sup_m \|K_\omega^{m,m}\| < \infty$ and $\|(\mathbf{K} - \mathbf{K}_\omega^{m,m})\mathbf{K}_\omega^{m,m}\|$ tends to zero [1, Lemma 4.1.2]. Consequently, we can claim that for m sufficiently large, the operator $(\mathbf{I} - \mathbf{K}_\omega^{m,m})^{-1}$ exists and it is uniformly bounded i.e. the method is stable.

About the well-conditioning of the matrix of system (27) we can use the same arguments in [1, p.113] only by replacing the usual infinity norm with the weighted uniform norm of $C_{\boldsymbol{\sigma}}$. Finally, estimate (29) follows taking into account that

$$\|(\mathbf{f} - \mathbf{f}^{m,m})\boldsymbol{\sigma}\|_\infty \le \|(\mathbf{I} + \mathbf{K}_\omega^{m,m})^{-1}\|\,\|(\mathbf{K} - \mathbf{K}_\omega^{m,m})\mathbf{f}\|_\infty$$

and by applying Theorem 3.2 to the last term. □

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