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On the generation of some Lie-type geometries

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Abstract

Let $X_n(\mathbb{K})$ be a building of Coxeter type $X_n = A_n$ or $X_n = D_n$ defined over a given division ring \mathbb{K} (a field when $X_n = D_n$). For a non-connected set J of nodes of the diagram X_n , let $\Gamma(\mathbb{K}) = \text{Gr}_J(X_n(\mathbb{K}))$ be the J -grassmannian of $X_n(\mathbb{K})$. We prove that $\Gamma(\mathbb{K})$ cannot be generated over any proper sub-division ring \mathbb{K}_0 of \mathbb{K} . As a consequence, the generating rank of $\Gamma(\mathbb{K})$ is infinite when \mathbb{K} is not finitely generated. In particular, if \mathbb{K} is the algebraic closure of a finite field of prime order then the generating rank of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ is infinite, although its embedding rank is either $(n+1)^2 - 1$ or $(n+1)^2$.

Keywords: Coxeter building, A_n , D_n , Grassmann geometry, Subgeometry, Generation
2010 MSC: 51E24, 51M30, 51M35, 14M15

1. Introduction

We presume that the reader has some acquaintance with buildings and is familiar with basics of point-line geometry. In case, we refer to Tits [14] for buildings and Shult [13] for point-line geometries. We only recall the notion of generation in point-line geometries. A *subspace* of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is a subset S of the point-set \mathcal{P} of Γ such that, for every line $\ell \in \mathcal{L}$, if $|S \cap \ell| > 1$ then $\ell \subseteq S$. We say that a subset X of a subspace S *generates* S , in symbols $S = \langle X \rangle$, if S is the minimum subspace of Γ containing X , namely S is the intersection of all subspaces of Γ which contain X . We also recall that the *generating rank* of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is the number $\text{gr}(\Gamma) := \min\{|X| : X \subseteq \mathcal{P}, \langle X \rangle = \mathcal{P}\}$.

1.1. Basic definitions and known results

Let \mathfrak{X} be a class of buildings such that, for every division ring \mathbb{K} , at most one (up to isomorphism) member of \mathfrak{X} is defined over \mathbb{K} . For instance, \mathfrak{X} can be the class of buildings belonging to a given simply laced Coxeter diagram or a given Dynkin diagram, possibly of twisted type. With \mathfrak{X} as above, let $\Delta(\mathbb{K})$ be the member of \mathfrak{X} defined over \mathbb{K} (provided it exists) and, for a nonempty subset J of the type-set of $\Delta(\mathbb{K})$, let $\text{Gr}_J(\Delta(\mathbb{K}))$ be the J -grassmannian of $\Delta(\mathbb{K})$, regarded as a point-line geometry. For a sub-division ring \mathbb{K}_0 of \mathbb{K} , suppose that \mathfrak{X} also contains a member $\Delta(\mathbb{K}_0)$

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defined over \mathbb{K}_0 and $\text{Gr}_J(\Delta(\mathbb{K}))$ contains $\text{Gr}_J(\Delta(\mathbb{K}_0))$ as a subgeometry (as it is always the case for the geometries to be considered in this paper). We say that $\text{Gr}_J(\Delta(\mathbb{K}))$ is *generated over* \mathbb{K}_0 (\mathbb{K}_0 -*generated* for short) if $\text{Gr}_J(\Delta(\mathbb{K}_0))$, as a set of points of the point-line geometry $\text{Gr}_J(\Delta(\mathbb{K}))$, generates $\text{Gr}_J(\Delta(\mathbb{K}))$.

Clearly, if $\text{Gr}_J(\Delta(\mathbb{K}))$ is \mathbb{K}_0 -generated and $\text{Gr}_J(\Delta(\mathbb{K}_0))$ is \mathbb{K}_1 -generated for a division ring $\mathbb{K}_1 < \mathbb{K}_0$, then $\text{Gr}_J(\Delta(\mathbb{K}))$ is \mathbb{K}_1 -generated too. It is also clear that if $\text{Gr}_J(\Delta(\mathbb{K}))$ is \mathbb{K}_0 -generated then the generating rank of $\text{Gr}_J(\Delta(\mathbb{K}))$ cannot be larger than that of $\text{Gr}_J(\Delta(\mathbb{K}_0))$. On the other hand, suppose that every finite set of points of $\text{Gr}_J(\Delta(\mathbb{K}))$ belongs to a subgeometry of $\text{Gr}_J(\Delta(\mathbb{K}))$ isomorphic to $\text{Gr}_J(\Delta(\mathbb{K}_0))$ for a finitely generated sub-division ring \mathbb{K}_0 of \mathbb{K} (as it is often the case). Suppose moreover that \mathbb{K} is not finitely generated as a division-ring and that $\text{Gr}_J(\Delta(\mathbb{K}))$ is not \mathbb{K}_0 -generated, for any $\mathbb{K}_0 < \mathbb{K}$. Then $\text{Gr}_J(\Delta(\mathbb{K}))$ has infinite generating rank, as we prove in Lemma 1.4. In short, obvious links exist between the \mathbb{K}_0 -generation problem and the computation of generating ranks. Less obviously, some relations also seem to exist between \mathbb{K}_0 -generability and the existence of the absolutely universal embedding. For instance, a number of grassmannians $\text{Gr}_J(\Delta(\mathbb{K}))$ for which the existence of the absolutely universal embedding is still an open problem, cannot be generated over any proper sub-division ring \mathbb{K}_0 of \mathbb{K} (see Section 1.3, Remark 1.6).

We shall now briefly summarize what is currently known about \mathbb{K}_0 -generation. For X_n a simply laced Coxeter diagram of rank n or a Dynkin diagram of rank n (but not of twisted type) and a division ring \mathbb{K} (a field if $X_n \neq A_n$), let $X_n(\mathbb{K})$ be the unique building of type X_n defined over \mathbb{K} . In particular, $B_n(\mathbb{K})$ and $C_n(\mathbb{K})$ are the buildings associated to the orthogonal group $O(2n+1, \mathbb{K})$ and the symplectic group $\text{Sp}(2n, \mathbb{K})$ respectively.

Suppose firstly that $\text{Gr}_J(X_n(\mathbb{K}))$ is spanned by $\text{Gr}_J(A)$ for an apartment A of $X_n(\mathbb{K})$ (for short, $\text{Gr}_J(X_n(\mathbb{K}))$ is *spanned by an apartment*). For every sub-division ring \mathbb{K}_0 of \mathbb{K} , the geometry $\text{Gr}_J(A)$ is contained in a subgeometry of $\text{Gr}_J(X_n(\mathbb{K}))$ isomorphic to $\text{Gr}_J(X_n(\mathbb{K}_0))$. Hence $\text{Gr}_J(X_n(\mathbb{K}))$ is \mathbb{K}_0 -generated for any $\mathbb{K}_0 \leq \mathbb{K}$. In particular, $\text{Gr}_J(X_n(\mathbb{K}))$ is generated over the prime subfield of \mathbb{K} .

It is known (Cooperstein and Shult [8], Blok and Brouwer [1]) that the following grassmannians are generated by apartments, where we take the integers $1, 2, \dots, n$ as types as usual but when $X_n = D_n$, according to the notation adopted in Section 2.1, we replace $n-1$ and n with $+$ and $-$: $\text{Gr}_k(A_n(\mathbb{K}))$ for $1 \leq k \leq n$; $\text{Gr}_1(D_n(\mathbb{K}))$ and $\text{Gr}_+(D_n(\mathbb{K}))$ as well as $\text{Gr}_-(D_n(\mathbb{K}))$; $\text{Gr}_1(C_n(\mathbb{K}))$ and $\text{Gr}_n(B_n(\mathbb{K}))$ but with $\text{char}(\mathbb{K}) \neq 2$ in both cases; $\text{Gr}_1(E_6(\mathbb{K}))$, $\text{Gr}_6(E_6(\mathbb{K}))$ and $\text{Gr}_1(E_7(\mathbb{K}))$ (the nodes of the E_7 -diagram being labeled as in [8]). Therefore, all above mentioned grassmannians are generated over the prime subfield of \mathbb{K} . It is easily seen that the same holds for $\text{Gr}_1(B_n(\mathbb{K}))$, even if this geometry is not spanned by any apartment. It is likely that if $\text{char}(\mathbb{K}) \neq 2$ then, for every $i \leq n$, the i -grassmannian $\text{Gr}_i(C_n(\mathbb{K}))$ is generated over the prime subfield of \mathbb{K} , but we are not aware of any explicit proof of this claim.

We now turn to $\text{Gr}_{1,n}(A_n(\mathbb{K}))$. This geometry is interesting in its own right. When \mathbb{K} is a field it is known as the *long root geometry* for $\text{SL}(n+1, \mathbb{K})$. In [2] it is proved that if $n > 2$ then $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ is not \mathbb{K}_0 -generated, for any proper sub-division ring \mathbb{K}_0 of \mathbb{K} (see also [6, Theorem 5.10] for an alternative proof in the special case where $n = 3$ and \mathbb{K} is a field). However, when \mathbb{K} is a field and $\mathbb{K} = \mathbb{K}_0(a_1, \dots, a_t)$ for suitable elements $a_1, \dots, a_t \in \mathbb{K} \setminus \mathbb{K}_0$, then $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ can be generated by adding at most t elements to $\text{Gr}_{1,n}(A_n(\mathbb{K}_0))$ (Blok and Pasini [2]). In particular, when \mathbb{K} is finite, $(n+1)^2$ points are enough to generate $\text{Gr}_{1,n}(A_n(\mathbb{K}_0))$. Indeed, in this case \mathbb{K} is a simple extension of its prime subfield \mathbb{K}_0 and the generating rank of $\text{Gr}_{1,n}(A_n(\mathbb{K}_0))$ is equal to $(n+1)^2 - 1$ (Cooperstein [7]).

Not much is known on $\text{Gr}_k(B_n(\mathbb{K}))$ for $1 < k < n$ and $\text{Gr}_k(D_n(\mathbb{K}))$ for $1 < k \leq n-2$. Probably,

what makes these cases so difficult to investigate is the fact that the special case $\text{Gr}_{1,3}(A_3(\mathbb{K})) \cong \text{Gr}_{+,-}(D_3(\mathbb{K}))$ of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ somehow enters the game in any attempt to compute the generating rank of $\text{Gr}_k(B_n(\mathbb{K}))$ or $\text{Gr}_k(D_n(\mathbb{K}))$ and, as we have seen above, as far as generation is concerned, $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ can behave wildly. Nevertheless, in [6] we have shown that for $\mathbb{K} = \mathbb{F}_4, \mathbb{F}_8$ or \mathbb{F}_9 the grassmannians $\text{Gr}_2(B_n(\mathbb{K}))$ ($n \geq 3$) and $\text{Gr}_2(D_n(\mathbb{K}))$ ($n > 3$) are generated over the corresponding prime subfields \mathbb{F}_2 or \mathbb{F}_3 . The generating ranks of $\text{Gr}_2(B_n(\mathbb{K}_0))$ and $\text{Gr}_2(D_n(\mathbb{K}_0))$, for \mathbb{K}_0 a finite field of prime order, are known to be equal to $\binom{2n+1}{2}$ and $\binom{2n}{2}$ respectively (Cooperstein [7]). Hence $\binom{2n+1}{2}$ and $\binom{2n}{2}$ are the generating ranks of $\text{Gr}_2(B_n(\mathbb{K}))$ and $\text{Gr}_2(D_n(\mathbb{K}))$ respectively, with \mathbb{K} as above.

1.2. Main results

We state in this subsection the main results of this paper. We refer to Section 2 or to [11] and [4] for the notation and further details on the grassmannians we are dealing with. The following, to be proved in Section 3, is our first main result in this paper:

Theorem 1.1. *For a division ring \mathbb{K} , let $\Gamma(\mathbb{K})$ be one of the following: $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ for $n \geq 3$; $\text{Gr}_{+,-}(D_n(\mathbb{K}))$, $n \geq 3$; $\text{Gr}_{1,+,-}(D_n(\mathbb{K}))$ with $n \geq 4$; $\text{Gr}_{1,-}(D_n(\mathbb{K}))$ for $n \geq 4$. Then $\Gamma(\mathbb{K})$ is not \mathbb{K}_0 -generated for any proper sub-division ring \mathbb{K}_0 of \mathbb{K} .*

As said in Section 1.1, the case of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ has been already considered in [2], but the proof we shall give in this paper is different and simpler than that of [2]. Theorem 1.1 also contains a proof of a conjecture presented in [6, Remark 5.11].

Corollary 1.2. *The $(n-1)$ -grassmannian $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ of the top-thin polar space $B_n^+(\mathbb{K}) = \text{Gr}_1(D_n(\mathbb{K}))$ is not \mathbb{K}_0 -generated for any proper subfield \mathbb{K}_0 of \mathbb{K} .*

Seemingly, this corollary is an obvious consequence of Theorem 1.1 and the isomorphism $\text{Gr}_{n-1}(B_n^+(\mathbb{K})) \cong \text{Gr}_{+,-}(D_n(\mathbb{K}))$. However its proof is not so trivial as one might believe; we will give it in Section 3.2. As we shall see in Section 3.3, Theorem 1.1 admits the following far reaching generalization:

Theorem 1.3. *Let $\Gamma(\mathbb{K})$ be either $\text{Gr}_J(A_n(\mathbb{K}))$ or $\text{Gr}_J(D_n(\mathbb{K}))$, with J a non-connected set of nodes of the diagram A_n or D_n respectively. Then $\Gamma(\mathbb{K})$ is not \mathbb{K}_0 -generated, for any proper sub-division ring \mathbb{K}_0 of \mathbb{K} .*

1.3. Applications to generating ranks and embeddings

Given a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, a (full) projective embedding $e : \Gamma \rightarrow \text{PG}(V)$ of Γ (henceforth often called simply an *embedding* of Γ , for short) is an injective map $e : \mathcal{P} \rightarrow \text{PG}(V)$ from the point-set \mathcal{P} of Γ to the set of points of the projective space $\text{PG}(V)$ of a vector space V , such that for every line $\ell \in \mathcal{L}$ of Γ the set $e(\ell) := \{e(p) : p \in \ell\}$ is a projective line of $\text{PG}(V)$ and $e(\mathcal{P})$ spans $\text{PG}(V)$. We put $\dim(e) := \dim(V)$, calling $\dim(e)$ the *dimension* of e . If \mathbb{K} is the underlying division ring of V , we say that e is *defined over \mathbb{K}* , also that e is a \mathbb{K} -*embedding*. If Γ admits a projective embedding we say that Γ is *projectively embeddable* (also *embeddable*, for short). If $e : \Gamma \rightarrow \text{PG}(V)$ and $e' : \Gamma \rightarrow \text{PG}(V')$ are two \mathbb{K} -embeddings of Γ we say that e *dominates* e' if there is a \mathbb{K} -semilinear mapping $\varphi : V \rightarrow V'$ such that $e' = \varphi \cdot e$. If φ is an isomorphism then we say that e and e' are *isomorphic*. Following Tits [14], we say that an embedding e is *dominant* if, modulo isomorphisms, it is not dominated by any embedding other than itself. Every \mathbb{K} -embedding e of Γ admits a *hull* \tilde{e} , uniquely determined up to isomorphism and characterized by the following property: \tilde{e} dominates all \mathbb{K} -embeddings of Γ which dominate e (see Ronan [12]). Accordingly, an

embedding is dominant if and only if it is the hull of at least one embedding; equivalently, if and only if it is its own hull. Finally, an embedding \tilde{e} of Γ is *absolutely universal* (henceforth called just *universal*, for short) if it dominates all embeddings of Γ . In other words, Γ admits the universal embedding if and only if all of its embeddings have the same hull, that common hull being the universal embedding of Γ . Note that this forces all embeddings of Γ to be defined over the same division ring. Note also that the universal embedding, if it exists, is homogeneous, an embedding e of Γ being *homogeneous* if $eg \cong e$ for every automorphism g of Γ . The *embedding rank* $\text{er}(\Gamma)$ of an embeddable geometry Γ is defined as follows:

$$\text{er}(\Gamma) := \sup\{\dim(\varepsilon) : \varepsilon \text{ projective embedding of } \Gamma\}.$$

Obviously, if Γ admits the universal embedding \tilde{e} then $\text{er}(\Gamma) = \dim(\tilde{e})$, but $\text{er}(\Gamma)$ is defined even if no embedding of Γ is universal. If $e : \Gamma \rightarrow \text{PG}(V)$ is an embedding of $\Gamma = (\mathcal{P}, \mathcal{L})$ then stretching a line in Γ through two collinear points $p, q \in \mathcal{P}$ corresponds to forming the span $\langle v, w \rangle \subseteq V$ of any two non-zero vectors $v \in e(p)$ and $w \in e(q)$. If $X \subseteq \mathcal{P}$ generates Γ then $\mathcal{P} = \cup_{n=0}^{\infty} X_n$ where $X_0 := X$ and $X_{n+1} := \cup_{p, q \in X_n} \langle p, q \rangle_{\Gamma}$. Consequently, if we select a non-zero vector $v_p \in e(p)$ for every point $p \in X$ then $\{v_p\}_{p \in X}$ spans V . This makes it clear that $|X| \geq \dim(e)$. Accordingly,

$$\dim(e) \leq \text{gr}(\Gamma). \quad (1)$$

Therefore, if $\text{gr}(\Gamma)$ is finite and $\dim(e) = \text{gr}(\Gamma)$ then e is dominant (hence universal, if Γ admits the universal embedding). In any case, (1) implies the following:

$$\text{er}(\Gamma) \leq \text{gr}(\Gamma). \quad (2)$$

In fact the equality $\text{er}(\Gamma) = \text{gr}(\Gamma)$ holds for many embeddable geometries, but not for all of them. For instance Heiss [9] gives an example where $\text{gr}(\Gamma) = \text{er}(\Gamma) + 1$. The example of [9] looks fairly artificial. A more natural example, where $\text{er}(\Gamma)$ is finite but $\text{gr}(\Gamma)$ is infinite, is given by Theorem 1.5, to be stated below. That theorem will be obtained in Section 4 with the help of the following lemma. In order to properly state it, we recall that a division ring \mathbb{K} is *finitely generated* if it is generated as a division ring by a finite subset $X \subseteq \mathbb{K}$, namely no proper sub-division ring of \mathbb{K} contains X . For instance, an algebraic extension of a finite field of prime order \mathbb{F}_p is finitely generated if and only if it is finite, in which case it is a simple extension of \mathbb{F}_p . On the other hand, no algebraically closed field is finitely generated.

Lemma 1.4. *Let $\Gamma(\mathbb{K})$ be either $\text{Gr}_J(A_n(\mathbb{K}))$ or $\text{Gr}_J(D_n(\mathbb{K}))$ for a set of types J non-connected as a set of nodes of A_n or D_n . Suppose that \mathbb{K} is not finitely generated. Then the generating rank of $\Gamma(\mathbb{K})$ is infinite.*

Lemma 1.4 will be obtained in Section 4 as a consequence of Theorem 1.3. By exploiting it we will obtain the following:

Theorem 1.5. *Let \mathbb{F}_p be a finite field of prime order and $\overline{\mathbb{F}_p}$ its algebraic closure. Then, for $n \geq 3$, the geometry $\text{Gr}_{1,n}(A_n(\overline{\mathbb{F}_p}))$ has infinite generating rank but its embedding rank is equal to either $(n+1)^2 - 1$ or $(n+1)^2$.*

Remark 1.6. It is well known that if \mathbb{K} is a field then $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ admits an $(n+1)^2 - 1$ dimensional embedding, say e_{Lie} , in (the projective space of) the space of square matrices of order $n+1$ with entries in \mathbb{K} and null trace (see e.g. Blok and Pasini [3]; the choice of the symbol e_{Lie} for

this embedding is motivated by the fact that it affords the representation of the group $\mathrm{SL}(n+1, \mathbb{K})$ in its action on its own Lie algebra). However $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ does not satisfy the sufficient conditions of Kasikova and Shult [10] for the existence of the universal embedding. So, we do not know if it always admits the universal embedding, let alone if e_{Lie} is universal. A complete answer is known only when \mathbb{K} is a prime field. In this case e_{Lie} is indeed universal (Blok and Pasini [3, Section 3]). A bit less is known when \mathbb{K} is a number field or a perfect field of positive characteristic; in this case e_{Lie} dominates all homogeneous embeddings of $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ (Völklein [15]). As for the remaining geometries of Theorem 1.1, namely $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$, $\mathrm{Gr}_{1,+,-}(D_n(\mathbb{K}))$ and $\mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$, they too are embeddable (see [3]) and, when \mathbb{K} is a prime field, they admit the universal embedding (Blok and Pasini [3, Section 4]), even if none of them satisfies the conditions of Kasikova and Shult [10].

Remark 1.7. The geometry Δ_2^+ of [5] with $n = 3$ is the same as $\mathrm{Gr}_{1,3}(A_3(\mathbb{F}))$. According to the above, Lemma 4.8 of [5], which deals with that geometry and its Weyl embedding ε_2^+ (which is the same as e_{Lie}), might possibly be wrong as stated. It should be corrected as follows: when $n = 3$ and \mathbb{F} is a perfect field of positive characteristic or a number field, then $\tilde{\varepsilon}_2^+$ dominates all homogeneous embeddings of Δ_2^+ .

Remark 1.8. In our survey of embeddings we have stuck to full projective embeddings, but in the proof of Theorem 1.5 we shall deal also with lax embeddings. *Lax projective embeddings* are defined in the same way as full projective embeddings but for replacing the condition that $e(\ell)$ is a line of $\mathrm{PG}(V)$ with the weaker condition that $e(\ell)$ spans a line of $\mathrm{PG}(V)$, for every line ℓ of Γ . Many authors also require that no two lines of Γ span the same line of $\mathrm{PG}(V)$, but in view of our needs in this paper we can safely renounce that requirement. The only fact relevant for us is that inequality (1) holds true even if e is lax, as it is clear from the way we have obtained it.

2. Preliminaries

2.1. Setting and notation

We refer to [11, Chapter 5] for the definition of the J -grassmannian $\mathrm{Gr}_J(\Delta)$ of a geometry Δ . We recall that when Δ satisfies the so-called Intersection Property (which is always the case when Δ is a building) then $\mathrm{Gr}_J(\Delta)$ is the same as the J -shadow space of Δ as defined by Tits [14, Chapter 12]. According to [11] (and [14]), the J -grassmannian of a geometry Δ is a geometry with a string-shaped diagram graph and the same rank as Δ , but in this paper, following Buekenhout and Cohen [4, §2.5], we shall mostly regard it as a point-line geometry, with the J -flags of Δ taken as points, while the lines are the flags of Δ of type $(J \setminus \{j\}) \cup \mathrm{fr}(j)$ for $j \in J$, where $\mathrm{fr}(j)$ stands for the set of types adjacent to j in the diagram of Δ ; a point and a line of $\mathrm{Gr}_J(\Delta)$ are incident precisely when they are incident as flags of Δ . So, the lines of $\mathrm{Gr}_J(\Delta)$ are particular flags of Δ . This setting will indeed be helpful in some respects but it forces to distinguish between a line and its set of points and this distinction often ends in a burden for the exposition; we will often neglect it. This is a harmless abuse. Indeed only grassmannians of buildings are considered in this paper; buildings satisfy the Intersection Property and, if that property holds in a geometry Δ , then no two lines of $\mathrm{Gr}_J(\Delta)$ have the same points (even better: no two lines of $\mathrm{Gr}_J(\Delta)$ have two points in common). As in Section 1.1, given a division ring \mathbb{K} , we denote by $A_n(\mathbb{K})$ the building of type A_n defined over \mathbb{K} . Similarly, if the division ring \mathbb{K} is a field (namely, is commutative) then $D_n(\mathbb{K})$ stands for the building of type D_n defined over \mathbb{K} . We allow $n = 3$ in D_n . So, $D_3 = A_3$. Nevertheless, when writing $D_3(\mathbb{K})$ we always understand that \mathbb{K} is a field, for consistency of notation. Let X_n stand for either A_n or D_n . It is well known that the elements of $X_n(\mathbb{K})$ can be identified with suitable vector

subspaces of a vector space V over \mathbb{K} of dimension either $n+1$ or $2n$ according to whether $X_n = A_n$ or $X_n = D_n$. Similarly, given a proper sub-division ring \mathbb{K}_0 of \mathbb{K} , the building $X_n(\mathbb{K}_0)$ is realized in a vector space V_0 over \mathbb{K}_0 , of the same dimension as V . We can always assume that V_0 is the set of \mathbb{K}_0 -linear combinations of the vectors of a selected basis E of V , so that V is obtained from V_0 by scalar extension from \mathbb{K}_0 to \mathbb{K} . Thus, with E suitably selected when $X_n = D_n$, the building $X_n(\mathbb{K}_0)$ is turned into a subgeometry of $X_n(\mathbb{K})$ (see Sections 2.4 and 2.5 for more details). Accordingly, for every subset J of the set of nodes of the diagram X_n , the J -grassmannian $\text{Gr}_J(X_n(\mathbb{K}_0))$ can be regarded as a subgeometry of $\text{Gr}_J(X_n(\mathbb{K}))$. Our main goal in this paper is to show that, if J consists of extremal nodes of X_n and $|J| > 1$ then $\text{Gr}_J(X_n(\mathbb{K}_0))$ does not generate $\text{Gr}_J(X_n(\mathbb{K}))$. We firstly consider the $\{1, n\}$ -grassmannian $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ of $A_n(\mathbb{K})$; see Fig. 1.

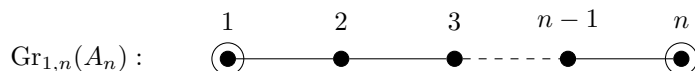


Figure 1: The $\{1, n\}$ -grassmannian of A_n

The points of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ are flags of type $\{1, n\}$ in $A_n(\mathbb{K})$; its lines are flags of type either $\{2, n\}$ or $\{1, n-1\}$; a point p and a line ℓ are incident if and only if $p \cup \ell$ is a flag of $A_n(\mathbb{K})$. Turning to D_n , we label the nodes of this diagram as in Fig. 2.

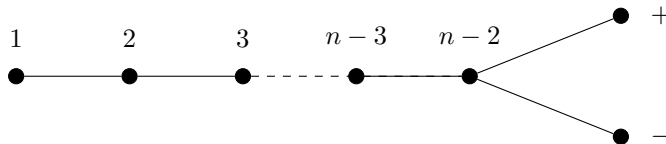


Figure 2: Labeling of types for buildings of type D_n

We are interested in the J -grassmannians $\text{Gr}_J(D_n(\mathbb{K}))$, where $J = \{+, -\}$ or $J = \{1, +, -\}$ or $J = \{1, -\}$ (we can omit the case $J = \{1, +\}$ since $\text{Gr}_{1,+}(D_n(\mathbb{K})) \cong \text{Gr}_{1,-}(D_n(\mathbb{K}))$); see Fig. 3. Explicitly, the points of $\text{Gr}_{+,-}(D_n(\mathbb{K}))$ are the flags of $D_n(\mathbb{K})$ of type $\{+, -\}$ while the lines are the flags of types $\{n-2, +\}$ and $\{n-2, -\}$ with incidence between a point p and a line ℓ given by the condition that $p \cup \ell$ must be a flag of $D_n(\mathbb{K})$. As for $\text{Gr}_{1,+,-}(D_n(\mathbb{K}))$, its points are the flags of type $\{1, +, -\}$, and the lines are the flags of type $\{2, +, -\}$, $\{1, n-2, +\}$ or $\{1, n-2, -\}$; incidence is defined as above. Finally, the points of $\text{Gr}_{1,-}(D_n(\mathbb{K}))$ are the flags of type $\{1, -\}$ and the lines are the flags of type either $\{2, -\}$ or $\{1, n-2\}$. Note that when $n = 3$, since $D_3(\mathbb{K}) \cong A_3(\mathbb{K})$, we have $\text{Gr}_{1,3}(A_3(\mathbb{K})) \cong \text{Gr}_{+,-}(D_3(\mathbb{K}))$. In any case, $\text{Gr}_{+,-}(D_n(\mathbb{K})) \cong \text{Gr}_{n-1}(B_n^+(\mathbb{K}))$, where $B_n^+(\mathbb{K}) := \text{Gr}_1(D_n(\mathbb{K}))$ is the 1-grassmannian of $D_n(\mathbb{K})$ (but regarded as a geometry of rank n), namely the top-thin polar space associated to the group $O^+(2n, \mathbb{K})$; see Fig. 4.

In the following we shall add more details on the grassmannians introduced before. In particular, we shall better explain in which sense $\text{Gr}_J(A_n(\mathbb{K}))$ and $\text{Gr}_J(D_n(\mathbb{K}))$ contain $\text{Gr}_J(A_n(\mathbb{K}_0))$ and $\text{Gr}_J(D_n(\mathbb{K}_0))$ for a sub-division ring \mathbb{K}_0 of \mathbb{K} .

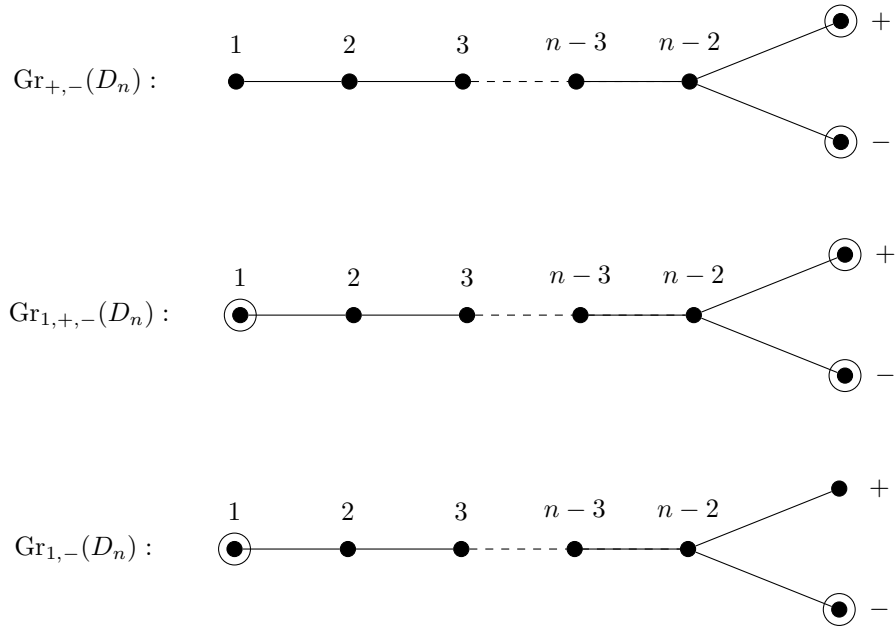


Figure 3: Geometries associated to buildings of type D_n

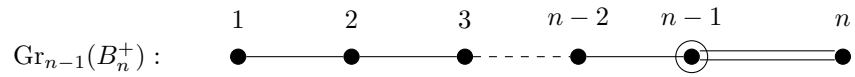


Figure 4: Geometry $\text{Gr}_{n-1}(B_n^+(\mathbb{K})) \cong \text{Gr}_{+,-}(D_n(\mathbb{K}))$

2.2. The geometry $A_n(\mathbb{K})$ and its grassmannian $\text{Gr}_{1,n}(A(\mathbb{K}))$

Let $A_n(\mathbb{K})$ be a geometry of type A_n defined over a division ring \mathbb{K} , with $n \geq 3$. Explicitly, $A_n(\mathbb{K}) \cong \text{PG}(V_{n+1}(\mathbb{K}))$ for a $(n+1)$ -dimensional right \mathbb{K} -vector space $V_{n+1}(\mathbb{K})$. For $i = 1, 2, \dots, n$ the elements of $A_n(\mathbb{K})$ of type i are the i -dimensional subspaces of $V_{n+1}(\mathbb{K})$, with symmetrized inclusion as the incidence relation. As customary we call the elements of $A_n(\mathbb{K})$ of type 1, 2 and n *points*, *lines* and *hyperplanes* respectively. The elements of type $n-1$ will be called *sub-hyperplanes*. Note that, when $n = 3$, lines and sub-hyperplanes are the same objects.

Turning to $\text{Gr}_{1,n}(A_n(\mathbb{K}))$, its points are the point-hyperplane flags (p, H) of $A_n(\mathbb{K})$. Its lines, regarded as sets of points, are of either of the following two types:

$$\ell_{p,S} := \{(p, X) : X \text{ hyperplane, } X \supset S\} \text{ for a (point, sub-hyperplane) flag } (p, S). \quad (3)$$

$$\ell_{L,H} := \{(x, H) : x \text{ a point, } x \subset L\}, \text{ for a line-hyperplane flag } (L, H). \quad (4)$$

2.3. $D_n(\mathbb{K})$ and $\text{Gr}_J(D_n(\mathbb{K}))$ for $J = \{+, -\}, \{1, -\}$ or $\{1, +, -\}$

Let \mathbb{K} be a field and $V_{2n}(\mathbb{K})$ a vector space of dimension $2n$ over \mathbb{K} , with $n \geq 3$. Consider a non-degenerate quadratic form q on $V_{2n}(\mathbb{K})$ of Witt index n . As in Section 2.1, let $B_n^+(\mathbb{K})$ be the polar space associated to q , namely the (weak) building of rank n whose elements are the vector subspaces of $V_{2n}(\mathbb{K})$ that are totally singular with respect to q , with their dimensions taken as types. The elements of $B_n^+(\mathbb{K})$ of dimension 1 are called *points* and those of dimension 2 *lines*.

It is well known that we can “unfold” $B_n^+(\mathbb{K})$ as to obtain a building $D_n(\mathbb{K})$ of type D_n (see e.g. Tits [14, Chapter 7]). Explicitly, let \sim be the equivalence relation on the set of all n -dimensional subspaces of $B_n^+(\mathbb{K})$ defined as follows: $X \sim Y$ if and only if $X \cap Y$ has even codimension in X (equivalently, in Y). Let \mathfrak{S}^+ and \mathfrak{S}^- be the two equivalence classes of \sim . Take $\{1, 2, \dots, n-2, +, -\}$ as the set of types. For $1 \leq i \leq n-2$ the i -elements of $B_n^+(\mathbb{K})$ are the elements of $D_n(\mathbb{K})$ of type i and the elements of \mathfrak{S}^+ and \mathfrak{S}^- are given types $+$ and $-$ respectively. The $(n-1)$ -elements of $B_n^+(\mathbb{K})$ are dropped (but we can recover them as flags of type $\{+, -\}$). Incidence between elements of different types $\{i, j\}$ with $\{i, j\} \neq \{+, -\}$ is symmetrized inclusion; if $X \in \mathfrak{S}^+$ and $Y \in \mathfrak{S}^-$ then X is incident with Y if and only if $\dim(X \cap Y) = n-1$.

It is clear from the way $D_n(\mathbb{K})$ is defined that the 1-grassmannian $\text{Gr}_1(D_n(\mathbb{K}))$ of $D_n(\mathbb{K})$, regarded as a geometry of rank n , is just the same as $B_n^+(\mathbb{K})$. So, we can go back and forth from $D_n(\mathbb{K})$ to $B_n^+(\mathbb{K})$ as if they were the same object. In the sequel we will sometimes avail ourselves of this opportunity, when profitable.

Turning to grassmannians, $\text{Gr}_{+,-}(D_n(\mathbb{K}))$ is the point-line geometry whose points are the flags (M_1, M_2) of $D_n(\mathbb{K})$ of type $(+, -)$ and whose lines are of the following two forms:

$$\ell_{U, M_1} := \{(M_1, X) : X \in \mathfrak{S}^-, M_1 \cap X \supset U\} \text{ with } (U, M_1) \text{ a flag of type } (n-2, +); \quad (5)$$

$$\ell_{U, M_2} := \{(X, M_2) : X \in \mathfrak{S}^+, X \cap M_2 \supset U\} \text{ with } (U, M_2) \text{ a flag of type } (n-2, -). \quad (6)$$

Recall that the points of the grassmannian $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ of $B_n^+(\mathbb{K})$ are the $(n-1)$ -dimensional subspaces of $V_{2n}(\mathbb{K})$ totally singular for the quadratic form q and the lines are the sets of the form $\ell_{X, M} := \{Y : X \subset Y \subset M\}$ where $\dim(X) = n-2$, $\dim(M) = n$, $X \subset M$ and M is totally singular. Every point X of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ is the intersection $X = M_1 \cap M_2$ of a unique pair $\{M_1, M_2\}$ of n -dimensional totally singular subspaces, which necessarily form a $(+, -)$ -flag of $D_n(\mathbb{K})$. Conversely, for every $(+, -)$ -flag (M_1, M_2) of $D_n(\mathbb{K})$, the intersection $X = M_1 \cap M_2$ is a point of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$. A bijective mapping ι is thus naturally defined from the set of points of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ onto the set of points of $\text{Gr}_{+,-}(D_n(\mathbb{K}))$. The mapping ι induces a bijection from the set of lines of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$

onto the set of lines of $\text{Gr}_{+,-}(D_n(\mathbb{K}))$. In fact, if $\ell_{X,M}$ is a line of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ then $\iota(\ell_{X,M})$ is the line of $\text{Gr}_{+,-}(D_n(\mathbb{K}))$ denoted by the very same symbol $\ell_{X,M}$ and it has either form (5) or (6) according to whether M belongs to \mathfrak{S}^+ or \mathfrak{S}^- . To sum up, $\text{Gr}_{n-1}(B_n^+(\mathbb{K})) \cong \text{Gr}_{+,-}(D_n(\mathbb{K}))$.

The grassmannian $\text{Gr}_{1,-}(D_n(\mathbb{K}))$ is the point-line geometry where the points are the flags (p, M) of $D_n(\mathbb{K})$ of type $(1, -)$ and the lines are as follows:

$$\ell_{p,U} := \{(p, X) : X \in \mathfrak{S}^-, X \supset U\} \text{ with } (p, U) \text{ a flag of type } (1, n-2); \quad (7)$$

$$\ell_{L,M} := \{(x, M) : \dim(x) = 1, x \subset L\} \text{ with } (L, M) \text{ a flag of type } (2, -). \quad (8)$$

The grassmannian $\text{Gr}_{1,+,-}(D_n(\mathbb{K}))$ is the point-line geometry where the points are the flags (p, M_1, M_2) of $D_n(\mathbb{K})$ of type $(1, +, -)$; the lines are as follows:

$$\ell_{L,M_1,M_2} := \{(p, M_1, M_2) : \dim(p) = 1, p \subset L\} \text{ with } (L, M_1, M_2) \text{ a flag of type } (2, +, -); \quad (9)$$

$$\ell_{p,U,M_1} := \{(p, M_1, X) : X \in \mathfrak{S}^-, X \supset U\} \text{ with } (p, U, M_1) \text{ a flag of type } (1, n-2, +); \quad (10)$$

$$\ell_{p,U,M_2} := \{(p, X, M_2) : X \in \mathfrak{S}^+, X \supset U\} \text{ with } (p, U, M_2) \text{ a flag of type } (1, n-2, -). \quad (11)$$

2.4. The subgeometry $\text{Gr}_J(A_n(\mathbb{K}_0))$ of $\text{Gr}_J(A_n(\mathbb{K}))$ for $\mathbb{K}_0 \leq \mathbb{K}$

Let E be a basis of $V_{n+1}(\mathbb{K})$ and \mathbb{K}_0 a sub-division ring of \mathbb{K} . We say that a vector $v \in V_{n+1}(\mathbb{K})$ is \mathbb{K}_0 -rational respect to E (also \mathbb{K}_0 -rational for short, when the basis E is clear from the context) if v is a linear combination of vectors of E with coefficients in \mathbb{K}_0 . The set of \mathbb{K}_0 -rational vectors (with respect to E) is a \mathbb{K}_0 -vector space, henceforth denoted $V_{n+1,E}(\mathbb{K}_0)$. For a subspace X of $V_{n+1}(\mathbb{K})$, let $X_0 := X \cap V_{n+1,E}(\mathbb{K}_0)$. Clearly, X_0 is a subspace of $V_{n+1,E}(\mathbb{K}_0)$. We say that X is \mathbb{K}_0 -rational with respect to E (also \mathbb{K}_0 -rational for short) if X_0 spans X (in $V_{n+1}(\mathbb{K})$); in other words X , as a subspace of $V_{n+1}(\mathbb{K})$, admits a basis formed by \mathbb{K}_0 -rational vectors. If this is the case, then X and X_0 have the same dimension (in $V_{n+1}(\mathbb{K})$ and $V_{n+1,E}(\mathbb{K}_0)$ respectively). Indeed the rank of a matrix M with entries in \mathbb{K}_0 does not change if M is regarded as matrix with entries in \mathbb{K} .

Clearly, the sum of two \mathbb{K}_0 -rational subspaces of $V_{n+1}(\mathbb{K})$ is still \mathbb{K}_0 -rational. Similarly,

Lemma 2.1. *The intersection of two \mathbb{K}_0 -rational subspaces is still \mathbb{K}_0 -rational.*

Proof. Let X_0, Y_0 be two subspaces of $V_{n+1,E}(\mathbb{K}_0)$ and X, Y be their spans in $V_{n+1}(\mathbb{K})$. Then $X \cap Y$ contains the span Z of $X_0 \cap Y_0$ in $V_{n+1}(\mathbb{K})$. We must prove that $X \cap Y = Z$. Clearly, $\dim(X) = \dim(X_0)$ and $\dim(Y) = \dim(Y_0)$. Moreover $\dim(X + Y) = \dim(X_0 + Y_0)$. Hence

$$\begin{aligned} \dim(X \cap Y) &= \dim(X) + \dim(Y) - \dim(X + Y) = \\ &= \dim(X_0) + \dim(Y_0) - \dim(X_0 + Y_0) = \dim(X_0 \cap Y_0) = \dim(Z). \end{aligned}$$

Therefore $X \cap Y = Z$. □

The following is now obvious:

Proposition 2.2. *The \mathbb{K}_0 -rational elements of $A_n(\mathbb{K})$ form a geometry $A_{n,E}(\mathbb{K}_0) \cong A_n(\mathbb{K}_0)$.*

In view of Proposition 2.2, we can freely identify $A_n(\mathbb{K}_0)$ with $A_{n,E}(\mathbb{K}_0)$, thus regarding $A_n(\mathbb{K}_0)$ as a subgeometry of $A_n(\mathbb{K})$. The flags of $A_n(\mathbb{K}_0)$ are thus identified with the \mathbb{K}_0 -rational flags of $A_n(\mathbb{K})$, namely the flags of $A_n(\mathbb{K})$ all elements of which are \mathbb{K}_0 -rational (with respect to the selected basis E of $V_{n+1}(\mathbb{K})$). Accordingly, for $\emptyset \neq J \subseteq \{1, 2, \dots, n\}$ the J -grassmannian $\text{Gr}_J(A_n(\mathbb{K}_0))$ of

$A_n(\mathbb{K}_0)$ is identified with the subgeometry $\text{Gr}_{J,E}(A_n(\mathbb{K}_0))$ of $\text{Gr}_J(A_n(\mathbb{K}))$ formed by the \mathbb{K}_0 -rational points and lines of $\text{Gr}_J(A_n(\mathbb{K}))$, namely the points and lines of $\text{Gr}_J(A_n(\mathbb{K}))$ which are \mathbb{K}_0 -rational as flags of $A_n(\mathbb{K})$.

Henceforth, by a harmless little abuse, we will always regard $\text{Gr}_J(A_n(\mathbb{K}_0))$ as the same as $\text{Gr}_{J,E}(A_n(\mathbb{K}_0))$, thus referring to the span of $\text{Gr}_J(A_n(\mathbb{K}_0))$ in $\text{Gr}_J(A_n(\mathbb{K}))$, as we have done in the Introduction, while in fact we mean the span of $\text{Gr}_{J,E}(A_n(\mathbb{K}_0))$.

The next proposition states that, regarding $\text{Gr}_J(A_n(\mathbb{K}_0))$ as a subgeometry of $\text{Gr}_J(A_n(\mathbb{K}))$, the collinearity graph of $\text{Gr}_J(A_n(\mathbb{K}_0))$ is just the graph induced on its point-set by the collinearity graph of $\text{Gr}_J(A_n(\mathbb{K}))$.

Proposition 2.3. *A line of $\text{Gr}_J(A_n(\mathbb{K}))$ is \mathbb{K}_0 -rational if and only if at least two of its points are \mathbb{K}_0 -rational.*

Proof. The ‘only if’ part of this claim easily follows from the isomorphism of the geometries $\text{Gr}_{J,E}(A_n(\mathbb{K}_0)) \cong \text{Gr}_J(A_n(\mathbb{K}_0))$. Turning to the ‘if’ part, given $j_0 \in J$, let L be a flag of $A_n(\mathbb{K})$ of type $(J \setminus \{j_0\}) \cup \text{fr}(j_0)$ and let P and P' be two distinct J -flags of $A_n(\mathbb{K})$ incident with L . We must prove that if both P and P' are \mathbb{K}_0 -rational then L is also \mathbb{K}_0 -rational. There are three cases to examine: J contains elements $j < j_0$ as well elements $j' > j_0$; $j_0 \leq j$ for every $j \in J$; $j_0 \geq j$ for every $j \in J$. We shall examine only the first case, leaving the remaining two (easier) cases to the reader.

With j_0 as in the first case, the flag L has type $(J \setminus \{j_0\}) \cup \{j_0 - 1, j_0 + 1\}$ and contains $Q := P \cap P'$, which is a flag of type $J \setminus \{j_0\}$. Moreover, there are distinct j_0 -subspaces S, S' of $V_{n+1}(\mathbb{K})$ incident with L such that $P = Q \cup \{S\}$ and $P' = Q \cup \{S'\}$. As S and S' are incident with L , the elements of L of type $j_0 - 1$ and $j_0 + 1$ coincide with $S \cap S'$ and $S + S'$ respectively, namely $L = Q \cup \{S \cap S', S + S'\}$. By assumption, P and P' are \mathbb{K}_0 -rational. Hence $Q = P \cap P'$ as well as S and S' are \mathbb{K}_0 -rational. If $j_0 - 1 \in J$ then $S \cap S' \in Q$, hence $S \cap S'$ is \mathbb{K}_0 -rational. Otherwise $S \cap S'$ is \mathbb{K}_0 -rational by Lemma 2.1. Similarly, $S + S'$ is \mathbb{K}_0 -rational. Thus, all elements of L are \mathbb{K}_0 -rational, namely L is \mathbb{K}_0 -rational. \square

2.5. The subgeometry $\text{Gr}_J(D_n(\mathbb{K}_0))$ of $\text{Gr}_J(D_n(\mathbb{K}))$ for $\mathbb{K}_0 \leq \mathbb{K}$

Let \mathbb{K}_0 be a subfield of \mathbb{K} . Let $q : V_{2n}(\mathbb{K}) \rightarrow \mathbb{K}$ be the quadratic form considered in Section 2.3. Without loss of generality we can assume to have chosen the basis $E = (e_1, \dots, e_{2n})$ of $V_{2n}(\mathbb{K})$ in such a way that q admits the following canonical expression with respect to E :

$$q(x_1, \dots, x_{2n}) = x_1x_2 + \dots + x_{2n-1}x_{2n}. \quad (12)$$

As in Section 2.4, we can consider the \mathbb{K}_0 -vector space $V_{2n,E}(\mathbb{K}_0)$ formed by the \mathbb{K}_0 -rational vectors (with respect to E). The form q induces a quadratic form q_0 on $V_{2n,E}(\mathbb{K}_0)$. Clearly, a \mathbb{K}_0 -rational subspace X of $V_{2n}(\mathbb{K})$ is totally singular for q if and only if $X \cap V_{2n,E}(\mathbb{K}_0)$ is totally singular for q_0 . Hence the polar space $B_n^+(\mathbb{K}_0)$ associated to q_0 can be identified with the subgeometry $B_{n,E}^+(\mathbb{K}_0)$ of $B_n^+(\mathbb{K})$ formed by the \mathbb{K}_0 -rational subspaces of $V_{2n}(\mathbb{K})$ which are totally singular for q . Similarly, $D_n(\mathbb{K}_0)$ can be identified with the subgeometry $D_{n,E}(\mathbb{K}_0)$ of $D_n(\mathbb{K})$ formed by the \mathbb{K}_0 -rational elements of $D_n(\mathbb{K})$.

A flag of $D_n(\mathbb{K})$ is \mathbb{K}_0 -rational if all of its elements are \mathbb{K}_0 -rational (with respect to E , of course). Given a nonempty subset J of the type-set $\{1, 2, \dots, n-2, +, -\}$ of $D_n(\mathbb{K})$, a point or a line of $\text{Gr}_J(D_n(\mathbb{K}))$ are said to be \mathbb{K}_0 -rational if they are \mathbb{K}_0 -rational as flags of $D_n(\mathbb{K})$. The \mathbb{K}_0 -rational points and lines of $\text{Gr}_J(D_n(\mathbb{K}))$ form a subgeometry $\text{Gr}_{J,E}(D_n(\mathbb{K}_0))$ of $\text{Gr}_J(D_n(\mathbb{K}))$ isomorphic to $\text{Gr}_J(D_n(\mathbb{K}_0))$. An analogue of Proposition 2.3 also holds:

Proposition 2.4. *A line of $\text{Gr}_J(D_n(\mathbb{K}))$ is \mathbb{K}_0 -rational if and only if at least two of its points are \mathbb{K}_0 -rational.*

Proof. This statement can be proved in the same way as Proposition 2.3 but for a couple of cases in the proof of the ‘only if’ part, which we shall now discuss.

1. Suppose that J contains at least one of the types $+$ and $-$, say $+$ $\in J$. Suppose moreover that $n - 2 \notin J$. Let L be a flag of $D_n(\mathbb{K})$ of type $(J \setminus \{+\}) \cup \text{fr}(+) = (J \setminus \{+\}) \cup \{n - 2\}$ and let P, P' be distinct \mathbb{K}_0 -rational flags of type J , both incident with L . Then $Q = P \cap P'$ is a \mathbb{K}_0 -rational flag, $P = Q \cup \{M\}$ and $P' = Q \cup \{M'\}$ for distinct \mathbb{K}_0 -rational element $M, M' \in \mathfrak{S}^+$. Also, $L = Q \cup S$ for an $(n - 2)$ -element S incident with Q . We have $S \subseteq M \cap M'$ since P and P' are incident with L . However, $\dim(M \cap M')$ has even codimension in M and M' , since M and M' belong to the same family of n -elements of $B_n^+(\mathbb{K})$, namely \mathfrak{S}^+ . Therefore $S = M \cap M'$. Hence S is \mathbb{K}_0 -rational by Lemma 2.1. Thus, L is \mathbb{K}_0 -rational.

2. The set J contains none of the types $+$ or $-$ but it contains $n - 2$. To fix ideas, suppose that $n > 3$. Let L be a flag of $D_n(\mathbb{K})$ of type $(J \setminus \{n - 2\}) \cup \text{fr}(n - 2) = (J \setminus \{n - 2\}) \cup \{n - 3, +, -\}$ and let P, P' be distinct \mathbb{K}_0 -rational flags of type J , both incident with L . Then $Q = P \cap P'$ is a \mathbb{K}_0 -rational flag, $P = Q \cup \{S\}$ and $P' = Q \cup \{S'\}$ for distinct \mathbb{K}_0 -rational $(n - 2)$ -elements S, S' of $D_n(\mathbb{K})$ and $L = Q \cup \{R, M_1, M_2\}$ for an $(n - 3, +, -)$ -flag (R, M_1, M_2) incident with Q . As both P and P' are incident with L , the sum $S + S'$ is contained in $M \cap M'$. However $\dim(M \cap M) = n - 1$ while $\dim(S + S') \geq n - 1$ since $S \neq S'$. Consequently, $M \cap M' = S + S'$. On the other hand, $S + S'$ is a \mathbb{K}_0 -rational subspace of $V_{2n}(\mathbb{K})$, since both S and S' are \mathbb{K}_0 -rational. Hence $M \cap M'$ is \mathbb{K}_0 -rational. Therefore $M \cap M'$ is an $(n - 1)$ -element of $B_{n,E}^+(\mathbb{K}_0) = \text{Gr}_1(D_{n,E}(\mathbb{K}_0))$. Accordingly, $M \cap M' = M_0 \cap M'_0$ for a $(+, -)$ -flag (M_0, M'_0) of $D_{n,E}(\mathbb{K}_0)$. On the other hand, all $(+, -)$ -flags of $D_{n,E}(\mathbb{K}_0)$ are $(+, -)$ -flags of $D_n(\mathbb{K})$ too and two $(+, -)$ -flags (M, M') and (M_0, M'_0) of $D_n(\mathbb{K})$ coincide if $M \cap M' = M_0 \cap M'_0$. It follows that $M = M_0$ and $M' = M'_0$, namely both M and M' are \mathbb{K}_0 -rational. It remains to prove that R too is \mathbb{K}_0 -rational. If $n - 3 \in J$ then $R \in Q$ and there is nothing to prove. Otherwise $R = S \cap S'$. Hence R is \mathbb{K}_0 -rational by Lemma 2.1. Therefore L is \mathbb{K}_0 -rational.

We have assumed that $n > 3$. When $n = 3$ we have $J = \{n - 2\}$ and $L = (M_1, M_2)$, of type $(+, -)$; we get the conclusion as above, but now with no R to take care of. \square

All we have said for $D_n(\mathbb{K}_0)$ and $\text{Gr}_J(D_n(\mathbb{K}_0))$ in this section holds for $B_n^+(\mathbb{K}_0)$ and $\text{Gr}_J(B_n^+(\mathbb{K}_0))$ as well.

3. Proof of Theorems 1.1 and 1.3

For X_n equal to A_n or D_n and a nonempty set of types J , let $\Gamma(\mathbb{K}) := \text{Gr}_J(X_n(\mathbb{K}))$ and $\Gamma(\mathbb{K}_0) := \text{Gr}_J(X_n(\mathbb{K}_0))$ be its \mathbb{K}_0 -rational subgeometry for a proper sub-division ring \mathbb{K}_0 of \mathbb{K} (Sections 2.4 and 2.5).

Definition 3.1. We say that a node t of X_n *splits* J if $t \notin J$ and J is not contained in one single connected component of $X_n \setminus \{t\}$. In other words, t separates at least two of the types of J .

Definition 3.2. We say that a J -flag F (point of $\Gamma(\mathbb{K})$) is *nearly \mathbb{K}_0 -rational* if either at least one of its elements is \mathbb{K}_0 -rational or there exists a \mathbb{K}_0 -rational element of $X_n(\mathbb{K})$ incident with F and such that its type splits J . We denote by $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ the set of all nearly \mathbb{K}_0 -rational points of $\Gamma(\mathbb{K})$.

Obviously, $\Gamma(\mathbb{K}_0) \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$. We shall prove the following:

Theorem 3.3. *If $\Gamma(\mathbb{K})$ is $\text{Gr}_{1,n}(A_n(\mathbb{K}))$, $\text{Gr}_{1,-}(D_n(\mathbb{K}))$, $\text{Gr}_{1,+,-}(D_n(\mathbb{K}))$ or $\text{Gr}_{+,-}(D_n(\mathbb{K}))$ then $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ is a proper subspace of $\Gamma(\mathbb{K})$.*

Theorem 1.1 then immediately follows from Theorem 3.3 and the inclusion $\Gamma(\mathbb{K}_0) \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

3.1. Proof of Theorem 3.3

We need a preliminary result from multi-linear algebra, to be exploited later, when discussing the case $\Gamma(\mathbb{K}) = \text{Gr}_{+,-}(D_n(\mathbb{K}))$.

Lemma 3.4. *Suppose that \mathbb{K} is a field and let $V := V_4(\mathbb{K})$. Given a basis $E = (e_1, e_2, e_3, e_4)$ of V , let $E \wedge E = (e_i \wedge e_j)_{i < j}$ be the corresponding basis of the second exterior power $V \wedge V$ of V . Then all the following hold:*

- (1) *The span $\langle v, w \rangle$ of two independent vectors $v, w \in V$ is \mathbb{K}_0 -rational with respect to E if and only if $v \wedge w$ is proportional to a vector of $V \wedge V$ which is \mathbb{K}_0 -rational with respect to $E \wedge E$.*
- (2) *A non-zero vector $v \in V$ is proportional to a \mathbb{K}_0 -rational vector if and only if the subspace $S_v := \langle v \wedge x \rangle_{x \in V}$ of $V \wedge V$ is \mathbb{K}_0 -rational with respect to $E \wedge E$.*
- (3) *The span $\langle u, v, w \rangle$ of three independent vectors $u, v, w \in V$ is \mathbb{K}_0 -rational (with respect to E) if and only if $\langle u \wedge v, u \wedge w, v \wedge w \rangle$ is \mathbb{K}_0 -rational with respect to $E \wedge E$.*

Proof. (1) Without loss of generality, we can assume $v = e_1 + e_3a_3 + e_4a_4$ and $w = e_2 + e_3b_3 + e_4b_4$ for $a_3, a_4, b_3, b_4 \in \mathbb{K}$. Hence $v \wedge w = e_{1,2} + e_{1,3}b_3 + e_{1,4}b_4 - e_{2,3}a_3 - e_{2,4}a_4 + e_{3,4}(a_3b_4 - a_4b_3)$, where we write $e_{i,j}$ for $e_i \wedge e_j$. Both parts of (1) are equivalent to the following single claim: $a_3, a_4, b_3, b_4 \in \mathbb{K}_0$. Hence they are mutually equivalent.

- (2) Without loss of generality, we can assume that $v = e_1 + e_2a_2 + e_3a_3 + e_4a_4$. Hence $S_v := \langle v \wedge e_2, v \wedge e_3, v \wedge e_4 \rangle$. We have

$$v \wedge e_2 = e_{1,2} - e_{2,3}a_3 - e_{2,4}a_4, \quad v \wedge e_3 = e_{1,3} + e_{2,3}a_2 - e_{3,4}a_4, \quad v \wedge e_4 = e_{1,4} + e_{2,4}a_2 + e_{3,4}a_3,$$

with $e_{i,j} := e_i \wedge e_j$, as above. Both parts of (2) are thus equivalent to this: $a_2, a_3, a_4 \in \mathbb{K}_0$. So Claim (2) is proved.

- (3) Without loss of generality, we can assume that $u = e_1 + e_4a$, $v = e_2 + e_4b$ and $w = e_3 + e_4c$. Hence $u \wedge v = e_{1,2} + e_{1,4}b - e_{2,4}a$, $u \wedge w = e_{1,3} - e_{1,4}c - e_{3,4}a$ and $v \wedge w = e_{2,3} + e_{2,4}c - e_{3,4}b$. Both parts of (3) are equivalent to this: $a, b, c \in \mathbb{K}_0$. Claim (3) follows. □

Lemma 3.5. *If $\Gamma(\mathbb{K})$ is as in the hypotheses of Theorem 3.3 then the set $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ is a subspace of $\Gamma(\mathbb{K})$.*

Proof. We must show that, for any two nearly \mathbb{K}_0 -rational collinear points F, F' of $\Gamma(\mathbb{K})$, the line $\langle F, F' \rangle_{\Gamma(\mathbb{K})}$ is fully contained in $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$. There are several cases to consider:

- 1) $\Gamma(\mathbb{K}) = \text{Gr}_{1,n}(A_n(\mathbb{K}))$. Let $F = (p, H)$ and $F' = (p', H')$ be two distinct collinear points of $\Gamma(\mathbb{K})$, namely two point-hyperplane flags with either $p \neq p'$ and $H = H'$ or $p = p'$ but $H \neq H'$. Suppose moreover that F and F' are nearly \mathbb{K}_0 -rational.

- (a) Let $p = p'$ and $H \neq H'$. Now $\langle F, F' \rangle_{\Gamma(\mathbb{K})} = \ell_{p,S} = \{(p, X): X \supset S, \dim(X) = n\}$ where $S = H \cap H' \supset p$ is a sub-hyperplane containing p . By assumption, there exist \mathbb{K}_0 -rational subspaces U_0, U'_0 of $V_{n+1}(\mathbb{K})$ such that $p \subseteq U_0 \subseteq H$ and $p \subseteq U'_0 \subseteq H'$. The subspace $U_0 \cap U'_0$ is \mathbb{K}_0 -rational by Lemma 2.1, it contains p and is contained in S . Hence it is contained in every hyperplane $X \supset S$. As $U_0 \cap U'_0$ is \mathbb{K}_0 -rational, the flag (p, X) is nearly \mathbb{K}_0 -rational for every hyperplane $X \subset S$, namely $\ell_{p,S} \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.
- (b) Let $H = H'$ but $p \neq p'$. Then $\langle F, F' \rangle_{\Gamma(\mathbb{K})} = \ell_{L,H} = \{(x, H): x \subset L, \dim(x) = 1\}$ where $L = p + p' \subset H$ is the span of $p \cup p'$ in $V_{n+1}(\mathbb{K})$. The argument used in case (a) above can be dualized as follows. By assumption, there exist \mathbb{K}_0 -rational subspaces U_0, U'_0 of $V_{n+1}(\mathbb{K})$ such that $p \subseteq U_0 \subseteq H$ and $p' \subseteq U'_0 \subseteq H'$. Clearly, $L \subseteq U_0 + U'_0 \subseteq H$. Hence $x \subseteq U_0 + U'_0 \subseteq H$ for every 1-subspace x of L . However $U_0 + U'_0$ is \mathbb{K}_0 -rational. Therefore (x, H) is nearly \mathbb{K}_0 -rational. It follows that $\ell_{L,H} \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.
- 2) $\Gamma(\mathbb{K}) = \text{Gr}_{1,-}(D_n(\mathbb{K}))$. Let $F = (p, M)$ and $F' = (p', M')$ be two collinear points of $\text{Gr}_{1,-}(D_n(\mathbb{K}))$. Since F and F' are collinear, either $p = p'$ or $M = M'$. The line $\langle F, F' \rangle_{\Gamma(\mathbb{K})}$ is as in (7) or (8) according to whether $p = p'$ or $M = M'$. When $p = p'$ then the same argument as in (a) of 1) does the job, with the only change that $M \cap M'$, which now plays the role of $H \cap H'$, has dimension $n - 2$ instead of $n - 1$. If $M = M'$ then an argument similar to that used for (b) of 1) yields the conclusion. We leave the details to the reader.
- 3) $\Gamma(\mathbb{K}) = \text{Gr}_{1,+,-}(D_n(\mathbb{K}))$. Let $F = (p, M_1, M_2)$ and $F' = (p', M'_1, M'_2)$ be two collinear points of $\Gamma(\mathbb{K})$ and suppose they both are nearly \mathbb{K}_0 -rational. Two subcases can occur:
- (a) $M_i = M'_i$ for $i = 1, 2$. If at least one of the n -spaces M_1 and M_2 is \mathbb{K}_0 -rational, there is nothing to prove. Suppose that neither of them is \mathbb{K}_0 -rational. Then, since F and F' are nearly \mathbb{K}_0 -rational by assumption, there are \mathbb{K}_0 -rational subspaces U_0 and U'_0 with $p \subseteq U_0 \subset M_1 \cap M_2$ and $p' \subseteq U'_0 \subset M_1 \cap M_2$. We have $\langle F, F' \rangle_{\Gamma(\mathbb{K})} = \ell_{L, M_1, M_2}$ as in (9) with $L = p + p'$. The sum $U_0 + U'_0$ is a \mathbb{K}_0 -rational subspace of $V_{2n}(\mathbb{K})$ and contains L .
If $\dim(U_0 + U'_0) < n - 1$ then $U_0 + U'_0$ is a \mathbb{K}_0 -rational element of $D_n(\mathbb{K})$ incident with the flag (L, M_1, M_2) , which corresponds to the line ℓ_{L, M_1, M_2} . As in (b) of 1), it follows that all points of ℓ_{L, M_1, M_2} are nearly \mathbb{K}_0 -rational.
If $\dim(U_0 + U'_0) > n - 2$ then necessarily $U_0 + U'_0 = M_1 \cap M_2$. In this case $U_0 + U'_0$ is not an element of $D_n(\mathbb{K})$, but it is a \mathbb{K}_0 -rational $(n - 1)$ -element of $B_n^+(\mathbb{K})$, hence an $(n - 1)$ -element of the subgeometry $B_n^+(\mathbb{K}_0)$ of $B_n^+(\mathbb{K})$. As such, $U_0 + U'_0$ is contained in just two n -elements N_1 and N_2 of $B_n^+(\mathbb{K}_0)$. However N_1 and N_2 also belong to $B_n^+(\mathbb{K})$. In fact, they are the unique two n -elements of $B_n^+(\mathbb{K})$ which contain $U_0 + U'_0$. On the other hand, $U_0 + U'_0$ is contained in M_1 and M_2 . Therefore $\{M_1, M_2\} = \{N_1, N_2\}$. However N_1 and N_2 are \mathbb{K}_0 -rational. Hence M_1 and M_2 are \mathbb{K}_0 -rational, contrary to our assumptions. We have reached a contradiction. The proof is complete, as far as the present subcase is concerned.
- (b) Let $p = p'$, $M_i = M'_i$ but $M_j \neq M'_j$, for $\{i, j\} = \{1, 2\}$. To fix ideas, assume that $M_1 = M'_1$ and $M_2 \neq M'_2$. If M_1 or p are \mathbb{K}_0 -rational, then there is nothing to prove. Suppose that neither M_1 nor p are \mathbb{K}_0 -rational. Recalling that F and F' are nearly \mathbb{K}_0 -rational, one of the following occurs:
- (b1) There are \mathbb{K}_0 -rational subspaces U_0, U'_0 of dimension at most $n - 2$ such that $p \subseteq U_0 \subset M_1 \cap M_2$ and $p \subseteq U'_0 \subset M_1 \cap M'_2$.

- (b2) Just one of M_2 and M'_2 is \mathbb{K}_0 -rational. To fix ideas, let M'_2 be the \mathbb{K}_0 -rational one. Then there exists a \mathbb{K}_0 -rational subspace U_0 of dimension $\dim(U_0) \leq n - 2$ such that $p \subseteq U_0 \subset M_1 \cap M_2$.
- (b3) Both M_2 and M'_2 are \mathbb{K}_0 -rational.

In subcases (b1) and (b2) we can consider the element $U_0 \cap U'_0$ or $U_0 \cap M'_2$ respectively. This element contains p and is \mathbb{K}_0 -rational by Lemma 2.1. So, we get the conclusion as in (a) of 1). In subcase (b3), the intersection $U_0 = M_2 \cap M'_2$ is \mathbb{K}_0 -rational by Lemma 2.1 and has dimension $\dim(U_0) = n - 2k$ for a positive integer $k < n/2$, since M_2 and M'_2 belong to the same class \mathfrak{S}^- . Hence $\dim(U_0) \leq n - 2$. Moreover, $U_0 \subset M_1$, since both M_2 and M'_2 are incident with M_1 in $D_n(\mathbb{K})$. Clearly, $p \subseteq U_0$. Again, the conclusion follows as in (a) of 1).

- 4) $\Gamma(\mathbb{K}) = \text{Gr}_{+,-}(D_n(\mathbb{K}))$. Assume firstly that $n = 3$. We have discussed this case in [6, Theorem 5.10] but we turn back to it here, using an argument different from that of [6].

By the Klein correspondence, $V_{2n}(\mathbb{K}) = V_6(\mathbb{K})$ can be regarded as the exterior square of $V_4(\mathbb{K})$, with the basis $E = (e_1, \dots, e_6)$ of $V_6(\mathbb{K})$, to be chosen as in Section 2.5, realized as the exterior square $E = E' \wedge E'$ of a suitable basis E' of $V_4(\mathbb{K})$. The elements of $D_3(\mathbb{K})$ of type $+$ or $-$ correspond to 1- and 3-dimensional subspaces of $V_4(\mathbb{K})$ and the 1-elements of $D_3(\mathbb{K})$ correspond to 2-subspaces of $V_4(\mathbb{K})$. By Lemma 3.4, an element of $D_3(\mathbb{K})$ is \mathbb{K}_0 -rational with respect to E if and only if the subspace corresponding to it in $V_4(\mathbb{K})$ is \mathbb{K}_0 -rational with respect to E' . Accordingly, a $(+, -)$ -flag of $D_3(\mathbb{K})$ is nearly \mathbb{K}_0 -rational if and only if the corresponding $(1, 3)$ -flag of $A_3(\mathbb{K})$ is nearly \mathbb{K}_0 -rational. Thus, we are driven back to the special case $\text{Gr}_{1,3}(A_3(\mathbb{K}))$ of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$, already discussed in 1) of this proof. It follows that $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ is a subspace of $\Gamma(\mathbb{K})$, as claimed.

Consider now $n > 3$. Let $F = (M_1, M_2)$ and $F' = (M'_1, M'_2)$ be two distinct nearly \mathbb{K}_0 -rational collinear points of $\Gamma(\mathbb{K})$. As F and F' are collinear, either $M_1 = M'_1$ or $M_2 = M'_2$. To fix ideas, let $M_2 = M'_2$. Hence

$$\ell_{U, M_2} = \{(M, M_2) : M \in \mathfrak{S}^+, M \cap M_2 \supset U\}$$

is the line of $\Gamma(\mathbb{K})$ through F and F' , where $U = M_1 \cap M'_1 \subset M_2$, $\dim(U) = n - 2$ (see (6)). If M_2 is \mathbb{K}_0 -rational, there is nothing to prove. Assuming that M_2 is not \mathbb{K}_0 -rational, there are still a number of subcases to examine:

- (a) Both M_1 and M'_1 are \mathbb{K}_0 -rational. Hence $U = M_1 \cap M'_1$ is \mathbb{K}_0 -rational. Accordingly, every $(+, -)$ -flag $(M, M_2) \in \ell_{U, M_2}$ is nearly \mathbb{K}_0 -rational.
- (b) Neither M_1 nor M'_1 are \mathbb{K}_0 -rational. Hence there exist \mathbb{K}_0 -rational $(n - 2)$ -elements U_0 and U'_0 such that $U_0 \subset M_1 \cap M_2$ and $U'_0 \subset M'_1 \cap M_2$.

If $U_0 = U'_0$ then $U_0 = M_1 \cap M'_1$. However $M_1 \cap M'_1 = U$. Hence $U = U_0$ is \mathbb{K}_0 -rational. In this case we are done: all $(+, -)$ -flags incident to U are nearly \mathbb{K}_0 -rational.

On the other hand, suppose $U_0 \neq U'_0$. Then $U_0 + U'_0$ is a \mathbb{K}_0 -rational $(n - 1)$ -dimensional subspace of M_2 . Being \mathbb{K}_0 -rational, $U_0 + U'_0$ is an $(n - 1)$ -element of $B_n^+(\mathbb{K}_0)$. As such, it is contained in just two n -elements of $B_n^+(\mathbb{K}_0)$. In other words, both n -elements of $B_n^+(\mathbb{K})$ containing $U_0 + U'_0$ are \mathbb{K}_0 -rational. However M_2 is indeed one of those two elements. Therefore M_2 is \mathbb{K}_0 -rational. This contradicts the assumptions made on M_2 . Consequently, this case we have now been considering cannot occur.

- (c) Just one of M_1 and M'_1 is \mathbb{K}_0 -rational. To fix ideas, let M_1 be the \mathbb{K}_0 -rational one. As (M'_1, M_2) is nearly \mathbb{K}_0 -rational by assumption, but neither M'_1 nor M_2 are \mathbb{K}_0 -rational, there exists a \mathbb{K}_0 -rational $(n-2)$ -element $U' \subset M'_1 \cap M_2$. If $U' = U$ then U is \mathbb{K}_0 -rational and we are done.

Suppose that $U' \neq U$. Therefore $U' \not\subseteq M_1$, otherwise $U' = M_1 \cap M'_1 = U$. As both M_1 and U' are \mathbb{K}_0 -rational, their intersection $W := M_1 \cap U'$ is \mathbb{K}_0 -rational. Note that $\dim(W) = n-3$, as one can see by noticing that $M_1 \cap U' = (M_1 \cap U') \cap U' = (M_1 \cap M_2) \cap U'$ and recalling that $M_1 \cap M_2$ is a hyperplane of M_2 .

Consider the orthogonal W^\perp of W with respect to the form q of Section 2.3. Taking equation (12) into account and recalling that W is \mathbb{K}_0 -rational, we see that W^\perp is a \mathbb{K}_0 -rational vector subspace of $V_{2n}(\mathbb{K})$. In fact W^\perp is the span in $V_{2n}(\mathbb{K})$ of the orthogonal $W_0^\perp \subset V_{2n,E}(\mathbb{K}_0)$ of $W_0 := W \cap V_{2n,E}(\mathbb{K}_0)$ with respect to the form q_0 induced by q on $V_{2n,E}(\mathbb{K}_0)$. All of the spaces M_1, M'_1, U, U' and M_2 contain W and are totally singular, hence they are contained in W^\perp . Moreover, M_1 and U' are \mathbb{K}_0 -rational.

As W^\perp is \mathbb{K}_0 -rational, we can choose a basis $B = (w_1, \dots, w_{n+3})$ of W^\perp formed by \mathbb{K}_0 -rational vectors. We can also assume that w_1, \dots, w_{n-3} span W . As B consists of \mathbb{K}_0 -rational vectors, a vector subspace of W^\perp is \mathbb{K}_0 -rational with respect to B if and only if it is \mathbb{K}_0 -rational with respect to E . Accordingly, M_1 and U' are \mathbb{K}_0 -rational with respect to B while M'_1, U and M_2 are not.

We now switch to the quotient W^\perp/W , taking the cosets $\bar{w}_i := w_{n-3+i} + W$ for $i = 1, 2, \dots, 6$ to form a basis \bar{B} of W^\perp/W . Since W is totally singular, the form q induces a quadratic form \bar{q} on W^\perp/W . Let $\bar{\Delta}$ be the D_3 -building associated to \bar{q} and let $\bar{\Gamma}$ be its $(+, -)$ -grassmannian. So, $(M_1/W, M_2/W)$ and $(M'_1/W, M_2/W)$ are points of the line $\ell_{U/W, M_2/W}$ of $\bar{\Gamma}$, with $U/W \neq U'/W$ and $U'/W \subset M'_1/W \cap M_2/W$. By the above, M_1/W and U'/W are \mathbb{K}_0 -rational while $M'_1/W, M_2/W$ and U/W are not. We now switch from the D_3 -building $\bar{\Delta}$ to the corresponding A_3 -geometry, with elements of type $+$ and $-$ realized as points and planes of $\text{PG}(V_4(\mathbb{K}))$. In this new perspective, the above situation looks as follows: we have two distinct points p and p' (corresponding to M_1/W and M'_1/W), two distinct lines L and L' (corresponding to U/W and U'/W) and a plane S (corresponding to M_2/W). Both p and p' belong to L , $p' \in L'$ but $p \notin L'$. Moreover, S contains both L and L' . Hence S is spanned by p and L' . However, M_1/W and U'/W are \mathbb{K}_0 -rational. Therefore, in view of Lemma 3.4, both p and L' are \mathbb{K}_0 -rational with respect to a suitable basis B' of $V_4(\mathbb{K})$. Hence S is \mathbb{K}_0 -rational with respect to B' , since it is spanned by p and L' . By exploiting Lemma 3.4 once again, we obtain that M_2/W is \mathbb{K}_0 -rational with respect to \bar{B} . We have reached a final contradiction, which shows that the case we have been considering cannot occur.

The proof is complete. □

Lemma 3.6. *Let V be a vector space over a division ring \mathbb{K} and $E = (e_1, \dots, e_n)$ a basis of V . Let \mathbb{K}_0 be a proper sub-division ring of \mathbb{K} and take $\eta \in \mathbb{K} \setminus \mathbb{K}_0$. Suppose S is a subspace of V containing $e_1 + e_2\eta$. If S is \mathbb{K}_0 -rational (with respect to E) then $e_1, e_2 \in S$.*

Proof. Following our conventions, we assume that V is a right vector space. Let V_0 be the \mathbb{K}_0 -vector space of the \mathbb{K}_0 -rational vectors of V (with respect to E). In order to avoid any confusion, we denote spans in V by the symbol $\langle \dots \rangle_V$ and spans in V_0 by the symbol $\langle \dots \rangle_{V_0}$.

Assuming that S is \mathbb{K}_0 -rational, let (v_1, \dots, v_k) be a basis of S consisting of \mathbb{K}_0 -rational vectors and suppose that $e_1 + e_2\eta \in S$. Then $\dim(S \cap \langle e_1, e_2 \rangle_V) \geq 1$. Note that the vector space $S_0 := \langle v_1, \dots, v_k \rangle_{V_0} = S \cap V_0$ has the same dimension as S . Thus, since $\dim(S \cap \langle e_1, e_2 \rangle_V) \geq 1$, we also have $\dim(S_0 \cap \langle e_1, e_2 \rangle_{V_0}) \geq 1$ by the well known Grassmann dimension formula. It follows that there exists a non-zero vector $w \in S_0$ which is a linear combination $w = e_1c_1 + e_2c_2$ with $c_1, c_2 \in \mathbb{K}_0$ and $(c_1, c_2) \neq (0, 0)$. If either $c_1 = 0$ or $c_2 = 0$, then we are done. So, we can assume that $c_1 \neq 0 \neq c_2$. Without loss of generality, we can put $c_1 = 1$, so that $w_1 = e_1 + e_2c_2$ with $c_2 \in \mathbb{K}_0$. Now we claim that there exists $j_0 \in \{1, \dots, k\}$ such that $v_{j_0} = e_1a_{1,j_0} + e_2a_{2,j_0} + \dots + e_na_{n,j_0}$ with $c_2a_{1,j_0} \neq a_{2,j_0}$. By way of contradiction, suppose that for all $v_j \in \{v_1, \dots, v_k\}$ we have

$$v_j = e_1a_{1,j} + e_2a_{2,j} + e_3a_{3,j} + \dots + e_na_{n,j}$$

with $c_2a_{1,j} = a_{2,j}$, i.e. $(a_{1,j}, a_{2,j}) = (1, c_2)d_j$ for some $d_j \in \mathbb{K}$. This implies that for all vectors $v \in S$ we have

$$v = v_1\lambda_1 + v_2\lambda_2 + \dots + v_k\lambda_k = e_1\left(\sum_{i=1}^k \lambda_i + \sum_{i=1}^k d_i\right) + e_1c_2\left(\sum_{i=1}^k \lambda_i + \sum_{i=1}^k d_i\right) + u$$

with $u \in \langle e_3, \dots, e_n \rangle$, $\lambda_i, d_i \in \mathbb{K}$ and $c_2 \in \mathbb{K}_0$. In particular, taking $v = e_1 + e_2\eta \in S$ we have $\sum_{i=1}^k \lambda_i + \sum_{i=1}^k d_i = 1$ and $c_2(\sum_{i=1}^k \lambda_i + \sum_{i=1}^k d_i) = \eta$, forcing $\eta = c_2 \in \mathbb{K}_0$ which is a contradiction. The claim is proved.

Consider the ordered pair (w, v_{j_0}) . As $w, v_{j_0} \in S_0$, we can complete this pair to an ordered basis B of S_0 by choosing $k - 2$ suitable vectors from the $k - 1$ vectors in $\{v_1, \dots, v_k\} \setminus \{v_{j_0}\}$. Without getting out of V_0 , we can now apply a full Gaussian reduction to the sequence of vectors of B to obtain another basis (v'_1, \dots, v'_k) of S_0 such that the $(n \times k)$ -matrix M of the coefficients of the vectors v'_1, \dots, v'_k with respect to e_1, \dots, e_n is in Column Reduced Echelon Form. (Note that, according to our convention to deal with right vector spaces, vectors should be represented as columns.) By construction, the matrix M contains the identity matrix I_k as a minor. Up to a permutation of the vectors e_3, \dots, e_n we can suppose that this minor encompasses the first k rows of the matrix M . The remaining $n - k$ rows form an $((n - k) \times k)$ -matrix

$$N = (b_{k+i,j})_{i,j=1}^{n-k,k}$$

with entries $b_{k+i,j} \in \mathbb{K}_0$. However $e_1 + e_2\eta \in S = \langle S_0 \rangle_V = \langle v'_1, v'_2, \dots, v'_k \rangle_V$. Hence there exist $\alpha_1, \dots, \alpha_k \in \mathbb{K}$ such that $e_1 + e_2\eta = v'_1\alpha_1 + v'_2\alpha_2 + \dots + v'_k\alpha_k$. For every $i = 1, \dots, k$ we have $v'_i = e_i + \sum_{j=k+1}^n e_j b_{j,i}$. Therefore

$$e_1 + e_2\eta = \sum_{i=1}^k e_i\alpha_i + e_{k+1}\left(\sum_{j=1}^k b_{k+1,j}\alpha_j\right) + e_{k+2}\left(\sum_{j=1}^k b_{k+2,j}\alpha_j\right) + \dots + e_n\left(\sum_{j=1}^k b_{n,j}\alpha_j\right),$$

which implies $\alpha_1 = 1, \alpha_2 = \eta, \alpha_3 = \alpha_4 = \dots = \alpha_k = 0$ and

$$\sum_{j=1}^k b_{k+1,j}\alpha_j = \sum_{j=1}^k b_{k+2,j}\alpha_j = \dots = \sum_{j=1}^k b_{n,j}\alpha_j = 0.$$

It follows that $e_1 + e_2\eta = v'_1 + v'_2\eta$, whence $(e_1 - v'_1) = (v'_2 - e_2)\eta$. However,

$$(e_1 - v'_1) = \sum_{i=k+1}^n e_i(-b_{i,1}), \quad (v'_2 - e_2) = \sum_{i=k+1}^n e_i(b_{i,2}\eta),$$

whence $-b_{i,1} = b_{i,2}\eta$ for all $i \geq k+1$. Since $b_{i,j} \in \mathbb{K}_0$ for all i, j and the elements $1, \eta \in \mathbb{K}$ are linearly independent over \mathbb{K}_0 , it follows that $b_{i,1} = b_{i,2} = 0$ for all $i \geq k+1$. So, $v'_1 = e_1$ and $v'_2 = e_2$. Since $v'_1, v'_2 \in S$, we obtain $e_1, e_2 \in S$, which proves the lemma. \square

Lemma 3.7. *If $\Gamma(\mathbb{K})$ is as in the hypotheses of Theorem 3.3 then not all points of $\Gamma(\mathbb{K})$ belong to $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.*

Proof. We first consider the case $\Gamma(\mathbb{K}) = \text{Gr}_{1,n}(A_n(\mathbb{K}))$. Pick $\eta \in \mathbb{K} \setminus \mathbb{K}_0$. With $E = (e_1, \dots, e_{n+1})$ as in Section 2.4, put $p = \langle e_1 + e_2\eta \rangle$ and $H = \langle e_1 + e_2\eta, e_3, \dots, e_n, e_{n+1} \rangle$. (Needless to say, the symbol $\langle \dots \rangle$ refers to spans in $V_{n+1}(\mathbb{K})$.) The flag (p, H) is a point of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$. Let S be a subspace of $V_{n+1}(\mathbb{K})$ such that $p \subseteq S \subseteq H$. Any such subspace contains the vector $e_1 + e_2\eta$ but neither e_1 nor e_2 . Hence S cannot be \mathbb{K}_0 -rational, by Lemma 3.6. Consequently, $(p, H) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

The case $\Gamma(\mathbb{K}) = \text{Gr}_{1,+,-}(D_n(\mathbb{K}))$ is entirely analogous. With $E = (e_1, e_2, \dots, e_{2n})$ as in Section 2.5 and η as above, put $p = \langle e_1 + e_3\eta \rangle$, $M_1 = \langle e_1 + e_3\eta, e_2\eta - e_4, e_5, e_7, \dots, e_{2n-1} \rangle$ and $M_2 = \langle e_1 + e_3\eta, e_2\eta - e_4, e_5, e_7, \dots, e_{2n-1} \rangle$. Taking equation (12) into account, it is straightforward to see that p, M_1 and M_2 belong to $D_n(\mathbb{K})$. It is also easy to see that they form a $(1, +, -)$ -flag of $D_n(\mathbb{K})$, namely a point of $\Gamma(\mathbb{K})$. Clearly, none of p, M_1 or M_2 is \mathbb{K}_0 -rational and Lemma 3.6 implies that none of the subspaces contained in $M_1 \cap M_2$ and containing p can be \mathbb{K}_0 -rational. Hence $(p, M_1, M_2) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

When $\Gamma(\mathbb{K}) = \text{Gr}_{1,-}(D_n(\mathbb{K}))$ we can consider the flag (p, M) where $p = \langle e_1 + e_3\eta \rangle$ and $M = \langle e_1 + e_3\eta, e_2\eta - e_4, e_5, e_7, \dots, e_{2n-1} \rangle$. The subspace M is n -dimensional and totally singular for q . We can also assume to have chosen the signs $+$ and $-$ in such a way that \mathfrak{S}^- is indeed the class which M belongs to. So, (p, M) is a point of $\Gamma(\mathbb{K})$. Once again, by Lemma 3.6 we see that $(p, M) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

Finally, let $\Gamma(\mathbb{K}) = \text{Gr}_{+,-}(D_n(\mathbb{K}))$. In view of Lemma 3.4, if $n = 3$ we are back to A_3 . So, assume $n > 3$. With $\eta \in \mathbb{K} \setminus \mathbb{K}_0$ and $E = (e_1, \dots, e_{2n})$ as in Section 2.5, put

$$\begin{aligned} M_1 &:= \langle e_1 + e_3, e_2 - e_4, e_5 + e_7\eta, e_6\eta - e_8, e_{10}, e_{12}, \dots, e_{2n} \rangle, \\ M_2 &:= \langle e_1 + e_4, e_2 - e_3, e_5 + e_7\eta, e_6\eta - e_8, e_{10}, e_{12}, \dots, e_{2n} \rangle. \end{aligned}$$

Then M_1 and M_2 are n -dimensional totally singular subspaces of $D_n(\mathbb{K})$ but neither of them is \mathbb{K}_0 -rational. Moreover $M_1 \cap M_2 = \langle e_1 - e_2 + e_3 + e_4, e_5 + e_7\eta, e_6\eta - e_8, e_{10}, e_{12}, \dots, e_{2n} \rangle$. Hence $\{M_1, M_2\}$ is a $\{+, -\}$ -flag of $D_n(\mathbb{K})$, necessarily not \mathbb{K}_0 -rational, since neither M_1 nor M_2 is \mathbb{K}_0 -rational. Accordingly, $M_1 \cap M_2$ is not \mathbb{K}_0 -rational. In fact all \mathbb{K}_0 -rational subspaces of $M_1 \cap M_2$ are contained in $\langle e_1 - e_2 + e_3 + e_4, e_{10}, e_{12}, \dots, e_{2n} \rangle$, which is $(n-3)$ -dimensional. Their dimensions are too small for them to split $(+, -)$. Therefore $(M_1, M_2) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$. \square

Lemmas 3.5 and 3.7 yield Theorem 3.3.

3.2. Proof of Corollary 1.2

As already remarked in Section 2.3, the function ι that maps every $(n-1)$ -element of $B_n^+(\mathbb{K})$ onto the pair of n -elements containing it is an isomorphism from $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ to $\text{Gr}_{+,-}(D_n(\mathbb{K}))$. We know from Theorem 1.1 that if $\mathbb{K}_0 < \mathbb{K}$ then $\text{Gr}_{+,-}(D_n(\mathbb{K}_0))$ spans a proper subspace of $\text{Gr}_{+,-}(D_n(\mathbb{K}))$. In order to show that the same holds for $\text{Gr}_{n-1}(B_n^+(\mathbb{K}_0))$ and $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$, as claimed in Corollary 1.2, we only need to prove the following:

Proposition 3.8. *The isomorphism ι maps the subgeometry $\text{Gr}_{n-1}(B_n^+(\mathbb{K}_0))$ of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ onto the subgeometry $\text{Gr}_{+,-}(D_n(\mathbb{K}_0))$ of $\text{Gr}_{+,-}(D_n(\mathbb{K}))$.*

Proof. It goes without saying that both $\text{Gr}_{n-1}(B_n^+(\mathbb{K}_0)) = \text{Gr}_{n-1,E}(B_n^+(\mathbb{K}_0))$ and $\text{Gr}_{+,-}(D_n(\mathbb{K}_0)) = \text{Gr}_{+,-,E}(D_n(\mathbb{K}_0))$ for the same basis E of $V_{2n}(\mathbb{K})$, chosen as in Section 2.5.

Let $U = M_1 \cap M_2$ for a $(+, -)$ -flag (M_1, M_2) of $D_n(\mathbb{K})$. If both M_1 and M_2 are \mathbb{K}_0 -rational then U is \mathbb{K}_0 -rational, by Lemma 2.1. Conversely, let U be \mathbb{K}_0 -rational. Let M'_1 and M'_2 be the two n -elements of $B_{n,E}^+(\mathbb{K}_0)$ containing U . Then M'_1 and M'_2 are \mathbb{K}_0 -rational, as they belong to $B_{n,E}^+(\mathbb{K}_0)$. However, they are the only two n -elements on U . Hence $\{M'_1, M'_2\} = \{M_1, M_2\}$. Accordingly, M_1 and M_2 are \mathbb{K}_0 -rational. \square

3.3. Proof of Theorem 1.3

We are not going to give a detailed proof of this theorem. We will only offer a sketch of it, leaving the details to reader.

As stated since the beginning of this section, \mathbb{K}_0 is a proper sub-division ring of \mathbb{K} and $\Gamma(\mathbb{K}) = \text{Gr}_J(X_n(\mathbb{K}))$, where X_n stands for A_n or D_n . According to the hypotheses of Theorem 1.3, we assume that J is not connected.

Suppose firstly that J contains two types j_1 and j_2 , with $j_1, j_2 \leq n - 2$ when $X_n = D_n$, such that $j_1 + 1 < j_2$ and $i \notin J$ for every type $i \in \{j_1 + 1, j_1 + 2, \dots, j_2 - 1\}$. We say that a J -flag F of $X_n(\mathbb{K})$ (point of $\Gamma(\mathbb{K})$) is *nearly \mathbb{K}_0 -rational at (j_1, j_2)* if there exists a \mathbb{K}_0 -rational element X of $X_n(\mathbb{K})$ incident to F and such that $j_1 \leq \dim(X) \leq j_2$. Let $\Omega_{\mathbb{K}_0, j_1, j_2}(\Gamma(\mathbb{K}))$ be the set of J -flags which are nearly \mathbb{K}_0 -rational at (j_1, j_2) . Using the same argument as in 1) in the proof of Lemma 3.5, with the roles of 1 and n respectively taken by j_1 and j_2 we see that $\Omega_{\mathbb{K}_0, j_1, j_2}(\Gamma(\mathbb{K}))$ is a subspace of $\Gamma(\mathbb{K})$. Next, by an argument similar to that used for $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ in the proof of Lemma 3.7, we obtain that $\Omega_{\mathbb{K}_0, j_1, j_2}(\Gamma(\mathbb{K})) \neq \Gamma(\mathbb{K})$, namely $\Omega_{\mathbb{K}_0, j_1, j_2}(\Gamma(\mathbb{K}))$ is a proper subspace of $\Gamma(\mathbb{K})$. However $\Gamma(\mathbb{K}_0) := \text{Gr}_J(X_n(\mathbb{K}_0))$ is contained in $\Omega_{\mathbb{K}_0, j_1, j_2}(\Gamma(\mathbb{K}))$. Hence $\Gamma(\mathbb{K}_0)$ spans a proper subspace of $\Gamma(\mathbb{K})$, as stated in Theorem 1.3.

Two more possibilities remain to examine, which are not considered in Theorem 1.1, namely $X_n(\mathbb{K}) = D_n(\mathbb{K})$ and J as follows:

1. $J = \{j, j + 1, \dots, j + k\} \cup \{+, -\}$ for $j \geq 1$, $j + k < n - 2$ and either $j > 1$ or $k > 0$. In this case we can use the same arguments as for $J = \{1, +, -\}$ in the proof of Theorem 1.1, with $j + k$ playing the role of 1.
2. $J = \{j, j + 1, \dots, j + k\} \cup \{-\}$ or $J = \{j, j + 1, \dots, j + k\} \cup \{+\}$, for $j \geq 1$, $j + k < n - 2$ and either $j > 1$ or $k > 0$. The arguments used for $J = \{1, -\}$ work for this case as well, with 1 replaced by $j + k$.

4. Proof of Lemma 1.4 and Theorem 1.5

4.1. Proof of Lemma 1.4

Assume that J is non-connected and \mathbb{K} is not finitely generated. Let S be a finite set of points of $\Gamma(\mathbb{K}) = \text{Gr}_J(X_n(\mathbb{K}))$, where X_n stands for A_n or D_n . Each element F of S is a J -flag $F = \{U_1, U_2, \dots, U_t\}$ of vector subspaces U_i of $V_N(\mathbb{K})$, where $t := |J|$ and N is $n + 1$ or $2n$ according as X_n is A_n or D_n . Fix a basis $B_{i,F}$ for each of the vector subspaces $U_i \in F$ and each $F \in S$ and let $C(S)$ be the set of all the coordinates of the vectors of $\cup_{F \in S} \cup_{i=1}^t B_{i,F}$ with respect to a given basis of $V_N(\mathbb{K})$ (chosen as in Section 2.5 when $X_n = D_n$).

Since S is finite, $C(S)$ is finite as well; in fact $|C(S)| \leq t \cdot N \cdot |S|$. Therefore, and since \mathbb{K} is not finitely generated, $C(S)$ generates a proper sub-division ring \mathbb{K}_0 of \mathbb{K} . Then $\Gamma(\mathbb{K}_0) := \text{Gr}_J(X_n(\mathbb{K}_0))$

spans a proper subspace of $\Gamma(\mathbb{K})$, by Theorem 1.3. Obviously, S is contained in $\Gamma(\mathbb{K}_0)$. Hence S spans a proper subspace of $\Gamma(\mathbb{K})$. Thus we have proved that no finite subset of $\Gamma(\mathbb{K})$ generates $\Gamma(\mathbb{K})$, as claimed in Lemma 1.4.

4.2. Proof of Theorem 1.5

Put $\Gamma := \text{Gr}_{1,n}(A_n(\overline{\mathbb{F}}_p))$. We have $\text{gr}(\Gamma) = \infty$ by Lemma 1.4, since $\overline{\mathbb{F}}_p$ is not finitely generated. The geometry Γ admits a (full) projective embedding of dimension $(n+1)^2 - 1$, namely the embedding e_{Lie} mentioned in Remark 1.6. Therefore $\text{er}(\Gamma) \geq (n+1)^2 - 1$.

By way of contradiction, suppose that $\text{er}(\Gamma) > (n+1)^2$. Then Γ admits a (full) projective embedding $e : \Gamma \rightarrow \text{PG}(V)$ of dimension $\dim(e) \geq (n+1)^2 + 1$. Consequently, there exists a set S of $(n+1)^2 + 1$ points of Γ such that $\cup_{x \in S} e(x) \subset V$ spans a subspace V_S of V of dimension $\dim(V_S) = (n+1)^2 + 1$.

Every point $x \in S$ is a point-hyperplane flag (p_x, H_x) of $A_n(\overline{\mathbb{F}}_p)$. For every $x \in S$ we choose a non-zero vector $v_x \in p_x$ and a basis B_x of H_x . Chosen a basis E of $V_{n+1}(\overline{\mathbb{F}}_p)$, let $C(S)$ be the set of all elements of $\overline{\mathbb{F}}_p$ which occur as coordinates (with respect to E) of either v_x or a vector of B_x , for $x \in S$. The set $C(S)$ is finite. Hence it generates a finite subfield \mathbb{L} of $\overline{\mathbb{F}}_p$. Every point $x \in S$ is obviously \mathbb{L} -rational. Therefore $S \subset \Gamma_{\mathbb{L}} := \text{Gr}_{1,n}(A_n(\mathbb{L})) \subset \Gamma$.

Let $V_{\mathbb{L}}$ be the subspace of V corresponding to the span of $e(\Gamma_{\mathbb{L}})$. Clearly $V_{\mathbb{L}} \supseteq V_S$. Hence $\dim(V_{\mathbb{L}}) \geq \dim(V_S) = (n+1)^2 + 1$. The restriction $e_{\mathbb{L}}$ of e to $\Gamma_{\mathbb{L}}$ is a lax embedding of $\Gamma_{\mathbb{L}}$ in $\text{PG}(V_{\mathbb{L}})$. As noticed in Remark 1.8, inequality (1) holds for lax embeddings too. Therefore $\Gamma_{\mathbb{L}}$ has generating rank $\text{gr}(\Gamma_{\mathbb{L}}) \geq \dim(e_{\mathbb{L}}) = \dim(V_{\mathbb{L}}) > (n+1)^2$.

On the other hand, the field \mathbb{L} is a simple extension of the prime field \mathbb{F}_p and $\text{Gr}_{1,n}(A_n(\mathbb{F}_p))$ has generating rank equal to $(n+1)^2 - 1$, by Cooperstein [7]. Therefore $\text{gr}(\Gamma_{\mathbb{L}}) \leq (n+1)^2$ by Blok and Pasini [2, Corollary 4.8]. We have reached a contradiction. Consequently, $\text{er}(\Gamma) \leq (n+1)^2$. The proof of Theorem 1.5 is complete.

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