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Psychological Nash Equilibria under Ambiguity

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Giuseppe De Marco*, Maria Romaniello†, and Alba Roviello ‡

Abstract

Psychological games aim to represent situations in which players have belief-dependent motivations or believe that their opponents have belief-dependent motivations. In this setting, utility functions are directly dependent on the entire hierarchy of beliefs of each player. On the other hand, the literature on strategic ambiguity in classical games highlights that players may have ambiguous (or imprecise) beliefs about opponents' strategy choices. In this paper, we look at the issue of strategic ambiguity in the framework of psychological games by taking into account ambiguous hierarchies of beliefs and we study the natural generalization of the psychological Nash equilibrium concept to this framework. We give an existence result for this new concept of equilibrium and provide examples that show that even an infinitesimal amount of ambiguity may alter significantly the equilibria of the game or can work as an equilibrium selection device. Finally, we look at the problem of stability of psychological equilibria with respect to ambiguous trembles on the entire hierarchy of correct beliefs and we provide a limit result that gives conditions so that sequences of psychological equilibria under ambiguous perturbation converge to psychological equilibria of the unperturbed game.

Keywords: Psychological Games, Ambiguous Beliefs, Equilibrium Existence, Equilibrium Selection.

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1 Introduction

Psychological games have been introduced to understand how emotions, opinions and intentions of the decision makers can affect a game. In the pioneering paper by Geanakoplos et al. (1989), this goal is tackled by assuming that payoffs are directly dependent not only on the strategies, but also on the beliefs of each player: players may have belief-dependent motivations or may believe that their opponents have belief-dependent motivations[3]. Geanakoplos et al. (1989) present an equilibrium concept for this class of games based on the idea that the entire hierarchy of beliefs of each player must be correct in equilibrium. Moreover, they provide an existence result for this notion of equilibrium.

There is another strand of literature that focuses on *strategic ambiguity* in classical games as it is well known that players may have ambiguous (or imprecise) beliefs about opponents' strategy choices. Many equilibrium concepts for games under strategic ambiguity have been introduced and used in applications (see for instance [Dow and Werlang 1994], [Eichberger and Kelsey 2000], [Lehrer 2012], [Riedel and Sass 2013], [De Marco and Romaniello 2015] and references therein). In addition, limit results provide conditions that guarantee the convergence of sequences of equilibria of ambiguous games to the equilibria of the unperturbed games when ambiguity converges to zero (see [De Marco and Romaniello 2013] and references therein).

In this paper, we look at the issue of strategic ambiguity in the framework of psychological games by taking into account ambiguous hierarchies of beliefs. In particular, we adapt the model of psychological games of Geanakoplos et al. (1989) to the ambiguity framework. The idea is that beliefs might be imprecise or ambiguous in equilibrium. More precisely, the function that maps strategy profiles to the *correct* hierarchies of beliefs, that is used in the classical definition of psychological Nash equilibria, is now replaced by the (so called) *ambiguous belief correspondence*, which maps strategy profiles to the subsets of those hierarchies of beliefs that players perceive to be consistent with the corresponding strategy profile. Following the standard approach, agents are assumed to have a pessimistic attitude towards ambiguity as they are endowed with the classical *maxmin* preferences to compare ambiguous alternatives. From the mathematical point of view, such *maxmin* preferences correspond to the maximization (with respect to the strategy of the corresponding player) of a marginal function computed along the graph of the ambiguous belief correspondence whose values, in turn, depend on the entire strategy profile. The equilibrium concept studied in this paper, called *psychological Nash equilibrium under ambiguity*, appears to be the natural generalization of the psychological Nash equilibrium notion in Geanakoplos et al. (1989). We give an existence result for this notion that is naturally based on continuity properties of the ambiguous belief correspondences. We provide also different examples in order to better

[3]The literature on psychological games has increased considerably in the past decades; we recall [Battigalli and Siniscalchi 1999] and [Battigalli and Dufwenberg 2009] for further theoretical findings, [Rabin 1993], [Battigalli and Dufwenberg 2007], [Attanasi et al. 2010] for some applications just to quote a few and [Attanasi and Nagel 2008] and [Battigalli and Dufwenberg n.d.] for surveys on psychological games and references.

illustrate this new concept of equilibrium: they show that even a little (infinitesimal) amount of ambiguity may alter significantly the equilibria of the game; however, the way in which the set of equilibria changes is not unequivocally determined but depends on the specific model. In fact, a first example shows that, when ambiguity is introduced, the set of equilibria might remain unaltered; in a second example, instead, the set of psychological equilibria under ambiguity is disjoint from the set of classical psychological equilibria. In a further example, ambiguity produces an equilibrium selection, that is, the set of psychological Nash equilibria under ambiguity is a proper subset of the standard psychological Nash equilibria set.

The issue of equilibrium selection that arises from the example previously mentioned relates this work with another relevant strand of literature that concerns the classical theory of refinements of Nash equilibria[4]. These equilibrium concepts are based on properties of stability with respect to some kind of perturbations: roughly speaking, an equilibrium is stable if a game nearby has an equilibrium nearby. In the seminal paper by [Selten 1975], the *trembling hand perfect equilibrium* concept selects equilibria that are stable with respect to the possibility that players believe that their opponents can make (infinitesimal) mistakes playing their equilibrium strategies: each equilibrium strategy should be *close* to the best reply against perturbed expectations about opponents' behavior, if the perturbation is small enough. In Geanakoplos et al. (1989) it is considered a notion of trembling hand perfect psychological equilibrium, that is constructed by perturbing the strategies as in [Selten 1975] and keeping the hierarchies of beliefs fixed along the perturbation and equal to those that are correct given the unperturbed strategies. In this paper we look at the problem of stability of psychological equilibria from another perspective as perturbations concern the entire hierarchy of correct beliefs and, as the literature on strategic ambiguity suggests, they (can) take the form of sets of hierarchies. However, our approach has an underlying problem that concerns the understanding in which way ambiguous beliefs should converge to correct beliefs so that sequences of psychological equilibria under perturbation converge to psychological equilibria of the unperturbed game. We give a general limit theorem that tackles this issue. Then, we show how to construct selection criteria for classical psychological equilibria based on *ambiguous trembles* of the correct belief function.

The paper is organized as follows: Section 2 presents the model of psychological games under ambiguity and the equilibrium concept. The three examples mentioned above are given in Section 3. Section 4 focuses on the equilibrium existence theorem. The problem of stability of psychological Nash equilibria with respect to ambiguous trembles is studied in Section 5.

[4]See, for example, [van Damme 1989] for an extensive survey and rich list of references.

2 Model and Equilibria

2.1 Beliefs

We consider a finite set of players $I = \{1, \dots n\}$, and for each player i, denote with $A_i = \{a_i^1, \dots, a_i^{k(i)}\}$ the (finite) pure strategy set of player i. As usual, the set of strategy profiles A is the cartesian product of the strategy sets of each player, that is $A = A_1 \times \dots \times A_n = \prod_{i \in I} A_i$ and $A_{-i} = A_1 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_n = \prod_{j \neq i} A_j$. Let Σ_i be the set of mixed strategies of player i, where each mixed strategy $\sigma_i \in \Sigma_i$ is a nonnegative vector $\sigma_i = (\sigma_i(a_i))_{a_i \in A_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$. Denote also with $\Sigma = \prod_{i \in I} \Sigma_i$ and with $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$. We use (σ_i, σ_{-i}) with $\sigma_i \in \Sigma_i$ and $\sigma_{-i} \in \Sigma_{-i}$ to represent $\sigma \in \Sigma$.

Hierarchies of beliefs

Hierarchies of beliefs are constructed as in [Geanakoplos et al. 1989]. For any set S, $\Delta(S)$ denotes the set of probability measures on S. Then, for every player i, $B_i^1 := \Delta(\Sigma_{-i})$ is the set of the first order beliefs of player i. Therefore, a first order belief of player i, $b_i^1 \in B_i^1$, is a probability measure over the product of the other players' mixed strategy sets. The set B_i^1 is endowed with the weak topology and it is a separable and compact metric space because so it is Σ_{-i}[5][6].

Higher order beliefs are defined inductively as follows:

$$B_i^k := \Delta(\Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}),$$

$$B_{-i}^k := \prod_{j \neq i} B_j^k, \quad B^k := \prod_{i \in I} B_i^k.$$

Moreover, for every k, B_i^k is compact and can be metrized as separable metric space as done for B_i^1.[7].

[5] This property is a consequence of the fact that $\Delta(S)$ can be metrized as a separable metric space if and only if S is a separable metric space. In particular, the metric is the Prokhorov distance (see Prokhorov (1956) or theorems 6.2 and 6.5 Chapter 2 in Parthasarathy (2005)). With this metric structure, the space is also compact (see theorem 6.4 Chapter 2 in Parthasarathy (2005)).

[6] Sometimes it can be useful to regard B_i^1 as a subset of the linear topological space of finite signed measures V_i^1, defined on the same σ-algebra. The space V_i^1 is endowed with the same weak topology and it is metrized as a separable metric space in the same way.

[7] More generally, the set of probability measures on a countable product of compact and separable metric spaces is still compact and separable [see Greever 1967, pp. 46, 61]. Moreover, B_i^k can be regarded as compact subset of the linear topological space of finite signed measures V_i^k, endowed with the weak topology.

Finally, the set of all *hierarchies of beliefs*[8] of player i is

$$B_i = \prod_{k=1}^{\infty} B_i^k.$$

The space B_i is a countable product of metric space so it is also metrizable in such a way that the topology induced by the corresponding metric is equivalent to the product topology. Moreover, under this topology, B_i is compact.

Coherent beliefs

As pointed out in [Geanakoplos et al. 1989], it is a common practice to restrict beliefs of each player i to the subset of *collectively coherent beliefs* $\overline{B}_i \subset B_i$, that is, the set of beliefs of player i in which he is sure that it is common knowledge that beliefs are coherent, where

DEFINITION 2.1: A belief $b_i = (b_i^1, b_i^2, \dots) \in B_i$ is said to be *coherent* if, for every $k \in \mathbb{N}$, the marginal probability of b_i^{k+1} on $\Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \cdots \times B_{-i}^{k-1}$ coincides with b_i^k, that is

$$\operatorname{marg}(b_i^{k+1}, \Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \cdots \times B_{-i}^{k-1}) = b_i^k.$$

More precisely, the set of collectively coherent beliefs is defined as follows.

DEFINITION 2.2: Let $\hat{B}_i(0)$ be the set of coherent beliefs of player i. Inductively, for every $\alpha > 0$ let $\hat{B}_i(\alpha)$ be the set

$$\hat{B}_i(\alpha) := \{b_i \in \hat{B}_i(\alpha - 1) \,|\, \forall k \geq 1,\, b_i^{k+1}(\Sigma_{-i} \times X_{-i}^k(\alpha - 1)) = 1\},$$

where

$$X_j^k(\alpha) := \text{projection of } \hat{B}_j(\alpha) \text{ into } \prod_{l=1}^{k} B_j^l, \qquad X_{-i}^k(\alpha) := \prod_{j \neq i} X_j^k(\alpha).$$

Then, the set of *collectively coherent beliefs* $\overline{B}_i$ is defined by

$$\overline{B}_i = \bigcap_{\alpha > 0} \hat{B}_i(\alpha).$$

The set $\overline{B}_i$ is compact (see [Battigalli and Dufwenberg 2009]). However, we give a self-contained proof below.

LEMMA 2.3: *The set of collectively coherent beliefs $\overline{B}_i$ is a compact subset of B_i for every i.*

[8]The notion of hierarchy of beliefs can be found in several other papers (see for example [Harsanyi 1967], [Mertens and Zamir 1985], [Brandenburger and Dekel 1993]).

Proof. Given that the weak topology is Hausdorff and that intersection of compact sets in an Hausdorff space is compact, it is sufficient to prove that each $\hat{B}_i(\alpha)$ is compact, which results in proving that $\hat{B}_i(\alpha)$ is closed[9].

We proceed by induction on α. Consider $\hat{B}_i(0)$ and let $\{b_{i,\nu}\}_{\nu\in\mathbb{N}} \subset \hat{B}_i(0)$ be a sequence converging in the product topology to a point $\tilde{b}_i$. Since B_i is compact then $\tilde{b}_i \in B_i$. Therefore, for every $k \geqslant 1$, the sequence $\{b^k_{i,\nu}\}_{\nu\in\mathbb{N}}$ weakly converges to $\tilde{b}^k_i \in B^k_i$. For every $k \geq 1$, we have to check that

$$\operatorname{marg}(\tilde{b}^{k+1}_i, \Sigma_{-i} \times B^1_{-1} \times \cdots \times B^{k-1}_{-i}) = \tilde{b}^k_i. \tag{1}$$

Now, for every measurable $A \subset \Sigma_{-i} \times B^1_{-1} \times \cdots \times B^{k-1}_{-i}$, weak convergence implies that

$$\operatorname{marg}(\tilde{b}^{k+1}_i, \Sigma_{-i} \times B^1_{-1} \times \cdots \times B^{k-1}_{-i})(A) = \int_{A\times B^k_i} \mathrm{d}\tilde{b}^{k+1}_i = \lim_{\nu\to\infty} \int_{A\times B^k_i} \mathrm{d}b^{k+1}_{i,\nu} =$$

$$\lim_{\nu\to\infty} \operatorname{marg}(b^{k+1}_{i,\nu}, \Sigma_{-i} \times B^1_{-1} \times \cdots \times B^{k-1}_{-i})(A) = \lim_{\nu\to\infty} \int_{A\times B^k_i} \mathrm{d}b^k_{i,\nu} = \int_{A\times B^k_i} \mathrm{d}\tilde{b}^k_i = b^k_i(A).$$

Hence (1) holds and $\hat{B}_i(0)$ is compact in B_i.
By induction, suppose that $\hat{B}_i(\alpha)$ is compact. Consider a sequence $\{b_{i,\nu}\}_{\nu\in\mathbb{N}} \subset \hat{B}_i(\alpha+1)$ converging in the product topology to $\tilde{b}_i$. Since $\hat{B}_i(\alpha+1) \subset \hat{B}_i(\alpha)$ and $\hat{B}_i(\alpha)$ is compact, then $\tilde{b}_i \in \hat{B}_i(\alpha)$. Moreover, by weak convergence we have

$$\tilde{b}^{k+1}_i(\Sigma_{-i} \times X^k_{-i}(\alpha)) = \int_{\Sigma_{-i}\times X^k_{-i}(\alpha)} \mathrm{d}\tilde{b}^{k+1}_i = \lim_{\nu\to\infty} \int_{\Sigma_{-i}\times X^k_{-i}(\alpha)} \mathrm{d}b^{k+1}_{i,\nu} =$$

$$\lim_{\nu\to\infty} b^{k+1}_{i,\nu}(\Sigma_{-i} \times X^k_{-i}(\alpha)) = 1.$$

Therefore, $\tilde{b}_i \in \hat{B}_i(\alpha+1)$ and $\hat{B}_i(\alpha+1)$ is compact.

□

In the remainder of the paper, with an abuse of notation we will denote with $\overline{B}_i$ the set of collectively coherent beliefs or any of its compact subsets.

Ambiguous Hierarchies

Differently from [Geanakoplos et al. 1989], where the beliefs of a player i are given by the elements $b_i \in \overline{B}_i$, we generalize the model and allow beliefs to be compact subsets[10] $K_i \subseteq \overline{B}_i$. We denote with $\mathscr{K}_i$ the set of all compact subsets of $\overline{B}_i$. This choice enables to consider the ambiguity players come up against during the game, due to the uncertainty about other players' actions and beliefs.

[9] In fact $\hat{B}_i(\alpha)$ is a subset of the compact space B_i for every $\alpha \geq 0$.

[10] The assumption of compactness of beliefs is rather standard (see for istance [Ahn 2007]) as it keeps the problem much more tractable from the mathematical point of view. Nevertheless it seems that non-compact beliefs might be realistic in some specific situation.

The interpretation is similar to the classical one of games under strategic ambiguity: the agent does not have a precise belief b_i but knows that the belief can be any $b_i \in K_i$. Trivially, if K_i is a singleton, then the belief is not ambiguous, leading the theory back to the standard case.

REMARK 2.4: We introduced ambiguity at the end of the process, representing ambiguous beliefs as compact subsets of the product space $\overline{B}_i$, but there is another natural approach to represent ambiguity on hierarchies of beliefs as shown in Ahn 2007[11]. Roughly speaking, Ahn's approach is to introduce ambiguity at each level of the hierarchy of beliefs, and then to take the product. However, Ahn himself proved the universality of the construction[12], i.e. our approach is actually equivalent to Ahn's one when coherency of beliefs is common knowledge.

2.2 Game and equilibria

Following the model in [Geanakoplos et al. 1989] each agent i is endowed with an utility function

$$u_i : \overline{B}_i \times \Sigma \to \mathbb{R}, \tag{2}$$

depending not only on the mixed strategy profile but also on agent's beliefs: $u_i(b_i, \sigma)$ represents the payoff to player i if he believed b_i and found out that σ was actually played. Indeed, fixing b_i, $u_i(b_i, \cdot)$ is a classical expected utility function. As agents face set-valued beliefs $K_i \in \mathscr{K}_i$, they have a set-valued payoff $\{u_i(b_i, \sigma)\}_{b_i \in K_i}$ for every given belief $K_i \in \mathscr{K}_i$ and strategy profile $\sigma \in \Sigma$. There are several ways in which agents' ambiguitymight be solved depending on the agents' attitudes towards ambiguity[13] . In this paper we focus on the so called *maxmin* preferences [see Gilboa and Schmeidler 1989]: each agent i has an utility function $U_i : \mathscr{K}_i \times \Sigma \to \mathbb{R}$ defined by

$$U_i\left(K_i, \sigma\right) = \inf_{b_i \in K_i} u_i(b_i, \sigma) \quad \forall (K_i, \sigma) \in \mathscr{K}_i \times \Sigma. \tag{3}$$

REMARK 2.5: In formula (3), we are implicitly assuming that the definition of U_i is well posed. This is obviously satisfied if the function u_i is continuous; in this case it immediately follows that $\inf u_i(b_i, \sigma) = \min u_i(b_i, \sigma)$.

Now, it is possible to define the game:

DEFINITION 2.6: A *normal form psychological game under ambiguity* is defined by

$$G = \{A_1, \cdots, A_n, U_1, \cdots, U_n\}$$

where the utility functions U_i are defined as in formula (3) for every $i \in N$.

[11]Similar results about the universality of unambiguous hierarchies of beliefs can be found in [Mariotti et al. 2005].

[12]Details are rather technical, we refer to Ahn's paper and in particular to Proposition 4 and the diagram in Figure 1 therein.

[13]See [Gilboa and Marinacci 2013] for a survey and many references.

In the classical models of strategic ambiguity, players have vague beliefs about their opponents' behavior and these beliefs might depend on the actual strategy; for instance, this is the case when players have partial knowledge of the strategies played by their opponents. In particular, when ambiguity is expressed by multiple probability distributions, each agent's beliefs take the form of a set-valued map (or correspondence) from the strategy profiles set to the set of probability distributions over opponents' strategies (see [Lehrer 2012], [De Marco and Romaniello 2012]). In this paper we generalize this approach to hierarchies of beliefs: we assume that each agent i is endowed with a set-valued map $\gamma_i : \Sigma \rightsquigarrow \overline{B}_i$, (that we call *belief correspondence* of player i), where each image $\gamma_i(\sigma)$ is not empty and compact, i.e.:

$$\emptyset \neq \gamma_i(\sigma) \in \mathscr{K}_i \quad \forall \sigma \in \Sigma.$$

Each subset $\gamma_i(\sigma) \subseteq \overline{B}_i$ provides the set of hierarchies of beliefs that player i perceives to be consistent given the strategy profile σ. Hence:

Definition 2.7: A *psychological Nash equilibrium under ambiguity* of the game G with belief correspondences $\gamma = (\gamma_1, \dots, \gamma_n)$ is a pair (K^*, σ^*), where $K^* = (K_1^*, \dots, K_n^*)$ with $K_i^* \subseteq \overline{B}_i$ and $\sigma^* \in \Sigma$, such that for every player i:

i) $K_i^* = \gamma_i(\sigma^*)$;

ii) $U_i(K_i^*, \sigma^*) \geqslant U_i(K_i^*, (\sigma_i, \sigma_{-i}^*))$ for every $\sigma_i \in \Sigma_i$.

In this case, we can also say that $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium under ambiguity.

We point out that the definition above captures, in a natural way, the main features of the classical equilibrium notion since condition *ii)* requires that the equilibrium strategy of each player is optimal given his beliefs and condition *i)* requires that beliefs must be consistent with the equilibrium strategy profile.

Similarly to [Geanakoplos et al. 1989], psychological equilibria have a characterization as Nash equilibria. Let $w_i : \Sigma \times \Sigma \to \mathbb{R}$ be the *summary utility function* defined by

$$w_i(\sigma, \tau) = U_i(\gamma_i(\sigma), \tau) = \inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma. \tag{4}$$

Then the *summary form* of the game G is $\hat{G} := (A_1, \cdots, A_n, w_1, \cdots, w_n)$. Now, it immediately follows from the definition that

Lemma 2.8: *The profile $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium under ambiguity if and only if, for every player i,*

$$w_i(\sigma^*, (\sigma_i^*, \sigma_{-i}^*)) \geqslant w_i(\sigma^*, (y_i, \sigma_{-i}^*)) \quad \forall y_i \in \Sigma_i. \tag{5}$$

Remark 2.9: Condition 5 above means that $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium if and only if σ^* is a mixed strategy equilibrium of a classical strategic form game where utility functions are specified by the strategy profile σ^*, that is the utility functions are $\sigma \in \Sigma \to w_i(\sigma^*, \sigma) \in \mathbb{R}$, for every player i.

Relation with psychological games without ambiguity

REMARK 2.10: From Definition 2.7 it is immediately clear why the concept of psychological Nash equilibrium under ambiguity is a natural generalization of the concept of psychological Nash equilibrium defined in [Geanakoplos et al. 1989]. In fact, when we replace, in Definition 2.7, the set-valued map γ_i with the (single-valued) map β_i as defined in [Geanakoplos et al. 1989], then we get the classical definition of psychological Nash equilibrium. More precisely,

DEFINITION 2.11: A *normal form psychological game* is determined by

$$G^{GPS} = \{A_1, \cdots, A_n, u_1, \cdots, u_n\},$$

where the utility functions u_i are defined as in (2) for every $i \in N$.

Recall that in [Geanakoplos et al. 1989] the function $\beta_i : \Sigma \to \overline{B}_i$ is used to represent, for every mixed strategy profile σ, the hierarchy of correct beliefs $\beta_i(\sigma)$. Therefore:

DEFINITION 2.12: A *psychological Nash equilibrium* of the game G^{GPS} is a pair (b^*, σ^*), where $b^* = (b_1^*, \ldots, b_n^*)$ with $b_i^* \in \overline{B}_i$ and $\sigma^* \in \Sigma$, such that for every player i,

i) $b_i^* = \beta_i(\sigma^*)$;

ii) $u_i(b_i^*, \sigma^*) \geqslant u_i(b_i^*, (\sigma_i, \sigma_{-i}^*))$ for every $\sigma_i \in \Sigma_i$.

In this case, we can also say that $(\beta(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium.

The GPS summary utility function of player i, that here we denote with w_i^{GPS}, takes the form

$$w_i^{GPS}(\sigma, \tau) = u_i(\beta_i(\sigma), \tau) \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma. \tag{6}$$

Hence, $(\beta(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium if and only if

$$w_i^{GPS}(\sigma^*, (\sigma_i^*, \sigma_{-i}^*)) \geqslant w_i^{GPS}(\sigma^*, (y_i, \sigma_{-i}^*)) \quad \forall y_i \in \Sigma_i, \forall i \in N. \tag{7}$$

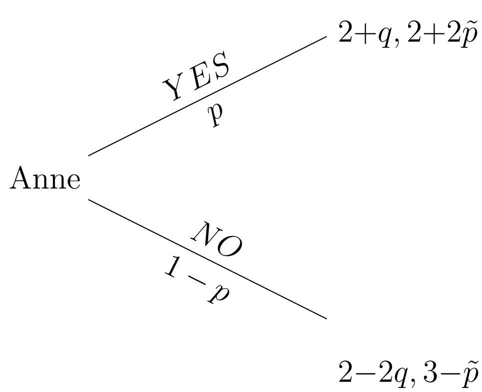
3 Examples

In this section we provide three different examples that show some possible effects of ambiguity on psychological equilibria. In Example 3.1 we look at the *Bravery Game* presented in [Geanakoplos et al. 1989] and show that when beliefs are perturbed by ambiguity, then the set of psychological equilibria remains unaltered. The Bravery Game is slightly modified in Example 3.2, where the unique psychological equilibrium under ambiguity differs from the unique equilibrium in the unambiguous case; however, when ambiguity converges to zero, equilibria of the ambiguous games converge to the equilibrium of the unambiguous one. Finally, Example 3.3 is a variation of the *Confidence Game* presented in [Geanakoplos et al. 1989]. In this case, the presence of ambiguity allows to refine the set of equilibria: roughly speaking, the psychological game without ambiguity has two equilibria while an infinitesimal amount of ambiguity destroys one (and only one) of them.

Example 3.1: We consider the *Bravery Game* presented in figure 2 in [Geanakoplos et al. 1989]. The game is the following: player 1 (say John) invites player 2 (say Anne) to go out for a dinner. Anne can accept or not, i.e. Anne's strategy set is $A_A = \{\text{YES, NO}\}$, while John is a non-active player, unsure about Anne's decision. We will firstly analyze the non-ambiguous case, on the line of [Geanakoplos et al. 1989]. Suppose that p is the probability that Anne says YES. John's first order beliefs about Anne's strategy are given by probability measures on the interval $[0, 1]$. However, in this example, it is considered the case in which only the mean of these measures plays a role in the utility functions; so, let q denote the mean of John's belief about Anne's mixed strategy. Anne prefers not to displease John, therefore if she believes that John believes she will accept, then she will be happier saying YES. Similarly to first order beliefs, only the mean of Anne's second order beliefs matter in the utility functions of this example; so, we denote with $\tilde{p}$ the mean of Anne's belief about the mean of John's belief . As John is a non-active player, only the mixed strategy profile is identified just by p. With an abuse of notation, in this example the correct belief functions simply map the strategy p to correct mean beliefs; more precisely, $\beta_J(p) = p$ tells that the correct mean of John's first order belief about Anne's strategy p must be equal to p and $\beta_A(p) = p$ tells that the correct mean of Anne's second order belief about John's first order belief about Anne's strategy p must be equal to p as well. The expected utility of Anne[14], that is proposed in [Geanakoplos et al. 1989], takes the following form for every belief $\tilde{p}$:

$$u_A(\tilde{p}, y) = y(2 + 2\tilde{p}) + (1 - y)(3 - \tilde{p}) = \tilde{p}(3y - 1) + 3 - y,$$

and the game is represented in the figure below:



As shown in [Geanakoplos et al. 1989], this game has three Nash equilibria:

- $p = 1 = \tilde{p} = q$: Anne says "YES";
- $p = 0 = \tilde{p} = q$: Anne says "NO";
- $p = 1/3 = \tilde{p} = q$: Anne randomizes with probabilities $p = 1/3$ and $1 - p = 2/3$.

[14] John's expected utility does not play any role in equilibrium so it is superfluous here.

Now, we introduce a specific form of ambiguous beliefs in the game. Suppose that Anne's belief is not a singleton anymore, but it is an interval: $\gamma_A^\varepsilon(p) = [p-\varepsilon, p+\varepsilon] \cap [0,1]$ with $\varepsilon > 0$ is the set-valued function that describes Anne's (second order) beliefs.

In order to compute Anne's summary utility function, we firstly compute

$$\arg\min_{\tilde{p}\in\gamma_A^\varepsilon(p)} u_A(\tilde{p}, y) = \left\{ \tilde{p}' \in [0,1] \;\middle|\; u_A(\tilde{p}', y) = \min_{\tilde{p}\in\gamma_A^\varepsilon(p)} u_A(\tilde{p}, y) \right\}.$$

We get

$$\arg\min_{\tilde{p}\in\gamma_A^\varepsilon(p)} u_A(\tilde{p}, y) = \arg\min_{\tilde{p}\in[p-\varepsilon,p+\varepsilon]\cap[0,1]} [\tilde{p}(3y-1)+3-y] = \begin{cases} \min\{p+\varepsilon, 1\} & \text{if } y \in [0, 1/3), \\ \gamma_A^\varepsilon(p) & \text{if } y = 1/3, \\ \max\{p-\varepsilon, 0\} & \text{if } y \in (1/3, 1]. \end{cases}$$

Denote with $p^- = \max\{p-\varepsilon, 0\}$ and $p^+ = \min\{p+\varepsilon, 1\}$ Therefore

$$w_A(p,y) = U_A(\gamma_A^\varepsilon(p), y) = \min_{\tilde{p}\in\gamma_A^\varepsilon(p)} [\tilde{p}(3y-1)+3-y] =$$

$$\begin{cases} p^+(3y-1)+3-y = y(3p^+-1)+3-p^+ & \text{if } y \in [0,1/3[, \\ 3-y & \text{if } y = 1/3, \\ p^-(3y-1)+3-y = y(3p^--1)+3-p^- & \text{if } y \in]1/3, 1]. \end{cases}$$

Recall that p gives a psychological Nash equilibrium under ambiguity if and only if

$$w_A(p,p) \geqslant w_A(p,y) \quad \forall y \in [0,1]$$

Now, three cases are possible:

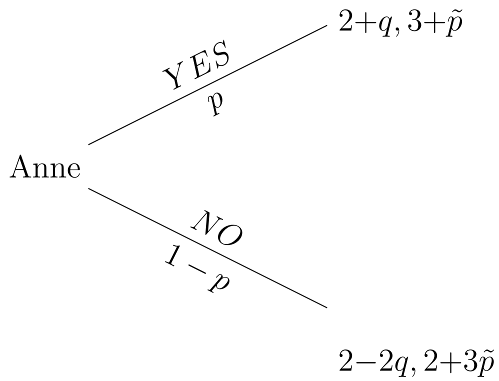
i) If p is such that $1/3 < p^- < p^+$, (that is $p > 1/3+\varepsilon$), then $w_A(p,y)$ is strictly increasing in $[0,1]$ and attains its maximum at $y=1$. So, in this case, there is only one equilibrium corresponding to $p=1$.

ii) If p is such that $p^- < p^+ < 1/3$, (that is $p < 1/3-\varepsilon$), then $w_A(p,y)$ is strictly decreasing in $[0,1]$ and attains its maximum at $y=0$. So, in this case, the unique equilibrium corresponds to $p=0$.

iii) If p is such that $p^- \leqslant 1/3 \leqslant p^+$, (that is $p \in [1/3-\varepsilon, 1/3+\varepsilon]$), then $w_A(p,y)$ is strictly increasing in $[0,1/3]$ and strictly decreasing in $[1/3,1]$. Therefore $y=1/3$ is the maximum point and there is only one equilibrium corresponding $p=1/3$.

Summarizing, we found out that the set of equilibria under ambiguity remains unaltered with respect to the nonambiguous case.

Example 3.2: We consider a variation of the previous example. Everything is unaltered except for Anne's payoff. The game is now the following:



The expected utility of Anne takes now the following form, for every belief $\tilde{p}$:

$$u_A(\tilde{p}, y) = y(3 + \tilde{p}) + (1 - y)(2 + 3\tilde{p}) = y(1 - 2\tilde{p}) + 3\tilde{p} + 2.$$

Firstly, let us look at psychological Nash equilibria (without ambiguity). Recall that p gives a psychological Nash equilibrium if and only if

$$w_A^{GPS}(p, p) \geqslant w_A^{GPS}(p, y) \quad \forall y \in [0, 1]$$

Now, we immediately get that

$$\begin{aligned} w_A^{GPS}(p, p) &< w_A^{GPS}(p, 1) \quad \forall p < 1/2, \\ w_A^{GPS}(p, p) &< w_A^{GPS}(p, 0) \quad \forall p > 1/2, \\ w_A^{GPS}(p, y) = 3p + 2 &\quad \forall y \in [0, 1] \quad \text{if } p = 1/2. \end{aligned}$$

Consequently, the unique psychological Nash equilibrium corresponds to $p = 1/2$.

We introduce ambiguity as done in the previous example: $\gamma_A^\varepsilon(p) = [p - \varepsilon, p + \varepsilon] \cap [0, 1]$, with $\varepsilon > 0$, is the set-valued function that describes Anne's (second order) beliefs. It follows that

$$\arg\min_{\tilde{p} \in \gamma_A^\varepsilon(p)} u_A(\tilde{p}, y) = \arg\min_{\tilde{p} \in [p-\varepsilon, p+\varepsilon] \cap [0,1]} [\tilde{p}(3 - 2y) + y + 2] = \max\{p - \varepsilon, 0\}.$$

Denote again with $p^- = \max\{p - \varepsilon, 0\}$. Therefore

$$w_A(p, y) = \min_{\tilde{p} \in \gamma_A^\varepsilon(p)} [y(1 - 2\tilde{p}) + 3\tilde{p} + 2] = y(1 - 2p^-) + 3p^- + 2$$

It can be immediately checked that

$$\begin{aligned} w_A(p, p) &< w_A(p, 1) \quad \forall p^- < 1/2 \\ w_A(p, p) &< w_A(p, 0), \quad \forall p^- > 1/2 \\ w_A(p, y) = 3p^- + 2 &\quad \forall y \in [0, 1], \quad \text{if } p^- = 1/2 \end{aligned}$$

So, the unique equilibrium that we get corresponds to p_ε such that $p^- = 1/2$, where this latter condition is equivalent to $p_\varepsilon = 1/2 + \varepsilon$. Finally, note that, as $\varepsilon \to 0$, the equilibrium p_ε converges to the unique equilibrium $p = 1/2$ of the game without ambiguity.

EXAMPLE 3.3: In this example, we consider a variation of the example constructed in [Geanakoplos et al. 1989], figure 3. There are two players, again Anne and John, acting simultaneously. Anne selects rows and John selects columns; the pure strategy set of Anne is $A_A = \{c, n\}$ and the pure strategy set of John is $A_J = \{C, N\}$. We denote with p the mixed strategy of Anne, where, with an abuse of notation, p is the probability of c and $1 - p$ is the probability of n. Similarly r is the mixed strategy of John; again, with an abuse of notation, r is the probability of C and $1-r$ is the probability of N. We assume that John's utility does not depend on beliefs while Anne's utility depends also on her second order beliefs. Moreover, as done in the previous examples, we consider the case in which only the mean of beliefs plays a role in Anne's utility function. We denote with $q \in [0, 1]$ the mean of John's first order belief about Anne's mixed strategy p and $\tilde{q} \in [0, 1]$ the mean of Anne's second order belief about the mean q of John's first order belief. So, the game is the following:

	C	N
c	$\tilde{q}, 0$	$\tilde{q}, 1$
n	$1, 0$	$1, 1$

A mixed strategy profile is identified by the pair (p, r). Also in this example, the correct belief functions simply map the strategy profiles (p, r) to correct mean beliefs; more precisely, $\beta_J(p, r) = p$ tells that the correct mean of John's first order belief about Anne's strategy p must be equal to p and $\beta_A(p, r) = p$ tells that the correct mean of Anne's second order belief about John's first order belief about Anne's strategy p must be equal to p as well.

A psychological Nash equilibrium is unequivocally determined by a mixed strategy profile (p^*, r^*) such that

$$w_A^{GPS}((p^*, r^*), (p^*, r^*)) \geqslant w_A^{GPS}((p^*, r^*), (y, r^*)) \quad \forall y \in [0, 1],$$
$$w_J^{GPS}((p^*, r^*), (p^*, r^*)) \geqslant w_J^{GPS}((p^*, r^*), (p^*, y)) \quad \forall y \in [0, 1].$$

Now, it is clear that strategy N (that is $r = 0$) is strictly dominant for John. So, in equilibrium r must be equal to 0. Hence, we need only to find the Anne's best reply, given that John plays N. In this case, the expected utility for Anne playing y and having second order belief $\tilde{q}$ is

$$u_A(\tilde{q}, y) = y(\tilde{q} - 1) + 1.$$

So, if $\tilde{q} < 1$ then Anne's best reply is $y = 0$. If $\tilde{q} = 1$, then every $y \in [0,1]$ is a best reply. It follows that

$$w_A^{GPS}((0,0),(0,0)) \geqslant w_A^{GPS}((0,0),(y,0)) \quad \forall y \in [0,1],$$
$$w_A^{GPS}((1,0),(1,0)) = w_A^{GPS}((1,0),(y,0)) \quad \forall y \in [0,1]$$

Therefore the strategy profiles $(p,r) = (0,0)$ and $(p,r) = (1,0)$ are psychological Nash equilibria. Note that there are no other Psychological Nash equilibria since

$$w_A^{GPS}((p,0),(p,0)) < w_A^{GPS}((p,0),(0,0)) \quad \forall p \in]0,1[,$$

and so the strategy profiles $(p,0)$ are not psychological Nash equilibria for every $p \in]0,1[$.

Now, suppose that Anne's second order belief is given by the interval $\gamma_A^{\varepsilon}(p) = [p-\varepsilon, p+\varepsilon] \cap [0,1]$ with $\varepsilon > 0$. For the sake of simplicity, we assume also that ε is small enough. Ambiguity does not affect John's utility so that $r = 0$ is again a strictly dominant strategy for John. It follows that every psychological equilibrium under ambiguity is given by a pair (p,r) with $r = 0$. So we only have to find Anne's best reply to $r = 0$. Again, given that $r = 0$, the expected utility for Anne playing y and having second order belief $\tilde{q}$ is

$$u_A(\tilde{q}, y) = y(\tilde{q} - 1) + 1.$$

We get

$$\arg\min_{\tilde{q} \in \gamma_A^{\varepsilon}(p)} u_A(\tilde{q}, y) = \arg\min_{\tilde{q} \in [p-\varepsilon, p+\varepsilon] \cap [0,1]} [\tilde{q}y + 1 - y] = \begin{cases} \gamma_A^{\varepsilon}(p) & \text{if } y = 0, \\ \max\{p - \varepsilon, 0\} & \text{if } y \in]0,1]. \end{cases}$$

Denote with $p^- = \max\{p - \varepsilon, 0\}$. Therefore

$$w_A((p,0),(y,0)) = U_A(\gamma_A^{\varepsilon}(p), y) =$$

$$\min_{\tilde{q} \in \gamma_A^{\varepsilon}(p)} [\tilde{q}y + 1 - y] = \begin{cases} 1 & \text{if } y = 0 \\ p^- y + 1 - y = y(p^- - 1) + 1 & \text{if } y \in]0,1] \end{cases}$$

Now, since

$$p^- \leqslant 1 - \varepsilon \implies p^- - 1 \leqslant -\varepsilon < 0,$$

then $U_A(\gamma_A^{\varepsilon}(p), y)$ is strictly decreasing with respect to y in the interval $[0,1]$. It follows that

$$w_A((0,0),(0,0)) \geqslant w_A((0,0),(y,0)) \quad \forall y \in [0,1],$$

which implies that $p = 0$ is a best reply to $r = 0$ given that beliefs are consistent with $p = 0$. Hence $(p,r) = (0,0)$ is a psychological Nash equilibrium under ambiguity. Moreover,

$$w_A((p,0),(p,0)) < w_A((p,0),(0,0)) \quad \forall p \in]0,1],$$

therefore every $p > 0$ cannot be an equilibrium strategy when $r = 0$. The unique psychological Nash equilibrium under ambiguity is $(p, r) = (0, 0)$.

This example shows that the presence of ambiguity detroys the psychological equilibrium $(p, r) = (1, 0)$ and selects only the equilibrium $(p, r) = (0, 0)$.

4 Equilibrium existence

This section is devoted to the issue of existence of psychological Nash equilibrium under ambiguity. To this purpose we need to recall some tools on set-valued maps[15].

Preliminaries about correspondences

Consider a set-valued map $\Gamma : X \rightsquigarrow Y$ between two metric spaces X and Y.
Then, the *upper limit* of Γ in $\overline{x} \in X$ is defined by

$$\operatorname*{Lim\,sup}_{x \to \overline{x}} \Gamma(x) = \left\{ y \in Y \;\middle|\; \liminf_{x \to \overline{x}} d(y, \Gamma(x)) = 0 \right\},$$

where $d(y, \Gamma(x))$ denotes the distance (in the metric space Y) between y and the set $\Gamma(x)$, while the *lower limit* of Γ in $\overline{x} \in X$ is defined by

$$\operatorname*{Lim\,inf}_{x \to \overline{x}} \Gamma(x) = \left\{ y \in Y \;\middle|\; \lim_{x \to \overline{x}} d(y, \Gamma(x)) = 0 \right\}.$$

Definition 4.1: The set-valued map $\Gamma : X \rightsquigarrow Y$ is said to be:

i) lower semicontinuous at $\overline{x} \in X$ if $\Gamma(\overline{x}) \subseteq \operatorname*{Lim\,inf}_{x \to \overline{x}} \Gamma(x)$, meaning that for any $y \in \Gamma(\overline{x})$ and for any sequence $(x_\nu)_\nu \subset X$ converging to $\overline{x}$, there exists a sequence of elements $(y_\nu)_\nu \subset Y$, with $y_\nu \in \Gamma(x_\nu)$ for every $\nu \in \mathbb{N}$, that converges to y. Γ is lower semicontinuous in X if it is so in every point $x \in X$;

ii) closed at $\overline{x} \in X$ if

$$\operatorname*{Lim\,sup}_{x \to \overline{x}} \Gamma(x) \subseteq \Gamma(\overline{x}),$$

that is, for every sequence $(x_\nu)_\nu \subset X$ converging to $\overline{x}$ and every sequence $(y_\nu)_\nu \subset Y$, with $y_\nu \in \Gamma(x_\nu)$ for every $\nu \in \mathbb{N}$, that converges to a point $y \in Y$, it follows that $y \in \Gamma(\overline{x})$. Γ is closed in X if it is so in every point $x \in X$. Moreover Γ is closed in X if and only if $\mathit{Graph}(\Gamma) = \{(x, y) \,|\, x \in X,\ y \in \Gamma(x)\}$ is a closed subset of $X \times Y$;

iii) upper semicontinuous at $\overline{x} \in X$ if for any neighborhood $\mathcal{U}$ of $\Gamma(\overline{x})$ there exists $\eta > 0$ such that $\Gamma(x) \subset \mathcal{U}$ for all $x \in B_X(\overline{x}, \eta) = \{x \in X : \|x - \overline{x}\|_X < \eta\}$. Γ is upper semicontinuous in X if it is so in every point $x \in X$;

[15] We refer mainly to [Aubin and Frankowska 1990] and references therein.

iv) continuous at $\overline{x} \in X$ if it is upper and lower semicontinuous at $\overline{x}$; Γ is continuous in X if it is so in every point $x \in X$.

Recall that if X is closed, Y is compact and Γ has closed values then Γ is upper semicontinuous if and only if it is closed (see Proposition 1.4.8 in [Aubin and Frankowska 1990]). We will see that every set-valued map introduced in this paper satisfies these properties, therefore, in our setting, upper semicontinuity and closedness are equivalent notions.

We conclude this section recalling some useful and well-known results. The first result is a version of the Berge's maximum theorem as presented in [Aubin and Frankowska 1990, Theorem 1.4.16].

THEOREM 4.2: *Let* X, Y *be two metric spaces,* $\Gamma : X \rightsquigarrow Y$ *a set-valued map and* $f : Graph(\Gamma) \to \mathbb{R}$ *a function. Let* $g : X \to \overline{\mathbb{R}}$ *be the marginal function defined by*

$$g(x) = \sup_{y \in \Gamma(x)} f(x, y) \quad \forall x \in X$$

Then,

i) If f *is a lower semicontinuous function and* Γ *a lower semicontinuous set-valued map then* g *is a lower semicontinuous function;*

ii) If f *is an upper semicontinuous function and* Γ *an upper semicontinuous set-valued map with compact images then* g *is an upper semicontinuous function.*

The second result is the well known Kakutani fixed point theorem.

THEOREM 4.3 (Kakutani fixed point theorem): *Let* X *be an Euclidean finite dimensional space. Let* K *be a non-empty compact convex subset of* X*. If* $\Gamma : K \rightsquigarrow K$ *is an upper semicontinuous mapping such that, for all* $x \in K$*, the set* $\Gamma(x)$ *is convex, closed and non-empty, then there exists a fixed point for* Γ*, that is a point* $\overline{x} \in K$ *such that* $\overline{x} \in \Gamma(\overline{x})$*.*

The existence theorem

THEOREM 4.4: *Consider a psychological game under ambiguity* $G = (A_1, \cdots, A_n, U_1, \cdots, U_n)$ *as presented in Definition 2.6. Assume that, for every player* $i \in I$*,*

i) $u_i : \overline{B}_i \times \Sigma \to \mathbb{R}$ *is a continuous function in* $\overline{B}_i \times \Sigma$*;*

ii) $u_i(b_i, (\cdot, \tau_{-i})) : \Sigma_i \to \mathbb{R}$ *is a quasi-concave function*[16] *in* Σ_i*, for every* $b_i \in \overline{B}_i$ *and every* $\tau_{-i} \in \Sigma_{-i}$*;*

[16]Here we refer to the classical definition of quasi-concavity: a function $g : X \to \mathbb{R}$ (where X is convex) is quasi-concave in X if and only the upper level sets are convex subsets of X.

iii) $\gamma_i : \Sigma \rightsquigarrow \overline{B}_i$ *is a continuous set-valued map in* Σ *with not empty, convex and compact images* $\gamma_i(\sigma)$ *for every* $\sigma \in \Sigma$.

Then there exists $\sigma^* \in \Sigma$ *such that* $(\gamma(\sigma^*), \sigma^*)$ *is a psychological Nash equilibrium under ambiguity for the game* G.

Proof. Consider the summary form $\hat{G} := (A_1, \cdots, A_n, w_1, \cdots, w_n)$ of the game G. Let $BR_i : \Sigma \rightsquigarrow \Sigma_i$ be the set-valued map defined by:

$$BR_i(\sigma) := \{\tau_i \in \Sigma_i \,|\, w_i(\sigma, (\tau_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})) \quad \forall y_i \in \Sigma_i\} \quad \forall \sigma \in \Sigma,$$

and $BR : \Sigma \rightsquigarrow \Sigma$ the set-valued map defined by:

$$BR(\sigma) = \Pi_{i=1}^n BR_i(\sigma) \quad \forall \sigma \in \Sigma.$$

Lemma 2.8 guarantees that σ^* is a fixed point for BR, i.e. $\sigma^* \in BR(\sigma^*)$, if and only if $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium under ambiguity for G. Therefore, our proof reduces to verify the existence of such a fixed point. To this aim we apply Theorem 4.3 to the correspondence $BR : \Sigma \rightsquigarrow \Sigma$.

From the assumptions it follows that the summary utility function w_i defined in (4) is well posed; moreover, $w_i(\sigma, \tau) = \min_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau)$, for all $(\sigma, \tau) \in \Sigma \times \Sigma$. Theorem 4.2 ensures that w_i is continuous in $\Sigma \times \Sigma$, hence the best reply correspondence BR_i is upper semicontinuous with not empty and compact images $BR(\sigma)$ for every $\sigma \in \Sigma$. It follows that BR is upper semicontinuous[17] with not empty and compact images $BR_i(\sigma)$ for every $\sigma \in \Sigma$.

Lastly, it remains to prove that $BR(\sigma)$ is a convex subset of Σ. It is sufficient to verify that each $BR_i(\sigma)$ is convex subset of Σ_i, as the finite product of convex sets is obviously convex. Take $\lambda \in [0, 1]$ and $\overline{\tau}_i$, $\hat{\tau}_i \in BR_i(\sigma)$. We will prove that $\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i \in BR_i(\sigma)$. Since $\overline{\tau}_i$, $\hat{\tau}_i \in BR_i(\sigma)$, then

$$w_i(\sigma, (\overline{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})), \qquad \forall y_i \in \Sigma_i,$$
$$w_i(\sigma, (\hat{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})), \qquad \forall y_i \in \Sigma_i,$$

which implies that

$$u_i(b_i, (\overline{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})), \qquad \forall y_i \in \Sigma_i, \forall b_i \in \gamma_i(\sigma)$$

$$u_i(b_i, (\hat{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})), \quad \forall y_i \in \Sigma_i, \forall b_i \in \gamma_i(\sigma).$$

Therefore, for every $b_i \in \gamma_i(\sigma)$, it follows that

$$\alpha_{b_i} := \min\{u_i(b_i, (\overline{\tau}_i, \sigma_{-i})), u_i(b_i, (\hat{\tau}_i, \tau_{-i}))\} \geqslant w_i(\sigma, (y_i, \sigma_{-i})) \qquad \forall y_i \in \Sigma_i.$$

[17][Berge 1997, Theorem 4' page 114] shows that the cartesian product of a finite number of upper semicontinuous set-valued map is an upper semicontinuous map.

Now, since the function $u_i(b_i, (\cdot, \tau_{-i}))$ is quasi-concave, it follows that

$$u_i(b_i, (\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i, \sigma_{-i})) \geqslant \alpha_{b_i} \geqslant w_i(\sigma, (y_i, \sigma_{-i})) \qquad \forall y_i \in \Sigma_i.$$

Since the previous inequality holds for every $b_i \in \gamma_i(\sigma)$, we finally get

$$w_i(\sigma, (\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i, \sigma_{-i})) = \inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, (\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})).$$

Hence, $\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i \in BR_i(\sigma)$.

□

5 Ambiguous trembles and stability

As already mentioned in the Introduction, the classical theory of refinements of Nash equilibria deals with the problem of equilibrium selection based on properties of stability of the equilibria ([van Damme 1989]). It is well known that, in case of games with multiple equilibria, some of them may not be robust with respect to perturbations on the strategies or on the payoffs of the players, so that it is possible to restrict significatively the set of equilibria on the basis of some stability property. This approach arises with the concept of *trembling hand perfect equilibrium* for Nash equilibria introduced in the seminal paper by Selten (1975). The main idea underlying this concept is that players believe that their opponents can make mistakes playing their equilibrium strategies, therefore each equilibrium strategy should be *close* to the best reply against perturbed expectations about opponents' behavior, if the perturbation is small enough. In Geanakoplos et al. (1989), the concept of trembling hand perfect equilibrium is extended to psychological games: the idea is that strategies are perturbed as in Selten (1975) and hierarchies of beliefs are consistent with the (unperturbed) equilibrium strategies, along the perturbations. In this paper, we take into account a different perspective as, on the one hand, we look at the stability with respect to perturbations on the entire hierarchies of beliefs and, on the other hand, we allow for ambiguous perturbations, that (can) take the form of sets of hierarchies of beliefs.

To better understand the problem, we look at the Example 3.3 in Section 3. It turns out that, when the correct belief function β_A is perturbed by ambiguous trembles so that beliefs are represented by the set-valued map γ_A^ε, the set of equilibria reduces to just one out of the two equilibria that we find in the non ambiguous case. Namely, the presence of ambiguity detroys the psychological equilibrium $(p, r) = (1, 0)$ and selects only the equilibrium $(p, r) = (0, 0)$. Now, when ε converges to 0, the set-valued map γ_A^ε converges (in a suitable way) to β_A. Taking the sequence of the corresponding psychological Nash equilibria under ambiguity (the constant sequence obtained for $p = 0$ and $r = 0$), we get that, as $\varepsilon \to 0$, the limit process obviously selects $(p, r) = (0, 0)$ and not $(p, r) = (1, 0)$. So we have constructed a selection mechanism for psychological equilibria based on ambiguous trembles.

The arguments above have an underlying problem that concerns the way ambiguous belief should converge to correct beliefs in such a way that sequences of psychological equilibria under perturbations converge to psychological equilibria of the unperturbed game. Below we look at this problem that we embody in a larger one in which the unperturbed game can be itself ambiguous[18] and utilities can be perturbed as well[19]. We give a general limit theorem that gives conditions on the convergence of psychological games under ambiguity to an unperturbed one in such a way that corresponding sequences of equilibria under perturbation converge to unperturbed equilibria. Then we apply the theorem to construct selection criteria for classical psychological equilibria.

5.1 The limit theorem

In this subsection we show what conditions must be imposed in order that sequences of psychological equilibria under ambiguity of perturbed games converge to psychological equilibria under ambiguity of the unpertubed game, as the perturbation vanishes. In order to state and prove this limit result, we need firstly to recall definitions on variational convergence of sequences of functions and set-valued maps.

Technical tools

We referred mainly to the paper [Lignola and Morgan 1992] for the following definitions and results.

Definition 5.1: Let X be a topological space. Consider a sequence of functions[20] $\{g_\nu\}_{\nu\in\mathbb{N}}$ with $g_\nu : X \subset \mathbb{R}^k \to \overline{\mathbb{R}}$ for every $\nu \in \mathbb{N}$ and a function $g : X \subset \mathbb{R}^k \to \overline{\mathbb{R}}$.

i) The sequence of functions $\{g_\nu\}_{\nu\in\mathbb{N}}$ *epiconverges* to the function g if:

(1) for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have

$$g(x) \leqslant \liminf_{\nu\to\infty} g_\nu(x_\nu);$$

(2) for every $x \in X$ there exists a sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X such that

$$\limsup_{\nu\to\infty} g_\nu(x_\nu) \leqslant g(x).$$

ii) The sequence $\{g_\nu\}_{\nu\in\mathbb{N}}$ *hypoconverges* to the function g if the sequence of functions $\{-g_\nu\}_{\nu\in\mathbb{N}}$ epiconverges to the function $-g$.

[18] This means that we allow for perturbations of ambiguous beliefs

[19] The theory of refinements of Nash equilibria studies stability with respect to perturbations on payoffs as well (see, for instance the property of *essentiality* in [van Damme 1989]).

[20] For technical reasons, we consider the case where functions take values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

iii) The sequence $\{g_\nu\}_{\nu\in\mathbb{N}}$ *continuously converges* to the function g if it epiconverges and hypoconverges to the function g, i.e. if for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have:

$$g(x) = \lim_{\nu\to\infty} g_\nu(x_\nu) = \limsup_{\nu\to\infty} g_\nu(x_\nu) = \liminf_{\nu\to\infty} g_\nu(x_\nu). \tag{8}$$

The next definition is devoted to set-valued maps.

Definition 5.2: Let X and Y be metric spaces. Let $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of set-valued maps, with $\Gamma_\nu : X \rightsquigarrow Y$ for every $\nu \in \mathbb{N}$ and let $\Gamma : X \rightsquigarrow Y$ be a set-valued map. Let $S(y,\varepsilon)$ be the ball in Y with center in y and radius ε and

$$\operatorname*{Lim\,inf}_{\nu\to\infty} \Gamma_\nu(x_\nu) = \{y \in Y \mid \forall\varepsilon > 0,\ \exists\overline{\nu} \text{ s.t. for all } \nu \geq \overline{\nu}\ S(y,\varepsilon) \cap \Gamma_\nu(x_\nu) \neq \emptyset\},$$

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \Gamma_\nu(x_\nu) = \{y \in Y \mid \forall\varepsilon > 0,\ \forall\overline{\nu}\ \exists\nu \geq \overline{\nu} \text{ s.t. } S(y,\varepsilon) \cap \Gamma_\nu(x_\nu) \neq \emptyset\}.$$

Then

i) $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ is *sequentially lower convergent* to Γ if for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have:

$$\Gamma(x) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \Gamma_\nu(x_\nu);$$

ii) $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ is *sequentially upper convergent* to Γ if for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have:

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \Gamma_\nu(x_\nu) \subseteq \Gamma(x);$$

iii) $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ is *sequentially convergent* to Γ if for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have:

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \Gamma_\nu(x_\nu) \subseteq \Gamma(x) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \Gamma_\nu(x_\nu).$$

The result

Now we can state the limit theorem.

Theorem 5.3: *Let $G = \{A_1, \cdots, A_n, U_1, \cdots, U_n\}$ be a psychological game under ambiguity*[21]*. For every player i, let*

[21] Recall definition 2.6.

a) $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ *be a sequence of functions with* $u_{i,\nu} : \overline{B}_i \times \Sigma \to \mathbb{R}$ *for every* $\nu \in \mathbb{N}$*;*

b) $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ *be a sequence of set-valued maps* $\gamma_{i,\nu} : \Sigma \rightsquigarrow \overline{B}_i$*, for every* $\nu \in \mathbb{N}$*;*

c) $\{U_{i,\nu}\}_{\nu\in\mathbb{N}}$ *be the sequence of functions* $U_{i,\nu} : \mathscr{K}_i \times \Sigma \to \mathbb{R}$ *defined by*

$$U_{i,\nu}(K_i, \sigma) = \inf_{b_i\in K_i} u_{i,\nu}(b_i, \sigma) \quad \forall (K_i, \sigma) \in \mathscr{K}_i \times \Sigma$$

for every $\nu \in \mathbb{N}$*;*

d) $\{G_\nu\}_{\nu\in\mathbb{N}}$ *be the sequence of games where* $G_\nu = \{A_1, \cdots, A_n, U_{1,\nu}, \cdots, U_{n,\nu}\}$ *for every* $\nu \in \mathbb{N}$*.*

Assume that, for every player i*,*

i) *the sequence* $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ *sequentially converges to the function* u_i*;*

ii) *each function* $u_{i,\nu}$ *and the function* u_i *are continuous in* $\overline{B}_i$*;*

ii) *the sequence* $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ *sequentially converges to the set-valued map* γ_i*. Suppose additionally that each* $\gamma_{i,\nu}$ *and* γ_i *have compact and not-empty values for every* $\sigma \in \Sigma$*.*

If the sequence $\{\sigma^*_\nu\}_{\nu\in\mathbb{N}} \subset \Sigma$ *converges to* $\sigma^* \in \Sigma$ *and* $(\gamma_\nu(\sigma^*_\nu), \sigma^*_\nu)$ *is a psychological Nash equilibrium of* G_ν *for every* $\nu \in \mathbb{N}$*, then it follows that* $(\gamma(\sigma^*), \sigma^*)$ *is a psychological Nash equilibrium under ambiguity of* G*.*

Proof. For every player i and every $\nu \in \mathbb{N}$ let $w_{i,\nu}$ be the summary utility function in the game G_ν, that is

$$w_{i,\nu}(\sigma, \tau) := \inf_{b_i\in\gamma_{i,\nu}(\sigma)} u_{i,\nu}(b_i, \tau) \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma,$$

and w_i be the summary utility function in the game G, that is

$$w_i(\sigma, \tau) := \inf_{b_i\in\gamma_i(\sigma)} u_i(b_i, \tau) \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma.$$

We begin by proving the continuous convergence of the sequence $\{w_{i,\nu}\}_{\nu\in\mathbb{N}}$ to w_i. As defined in (8), we need to check that for every $(\sigma, \tau) \in \Sigma \times \Sigma$ and for every sequence $\{(\sigma_\nu, \tau_\nu)\}_{\nu\in\mathbb{N}}$ converging to (σ, τ) we get the inequalities

$$\limsup_{\nu\to\infty} w_{i,\nu}(\sigma_\nu, \tau_\nu) \leqslant w_i(\sigma, \tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}(\sigma_\nu, \tau_\nu).$$

Step 1: $w_i(\sigma,\tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau_\nu)$.

Continuity of u_i and $u_{i,\nu}$, for every ν, and compactness of the images of γ_i and $\gamma_{i,\nu}$, for every ν, guarantee that there exist $b_i^* \in \gamma_i(\sigma)$ and $b_{i,\nu}^* \in \gamma_{i,\nu}(\sigma_\nu)$, for every ν, such that

$$u_i(b_i^*,\tau) = \inf_{b_i\in\gamma_i(\sigma)} u_i(b_i,\tau) = w_i(\sigma,\tau), \quad u_{i,\nu}(b_{i,\nu}^*,\tau_\nu) = \inf_{b_{i,\nu}\in\gamma_{i,\nu}(\sigma_\nu)} u_{i,\nu}(b_{i,\nu},\tau_\nu) = w_{i,\nu}(\sigma_\nu,\tau_\nu).$$

Suppose by contradiction that

$$w_i(\sigma,\tau) > \liminf_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau_\nu). \tag{9}$$

This means that there exists a converging subsequence $\{(\sigma_{\nu_k},\tau_{\nu_k})\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} w_{i,\nu_k}(\sigma_{\nu_k},\tau_{\nu_k}) < w_i(\sigma,\tau).$$

The corresponding sequence of beliefs $\{b_{i,\nu_k}^*\}_{k\in\mathbb{N}}$ has a subsequence $\{b_{i,\nu_h}^*\}_{h\in\mathbb{N}}$ which is converging[22] to a point $\hat{b}_i \in \overline{B}_i$. Now, by definition, the upper limit $\operatorname{Lim\,sup}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu)$ contains the limits of every converging subsequence of $\{b_{i,\nu}^*\}_{\nu\in\mathbb{N}}$; then

$$\hat{b}_i \in \operatorname*{Lim\,sup}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu).$$

Moreover, from the assumptions it follows that $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ is sequentially upper convergent to γ_i, that is,

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu) \subseteq \gamma_i(\sigma).$$

Therefore, $\hat{b}_i \in \gamma_i(\sigma)$ and $u_i(b_i^*,\tau) \leqslant u_i(\hat{b}_i,\tau)$.

On the other hand, the sequence $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ epiconverges to u_i; in particular, it follows that

$$u_i(\hat{b}_i,\tau) \leqslant \liminf_{h\to\infty} u_{i,\nu_h}(b_{i,\nu_h}^*,\tau_{\nu_h}).$$

Since $u_i(b_i^*,\tau) = w_i(\sigma,\tau)$ and $u_{i,\nu}(b_{i,\nu}^*,\tau_\nu) = w_{i,\nu}(\sigma_\nu,\tau_\nu)$, for every $\nu \in \mathbb{N}$, we finally get

$$w_i(\sigma,\tau) = u_i(b_i^*,\tau) \leqslant u_i(\hat{b}_i,\tau) \leqslant \liminf_{h\to\infty} u_{i,\nu_h}(b_{i,\nu_h}^*,\tau_{\nu_h}) =$$

$$\liminf_{h\to\infty} w_{i,\nu_h}(\sigma_{\nu_h},\tau_{\nu_h}) = \lim_{h\to\infty} w_{i,\nu_h}(\sigma_{\nu_h},\tau_{\nu_h}).$$

This implies, using inequality (9), that

$$w_i(\sigma,\tau) \leqslant \lim_{h\to\infty} w_{i,\nu_h}(\sigma_{\nu_h},\tau_{\nu_h}) < w_i(\sigma,\tau),$$

which results in a contradiction. So

$$w_i(\sigma,\tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau_\nu).$$

[22] This is true because $\overline{B}_i$ is compact.

Step 2: $w_i(\sigma, \tau) \geqslant \limsup_{\nu\to\infty} w_{i,\nu}(\sigma_v, \tau)$.

Let $b_i^* \in \gamma_i(\sigma)$ be such that

$$u_i(b_i^*, \tau) = \inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) = w_i(\sigma, \tau).$$

The sequence $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ is sequentially lower convergent to γ_i, that is,

$$\gamma_i(\sigma) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu).$$

Then there exists a sequence $\{b_{i,\nu}\}_{\nu\in\mathbb{N}}$ converging to b_i^* such that $b_{i,\nu} \in \gamma_{i,\nu}(\sigma_\nu)$, for every ν.

The sequence $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ hypoconverges to u_i; it follows that

$$\limsup_{\nu\to\infty} u_{i,\nu}(b_{i,\nu}, \tau_\nu) \leqslant u_i(b_i^*, \tau).$$

Moreover, by construction $w_{i,\nu}(\sigma_\nu, \tau_\nu) \leqslant u_{i,\nu}(b_{i,\nu}, \tau_\nu)$ for every $\nu \in \mathbb{N}$. This finally implies that

$$\limsup_{\nu\to\infty} w_{i,\nu}(\sigma_\nu, \tau_\nu) \leqslant \limsup_{\nu\to\infty} u_{i,\nu}(b_{i,\nu}, \tau_\nu) \leqslant u_i(b_i, \tau) = w_i(\sigma, \tau).$$

As a consequence of the continuous convergence of the sequence $\{w_{i,\nu}\}_{\nu\in\mathbb{N}}$ to w_i, for every player i, we immediately get the result. In fact, let $\{\sigma_\nu^*\}_{\nu\in\mathbb{N}} \subset \Sigma$ be a sequence converging to $\sigma^* \in \Sigma$ such that $(\gamma_\nu(\sigma_\nu^*), \sigma_\nu^*)$ is a psychological Nash equilibrium of G_ν for every $\nu \in \mathbb{N}$. From the characterization of psychological Nash equilibria under ambiguity it follows that, for every player i,

$$w_{i,\nu}(\sigma_\nu^*, \sigma_\nu^*) \geqslant w_{i,\nu}(\sigma_\nu^*, (y_i, \sigma_{-i,\nu}^*)) \quad \forall y_i \in \Sigma_i.$$

Applying the continuous convergence of $\{w_{i,\nu}\}_{\nu\in\mathbb{N}}$ we get

$$w_i(\sigma^*, \sigma^*) = \lim_{\nu\to\infty} w_{i,\nu}(\sigma_\nu^*, \sigma_\nu^*) \geqslant \lim_{\nu\to\infty} w_{i,\nu}(\sigma_\nu^*, (y_i, \sigma_{-i,\nu}^*)) = w_i(\sigma^*, (y_i, \sigma_{-i}^*)) \quad \forall y_i \in \Sigma_i.$$

This latter inequality implies that $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium under ambiguity of G. □

Remark 5.4: The proof of the previous theorem is self contained. An alternative proof could be obtained by applying the stability results for marginal functions under constraints as considered in Lignola and Morgan (1992).

5.2 Equilibrium selection

Building upon the previous result, in this subsection we show how to construct selection mechanism for psychological Nash equilibria based on ambiguous trembles. Let $G^{GPS} = \{A_1, \cdots, A_n, u_1, \cdots, u_n\}$ be an ambiguous game having GPS psychological Nash equilibria. The selection mechanism works as follows:

- for every player i, choose a sequence of beliefs correspondences $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$, with $\gamma_{i,\nu}:\Sigma\rightsquigarrow\overline{B}_i$ that sequentially converges to the function β_i;

- for every player i, choose a sequence of utility functions $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$,with $u_{i,\nu}:\overline{B}_i\times\Sigma\to\mathbb{R}$ that sequentially converges to the function u_i;

- let $\{U_{i,\nu}\}_{\nu\in\mathbb{N}}$ be the sequence of functions $U_{i,\nu}:\mathscr{K}_i\times\Sigma\to\mathbb{R}$ defined by

$$U_{i,\nu}\left(K_i,\sigma\right)=\inf_{b_i\in K_i}u_{i,\nu}(b_i,\sigma)\quad\forall(K_i,\sigma)\in\mathscr{K}_i\times\Sigma,$$

  and consider the corresponding sequence of games $\{G_\nu\}_{\nu\in\mathbb{N}}$ where $G_\nu=\{A_1,\cdots,A_n,U_{1,\nu},\cdots,U_{n,\nu}\}$ for every $\nu\in\mathbb{N}$;

- let $\{(\gamma_\nu(\sigma_\nu),\sigma_\nu)\}_{\nu\in\mathbb{N}}$ be a sequence where each $(\gamma_\nu(\sigma_\nu),\sigma_\nu)$ is a psychological Nash equilibrium under ambiguity of G_ν. Since Σ is compact, then $\{\sigma_\nu\}_\nu$ has a converging subsequence $\{\sigma_{\nu_k}\}_{k\in\mathbb{N}}$ whose limit is σ^*. Consequently, the subsequence $\{(\gamma_{\nu_k}(\sigma_{\nu_k}),\sigma_{\nu_k})\}_{k\in\mathbb{N}}$ converges to $(\beta(\sigma^*),\sigma^*)$, which is a psychological Nash equilibrium of G^{GPS} in light of Theorem 5.3 above. Hence, the psychological Nash equilibrium $(\beta(\sigma^*),\sigma^*)$ is *stable* with respect to the perturbation given by the sequence of games $\{G_{\nu_k}\}_{k\in\mathbb{N}}$;

- if the set of limit points of all the sequences of equilibria corresponding to the sequence of games $\{G_{\nu_k}\}_{k\in\mathbb{N}}$ is a proper subset of the set of equilibria of G^{GPS} then the selection method is *effective*.

REMARK 5.5: An underlying assumption is required so that the selection mechanism previously presented makes sense: it consists in the existence of psychological Nash equilibria at least for a subsequence of the sequence games $\{G_\nu\}_{\nu\in\mathbb{N}}$. Nevertheless, the examples in Section 3 show that it is reasonably simple to construct sequences of psychological games under ambiguity with nonempty sets of equilibria.

REMARK 5.6: At first sight, it might seem surprising that an equilibrium is selected if it is a limit point for just one sequence of perturbed equilibria. However, this is precisely what happens for trembling hand perfect equilibria. In fact, even in the classical game theory (with no psychological effects) it turns out that there exist entire classes of games in which no equilibrium is stable with respect to every possible perturbation. Therefore, the weaker assumption that we use is much more likely to be applicable; moreover, Example 3.3 above shows that it can provide an effective selection mechanism in simple games.

6 Conclusions

The present paper aims to jointly take into account two issues that arise from different strands of literature. On the one hand, the studies on psychological games point out that players' preferences might depend on the hierarchies of beliefs. On the other hand, the literature on *strategic ambiguity* in classical games suggests that beliefs might be ambiguous (or imprecise) in equilibrium. In this paper we deal with psychological games characterized by ambiguous beliefs that are represented as multiple hierarchies of beliefs. In the new concept of psychological Nash equilibrium under ambiguity, the correct belief function of each player is replaced by a set-valued map that specifies the set of hierarchies of beliefs that the corresponding player perceives to be consistent with the equilibrium played; moreover ambiguity is solved by considering the classical *maxmin* preferences. It follows that this concept generalizes the standard psychological Nash equilibrium defined in [Geanakoplos et al. 1989] in a natural way. The theory shows that continuity of the beliefs correspondences is the key for equilibrium existence. In addition, examples highlight that the presence of ambiguity may alter significantly the equilibria of the game: either they can be totally different from the unambiguous case or we can run into equilibrium selection.

The role of ambiguity as equilibrium selector put our paper in relation with the theory of Nash equilibria refinements: we look at the problem of stability of psychological equilibria when perturbations affect the entire hierarchy of correct beliefs. Firstly we show that, under suitable assumptions, we can obtain the convergence of equilibria of perturbed game to those of the unperturbed one. As a consequence, it is possible to refine psychological Nash equilibria by constructing selection mechanisms based on properties of stability with respect to ambiguous trembles on the hierarchies of beliefs. As suggested also in [Geanakoplos et al. 1989], psychological equilibrium refinement is a relevant issue from both theoretical and applicative point of view. Therefore, future works will include generalization of the results contained in this paper to dynamic psychological games (as studied in [Battigalli and Dufwenberg 2009]) as well as applications to specific models of psychological games.

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