DOI: 10.1478/AAPP.98S2A7

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 98, No. S2, A7 (2020)

DUALITY AND $o-O$ STRUCTURE IN NON REFLEXIVE BANACH SPACES

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ABSTRACT. Let E be a Banach space with a supremum type norm induced by a collection of functionals $\mathscr{L} \subset X^*$ where X is a reflexive Banach space. Familiar spaces of this type are BMO, BV, $C^{0,\alpha}$ $(0 < \alpha \leq 1)$, $L^{q,\infty}$, for $q > 1$. For most of these spaces E, the predual E_* exists and can be defined by atomic decomposition of its elements.
Another typical result, when it is possible to define a rich vanishing subspace $E_0 \subset E$ is the "two star theorem ", namely $(E_0)^* = E_*$. This fails for $E = BV$ and $E = C^{0,1} = Lip$.

1. Introduction

Suppose that a nonreflexive Banach space E is defined in terms of a supremum type norm induced by a subset $\mathscr{L}$ of the dual $X^\star$ of a reflexive Banach space X. Namely we have

$$E = \{x \in X : \sup |\langle L, x\rangle| < \infty, L \in \mathscr{L}\} \tag{1}$$

and assume that $E \subset X$ continuosly with $\bar{E} = X$. Spaces of this kind include the space BMO of functions with bounded mean oscillation, the space $C^{0,\alpha}$ $(0 < \alpha < 1)$ of Hölder continuos functions, the weak-L^q space $L^{q,\infty}$ $(q > 1)$ of Marcinkiewicz, the space BV of functions of bounded variation, the space Lip of Lipschitz functions and L^∞. Also the BMO type space B, introduced by Bourgain *et al.* (2015), can be framed in the same way. D'Onofrio *et al.* (2020) proved that such a space E has a predual $E_\star$ enjoying *atomic decomposition*. This is well known for some of the listed spaces but D'Onofrio *et al.* (2020) provide a completely functional analytic proof independent of any special structure of $\mathscr{L}$. For the case L^∞ see Manzo (2020), indeed for $L^{q,\infty}$ see Angrisani *et al.* (2019). The advantage of this approach is to have an auxiliary reflexive Banach space $X \supset E$, with weaker norm governing the duality.

This was already noted by Kaijser (1977) in which $BMO(\mathbb{R})$ is proved to be dual of a Banach space H with atomic decomposition and is defined as the set of $f \in L^2_{loc}(\mathbb{R})$ such

Dedicated to Nino Maugeri on the occasion of his 75th birthday that

$$||f||_{BMO} = \sup_I \left(\fint_I |f - f_I|^2 dx \right)^{\frac{1}{2}} < \infty$$

where for $I \subset \mathbb{R}$ interval f_I denotes $\frac{1}{|I|} \int_I f dx$. Since constant functions have 0 norm, BMO is considered a subspace of $X = L^2_{loc}$ modulo constants, and since the unit ball B_{BMO} of BMO is a bounded and closed subset of X, BMO is a dual space. For every interval I we have

$$\left(\fint_I |f - f_I|^2 dx \right)^{\frac{1}{2}} = \sup_{g \in P_I} \fint fg$$

where

$$P_I = \left\{ g \in L^2(I) : \fint_I g = 0 \text{ and } \fint_I |g|^2 \leq 1 \right\}$$

this means that

$$||f||_{BMO} = \sup_{g \in P} \left| \int_{\mathbb{R}} fg \right|$$

where

$$P = \left\{ g \in L^2(\mathbb{R}) : supp\, g \subset\subset I, \fint_I g = 0 \text{ and } ||g||_{L^2(I)} \leq \frac{1}{|I|^{\frac{1}{2}}} \right\}.$$

Hence the elements of predual $BMO_\star$ have representation

$$h = \sum_j \alpha_j g_j$$

where

$$\sum_j |\alpha_j| < \infty, \;\; g_j \in P.$$

Let us mention the papers of Perfekt (2015, 2017) which enlight the connections with the $o - O$ theory of non reflexive Banach spaces which deals with a *large* space E defined by a big-O condition like in case $E = BMO(Q_0)$, $Q_0 =]0,1[^n$, the space of John-Nirenberg of $u \in L^1(Q_0)$ such that :

$$||u||_E = \sup_{Q \subset Q_0} \fint |u - u_Q| < \infty \tag{2}$$

Q is a cube with side parallel to the axis, $u_Q = \fint_Q u$ is the mean value of u on Q. Associated to E there is a *small* space $E_0 \subset E$ given by the corresponding little-o condition

$$\lim_{|Q| \to 0} \sup \fint_Q |v - v_Q| = 0 \tag{3}$$

which characterizes the space $E_0 = VMO(Q_0)$ of functions of vanishing mean oscillation according to Sarason (1975). In this case the predual $E_\star$ is the Hardy space $\mathscr{H}^1$. A typical result holds true in general when the *vanishing* space E_0 is sufficiently rich, namely the so called *two star* theorem

$$E_\star \simeq E_0^\star \tag{4}$$

isometrically. All mentioned spaces enjoy this property except

$$E = L^\infty(Q_0), \; E = BV(Q_0), \; E = Lip(Q_0)$$

for which E_0 is always trivial.

Let us notice that in the other cases we have always three spaces in duality E, $E_\star$, E_0 with the general property

$$E^\star = E_0^\star \oplus E_0^\perp, \tag{5}$$

where $F^\perp$ is the orthogonal space and (5) is an l^1-decomposition. This will be a consequence of a strong opposition to the *triviality* of E_0, consisting on the approximability of all vectors $u \in E$ by mean of sequences of vectors $v_j \in E_0$ in the X-topology. More precisely

(AP) *For every $u \in E$ there exists $(v_j) \subset E_0$ such that*

$$||v_j||_E \leq ||u||_E. \tag{6}$$

and $v_j \rightharpoonup u$ in X.

Let us finally notice that if (AP) holds (4) can be precisely stated as follows once we indicate by $i : E_0 \to E_0^{\star\star}$ the canonical embedding.

Theorem 1.1. *$(E_0)^{\star\star} \simeq E$ isometrically via the $X - X^\star$ pairing in the sense that, if*

$$I : E_0 \to X$$

denotes the inclusion operator, and we define

$$U = I^{\star\star}$$

then

$$U(E_0^{\star\star}) = E$$

and considered as an operator $U : E_0^{\star\star} \to E$, U is the unique isometric isomorphism such that $U(i(v)) = v$, $\forall v \in E_0$.

2. The concrete $o-O$ type pairs

In this Section we consider some interesting cases of $o-O$ pairs (E_0, E).

Example **2.1**. (BMO space)

The BMO space was first identified by John and Nirenberg (1961). BMO functions appear in the theory of solutions to second order elliptic PDE's of non variational type both as solutions and as coefficients for existence–regularity in Dirichlet problem, which involved italian school on PDE's.

We mention here the pioneering paper of Chiarenza *et al.* (1993) which originated many papers, see Maugeri *et al.* (2000).

Also the Marcinkiewicz space was useful as an adequate for gradients of coefficients in the spirit of papers of Miranda (1963), Guglielmino (1964), Alvino and Trombetti (1984).

Let us show that for $E = BMO(Q_0)$ a good choice of the reflexive Banach space is $X = L^2(Q_0)/\mathbb{R}$. We consider for simplicity $n = 1$. Following Manzo (2020) we define

$$\mathscr{L} = \{f_{[a,b],h} : 0 \leq a < b \leq 1, h \in B_{L^\infty(0,1)} \ a.e.\}$$

where B_F denotes the unit ball of Banach space F and

$$f_{[a,b],h} u = \fint_a^b h(x)(u(x) - u_{[a,b]})dx.$$

We will equip $\mathscr{L} \subset X^\star$ with the topology induced by the product of the natural interval topology and the weak-$\star$ topology $\sigma(L^\infty, L^1)$. Notice that since $u \in L^1(0,1)$ and $[a,b] \subset]0,1[$, the set

$$\pi(u) = \{x \in]0,1[: u - u_{[a,b]} \geq 0\}$$

is measurable and we can consider $h = \chi_{\pi(u)} - \chi_{[0,1]\setminus\pi(u)}$ to obtain the mean oscillation of u in $[a,b]$. The (AP) condition was proved by Perfekt (2013).

Example **2.2**. (Rectangular BMO space)

In similar way we can treat the case of functions with *rectangular* bounded mean oscillation on $Q_0 \subset \mathbb{R}^2$. Recall that $u \in BMO_{rect}(Q_0)$ if $u \in L^2(Q_0)$ and over all I, J open intervals in $\mathbb{R}$

$$\sup \fint_I \fint_J |u(z,w) - u_J(z) - u_I(w) + u_{I\times J}|^2 d\tau d\omega < \infty.$$

Example **2.3**. (Space B of Bourgain-Brezis-Mironescu) For $u \in L^1(Q_0)$ and a cube $Q_\varepsilon \subset Q_0$ with sides of lenght ε and parallel to the co-ordinate axes define the norm (modulo constants)

$$||u||_B = \sup_{0<\varepsilon<1} [u]_\varepsilon$$

where

$$[u]_\varepsilon = \varepsilon^{n-1} \sup_{\mathscr{F}_\varepsilon} \sum_{Q_\varepsilon \in \mathscr{F}_\varepsilon} \fint_{Q_\varepsilon} |u(x) - u_{Q_\varepsilon}|$$

where $\mathscr{F}_\varepsilon = (Q_\varepsilon(a_j)_{j\in J})$ denotes a collection of mutually disjoint ε-cubes $Q_\varepsilon(a_j) \subset Q_0$ centered at a_j such that the cardinality $\sharp\mathscr{F}_\varepsilon \leq \varepsilon^{1-n}$. The space B is then defined as

$$B = \{u \in L^1(Q_0) : ||u||_B < \infty\}.$$

For $n = 1$, $BMO = B$. For $n \geq 2$ the B-norm is strictly weaker then the BMO-norm. Actually BMO and BV (the space of functions of bounded variations) are continuously contained in B. The separable vanishing subspace B_0 consists of those $v \in B$ such that

$$\limsup_{\varepsilon\to 0} [v]_\varepsilon = 0.$$

In this case we choose (see D'Onofrio *et al.* (2020)) $X = L^2(Q_0)/\mathbb{R}$ and

$$\mathscr{L} = \{L_{\mathscr{F}_{\varepsilon,h}} : \sharp\mathscr{F}_\varepsilon \leq \varepsilon^{1-n}, h \in L^\infty(Q_0)\} \subset X^\star \tag{7}$$

with

$$L_{\mathscr{F}_{\varepsilon,h}} u = \varepsilon^{n-1} \sum_{j=1}^{\sharp\mathscr{F}_{\varepsilon,h}} \fint_{Q_\varepsilon(a_j)} h(x)[u - u_{Q_\varepsilon(a_j)}]$$

The (AP) condition is proved by D'Onofrio *et al.* (2020). For the sake of completeness, we mention the following theorem which generalizes a result of Leibov (1990).

Theorem 2.4. *If $v \in B_0$, then there exists $\bar\varepsilon \in (0,1)$ such that*

$$||v||_B = [v]_{\bar\varepsilon}$$

which shows that the supremum in the norm is attained when the functions belongs to the vanishing subspace of B.

Example **2.5**. (Marcinkiewicz space $L^{q,\infty}$)

Let $1 < p < q$ and consider weak-L^q space of Marcinkiewicz $L^{q,\infty}(Q_0)$ of functions $u \in L^1(Q_0)$ such that

$$||u||_{L^{q,\infty}} = \sup_{A \subset Q_0} |A|^{\frac{1}{q}} \fint_A |u|$$

with the corresponding small-o condition $v \in L_0^{q,\infty}$

$$\lim_{|A| \to 0} |A|^{\frac{1}{p}} \fint_A |v| = 0.$$

It is well known (Carozza and Sbordone 1997) that $L_0^{q,\infty}$ is the closure of $L^\infty(Q_0)$ in $L^{p,\infty}(Q_0)$. Moreover

$$(L_0^{q,\infty})^\star \simeq L^{p,1} \quad \frac{1}{p} + \frac{1}{q} = 1,$$
$$(L^{p,1})^\star \simeq L^{p,\infty},$$
$$(L_0^{q,\infty})^{\star\star} \simeq L^{q,\infty}$$

and $L^{p,1}$ is a Lorentz space whose functions enjoy atomic decomposition (De Souza 2010).

Example **2.6**. (Lipschitz spaces $Lip_\alpha(\bar{\Omega})$, $0 < \alpha < 1$, Ω bounded domain of $\mathbb{R}^n$)

A function $u : \Omega \to \mathbb{R}$ belongs to Lip_α if and only if

$$||u||_{Lip_\alpha(\Omega)} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty. \tag{8}$$

We identify u and $u + C$, C constant in order to obtain a norm. In this case we choose $X = W^{\ell,p} \setminus A_\Omega$ as a quotient space of a fractional Sobolev space $W^{\ell,p}$ where $0 < \ell < 1$, $p > 1$ that consists of $u \in L^p$ such that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{p\ell+n}} \mathrm{d}x\mathrm{d}y < \infty.$$

If we assume $0 < \ell < \alpha$, $p\ell > n$ by a Sobolev type theorem $W^{\ell,p}$ continuously embeds into C_b, the space of bounded continuous functions. Let

$$A_\Omega = \{u \in W^{\ell,p} : \exists x_0 \in \Omega : u(x_0) = 0\}$$

and set

$$X = W^{\ell,p} \setminus A_\Omega. \tag{9}$$

The family $\mathscr{L} \subset X^\star$ consists of functional of the form

$$L_{x,y} u = \frac{u(x) - u(y)}{|x-y|^\alpha} \tag{10}$$

with the topology of

$$\{(x,y) \in \Omega \times \Omega : x \neq y\}.$$

Jonsson (1990) proved for $0 < \alpha < 1$ that the predual of $Lip_\alpha(\Omega)$ is the space $H_\alpha(\Omega)$ of all $f \in Lip_\alpha(\Omega)^\star$ which may be represented as ($\gamma_i \in \mathbb{R}$)

$$f(x) = \sum_{i=1}^{\infty} \gamma_i a_i(x) \tag{11}$$

where $\sum_{i=1}^{\infty}|\gamma_i| < \infty$ and $a_i(x)$ are *atoms* with supporting balls $B_{r_i}(x_i)$. Notice that the action of f on $\varphi \in Lip_\alpha$ is given by

$$\langle \sum_j \gamma_i a_i, \varphi \rangle = \sum_i \gamma_i \int a_i \varphi d\mu.$$

The norm in H_α is the infimum of $\sum_i |\gamma_i|$ over all representations.

Definition 2.7. By an α-atom with respect to a doubling measure μ we mean a function $a \in L^\infty(\mu)$ such that for some ball $B = B(x,r)$, $x \in \bar{\Omega}$, $0 < r \leq 1$ (a supporting ball for a) we have

$$\operatorname{supp} a \subset B \tag{12}$$

the cancellation

$$|a| \leq \frac{1}{r^\alpha}\frac{1}{|\mu(B)|} \tag{13}$$

and the size condition

$$\int aPd\mu = 0 \ \text{ if } r < 1 \tag{14}$$

for all polynomials P of degree less or equal then $[\alpha]$. The norm in $H_\alpha(\bar{\Omega})$ is

$$||a||_{H_\alpha(\bar{\Omega})} = \inf \sum |\gamma_i| \tag{15}$$

over all representation.

Another atomic representation of the predual of Lip_α was recently established by Angrisani *et al.* (2020). Indeed, since we have shown that the pair (lip_α, Lip_α) admits a *o–O* structure whenever $\alpha \in (0,1)$, we know that $(lip_\alpha(K,\rho))^*$ is the strongly unique predual of $Lip_\alpha(K,\rho)$ and $(lip_\alpha(K,\rho))^* \simeq (M(K))^c$ where $M(K)$ is the normed space of finite (signed) Borel measures on K endowed with the Kantorovich-Rubinstein norm. Now we are ready to give an atomic decomposition of the space $M(K)$.

Theorem 2.8. *Fix $\alpha \in (0,1)$. Let $\mu \in M(K)$. Then there exists a sequence of atomic measures $(\mu_n)_{n\in\mathbb{N}} \subset M(K)$ with $card(supp(\mu_n)) \leq 3$ and a sequence $(\gamma_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{R})$ with $\gamma_n \geq 0$ such that*

$$\mu = \sum_{n=1}^{+\infty} \gamma_n \mu_n$$

where the convergence is intended in the Kantorovich-Rubinstein norm. Moreover there is $C > 0$ such that

$$C\sum_{n=1}^{+\infty} \gamma_n \leq ||\mu||_{M(K)} \leq \sum_{n=1}^{+\infty} \gamma_n. \tag{16}$$

Example **2.9**. (BV space)

Assume $E = BV(Q_0)$, $Q_0 =]0,1[^n$, is the space of functions u of bounded variation, that is $u \in L^1(Q_0)$ and the total variation

$$|Du|(Q_0) = \sup_{||\varphi||_{C^0(Q_0,\mathbb{R}^n)} \leq 1} \int_{Q_0} u \, div \varphi$$

is finite, equipped with the norm

$$||u||_{BV(Q_0)} = ||u||_{L^1(Q_0)} + |Du|(Q_0)$$

This is a non separable space where smooth compactly supported functions fail to be norm-dense and is dual of a separable Banach space

$$BV_* = \left\{ T \in D'(Q_0) : T = \varphi_0 + \sum_{j=1}^{n} \frac{\partial \varphi_j}{\partial x_j} : \varphi_j \in C_0(Q_0) \right\}$$

(see Ambrosio *et al.* (2000) and Fusco and Spector (2018)). We find with our methods atomic decomposition similar to the one described in the example **2.3**.

Acknowledgments

The authors are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of INdAM. The research of R.S. has been funded by PRIN Project 2017JFFHSH.

References

Alvino, A. and Trombetti, G. (1984). "Second order elliptic equations whose coefficients have their first derivatives weakly-L^n". *Annali di Matematica Pura ed Applicata* **138**, 331–340. DOI: 10.1007/BF01762551.

Ambrosio, L., Fusco, N., and Pallara, D. (2000). *Functions of bounded variations and free discontinuity problems*. Oxford Mathematical Monographs.

Angrisani, F., Ascione, G., D'Onofrio, L., and Manzo, G. (2020). "Duality and distance formulas in Lipschitz-Hölder spaces". *Rendiconti Lincei. Matematica e Applicazioni* **31**(2), 401–419. DOI: 10.4171/RLM/897.

Angrisani, F., Ascione, G., and Manzo, G. (2019). "Orlicz spaces with a $o-O$ type structure". *Ricerche di Matematica* **68**, 841–857. DOI: 10.1007/s11587-019-00441-3.

Bourgain, J., Brezis, H., and Mironescu, P. (2015). "A new function space and applications". *Journal of the European Mathematical Society* **17**(9), 2083–2101. DOI: 10.4171/JEMS/551.

Carozza, M. and Sbordone, C. (1997). "The distance to L^∞ in some function spaces and applications". *Differential Integral Equations* **10**(4), 599–607. URL: https://projecteuclid.org/euclid.die/1367438633.

Chiarenza, F., Frasca, M., and Longo, P. (1993). "$W^{2,p}$-Solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients". *Transactions of the American Mathematical Society* **336**(2), 841–853. URL: www.jstor.org/stable/2154379?origin=crossref&seq=1#metadata_info_tab_contents.

D'Onofrio, L., Greco, L., Perfekt, K.-M., Sbordone, C., and Schiattarella, R. (2020). "Atomic decompositions, two stars theorems, and distances for the Bourgain-Brezis-Mironescu space and other big spaces". *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* **37**(3), 653–661. DOI: 10.1016/j.anihpc.2020.01.004.

De Souza, G. S. (2010). "A new characterization of the Lorentz spaces $L(p,1)$ for $p>1$ and applications". *Real Analysis Exchange. Summer Symposium 2010*, 55–58. URL: https://www.stolaf.edu/analysis/Wooster2010/16-DeSouza.pdf.

Fusco, N. and Spector, D. (2018). "A remark on an integral characterization of the dual of BV". *Journal of Mathematical Analysis and Applications* **457**(2), 1370–1375. DOI: 10.1016/j.jmaa.2017.01.092.

Guglielmino, F. (1964). "Sulle equazioni paraboliche del secondo ordine di tipo non variazionale". *Annali di Matematica Pura ed Applicata* **65**, 127–151. DOI: 10.1007/BF02418223.

John, F. and Nirenberg, L. (1961). "On functions of bounded mean oscillation". *Communications on Pure and Applied Mathematics* **14**, 415–426. DOI: 10.1002/cpa.3160140317.

Jonsson, A. (1990). "The duals of Lipschitz spaces defined on closed sets". *Indiana University Mathematics Journal* **39**(2), 467–476. URL: file:///C:/Users/monic/Downloads/39025.pdf.

Kaijser, S. (1977). "A note on dual Banach spaces". *Mathematica Scandinavica* **41**, 325–330. DOI: 10.7146/math.scand.a-11725.

Leibov, M. (1990). "Subspaces of the VMO space". *Journal of Soviet Mathematics* **48**, 536–538. DOI: 10.1007/BF01095622.

Manzo, G. (2020). "A way to obtain an atomic decomposition of $L^1(\Omega)$". *preprint*.

Maugeri, A., Palagachev, D. K., and Softova, L. G. (2000). *Elliptic and Parabolic Equations with discontinuous coeffcients*. Vol. 109. Wiley-VCH, Berlin. DOI: 10.1002/3527600868.

Miranda, C. (1963). "Sulle equazioni ellittiche del secondo ordine di tipo non variazionale, a coefficienti discontinui". *Annali di Matematica Pura ed Applicata* **63**(4), 353–386. DOI: 10.1007/BF02412185.

Perfekt, K.-M. (2013). "Duality and distance formulas in spaces defined by means of oscillation". *Arkiv för Matematik* **51**(2), 345–361. URL: https://projecteuclid.org/euclid.afm/1485907220.

Perfekt, K.-M. (2015). "Weak compactness of operators acting on o–O type spaces". *Bulletin of the London Mathematical Society* **47**(4), 677–685. DOI: 10.1112/blms/bdv031.

Perfekt, K.-M. (2017). "On M-ideals and o-O type spaces". *Mathematica Scandinavica* **121**(1), 151–160. DOI: 10.7146/math.scand.a-96626.

Sarason, D. (1975). "Functions of vanishing mean oscillation". *Transactions of the American Mathematical Society* **207**, 391–405. DOI: 10.1090/S0002-9947-1975-0377518-3.

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Paper contributed to the meeting on “Variational Analysis, PDEs and Mathematical Economics“, held in Messina, Italy (19–20 September 2019), on the occasion of Prof. Antonino Maugeri’s 75th birthday, under the patronage of the *Accademia Peloritana dei Pericolanti*

Manuscript received 17 February 2020; published online 13 December 2020