

On the isomorphism of certain primitive Q -polynomial not P -polynomial association schemes

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Abstract

In 2011, Penttila and Williford constructed an infinite new family of primitive Q -polynomial 3-class association schemes, not arising from distance regular graphs, by exploring the geometry of the lines of the unitary polar space $H(3, q^2)$, q even, with respect to a symplectic polar space $W(3, q)$ embedded in it.

In a private communication to Penttila and Williford, H. Tanaka pointed out that these schemes have the same parameters as the 3-class schemes found by Hollmann and Xiang in 2006 by considering the action of $\text{PGL}(2, q^2)$, q even, on a non-degenerate conic of $\text{PG}(2, q^2)$ extended in $\text{PG}(2, q^4)$. Therefore, the question arises whether the above association schemes are isomorphic. In this paper we provide the positive answer. As by product, we get an isomorphism of strongly regular graphs.

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1 Introduction

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a (symmetric) association scheme with d classes. For $0 \leq i \leq d$, let A_i be the adjacency matrix of the relation R_i , and E_i the i -th primitive idempotent of the Bose-Mesner algebra of \mathfrak{X} which projects on the i -th maximal common eigenspace of A_0, \dots, A_d . The matrices P and Q defined by

$$(A_0 \ A_1 \ \dots \ A_d) = (E_0 \ E_1 \ \dots \ E_d)P$$

and

$$(E_0 \ E_1 \ \dots \ E_d) = |X|^{-1}(A_0 \ A_1 \ \dots \ A_d)Q$$

are the *first* and the *second eigenmatrix* of \mathfrak{X} , respectively.

An association scheme is said to be *P-polynomial*, or *metric*, if, after a reordering of the relations, there are polynomials p_i of degree i such that $A_i = p_i(A_1)$; an association scheme is called *Q-polynomial*, or *cometric*, if, after a reordering of the eigenspaces, there are polynomials q_i of degree i such that $E_i = q_i(E_1)$, where multiplication is done entrywise. The reader is referred to [1, 3] for further information on association schemes.

Two association schemes $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ and $\mathfrak{X}' = (X', \{R'_i\}_{0 \leq i \leq d})$ are *isomorphic* if there exists a bijection φ from X to X' such that for each $i \in \{0, \dots, d\}$ there exists $j \in \{0, \dots, d\}$ satisfying $\{(\varphi(x), \varphi(y)) : (x, y) \in R_i\} = R'_j$; the mapping φ is called an *isomorphism* from \mathfrak{X} to \mathfrak{X}' .

The idea of *P-polynomial* and *Q-polynomial* schemes was introduced by Delsarte in [9], who observed a formal duality between the two notions. Delsarte also noted that \mathfrak{X} is *P-polynomial* if and only if, after a proper re-ordering of the relations, (X, R_1) is a distance-regular graph [9, Theorems 5.6 and 5.16]. On the other hand, *Q-polynomial* schemes which are neither *P-polynomial* nor duals of *P-polynomial* schemes seem to be quite rare. In [7] van Dam, Martin and Muzychuk constructed an infinite family of such schemes from hemisystems of the unitary polar space $H(3, q^2)$ provided in [16]. In 2011, Penttila and Williford [14] constructed another infinite family of *Q-polynomial* 3-class association schemes, not *P-polynomial* nor the dual of a *P-polynomial*, by considering a relative hemisystem of $H(3, q^2)$, q even, with respect to a symplectic polar space $W(3, q)$ embedded in it. These schemes differ from all those previously known, they being primitive. The known examples of *Q-polynomial* schemes which are not *P-polynomial* are listed in [13, 16].

We underline that the Penttila-Williford 3-class schemes are obtained by applying [14, Theorem 2] which provides primitive *Q-polynomial* subschemes of *Q-polynomial* *Q*-bipartite schemes defined on certain generalized quadrangles. This result can be

viewed as a reversal of the so-called “extended Q -bipartite double” construction given in [13]. On the other hand, looking at the Krein array of the generic Penttilla-Williford scheme, we may note that it comes from a strongly regular graph after splitting one of its relations in two.

In a private communication to the authors of [14], H. Tanaka pointed out that their 3-class schemes have the same parameters as the 3-class schemes provided by Hollmann and Xiang in [11]. The latter, which were previously not noticed to be Q -polynomial, are obtained as fusion of association schemes constructed from the action of the projective group $\text{PGL}(2, q^2)$, q even, on a non-degenerate conic in the Desarguesian projective plane $\text{PG}(2, q^2)$ extended in $\text{PG}(2, q^4)$.

Therefore the question arises whether there exists an isomorphism that takes the Penttilla-Williford association schemes to the Hollmann-Xiang fusion schemes. In this paper, we provide the answer by proving the following result:

Main Theorem. *The Penttilla-Williford 3-class association schemes and the Hollmann-Xiang fusion association schemes are isomorphic.*

The proof essentially uses geometric arguments. We start off with an explicit description of the Penttilla-Williford relative hemisystems in terms of coordinates in the projective space $\text{PG}(3, q^2)$. Via the Klein correspondence from the lines of $\text{PG}(3, q^2)$ to the points of the Klein quadric of $\text{PG}(5, q^2)$, we obtain a geometric representation of the Penttilla-Williford association schemes in the orthogonal polar space $Q^-(5, q)$ whose points are the image of the lines in $H(3, q^2)$ [10]. Thanks to this representation we are able to find a desired isomorphism.

In [11] it was pointed out that a further fusion scheme of the 3-class Hollmann-Xiang scheme produces a strongly regular graph with parameters $v = q^2(q^2 - 1)/2$, $k = (q^2 + 1)(q - 1)$, $\lambda = q^2 + q - 2$, $\nu = 2(q^2 - q)$. These graphs have the same parameters of the ones found by R. Metz [8], which can be also constructed from a fusion of the Penttilla-Williford schemes; see also [4, p.189]. These graphs are denoted by $NO^-(5, q)$ in Brouwer’s table of strongly regular graphs [2].

The paper [11] announced an alleged isomorphism between the above graphs in a forthcoming paper. To the best of our knowledge, such a paper appears to have never been published. Anyway, the Main Theorem confirms the conjectured isomorphism.

The paper is structured as follows: in Section 2 we recall the construction of the Hollmann-Xiang and Penttilla-Williford association schemes. In Sections 3 we give a coordinatization of the relative hemisystems of Penttilla and Williford together with their representation in $Q^-(5, q)$. Finally, Section 4 contains the proof of the Main Theorem.

2 Preliminaries

For any given n -dimensional vector space $V = V(n, F)$ over a field F , the *projective geometry defined by V* is the partially ordered set of all subspaces of V , and it will be denoted by $\text{PG}(V)$. If F is the finite field \mathbb{F}_q with q elements, then we may write $V = V(n, q)$ and $\text{PG}(n-1, q)$ instead of $\text{PG}(V)$. The 1-dimensional subspaces are called *points*, the 2-dimensional subspaces are called *lines*, and the $(n-1)$ -dimensional subspaces are called *hyperplanes* of $\text{PG}(V)$. For a nonzero $v \in V$, $\langle v \rangle$ will denote the point of $\text{PG}(V)$ spanned by v . In order to simplify notation, for each subspace U of V , that is an element of $\text{PG}(V)$, we will use the same letter for the projective geometry defined by U . If V is endowed with a non-degenerate alternating, quadratic or hermitian form of Witt index m , the set \mathcal{P} of totally isotropic (or singular, in the case of quadratic form) subspaces of V is a *polar space of rank m* of $\text{PG}(V)$, which is called *symplectic*, *orthogonal* or *unitary*, respectively. Our principal reference on projective geometries and polar spaces is [15].

2.1 The Hollmann-Xiang association schemes

A *non-degenerate conic* \mathcal{C} of $\text{PG}(2, q^2)$ is an orthogonal polar space (of rank 1) arising from a non-degenerate quadratic form Q on $V(3, q^2)$. A line ℓ of $\text{PG}(2, q^2)$ is called a *passant*, *tangent* or *secant* of \mathcal{C} according as $|\ell \cap \mathcal{C}| = 0, 1$ or 2 .

Embed $\text{PG}(2, q^2)$ in $\text{PG}(2, q^4)$. Concretely this can be done by extending the scalars in $V(3, q^2)$. It follows that each point of $\text{PG}(2, q^2)$ extends to a point of $\text{PG}(2, q^4)$. Similarly, each line ℓ of $\text{PG}(2, q^2)$ extends to a line $\bar{\ell}$ of $\text{PG}(2, q^4)$. The extension \bar{Q} of Q in $V(3, q^4)$ is a non-degenerate quadratic form, and it defines a (non-degenerate) conic $\bar{\mathcal{C}}$ in $\text{PG}(2, q^4)$. While the extension $\bar{\ell}$ of a tangent (or secant) line ℓ of \mathcal{C} is a tangent (or secant) of $\bar{\mathcal{C}}$, the extension of a passant line of \mathcal{C} is a secant of $\bar{\mathcal{C}}$. Such a line is called an *elliptic* line of $\bar{\mathcal{C}}$, and we will denote by \mathcal{E} the set of these lines. Note that \mathcal{E} has size $(q^4 - q^2)/2$.

Since all non-degenerate quadratic forms on $V(3, q^2)$ are equivalent, we may assume

$$\begin{aligned} \bar{Q} : V(3, q^4) &\rightarrow \mathbb{F}_{q^4} \\ (x, y, z) &\mapsto y^2 - xz. \end{aligned}$$

Therefore,

$$\bar{\mathcal{C}} = \{ \langle (1, t, t^2) \rangle : t \in \mathbb{F}_{q^4} \} \cup \{ \langle (0, 0, 1) \rangle \}$$

and

$$\mathcal{C} = \{ \langle (1, t, t^2) \rangle : t \in \mathbb{F}_{q^2} \} \cup \{ \langle (0, 0, 1) \rangle \}.$$

Therefore, for every elliptic line $\bar{\ell}$ of $\bar{\mathcal{C}}$ we have $\bar{\ell} \cap \bar{\mathcal{C}} = \{\langle(1, t, t^2)\rangle, \langle(1, t^{q^2}, t^{2q^2})\rangle\}$, for some $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$. The reader is referred to [11] for more details.

Under the identification of $\mathbb{F}_{q^4} \cup \{\infty\}$ with $\bar{\mathcal{C}}$ given by

$$\xi : t \leftrightarrow \langle(1, t, t^2)\rangle, \quad \infty \leftrightarrow \langle(0, 0, 1)\rangle, \quad (1)$$

the pair $\mathbf{t} = \{t, t^{q^2}\}$, with $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, may be associated with the elliptic line intersecting $\bar{\mathcal{C}}$ at $\{\langle(1, t, t^2)\rangle, \langle(1, t^{q^2}, t^{2q^2})\rangle\}$. We will use $\bar{\ell}_{\mathbf{t}}$ to denote this line.

We assume q is even. For any given pair of distinct elliptic lines $\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}$, let

$$\widehat{\rho}(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) = \widehat{\rho}(\mathbf{s}, \mathbf{t}) = \frac{1}{\rho(s, t) + \rho(s, t)^{-1}}, \quad (2)$$

where

$$\rho(s, t) = \frac{(s+t)(s^{q^2} + t^{q^2})}{(s+t^{q^2})(s^{q^2} + t)}. \quad (3)$$

It is evident that $\text{Im } \widehat{\rho}$ is a subset of \mathbb{F}_{q^2} . The following result is straightforward.

Lemma 2.1. [11, Lemma 5.1]

$$\widehat{\rho}(\mathbf{s}, \mathbf{t}) = \frac{(s+t)(s^{q^2} + t^{q^2})(s+t^{q^2})(s^{q^2} + t)}{(s+s^{q^2})^2(t+t^{q^2})^2} = \left(\frac{1}{\rho(s, t) + 1}\right)^2 + \left(\frac{1}{\rho(s, t) + 1}\right).$$

Set $q = 2^h$. For $r \in \{1, 2\}$, let $\mathbf{T}_0(q^r)$ be the set of elements of \mathbb{F}_{q^r} with absolute trace zero:

$$\mathbf{T}_0(q^r) = \left\{ x \in \mathbb{F}_{q^r} : \sum_{i=0}^{rh-1} x^{2^i} = 0 \right\}.$$

In [11] Hollmann and Xiang consider the following sets to construct a 3-class association scheme:

$$\mathbf{T}_0 = \mathbf{T}_0(q^2), \quad \mathbf{S}_0^* = \mathbf{T}_0(q) \setminus \{0\}, \quad \mathbf{S}_1 = \mathbb{F}_q \setminus \mathbf{S}_0.$$

Note that $\mathbf{T}_0 = \{\alpha + \alpha^2 : \alpha \in \mathbb{F}_{q^2}\}$ as q is even. By Lemma 2.1, $\text{Im } \widehat{\rho}$ is contained in \mathbf{T}_0 .

Theorem 2.2. [11] *On the set of the elliptic lines \mathcal{E} define the following relations:*

$$R_1: (\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in R_1 \text{ if and only } \widehat{\rho}(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in \mathbf{S}_0^*;$$

$$R_2: (\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in R_2 \text{ if and only } \widehat{\rho}(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in \mathbf{S}_1;$$

R_3 : $(\bar{\ell}_s, \bar{\ell}_t) \in R_3$ if and only $\widehat{\rho}(\bar{\ell}_s, \bar{\ell}_t) \in \mathbf{T}_0 \setminus \mathbb{F}_q$.

Then the pair $(\mathcal{E}, \{R_i\}_{i=0}^3)$, where R_0 is the identity relation, is a 3-class association scheme.

The first eigen-matrix of the scheme is

$$P = \begin{pmatrix} 1 & (q-2)(q^2+1)/2 & q(q^2+1)/2 & q(q-2)(q^2+1)/2 \\ 1 & -(q-1)(q-2)/2 & -q(q-1)/2 & q(q-2) \\ 1 & -(q^2-q+2)/2 & q(q+1)/2 & -q \\ 1 & q-1 & 0 & -q \end{pmatrix}; \quad (4)$$

see [11, Section 7].

Remark 2.3. By identification (1), the set \mathcal{E} may be replaced by the set $\mathcal{X} = \{\mathbf{t} = \{t, t^{q^2}\} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ and the relations R_i , $i = 1, 2, 3$, replaced by

R'_1 : $(\mathbf{s}, \mathbf{t}) \in R'_1$ if and only $\widehat{\rho}(\mathbf{s}, \mathbf{t}) \in \mathbf{S}_0^*$;

R'_2 : $(\mathbf{s}, \mathbf{t}) \in R'_2$ if and only $\widehat{\rho}(\mathbf{s}, \mathbf{t}) \in \mathbf{S}_1$;

R'_3 : $(\mathbf{s}, \mathbf{t}) \in R'_3$ if and only $\widehat{\rho}(\mathbf{s}, \mathbf{t}) \in \mathbf{T}_0 \setminus \mathbb{F}_q$;

here $\widehat{\rho}(\mathbf{s}, \mathbf{t})$ is the quantity defined in (2). Hence, $(\mathcal{X}, \{R'_i\}_{i=0}^3)$ is an association scheme isomorphic to $(\mathcal{E}, \{R_i\}_{i=0}^3)$.

Remark 2.4. Actually, the scheme $(\mathcal{X}, \{R'_i\}_{i=0}^3)$ arises as a fusion of the one given by the following result [11].

Theorem. *Under the identification ξ , the action of $\mathrm{PGL}(2, q^2)$ on $\mathcal{E} \times \mathcal{E}$ gives rise to an association scheme on \mathcal{X} with $q^2/2 - 1$ classes $R_{\{\lambda, \lambda^{-1}\}}$, $\lambda \in \mathbb{F}_{q^2} \setminus \{0, 1\}$, where $(\mathbf{s}, \mathbf{t}) \in R_{\{\lambda, \lambda^{-1}\}}$ if and only if $\{\rho(s, t), \rho(s, t)^{-1}\} = \{\lambda, \lambda^{-1}\}$.*

2.2 The Penttila-Williford association schemes

Up to isometries, the vector space $V(4, q^2)$ has precisely one non-degenerate hermitian form, and its Witt index is 2. As usual, $H(3, q^2)$ denotes the unitary polar space of rank 2 defined by it. A *point* (resp. *line*) of $H(3, q^2)$ is a 1-dimensional (resp. 2-dimensional) subspace in $H(3, q^2)$.

Assume q even for the rest of the current section. In $V(4, q^2)$ there is a 4-dimensional \mathbb{F}_q -vector space \widehat{V} such that the restriction of the hermitian form on

it induces a non-degenerate alternating form \widehat{b} which defines a symplectic polar space $W(3, q)$ of rank 2 of $\text{PG}(\widehat{V})$ [5]. In addition, \widehat{b} is the polar of a non-degenerate quadratic form \widehat{Q} of Witt index 1, whose set of singular point is denoted by $Q^-(3, q)$. By $\widehat{\mathcal{W}}$ (resp. $\widehat{\mathcal{Q}}$) we denote the set of all the totally isotropic (resp. singular) subspaces of $W(3, q)$ (resp. $Q^-(3, q)$) extended over \mathbb{F}_{q^2} . As a consequence, for every point of $H(3, q^2)$ not in $\widehat{\mathcal{W}}$ there are exactly q lines of $H(3, q^2)$ disjoint from $\widehat{\mathcal{W}}$ and one in $\widehat{\mathcal{W}}$. Note that $\widehat{\mathcal{W}}$ is an embedding of $W(3, q)$ in $H(3, q^2)$.

The following definition was introduced in [14]. A *relative hemisystem of $H(3, q^2)$ with respect to $W(3, q)$* is a set \mathcal{H} of lines of $H(3, q^2)$ disjoint from $\widehat{\mathcal{W}}$ such that every point of $H(3, q^2)$ not in $\widehat{\mathcal{W}}$ lies on exactly $q/2$ lines of \mathcal{H} . For any given line l of $H(3, q^2)$ disjoint from $\widehat{\mathcal{W}}$, let \mathcal{S}_l denote the set of lines of $H(3, q^2)$ which meet both l and $\widehat{\mathcal{W}}$. We stress the fact that \mathcal{S}_l consists of the lines of $\widehat{\mathcal{W}}$ that extend elements of a regular spread of $W(3, q)$ ¹, and refer to \mathcal{S}_l as the *spread subtended by l* .

Theorem 2.5. [14, Theorem 4] *Let \mathcal{H} be a relative hemisystem of $H(3, q^2)$ with respect to $W(3, q)$. Then a primitive Q -polynomial 3-class association scheme can be constructed on \mathcal{H} by the defining the following relations:*

$$\widetilde{R}_1: (l, m) \in \widetilde{R}_1 \text{ if and only } |l \cap m| = 1;$$

$$\widetilde{R}_2: (l, m) \in \widetilde{R}_2 \text{ if and only } l \cap m = \emptyset \text{ and } |\mathcal{S}_l \cap \mathcal{S}_m| = 1;$$

$$\widetilde{R}_3: (l, m) \in \widetilde{R}_3 \text{ if and only } l \cap m = \emptyset \text{ are } |\mathcal{S}_l \cap \mathcal{S}_m| = q + 1.$$

Let $\text{PO}^-(\widehat{V})$ be the stabilizer of \widehat{Q} in the projective unitary group $\text{PGU}(4, q^2)$. By looking at the action of the commutator subgroup $\text{P}\Omega^-(\widehat{V})$ of $\text{PO}^-(\widehat{V})$ on the lines of $H(3, q^2)$, the following result was proved in [14].

Theorem 2.6. $\text{P}\Omega^-(\widehat{V})$ *has two orbits on the lines of $H(3, q^2)$ disjoint from $\widehat{\mathcal{W}}$, and each orbit is a relative hemisystem with respect to $W(3, q)$.*

We consider an association scheme $(\mathcal{H}, \{\widetilde{R}_i\}_{i=0}^3)$ as in Theorem 2.5 by using the hemisystems from Theorem 2.6. As expected, the first eigen-matrix of the scheme is precisely the matrix in (4).

¹A spread of $W(3, q)$ in $\text{PG}(3, q)$ is a set \mathcal{S} of totally isotropic lines which partition the pointset of $\text{PG}(3, q)$. \mathcal{S} is *regular* if for any three distinct lines of \mathcal{S} there is a set R of $q + 1$ lines of \mathcal{S} containing them, with the following property: any line of $\text{PG}(3, q)$ intersecting three lines in R meets all the lines of R .

3 The explicit construction of the relative hemisystem of Penttila-Williford

Let G and H be groups acting on the sets Ω and Δ , respectively. The two actions are said to be *permutationally isomorphic* if there exist a bijection $\theta : \Omega \rightarrow \Delta$ and an isomorphism $\chi : G \rightarrow H$ such that the following diagram commutes:

$$\begin{array}{ccc}
 G \times \Omega & \xrightarrow{\phi} & \Omega \\
 \chi \downarrow & \theta \downarrow & \downarrow \theta \\
 H \times \Delta & \xrightarrow{\tilde{\phi}} & \Delta
 \end{array}$$

Here ϕ and $\tilde{\phi}$ are the maps defining the action of G and H on Ω and Δ , respectively.

Let $Q^-(3, q)$ be the orthogonal polar space (of rank 1) defined by \widehat{Q} on the 4-dimensional \mathbb{F}_q -vector space \widehat{V} introduced in Section 2.2.

It is known that $(\text{PSL}(2, q^2), \text{PG}(1, q^2))$ and $(\text{P}\Omega^-(\widehat{V}), Q^-(3, q))$ are permutationally isomorphic for all prime powers q . For sake of completeness, we give an explicit description of the above isomorphism which is more suitable for our computation.

In $V(4, q^2) = \{(X_1, X_2, X_3, X_4) : X_i \in \mathbb{F}_{q^2}\}$, let \widehat{V} be the set of all vectors $v = (\alpha, x^q, x, \beta)$ with $\alpha, \beta \in \mathbb{F}_q$, $x \in \mathbb{F}_{q^2}$. With the usual sum and multiplication by scalars from \mathbb{F}_q , \widehat{V} is a 4-dimensional vector space over \mathbb{F}_q .

As usual we identify $\text{PG}(1, q^2)$ with $\mathbb{F}_{q^2} \cup \{\infty\}$ and we consider the following injective map:

$$\begin{array}{ccc}
 \theta : \mathbb{F}_{q^2} \cup \{\infty\} & \longrightarrow & \text{PG}(\widehat{V}) \\
 t & \mapsto & \langle (1, t^q, t, t^{q+1}) \rangle \\
 \infty & \mapsto & \langle (0, 0, 0, 1) \rangle
 \end{array} \quad . \quad (5)$$

Proposition 3.1. [6] *The image of θ is an orthogonal polar space of rank 1 of $\text{PG}(\widehat{V})$.*

Proof. Let Q be the quadratic form on $V(4, q^2)$ defined by

$$Q(\mathbf{X}) = X_1X_4 - X_2X_3,$$

which has $b(\mathbf{X}, \mathbf{Y}) = X_1Y_4 + X_4Y_1 - X_2Y_3 - X_3Y_2$ as the associated non-degenerate bilinear form. The restriction $\widehat{Q} = Q|_{\widehat{V}}$ is the quadratic form given by

$$\widehat{Q}(v) = \alpha\beta - x^{q+1},$$

which has

$$\widehat{b}(v, v') = \alpha\beta' + \beta\alpha' - xx'^q - x^qx' \quad (6)$$

as the associated bilinear form. Let $v = (\alpha, x^q, x, \beta) \in \text{Rad}(\widehat{V})$, that is $\widehat{b}(v, v') = 0$, for all $v' \in \widehat{V}$. If $\alpha' = \beta' = 0$, a necessary condition for $v \in \text{Rad}(\widehat{V})$ is

$$x^qx' + xx'^q = 0,$$

for all $x' \in \mathbb{F}_{q^2}$. This shows that the polynomial in x' of degree q on the left hand side has at least q^2 roots. Therefore, it must be the zero polynomial, and $x = 0$. We repeat the above argument for $\alpha' = x' = 0$ and for $x' = \beta' = 0$ to show that $v = 0$. This yields that \widehat{b} , and hence \widehat{Q} , is non-degenerate. Let u be a singular vector for \widehat{Q} . Without loss of generality we may take $u = (1, 0, 0, 0)$. Therefore, the subspace $U = \{v \in \widehat{V} : \widehat{b}(v, u) = 0\}$ coincides with $\{(\alpha, x^q, x, 0) : \beta \in \mathbb{F}_q, x \in \mathbb{F}_{q^2}\}$. It is easily seen that $U \cap \ker \widehat{Q} = \{\alpha u : \alpha \in \mathbb{F}_q\}$. Thus, \widehat{Q} is a quadratic form of Witt index 1 giving rise to the orthogonal polar space

$$Q^-(3, q) = \{\langle(1, t^q, t, t^{q+1})\rangle : t \in \mathbb{F}_{q^2}\} \cup \{\langle(0, 0, 0, 1)\rangle\},$$

which is precisely $\text{Im } \theta$. □

Let χ be the monomorphism defined by

$$\chi : \begin{array}{ccc} \text{SL}(2, q^2) & \longrightarrow & \text{SL}(4, q^2) \\ g & \longmapsto & g \otimes g^q \end{array},$$

where \otimes is the Kronecker product and g^q denotes the matrix g with its entries raised to the q -th power. It is straightforward to check that $\chi(g)$ is a \widehat{Q} -isometry, for every $g \in \text{SL}(2, q^2)$. Therefore, χ can be regarded as a monomorphism from $\text{PSL}(2, q^2)$ to $\text{PO}^-(\widehat{V})$. It is actually an isomorphism from $\text{PSL}(2, q^2)$ to $\text{P}\Omega^-(\widehat{V})$, as it will be shown below.

Let t_a be the transvection in $\text{SL}(2, q^2)$ with matrix $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, for some $a \in \mathbb{F}_{q^2}^*$. The isometry $\chi(t_a)$ maps (α, x^q, x, β) to $(\alpha, a^q\alpha + x^q, a\alpha + x, a^{q+1}\alpha + ax^q + a^qx + \beta)$. Its restriction on the hyperplane of all \widehat{b} -orthogonal vectors to $u = (0, 0, 0, 1)$ is the map

$$\eta_{u,y}(w) = w + \widehat{b}(w, y)u,$$

where $y = (0, -a^q, -a, 0)$. This yields that $\chi(t_a)$ is actually the unique Siegel transformation $\rho_{u,y}$ which extends $\eta_{u,y}$ [15, Theorem 11.18]. By using [15, Theorem 11.19 (ii)] it is possible to show that as a varies in $\mathbb{F}_{q^2}^*$, $\rho_{u,y}$ describes all the Siegel transformations centered at u .

Every transvection g is conjugate in $\mathrm{SL}(2, q^2)$ to a transvection of type t_a . This implies that $\chi(g)$ is also a Siegel transformation [15, Theorem 11.19 (iii)]. Therefore, χ gives rise to a bijection from the set of all transvections in $\mathrm{SL}(2, q^2)$ to all Siegel transformations of \widehat{V} . Since transvections generate $\mathrm{SL}(2, q^2)$, and Siegel transformations generate $\Omega^-(\widehat{V})$, we achieve $\chi(\mathrm{PSL}(2, q^2)) \leq \mathrm{P}\Omega^-(\widehat{V})$. As $|\mathrm{PSL}(2, q^2)| = |\mathrm{P}\Omega^-(\widehat{V})|$, χ is actually the desired isomorphism. It is a matter of fact that the diagram

$$\begin{array}{ccc}
\mathrm{PSL}(2, q^2) \times \mathbb{F}_{q^2} \cup \{\infty\} & \xrightarrow{\phi} & \mathbb{F}_{q^2} \cup \{\infty\} \\
\chi \downarrow & & \theta \downarrow \\
\mathrm{P}\Omega^-(\widehat{V}) \times Q^-(3, q) & \xrightarrow{\tilde{\phi}} & Q^-(3, q)
\end{array}$$

commutes.

For the rest of this section, assume q is even. The bilinear form \widehat{b} defined by (6) is a (non-degenerate) alternating form on \widehat{V} . Let h be the non-degenerate hermitian form on $V(4, q^2)$ given by

$$h(\mathbf{X}, \mathbf{Y}) = X_1 Y_4^q + X_2 Y_2^q + X_3 Y_3^q + X_4 Y_1^q,$$

with associated unitary polar space $H(3, q^2)$. It is evident that $h|_{\widehat{V}} = \widehat{b}$. Therefore, the symplectic polar space $W(3, q)$ defined by \widehat{b} , as well as the orthogonal polar space $Q^-(3, q)$, can be embedded in $H(3, q^2)$ by extending the scalars, so getting \widehat{W} and \widehat{Q} introduced in Section 2.2. This also implies that $\mathrm{P}\Omega^-(\widehat{V})$ is a subgroup of the projective symplectic group $\mathrm{PSp}(\widehat{V})$ which is in turn a subgroup of the projective unitary group $\mathrm{PGU}(4, q^2)$.

The semilinear involutorial transformation τ of $V(4, q^2)$ given by

$$\begin{array}{ccc}
\tau : & V(4, q^2) & \longrightarrow & V(4, q^2) \\
& (X_1, X_2, X_3, X_4) & \longmapsto & (X_1^q, X_3^q, X_2^q, X_4^q).
\end{array}$$

fixes $H(3, q^2)$ and acts as the identity on \widehat{W} .

If we embed $V(4, q^2)$ in $V(4, q^4)$ by extending the scalars, then $\mathrm{PG}(3, q^4)$ embeds $\mathrm{PG}(\widehat{V})$. Therefore, θ defined by (5) can be naturally thought as the restriction of the following map:

$$\begin{array}{ccc}
\theta : \mathbb{F}_{q^4} \cup \{\infty\} & \longrightarrow & \mathrm{PG}(3, q^4) \\
t & \longmapsto & \langle (1, t^q, t, t^{q+1}) \rangle \\
\infty & \longmapsto & \langle (0, 0, 0, 1) \rangle.
\end{array}$$

Note that, for any $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, $\theta(t)$ is not the span of a vector of $V(4, q^2)$. Moreover,

$$\theta(t^{q^2}) = \langle (1, t^{q^3}, t^{q^2}, t^{q^3+q^2}) \rangle = \theta(t)^{\tau^2} \neq \theta(t).$$

For each $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, we associate the pair $\mathbf{t} = \{t, t^{q^2}\}$ with the line $M_{\mathbf{t}}$ of $\text{PG}(3, q^4)$ spanned by $\theta(t)$ and $\theta(t^{q^2})$, which is distinct from $M_{\mathbf{t}^\tau}$.

Lemma 3.2. *For each pair \mathbf{t} , $M_{\mathbf{t}} \cap V(4, q^2)$ is a line of $H(3, q^2)$, say $m_{\mathbf{t}}$, which is disjoint from $\widehat{\mathcal{W}}$.*

Proof. A straightforward computation shows that the vectors in $M_{\mathbf{t}} \cap V(4, q^2)$ are precisely $\mathbf{X}_\lambda = (\lambda + \lambda^{q^2}, \lambda t^q + \lambda^{q^2} t^{q^3}, \lambda t + \lambda^{q^2} t^{q^2}, \lambda t^{q+1} + \lambda^{q^2} t^{q^3+q^2})$, for all $\lambda \in \mathbb{F}_{q^4}$, and they form a line $m_{\mathbf{t}}$ of $\text{PG}(3, q^2)$. Since $h(\mathbf{X}_\lambda, \mathbf{X}_\lambda) = 0$ for all $\lambda \in \mathbb{F}_{q^4}$, $m_{\mathbf{t}}$ is a line of $H(3, q^2)$. Finally, in order to prove that $m_{\mathbf{t}}$ is disjoint from $\widehat{\mathcal{W}}$, consider the following system:

$$\begin{aligned} a\alpha &= \lambda + \lambda^{q^2} \\ ax^q &= \lambda t^q + \lambda^{q^2} t^{q^3} \\ ax &= \lambda t + \lambda^{q^2} t^{q^2} \\ a\beta &= \lambda t^{q+1} + \lambda^{q^2} t^{q^3+q^2}, \end{aligned} \tag{7}$$

with $\alpha, \beta \in \mathbb{F}_q$, $x, a \in \mathbb{F}_{q^2}$, $\lambda \in \mathbb{F}_{q^4}$, $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$. The existence of a solution for (7), or rather the existence of $a \in \mathbb{F}_{q^2}$, makes the system inconsistent. This concludes the proof. \square

Proposition 3.3. *The sets $\{m_{\mathbf{t}} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ and $\{m_{\mathbf{t}^\tau} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ are precisely the two orbits of $\text{P}\Omega^-(\widehat{V})$ on the lines of $H(3, q^2)$ disjoint from $\widehat{\mathcal{W}}$.*

Proof. From the proof of [14, Theorem 5], $\text{P}\Omega^-(\widehat{V})$ has two orbits on the lines of $H(3, q^2)$ disjoint from $\widehat{\mathcal{W}}$, and these two orbits are interchanged by τ . We recall that $m_{\mathbf{t}}$ is uniquely defined by the line $M_{\mathbf{t}}$ of $\text{PG}(3, q^4)$, which is spanned by $\theta(t)$ and $\theta(t^{q^2})$. Hence, it suffices to prove that $\{M_{\mathbf{t}} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ is an orbit of $\text{P}\Omega^-(\widehat{V})$.

Let $\omega \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ such that $\omega^{q^2} = \omega + 1$ and $\omega^2 + \omega = \delta$, with $\delta \in \mathbb{F}_{q^2} \setminus \mathbf{T}_0$, $\delta \neq 1$. For all $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, write $t = x + y\omega$, with $x, y \in \mathbb{F}_{q^2}$, $y \neq 0$.

As a group acting on the projective line $\text{PG}(1, q^2)$ assimilated to the set $\mathbb{F}_{q^2} \cup \{\infty\}$, $\text{PSL}(2, q^2)$ may be identified with the group of linear fractional transformations

$$z \mapsto \frac{az + b}{cz + d},$$

where $ad - bc$ is a non-zero square in \mathbb{F}_{q^2} [15]. For any given $t = x + y\omega \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, let $g \in \text{PSL}(2, q^2)$ with matrix $[1, 0; x, y]$. Then, $\chi(g) = g \otimes g^q$ maps $(\theta(\omega), \theta(\omega^{q^2}))$ to $(\theta(t), \theta(t^{q^2}))$ by taking into account $\omega^{q^2} = \omega + 1$. This implies that $\chi(g) \in \text{P}\Omega(\widehat{V})$ maps the line $M_{\{\omega, \omega^{q^2}\}}$ to $M_{\mathbf{t}}$. \square

Corollary 3.4. *The sets $\{m_{\mathbf{t}} : \mathbf{t} \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ and $\{m_{\mathbf{t}}^r : \mathbf{t} \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ are the Penttila-Williford relative hemisystems.*

4 The proof of the Main Theorem

Define $T = \{\{t, t^{q^2}\} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$, and put $\mathcal{X} = T$. By Remark 2.3, we need to find a bijection between the set \mathcal{X} and the relative hemisystem $\mathcal{H} = \{m_{\mathbf{t}} : \mathbf{t} \in T\}$ preserving the relations defined on them.

From the arguments in Section 3, we may associate the pair $\mathbf{t} = \{t, t^{q^2}\} \in \mathcal{X}$ with the line $m_{\mathbf{t}} \in \mathcal{H}$. Moreover, Corollary 3.4 gives $|\mathcal{H}| = (q^4 - q^2)/2 = |\mathcal{X}|$, and this contributes to make the mapping $\varphi : \mathcal{X} \rightarrow \mathcal{H}, \mathbf{t} \mapsto m_{\mathbf{t}}$ a bijection. In order to show that φ , in fact, preserves the relations, we will move into a different geometric setting. More precisely, we will use the following dual representation of $H(3, q^2)$. Via the Klein correspondence κ , the lines of $\text{PG}(3, q^2)$ are mapped to the points of an orthogonal polar space $Q^+(5, q^2)$ of rank 3 of $\text{PG}(5, q^2)$, which is the so-called the *Klein quadric*. In particular, the lines of $H(3, q^2)$ are mapped to the points of an orthogonal polar space $Q^-(5, q)$ of rank 2 in a $\text{PG}(5, q)$ embedded in $\text{PG}(5, q^2)$. When q is even, κ maps the lines of any symplectic polar space of rank 2 embedded in $H(3, q^2)$ to the points of an orthogonal polar space of rank 2, which is the intersection of $Q^-(5, q)$ with a hyperplane of $\text{PG}(5, q)$. The reader is referred to [10] for more details on the Klein correspondence.

Assume q even. In $V(6, q^2)$ consider the 6-dimensional \mathbb{F}_q -subspace $\widetilde{V} = \{(x, x^q, y, y^q, z, z^q) : x, y, z \in \mathbb{F}_{q^2}\}$. Let $\text{PG}(\widetilde{V})$ be the projective geometry defined by \widetilde{V} .

We consider the Klein quadric $Q^+(5, q^2)$ defined by the (non-degenerate) quadratic form $Q(\mathbf{X}) = X_1X_6 + X_2X_5 + X_3X_4$ on $V(6, q^2)$. For any given $w = (x, x^q, y, y^q, z, z^q) \in \widetilde{V}$,

$$\widetilde{Q}(w) = Q|_{\widetilde{V}}(w) = xz^q + x^qz + y^{q+1}.$$

From [12, Proposition 2.4], \widetilde{Q} is a non-degenerate quadratic form of Witt index 2 on \widetilde{V} with associated alternating form

$$\widetilde{b}(w, w') = xz'^q + x^qz' + yy'^q + y^qy' + zx'^q + z^qx'.$$

Therefore, \tilde{Q} gives rise to an orthogonal polar space $Q^-(5, q)$ of $\text{PG}(\tilde{V})$ embedded in $Q^+(5, q^2)$.

For any subspace X of \tilde{V} , set

$$X^\perp = \{w \in \tilde{V} : \tilde{b}(w, u) = 0, \text{ for all } u \in X\}.$$

Let $Q(4, q)$ be the polar space whose points are the κ -image of the lines of $\widehat{\mathcal{W}}$, and Γ be the hyperplane of $\text{PG}(\tilde{V})$ containing $Q(4, q)$. For a complete description of Γ observe that the pairs

$$\{(1, 0, 0, 0), (0, x, x^q, 0)\}, \quad \{(0, 0, 0, 1), (0, x, x^q, 0)\}, \quad \{(1, 1, 1, 1), (x + x^q, x, x^q, 0)\},$$

with $x \in \mathbb{F}_{q^2}$, span lines of $\widehat{\mathcal{W}}$ which give three skew lines of $Q(4, q)$ under κ generating Γ . It follows that $\Gamma = \{(x, x^q, c, c, z, z^q) : x, z \in \mathbb{F}_{q^2}, c \in \mathbb{F}_q\}$.

Under κ , the line $m_{\mathbf{t}}$ of \mathcal{H} is mapped to the point $P_{\mathbf{t}} = \langle w_{\mathbf{t}} \rangle$ of $\text{PG}(\tilde{V})$, where

$$w_{\mathbf{t}} = (t^q + t^{q^3}, t + t^{q^2}, t^{1+q} + t^{q^2+q^3}, t^{1+q^3} + t^{q+q^2}, t^{1+q+q^3} + t^{q+q^2+q^3}, t^{1+q+q^2} + t^{1+q^2+q^3}).$$

Note that $P_{\mathbf{t}}$ is in $Q^-(5, q)$, but not in $Q(4, q)$. Let $P'_{\mathbf{t}} = \kappa(m_{\mathbf{t}}^\tau)$. Since $m_{\mathbf{t}}$ and $m_{\mathbf{t}}^\tau$ are disjoint lines of $H(3, q^2)$, the line $L_{\mathbf{t}}$ spanned by $P_{\mathbf{t}}$ and $P'_{\mathbf{t}}$ intersects $Q^-(5, q)$ just at $P_{\mathbf{t}}$ and $P'_{\mathbf{t}}$. On the other hand, $m_{\mathbf{t}}$ and $m_{\mathbf{t}}^\tau$ subtend the same spread $\mathcal{S}_{\mathbf{t}} = \mathcal{S}_{m_{\mathbf{t}}}$ in $\widehat{\mathcal{W}}$. The κ -image of $\mathcal{S}_{\mathbf{t}}$ is an orthogonal polar space of rank 1 contained in $Q(4, q)$ [10], and it turns out this is precisely $Q(4, q) \cap L_{\mathbf{t}}^\perp$. Consequently, $L_{\mathbf{t}}^\perp$ is in Γ and $\Gamma^\perp = \langle (0, 0, 1, 1, 0, 0) \rangle = \langle w_0 \rangle$ is a point of $L_{\mathbf{t}}$, for all $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$. The symbol $\tilde{\mathcal{O}}_{\mathbf{t}}$ will be used to indicate $Q(4, q) \cap L_{\mathbf{t}}^\perp$.

For any given distinct pairs \mathbf{s} and \mathbf{t} , let $\Pi_{\mathbf{s}, \mathbf{t}}$ be the plane of $\text{PG}(\tilde{V})$ spanned by Γ^\perp , $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$. The restriction of \tilde{Q} and \tilde{b} on $\Pi_{\mathbf{s}, \mathbf{t}}$ will be denoted by $\tilde{Q}_{\mathbf{s}, \mathbf{t}}$ and $\tilde{b}_{\mathbf{s}, \mathbf{t}}$, respectively. Identifying a triple $(a, b, c) \in \mathbb{F}_q^3$ with the vector $v = aw_{\mathbf{s}} + bw_0 + cw_{\mathbf{t}} \in \Pi_{\mathbf{s}, \mathbf{t}}$, we obtain that the action of $\tilde{b}_{\mathbf{s}, \mathbf{t}}$ induced on \mathbb{F}_q^3 is given by the matrix

$$B = \begin{pmatrix} \tilde{b}(w_{\mathbf{s}}, w_{\mathbf{s}}) & \tilde{b}(w_{\mathbf{s}}, w_0) & \tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) \\ \tilde{b}(w_0, w_{\mathbf{s}}) & \tilde{b}(w_0, w_0) & \tilde{b}(w_0, w_{\mathbf{t}}) \\ \tilde{b}(w_{\mathbf{t}}, w_{\mathbf{s}}) & \tilde{b}(w_{\mathbf{t}}, w_0) & \tilde{b}(w_{\mathbf{t}}, w_{\mathbf{t}}) \end{pmatrix} = \begin{pmatrix} 0 & \text{Tr}(s^{q+1}) & \tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) \\ \text{Tr}(s^{q+1}) & 0 & \text{Tr}(t^{q+1}) \\ \tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) & \text{Tr}(t^{q+1}) & 0 \end{pmatrix};$$

here Tr is the trace map from \mathbb{F}_{q^4} on \mathbb{F}_q . A straightforward calculation shows that $\Pi_{\mathbf{s}, \mathbf{t}}$ is degenerate as $\text{Rad}(\Pi_{\mathbf{s}, \mathbf{t}}) = \langle v_{\mathbf{s}, \mathbf{t}} \rangle$, where

$$v_{\mathbf{s}, \mathbf{t}} = \text{Tr}(t^{q+1})w_{\mathbf{s}} + \tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}})w_0 + \text{Tr}(s^{q+1})w_{\mathbf{t}}.$$

It is easily seen that

$$\begin{aligned}\tilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) &= \tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) \left(\tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) + \text{Tr}(s^{q+1})\text{Tr}(t^{q+1}) \right) \\ &= \tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) \tilde{b}(w_{\mathbf{s}}, w'_{\mathbf{t}}),\end{aligned}\tag{8}$$

where

$$w'_{\mathbf{t}} = (t^q + t^{q^3}, t + t^{q^2}, t^{q+q^2} + t^{1+q^3}, t^{1+q} + t^{q^2+q^3}, t^{1+q+q^3} + t^{q+q^2+q^3}, t^{1+q+q^2} + t^{1+q^2+q^3}).$$

Note that $P'_{\mathbf{t}} = \kappa(m_{\mathbf{t}}) = \langle w'_{\mathbf{t}} \rangle$.

Now two cases are possible according as $v_{\mathbf{s},\mathbf{t}}$ is singular or not.

If $\tilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) = 0$, then $\tilde{Q}_{\mathbf{s},\mathbf{t}}$ is degenerate, and $\mathcal{C}_{\mathbf{s},\mathbf{t}} = \Pi_{\mathbf{s},\mathbf{t}} \cap Q^-(5, q)$ consists of two distinct lines through $\langle v_{\mathbf{s},\mathbf{t}} \rangle$, as $P_{\mathbf{s}}, P'_{\mathbf{s}}, P_{\mathbf{t}}$ and $P'_{\mathbf{t}}$ are distinct points no three of them collinear. This yields that $L_{\mathbf{s}}^{\perp}$ meets $L_{\mathbf{t}}^{\perp}$ in the plane $\Pi_{\mathbf{s},\mathbf{t}}^{\perp}$ of Γ , with $\tilde{Q}|_{\Pi_{\mathbf{s},\mathbf{t}}^{\perp}}$ degenerate, and $\tilde{\mathcal{O}}_{\mathbf{s}} \cap \tilde{\mathcal{O}}_{\mathbf{t}} = \Pi_{\mathbf{s},\mathbf{t}}^{\perp} \cap Q^-(5, q) = \langle v_{\mathbf{s},\mathbf{t}} \rangle$. By taking into account (8), there are two possibilities of obtaining zero for $\tilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}})$: either $\tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) = 0$ or $\tilde{b}(w_{\mathbf{s}}, w'_{\mathbf{t}}) = 0$.

If $\tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) = 0$, then $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are collinear in $Q^-(5, q)$, or equivalently, the lines $m_{\mathbf{s}}$ and $m_{\mathbf{t}}$ are concurrent, that is $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \tilde{R}_1$ (see Theorem 2.5). On the other hand, by taking into account (3),

$$\begin{aligned}0 &= \tilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) \\ &= (s^{q^2} + s)^q (t^{q^2} + t)^q (s + t^{q^2})(s^{q^2} + t) + (s^{q^2} + s)(t^{q^2} + t)(s + t^{q^2})^q (s^{q^2} + t)^q\end{aligned}$$

if and only if

$$\nu = \frac{1}{\rho(s, t) + 1} = \frac{(s + t^{q^2})(s^{q^2} + t)}{(s^{q^2} + s)(t^{q^2} + t)} \in \mathbb{F}_q.$$

When $\nu \in \mathbb{F}_q$, $\hat{\rho}(s, t) \in \mathbf{S}_0^*$ by Lemma 2.1, that is $(\mathbf{s}, \mathbf{t}) \in R'_1$ (see Remark 2.3). On the other hand, if $\hat{\rho}(s, t) = \nu^2 + \nu \in \mathbf{S}_0^*$, then there exists $z \in \mathbb{F}_q$ such that $(z + \nu)^2 + (z + \nu) = 0$, which implies either $z = \nu$ or $z + 1 = \nu$. In both cases $\nu \in \mathbb{F}_q$. Therefore, $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \tilde{R}_1$ if and only if $(\mathbf{s}, \mathbf{t}) \in R'_1$.

If $\tilde{b}(w_{\mathbf{s}}, w'_{\mathbf{t}}) = 0$, then $P_{\mathbf{s}}$ and $P'_{\mathbf{t}}$ are collinear in $Q^-(5, q)$, and this leads to the non-collinearity of $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$. This means that $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \tilde{R}_2$ on one side, and $(\mathbf{s}, \mathbf{t}) \in R'_2$ on the other one. In fact,

$$\begin{aligned}0 &= \tilde{b}(w_{\mathbf{s}}, w'_{\mathbf{t}}) \\ &= (s^{q^2} + s)^q (t^{q^2} + t)^q (s + t^{q^2})(s^{q^2} + t) + (s^{q^2} + s)(t^{q^2} + t)(s + t^{q^2})^q (s^{q^2} + t)^q + \\ &\quad + (s^{q^2} + s)^{q+1} (t^{q^2} + t)^{q+1}\end{aligned}$$

if and only if

$$\nu^q + \nu = \left(\frac{1}{\rho(s, t) + 1} \right)^q + \frac{1}{\rho(s, t) + 1} = 1.$$

When $\nu^q + \nu = 1$, then $\nu \notin \mathbb{F}_q$. This implies that the equation $Z^2 + Z = \widehat{\rho}(s, t)$ has no solutions in \mathbb{F}_q , that is $\widehat{\rho}(s, t) \in \mathbf{S}_1$, i.e. $(\mathbf{s}, \mathbf{t}) \in R'_2$. On the other hand, $\widehat{\rho}(s, t) = \nu^2 + \nu \in \mathbf{S}_1 \subset \mathbb{F}_q$ implies $\nu \notin \mathbb{F}_q$. As $\widehat{\rho}(s, t) \in \mathbb{F}_q$, then $(\nu^q + \nu)^2 + (\nu^q + \nu) = 0$ holds, whence $\nu^q + \nu = 1$.

Finally, if $\widetilde{Q}_{\mathbf{s}, \mathbf{t}}(v_{\mathbf{s}, \mathbf{t}}) \neq 0$, then $\widetilde{Q}_{\mathbf{s}, \mathbf{t}}$ is non-degenerate and $\langle v_{\mathbf{s}, \mathbf{t}} \rangle$ is the nucleus of the (non-degenerate) conic $\mathcal{C}_{\mathbf{s}, \mathbf{t}}$. Therefore, $\widetilde{\mathcal{O}}_{\mathbf{t}}$ and $\widetilde{\mathcal{O}}_{\mathbf{s}}$ meet in $q + 1$ points of $\Pi_{\mathbf{s}, \mathbf{t}}^\perp \cap Q(4, q)$. Then, $\mathcal{S}_{m_{\mathbf{t}}} = \kappa^{-1}(\widetilde{\mathcal{O}}_{\mathbf{t}})$ and $\mathcal{S}_{m_{\mathbf{s}}} = \kappa^{-1}(\widetilde{\mathcal{O}}_{\mathbf{s}})$ meet in exactly $q + 1$ lines in $\widehat{\mathcal{W}}$, that is $(m_{\mathbf{t}}, m_{\mathbf{s}}) \in \widetilde{R}_3$. It is clear that $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \widetilde{R}_3$ if and only if $(\mathbf{s}, \mathbf{t}) \in R'_3$ by exclusion.

Summing up, for each $i = 0, \dots, 3$, we have

$$(\mathbf{s}, \mathbf{t}) \in R'_i \quad \text{if and only if} \quad (m_{\mathbf{s}}, m_{\mathbf{t}}) = \varphi(\mathbf{s}, \mathbf{t}) \in \widetilde{R}_i,$$

i.e. φ induces a bijection between R'_i and \widetilde{R}_i , thus achieving our aim.

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