Local generic behavior of a planar Filippov system with non-smooth switching curve

J. Larrosa · T. M. Seara · M. A. Teixeira

Abstract This paper discusses the generic local classification of typical singularities of $2D$ piecewise smooth vector fields when the switching set is an algebraic variety. The main goal is to obtain classification results concerning structural stability and generic codimension one local bifurcations.

Keywords Discontinuous systems · Structural tability · Generic bifurcations · Versal unfoldings

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1 Introduction

First of all, we observe that this paper is part of a general program involving the study of discontinuous piecewise smooth systems in $\mathbb{R}^n$ of the form

$$\dot{\mathbf{x}} = F(\mathbf{x}) + \operatorname{sgn}(f(\mathbf{x}))\, G(\mathbf{x}); \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $F, G : \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ are smooth functions on $\mathbb{R}^n$. Note that we have two different smooth systems defined in all $\mathbb{R}^n$, one,

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$X = F + G$, that will be considered in the region where $f(\mathbf{x}) > 0$ and the other, $Y = F - G$, where $f(\mathbf{x}) < 0$.

In this direction, let $X, Y : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ be sufficiently smooth vector fields defined in a bounded neighborhood $\mathcal{U}$ of the origin. Consider $f : (x_1, x_2) \in \mathcal{U} \subset \mathbb{R}^2 \to x_2 \in \mathbb{R}$ having 0 as a regular value and let $\Sigma = f^{-1}(0)$. Thus Σ is a regular codimension one submanifold of $\mathcal{U} \subset \mathbb{R}^2$. The submanifold Σ splits the open set $\mathcal{U}$ into two open sets $\mathcal{U}^+ = \{p \in \mathcal{U} : x_2 > 0\}$ and $\mathcal{U}^- = \{p \in \mathcal{U} : x_2 < 0\}$.

Let $\mathcal{Z}$ be the space of all piecewise smooth vector fields $Z = (X, Y)$ defined as

$$Z(p) = \begin{cases} X(p), & \text{if } p \in \mathcal{U}^+ \\ Y(p), & \text{if } p \in \mathcal{U}^- \end{cases}. \tag{2}$$

The dynamics on each open set $\mathcal{U}^\pm$ is given by the smooth vector fields X and Y, respectively. The submanifold Σ is called *discontinuity curve* or *switching curve* and we assume that the dynamics over Σ is given by the Filippov's convention. More details about the Filippov's convention can be found in [13]. The piecewise smooth, also called non-smooth, vector field defined in this way is called a *Filippov system.*

Due to their importance in applications, non-smooth Systems have been largely studied in recent years. There are a huge number of works focusing on the local and global dynamics of these systems. We refer the reader to [4] and references therein for an extensive survey about general non-smooth systems.

In the case of Filippov systems, many authors have contributed to their study. See, for instance, [13, 19, 26, 20, 15, 11, 2, 6, 16, 7].

Concerning the study of the generic behavior for planar Filippov systems, it was firstly made by Kozlova ([19]). In [20], Kuznetsov at al., classified and studied all codimension one bifurcations and also some global bifurcations using suitable normal forms. Following the same approach, in [5] the authors study some higher codimension bifurcations.

One of the starting points for our approach in the study of bifurcations in planar Filippov systems was the work of M.A. Teixeira [25] about smooth systems in 2-dimensional manifolds with boundary. This work was generalized in [6] to the study of structurally stable Filippov Systems defined in 2-dimensional manifolds.

Using the concepts of local $\Sigma-$equivalence and weak equivalence of unfoldings, in [15] the authors gave a rigorous proof of the theorem which classifies the set of the local $\Sigma-$structural stable Filippov systems and revisited the codimension one bifurcations. In addition, they gave a preliminary classification of codimension two bifurcations. In [23] the authors revisited the codimension one generic local bifurcations of Filippov systems, presenting a rigorous classification of the generic fold-fold singularities set Λ^F, as well as a formal study of the versal unfoldings of each singularity. Moreover, they have proved that Λ^F is a codimension one embedded submanifold of $\mathcal{Z}$. There are also results about global bifurcations involving periodic orbits of codimension one or higher, see for instance [18, 14, 4, 22, 21]. Concerning Filippov systems in higher dimensions [8, 9, 17, 10] study local and global bifurcations involving periodic orbits in $\mathbb{R}^n$ for $n \geq 3$.

Most works on piecewise smooth systems are devoted to study switching manifold which are regular curves or manifolds. However, using the same definition of piecewise smooth vector fields, some works have considered Σ as the union of two codimension one submanifolds which intersect transversely at the origin. For

instace, in [12], the authors propose a special choice for the sliding vector field at the intersection of the two manifolds, and the work [3] studies some global bifurcations related to "corner collision" bifurcations where a periodic orbit of one of the systems intersects the non- regular point of the boundary.

In this paper, we are interested in the particular subject: $2D$ systems as in (1) for which $f(x_1, x_2) = x_1 \cdot x_2$. We understand that a systematic program towards the bifurcation theory for such systems is currently emergent.

In this case, the origin is a non-degenerated critical point of f and $\Sigma = f^{-1}(0)$ is a degenerate hyperbole. Thus Σ can be seen as the union of two lines that intersect transversally at the origin. The piecewise smooth vector field $Z = (X, Y)$ is given exactly as in (2) and in this case, we assume that the Filippov's convention is valid in $\Sigma \setminus \{\mathbf{0}\}$.

Piecewise dynamical systems naturally arise in the context of many applications. In this direction our approach is motivated by equations expressed as

$$\ddot{\mathbf{x}} + a\dot{\mathbf{x}} = \mathrm{sgn}\,(\mathbf{x} \cdot f(\mathbf{x}, \dot{\mathbf{x}})));$$

that are commonly found in many fields such as Control Theory and Engineering. In [1] problems involving asymptotic stability of such systems are fairly discussed. In this book the author presents an stabilization problem that can be solved provided a discontinuity of this type is introduced in the system.

Let Ω be the set of all piecewise vector fields defined as above. In this work, our aim is to describe rigorously the set of the locally $\Sigma-$structurally stable vector fields having this kind of switching set and their codimension one bifurcations. The structure of this work is as follows:

In section 2 we describe the objects we are going to work with, as the trajectories, tangencies, notion of local $\Sigma-$equivalence and weak equivalences of unfoldings.

Section 3 is devoted to classify the set Ω_0 composed by the locally $\Sigma-$structurally stable vector fields in Ω. In this way we establish the generic conditions which are necessary to $Z \in \Omega$ be locally $\Sigma-$structurally stable. Moreover, in each case we construct the $\Sigma-$equivalence between $Z \in \Omega_0$ and its corresponding "normal form" which describes the dynamics.

Once we have classified the set $\Omega_0 \subset \Omega$, in section 4 we describe the set $\Xi_1 \subset \Omega_1 = \Omega \setminus \Omega_0$ composed by the vector fields Z which have a codimension one bifurcation at the origin. In order to get this result, we establish some conditions for $Z \in \Xi_1$ and show that Ξ_1 is open in Ω_1 (endowed with the induced topology of Ω). More precisely, Ξ_1 is an embedded codimension one submanifold of Ω. Finally, we show that for each $Z \in \Xi_1$ the unfoldings which are transverse to Ξ_1 at Z are weakly equivalent and its dynamic is then described. One special feature of these systems is that, contrarily to what happens with the unfoldings of codimension one singularities of Filippov systems, the unfoldings in this case no periodic orbits appear nearby the singularity.

2 First definitions and results

Let $f : \mathbb{R}^2, \mathbf{0} \to \mathbb{R}, 0$ be a C^r smooth function such that $f(\mathbf{0}) = 0$ and that 0 a non degenerate critical point. The purpose of this work is to study the generic

singularities of planar piecewise vector fields Z which discontinuity set is given by the zeros of the map $f(x_1, x_2)$.

As it is known that there are coordinates around the origin such that f can be written as $f(x_1, x_2) = x_1^2 \pm x_2^2$. In this paper we will study the case $f(x_1, x_2) = x_1^2 - x_2^2$, or equivalently, $f(x_1, x_2) = x_1 \cdot x_2$.

In this direction, let X and Y be smooth $\mathcal{C}^r$, $r \geq 1$ vector fields defined in a bounded neighborhood $\mathcal{U}$ of the origin.

Let $f : p = (x_1, x_2) \in \mathcal{U} \mapsto f(p) = x_1 x_2 \in \mathbb{R}$. The set $\Sigma = f^{-1}(\mathbf{0})$ is an algebraic variety and splits $\mathcal{U}$ as the closure of the regions $\mathcal{U}^+ = \{p \in \mathbb{R}^2 : f(p) > 0\}$ and $\mathcal{U}^- = \{p \in \mathbb{R}^2 : f(p) < 0\}$. Moreover, the region $\mathcal{U}^+$ can be decomposed as $\mathcal{U}_+^+ = \mathcal{U}^+ \cap \{x_2 > 0\}$ and $\mathcal{U}_-^+ = \mathcal{U}^+ \cap \{x_2 < 0\}$, as can be seen in Figure 1. Analogously, we define the regions $\mathcal{U}_+^-$ and $\mathcal{U}_-^-$.

Let Ω be the set of all piecewise vector fields defined as:

$$Z(p) = \begin{cases} X(p), & \text{if} \quad p \in \mathcal{U}^+ \\ Y(p), & \text{if} \quad p \in \mathcal{U}^- \end{cases}. \tag{3}$$

The set Σ is the discontinuity set or switching set. Observe that we can write $\Sigma = \Sigma_1 \cup \Sigma_2$ with $\Sigma_1 = \{(x_1, x_2) \in \Sigma : x_1 = 0\}$ and $\Sigma_2 = \{(x_1, x_2) \in \Sigma : x_2 = 0\}$. Moreover, $\Sigma_1 = \overline{\Sigma_1^+ \cup \Sigma_1^-}$ where $\Sigma_1^+ = \{(0, x_2) \in \Sigma_1 : x_2 > 0\}$ and $\Sigma_1^- = \{(0, x_2) \in \Sigma_1 : x_2 < 0\}$. Similarly, one can write $\Sigma_2 = \overline{\Sigma_2^+ \cup \Sigma_2^-}$.
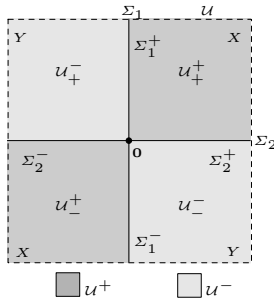

Fig. 1 The decomposition of the domain $\mathcal{U}$.

Each Σ_i, for $i = 1, 2$, can be decomposed as the closure of the crossing, sliding and escaping regions as follows:

$$\Sigma_i^c = \{p \in \Sigma_i : X_i \cdot Y_i(p) > 0\},$$

$$\Sigma_i^s = \left\{p \in \Sigma_i^+ : X_i(p) < 0, Y_i(p) > 0\right\} \cup \left\{p \in \Sigma_i^- : X_i(p) > 0, Y_i(p) < 0\right\},$$

$$\Sigma_i^e = \left\{p \in \Sigma_i^+ : X_i(p) > 0, Y_i(p) < 0\right\} \cup \left\{p \in \Sigma_i^- : X_i(p) < 0, Y_i(p) > 0\right\}.$$

Then the crossing, sliding and escaping regions in Σ are the union of the corresponding regions in Σ_1 and Σ_2.

In the regions $\Sigma_i^{s,e}$, for $i = 1, 2$, we define the sliding vector field:

$$Z_i^s(p) = \frac{1}{Y_i(p) - X_i(p)} [Y_i(p)X_j(p) - X_i(p)Y_j(p)]\Big|_{\Sigma_i}, \tag{4}$$

where $i, j = 1, 2$ and $i \neq j$.

Definition 1 Fix $i = 1, 2$. The point $p \in \Sigma_i^{s,e}$ is a pseudo-equilibrium if $Z_i^s(p) = 0$ and it is hyperbolic if $(Z_i^s)'(p) \neq 0$.

Since we are interested on low codimension singularities, we focus our attention just to one kind of tangency, the fold point, which is defined below.

Definition 2 The point $p \in \Sigma_i$ is a **fold point** of X in Σ_i if $X_i(p) = 0$ and $X_j(p) \cdot \dfrac{\partial}{\partial x_j} X_i(p) \neq 0$ for $i, j = 1, 2$ and $i \neq j$. Moreover, $p \in \Sigma$ is a **regular-fold** of X in Σ_i if is a fold point for X in Σ_i and Y is transverse to Σ at the origin, that is, $Y_l(\mathbf{0}) \neq 0$ for $l = i, 2$.

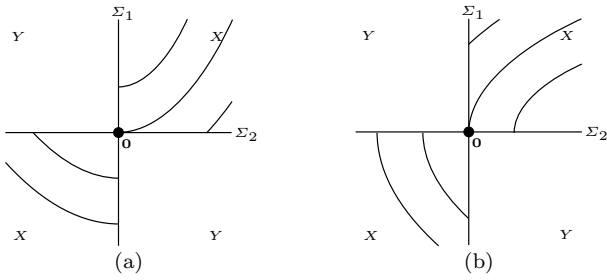Analogously, we define a fold point and a regular fold for Y.

Fig. 2 The origin is a fold point of X. (a) $X_2(\mathbf{0}) = 0$ and $X_1 \cdot \dfrac{\partial}{\partial x_1} X_2(\mathbf{0}) > 0$; (b) $X_1(\mathbf{0}) = 0$ and $X_2 \cdot \dfrac{\partial}{\partial x_2} X_1(\mathbf{0}) > 0$.

Now we define the trajectories of Z through points of $\mathcal{U}$, following [15]. Let us denote by $\varphi_X(t; p)$ the flow of a regular vector field X. In order to preserve the uniqueness of orbits, we assume that, if $p \in \mathcal{U}^{\pm}$ is such that the curve $\{\varphi_{X,Y}(t; p); t \in \mathbb{R}\} \pitchfork \Sigma^{s,e}$, then the trajectories of X and Y through p are relatively open, that is, they do not reach $\Sigma^e \cup \Sigma^s$ in finite time.

Next definition gives the trajectories through a point of $\mathcal{U} \setminus \{0\}$.

Definition 3 Let $p \in \mathcal{U} \setminus \{\mathbf{0}\}$, then its trajectory $\varphi_Z(t; p)$ is given by:

- If $p \in \mathcal{U}^+ \cup \mathcal{U}^-$ then its trajectory is given by the trajectory of X or Y, respectively.
- If $p \in \Sigma^c = \Sigma_1^c \cup \Sigma_2^c$ then its trajectory is the concatenation of its respective trajectories in $\mathcal{U}^+$ and $\mathcal{U}^-$;
- If $p \in (\Sigma_i^s \cup \Sigma_i^e) \setminus \{0\}$ for $i = 1$ or 2, then its trajectory is given by $\varphi_{Z_i^s}(t; p)$, where Z_i^s is given in (4);
- If $p \in \partial \Sigma_i^e \cup \partial \Sigma_i^s \cup \partial \Sigma_i^c$ and if $\lim\limits_{q \to p^-} \varphi_Z(t; q) = \lim\limits_{q \to p^+} \varphi_Z(t; q)$, then $\varphi_Z(t; p) = \lim\limits_{q \to p} \varphi_Z(t; q)$. These points are **regular tangency points**.
- If $p \in \partial \Sigma_i^e \cup \partial \Sigma_i^s \cup \partial \Sigma^c$ does not satisfy the last condition, then $\varphi_Z(t; p) = \{p\}$ and it is called by **singular tangency points**.

Next definition gives the trajectory of $p = \mathbf{0}$.

Definition 4 Let $\{\mathbf{0}\} \in \Sigma$, then its trajectory $\varphi_Z(t; \mathbf{0})$ is described below:

– If $\{\mathbf{0}\} \in \Sigma = \overline{\Sigma^c} = \overline{\Sigma_1^c \cap \Sigma_2^c}$ then there is only one trajectory of X or Y which cross the origin and we define $\varphi_Z(t; \mathbf{0})$ as being this trajectory;
– If $\{\mathbf{0}\} \in \overline{\Sigma_i^e \cup \Sigma_i^s} \cap \overline{\Sigma_j^c}$, $i, j = 1, 2$ and $i \neq j$, then the trajectory $\varphi_Z(t; \mathbf{0}) = \varphi_{Z_i^s}(t; \mathbf{0})$;
– If $\{\mathbf{0}\} \in \cap_{i=1}^2 \overline{\Sigma_i^e \cup \Sigma_i^s}$ then $\varphi_Z(t; \mathbf{0}) = \{\mathbf{0}\}$;

After the last two definitions, we can define the singularities of $Z \in \Omega$.

Definition 5 The singularities of $Z \in \Omega$ are:

– $p \in \overline{\mathcal{U}^{\pm}}$ which are singularities of X or Y, respectively;
– $p \in \Sigma_i^{e,s}$ such that $Z_i^s(p) = 0$;
– $p \in \partial \Sigma_i^e \cup \partial \Sigma_i^s \cup \partial \Sigma_i^c$, for $i = 1, 2$, which are singular tangency points;
– $\{\mathbf{0}\} \in \Sigma$ when both Z_i^s are defined in a neighborhood of the origin, $i = 1, 2$.

In the sequel we define some different types of "periodic" orbits which can appear in piecewise smooth systems, once again we follow the definitions given in [15].

Definition 6 A regular periodic orbit is a regular orbit $\gamma = \{\phi_Z(t; p) : t \in R\}$, which belongs to $\mathcal{U}^+ \cup \mathcal{U}^- \cup \Sigma^c$ and satisfies $\phi_Z(t + T; p) = \phi_Z(t; p)$ for some $T > 0$.
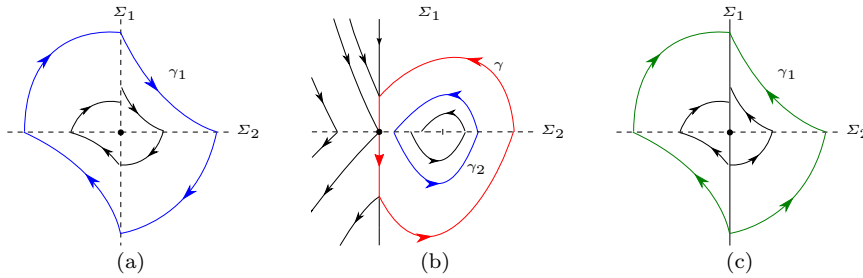

Fig. 3 Examples of periodic orbits containing the origin: (a) regular periodic cycle (b) periodic cycle (c) closed contour

Definition 7 A cycle is a closed curve formed by a finite set of pieces of orbits $\gamma_1, \ldots, \gamma_n$ such that γ_{2k} is a piece of sliding orbit, γ_{2k+1} is a maximal regular orbit and the departing and arrival points of γ_{2k+1} belong to γ_{2k} and γ_{2k+2}, respectively. We define the period of the cycle as the sum of the times that are spent in each of the pieces of orbit γ_i, $i = 1, \ldots, n$.

Definition 8 We define a closed contour (also called a pseudo-cycle, see [15]) as the closure of a set of regular orbits $\gamma_1, \ldots, \gamma_n$ such that their edges, that is the arrival and departing points, of any γ_i coincide with one of the edges of γ_{i+1} and

one of the edges of γ_{i+1} (and also between γ_1 and γ_n) forming a curve homeomorphic to $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, in such a way that in some point coincide two departing or two arrival points.

In Figures 3 (a) and (b) the curves γ_1 and γ_2 are examples of regular periodic orbits. As we will see in section 3 and 4, this kind of orbits do not appear in low codimension bifurcations. For example, the phase portrait sketched in Figure 3 (a), can happen when the origin is a saddle for X (with non admissible eigenspaces, that is, the eigenspaces $V_{1,2} \subset \mathcal{U}^-$) and a focus for Y, which is a bifurcation of codimension at least four. As an example of vector field having infinitely many periodic cycles, one can take $Z = (X, Y)$ where $X(x,y) = (-8x-5y, 10x+7y)$ and $Y(x,y) = (8x-5y, 10x-7y)$ are both linear vector fields. Observe that both vector fields have a saddle point at the origin and this is a codimension four phenomena.

The orbit γ in Figure 3(b), gives us an example of a periodic cycle. These orbits can appear, for example, in the unfolding a codimension 2 fold-fold bifurcation.

In figure 3(c), γ_3 is an example of closed contour. Observe that closed contours are not real closed orbits but they are preserved by Σ−equivalences (see definitions 11 and 12) and it is the unique type of recurrence containing the origin which appears in the unfoldings of some codimension one bifurcations.
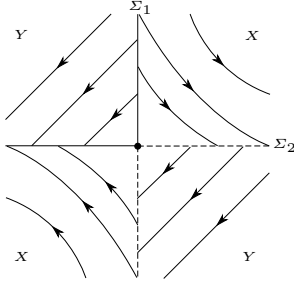

Fig. 4 Z is transient, in this picture, the origin can be a saddle point for X with non admissible eigenspaces and Y is transverse to Σ at the origin.

Definition 9 We say that $Z \in \Omega$ is transient in $\mathcal{U}^{\pm}$ if for every $p \in \overline{\mathcal{U}^{\pm}}$ there exist $t_1(p), t_2(p) \in \mathbb{R}$ satisfying $\varphi_Z(t_1(p); p) \in \Sigma_1$, $\varphi_Z(t_2(p); p) \in \Sigma_2$ and $\varphi_Z(t; p) \in \mathcal{U}^{\pm}$ for all $t \in [\min\{t_1(p), t_2(p)\}, \max\{t_1(p), t_2(p)\}]$. We say that Z is transient if it is transient in $\mathcal{U}^+$ and $\mathcal{U}^-$.

Let $Z \in \Omega$ transient. For each $p \in \Sigma_1$ there exist a unique $t_X(p) \in \mathbb{R}$ such that $\varphi_X(p; t) \in \overline{\mathcal{U}^+}$ for all $t \in [\min\{0, t_X(p)\}, \max\{0, t_X(p)\}]$ satisfying $\varphi_X(t_X(p); p) \in \Sigma_2$. We have defined a diffeomorphism

$$\phi_X : p \in \Sigma_1 \mapsto \varphi_X(t_X(p); p) \in \Sigma_2.$$

By the Implicit Function Theorem, the function $t_X : p \in \Sigma_1 \mapsto t_X(p) \in \mathbb{R}$ is a differentiable map. Analogously, we define

$$\phi_Y : p \in \Sigma_2 \mapsto \varphi_Y(t_Y(p); p) \in \Sigma_1.$$

And finally we define the *first return map of Z* by

$$\begin{aligned}\phi_Z : \Sigma_2^- &\to \Sigma_2^- \\ p &\mapsto (\phi_X \circ \phi_Y)^2(p)\end{aligned} \tag{5}$$

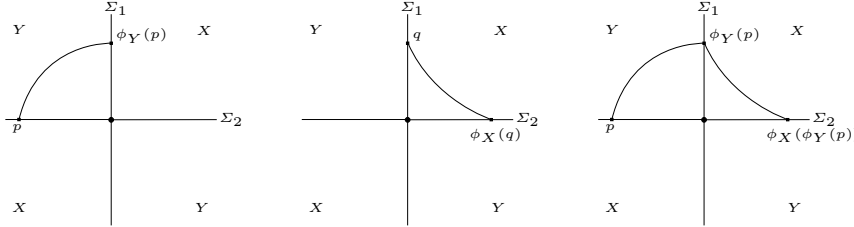which is clearly a diffeomorphism, since it is the restriction of a diffeomorphism to the cross section Σ_2^-.

Fig. 5 The first return map associated to the vector field Z.

Remark 1 In fact, the map ϕ_Z is an one dimensional map, since its second coordinate is always zero. Thus we will write ϕ_Z as a projection to the first coordinate of the first return map.

Observe that if Z is transient, the origin is always a fixed point for ϕ_Z since $t_X(\mathbf{0}) = t_Y(\mathbf{0}) = 0$. Moreover, by unicity of solutions for flows in the plane, ϕ_Z is an increasing function and therefore, ϕ_Z' is always positive.

In our context, sometimes the first return map will not have a real dynamical meaning, since it is defined for any transient vector field Z. It can happen that a trajectory of Z through a point $p \in \Sigma_2^-$ do not reach again the cross section Σ_2^- neither in backward nor in forward time, see Figure 4. But even in these cases the first return map will be important in order to detect the appearance of closed contours.

Definition 10 Let $Z \in \Omega$ be transient and let ϕ_Z be its first return map associated to Z at the origin. Then the origin is "geometrically stable" if $0 < \phi_Z'(0) < 1$ and it is "geometrically unstable" if $\phi_Z'(0) > 1$. When $\Sigma = \overline{\Sigma^c}$ then the dynamics of Z around the origin is similar to a focus, in this case we say that the origin is a "focus". Otherwise, $\mathbf{0} \in \overline{\Sigma_i^{e,s}} \cap \Sigma_j^c$ the trajectories of an initial condition does not reach Σ_2^- again and the origin will be called "geometric-focus".

Now we are going to start with the definitions of local $\Sigma-$structural stability and codimension k bifurcations. It is well known that the set $\mathfrak{X} = \mathfrak{X}^r(\mathcal{U})$, $\bar{\mathcal{U}}$ compact, of the germs of vector fields of class $\mathcal{C}^r, r \geq 1$ endowed with the $\mathcal{C}^r-$topology is a Banach space. Therefore, $\Omega = \mathfrak{X} \times \mathfrak{X}$ is also a Banach space. Consequently, Ω is a Banach manifold.

In the sequel we will establish a relation between local $\Sigma-$structural stability in our context with some special submanifolds of Ω.

Definition 11 Let Z and $\tilde{Z} \in \Omega$, defined in $\mathcal{U}$ and $\tilde{\mathcal{U}}$ neighborhoods of the origin, with discontinuity sets Σ and $\tilde{\Sigma}$, respectively. We say that Z and $\tilde{Z}$ are locally $\Sigma-$equivalent if there exist neighborhoods $\mathcal{U}_0, \tilde{\mathcal{U}}_0$ of the origin and an orientation preserving homeomorphism $h : \mathcal{U}_0 \to \tilde{\mathcal{U}}_0$ which maps trajectories of Z in trajectories of $\tilde{Z}$ and sends Σ in $\tilde{\Sigma}$.

Definition 12 We say that $Z \in \Omega$ is locally $\Sigma-$structurally stable at the origin if there exists a neighborhood $\mathcal{V}_Z \subset \Omega$ such that if $\tilde{Z} \in \mathcal{V}_Z$ then is locally $\Sigma-$equivalent to Z.

Let Ω_0 denote the set of all piecewise systems in Ω which are locally $\Sigma-$structurally stable.

When Z is not locally $\Sigma-$structurally stable at the origin, we say that Z belongs to the bifurcation set $\Omega_1 = \Omega \setminus \Omega_0$.

Definition 13 Let $Z \in \Omega$. A $m-$parameter unfolding of Z is a smooth map $\gamma : \delta = (\delta_1, \cdots, \delta_m) \in (-\delta_0, \delta_0)^m \mapsto Z_\delta \in \Omega$ with $\delta_0 \ll 1$, $m \geq 1$ and satisfying $\gamma(0) = Z_0 = Z$. We usually denote an unfolding of Z by Z_δ.

Definition 14 Let $Z, \tilde{Z} \in \Omega$. We say that two unfoldings of Z_δ and $\tilde{Z}_{\tilde{\delta}}$ are locally weak equivalent if there exists a homeomorphic change of parameters $\mu(\delta)$, such that, for each δ the vector fields Z_δ and $\tilde{Z}_{\mu(\delta)}$ are locally $\Sigma-$equivalent. Moreover, given an unfolding Z_δ of Z it is said to be a versal unfolding if every other unfolding Z_α of Z is locally weak equivalent to Z_δ.

Definition 15 A piecewise smooth vector field $Z \in \Omega_1$ has a codimension one singularity at the origin if it is locally $\Sigma-$structural stable in the induced topology of Ω_1. That is, if there exists an open set $\mathcal{V}_Z \subset \Omega$ such that $\tilde{Z} \in \mathcal{V}_Z \cap \Omega_1$ then $\tilde{Z}$ is locally $\Sigma-$equivalent to Z and any $1-$parameter unfoldings of Z and $\tilde{Z}$ are locally weak equivalent. We denote by Ξ_1 the set of all codimension one bifurcations in Ω.

One can define Ξ_k, the set of all $Z \in \Omega$ having a codimension k bifurcation at the origin, recursively. Let $\Omega_k = \Omega_{k-1} \setminus \Xi_{k-1}$ then Ξ_k is the subset of Ω_k composed by the $\Sigma-$structurally stable in Ω_k.

3 Local $\Sigma-$structural stability

The aim of this section is to describe the set $\Omega_0 \subset \Omega$ of all the vector fields which are locally $\Sigma-$structurally stable near the origin. We use the definitions of local $\Sigma-$equivalence and local $\Sigma-$structural stability stated previously. We will prove the following theorem:

Theorem 1 ($\Sigma-$structural stability on Ω) *Denote by $\Omega_0 \subset \Omega$ the set of all the vector fields which are locally $\Sigma-$structurally stable near the origin. Let $Z \in \Omega$. Then $Z \in \Omega_0$ if, and only if, Z satisfies one of the following conditions:*

A. $X_i(\mathbf{0}).Y_i(\mathbf{0}) > 0$, for $i = 1, 2$;
B. $X_i(\mathbf{0}).Y_i(\mathbf{0}) < 0$, for $i = 1, 2$ and $\det Z(\mathbf{0}) = (X_1 \cdot Y_2 - X_2 \cdot Y_1)(\mathbf{0}) \neq 0$;

C. $X_i(\mathbf{0}).Y_i(\mathbf{0}) > 0$, $X_j(\mathbf{0}).Y_j(\mathbf{0}) < 0$ for $i = 1, 2$, $i \neq j$. In addition, when Z is transient, it satisfies $\alpha_Z = \left(\dfrac{X_1 \cdot Y_2(\mathbf{0})}{X_2 \cdot Y_1(\mathbf{0})} \right)^2 \neq 1$.

Moreover, the subset Ω_0 is an open and dense in Ω, therefore local $\Sigma-$structural stability is a generic property in Ω.

We devote the rest of this section to prove this theorem.

Lets consider $Z \in \Omega$, with X and Y transverse to Σ_1 and Σ_2 at the origin. It is clear that the transversality of the vector field with Σ at the origin is a necessary condition for local $\Sigma-$structural stability of Z. In the sequel we will see that it is not a sufficient condition.

Before we start the analysis of the behavior near the origin, we summarize some important facts about the sliding vector fields. By definition of the sliding vector fields expressed in (4), as it was observed in [24], we have

$$Z_i^s(p) = h_i(p) \cdot \det Z(p), \quad p \in \Sigma_i, \tag{6}$$

where $\det Z(p) = X_1(p) \cdot Y_2(p) - X_2(p) \cdot Y_1(p)$ and

$$h_i(p) = [(-1)^{i-1}(X_i(p) - Y_i(p))]^{-1}, \tag{7}$$

Since each Z_i^s is defined on sliding and escaping regions of Σ_i, it follows that h_i does not vanish on these intervals since $X_i \cdot Y_i(p) < 0$ if $p \in \Sigma_i^{e,s}$. Then, we have the next proposition which proof follows directly.

Proposition 1 *Let $Z = (X, Y) \in \Omega$. Then $p \in \Sigma_i^{e,s}$ is a pseudo-equilibrium of Z_i^s, $i = 1, 2$, if and only if, $\det Z(p) = 0$. In addition, $p \in \Sigma_i^{e,s}$ is a hyperbolic pseudo-equilibrium to Z_i^s provided $\dfrac{\partial}{\partial x_j} \det Z(p) \neq 0$, for $i = 1, 2$ and $i \neq j$.*

When both sliding vector fields are defined in a neighborhood of the origin, using (6) we obtain that $Z_1^s(0) = 0$ if, and only if, $Z_2^s(0) = 0$.

The next proposition will be important when we construct the homeomorphisms in order to prove the local $\Sigma-$equivalence between the $\Sigma-$structural stable vector fields.

Proposition 2 *Suppose that $p_0 \in \Sigma_i^{s,e}$ is a regular point of the sliding vector field Z_i^s. Then Z_i^s is locally conjugated to the constant vector field $\tilde{Z}_i^s(p) = (-1)^{i-1} sgn(X_i(p_0)) \cdot sgn(\det Z(p_0))$ by the identity map.*

Proof A straightforward computation gives us that around p_0 we have sign $\operatorname{sgn}(Z)_i^s(p) = (-1)^{i-1} \operatorname{sgn}(X_i(p_0))$. Since Z_i^s and $\tilde{Z}_i^s$ are one dimensional vector fields, they are conjugated by the identity map.

Corollary 1 *Let $Z = (X, Y) \in \Omega$. Suppose that both sliding vector fields are defined around the origin, then $sgn(Z_1^s(0)) = sgn(Z_2^s(0))$, if $X_1(\mathbf{0}) \cdot X_2(\mathbf{0}) < 0$ or $sgn(Z_1^s(\mathbf{0})) = -sgn(Z_2^s(\mathbf{0}))$, if $X_1(\mathbf{0}) \cdot X_2(\mathbf{0}) > 0$.*

Proposition 3 *Given $Z = (X, Y) \in \Omega$ suppose that the origin belongs to $\Sigma_i^{e,s} \cap \Sigma_j^c$ for $i, j = 1, 2$ and $i \neq j$. Then the origin is a regular point of Z_i^s.*

Proof Since we have $Z_i^s(0) = 0$ if and only if, $\det Z(p) = 0$, it is enough to prove that $\det Z(p) \neq 0$. As $0 \in \Sigma_i^{e,s}$, we have $X_i \cdot Y_i(\mathbf{0}) < 0$ and $0 \in \Sigma_j^c$ therefore $X_j \cdot Y_j(\mathbf{0}) > 0$. Consequently, $sgn(X_i(\mathbf{0}).Y_j(\mathbf{0})) = sgn(-Y_i(\mathbf{0}).X_j(\mathbf{0}))$ and then $\det Z(\mathbf{0}) \neq 0$.

From now on, our goal is to classify the local $\Sigma-$structural stable behavior in Ω. We give a normal form for each equivalence class and construct the respective local $\Sigma-$equivalences between an arbitrary vector field and its corresponding normal form.

Observe that being X and Y transverse to Σ at the origin, we have the following configurations for Σ:

C1. $X_i \cdot Y_i(\mathbf{0}) > 0$, for $i = 1, 2$ and then $\Sigma = \Sigma^c$;
C2. $X_i \cdot Y_i(\mathbf{0}) < 0$, for $i = 1, 2$ and we have $\Sigma = \Sigma^s \cup \Sigma^e$;
C3. $X_i \cdot Y_i(\mathbf{0}) > 0$ and $X_j \cdot Y_j(\mathbf{0}) < 0$, then $\Sigma_i = \Sigma_i^c$ and $\Sigma_j = \Sigma_j^s \cup \Sigma_j^e$ for $i, j = 1, 2$ and $i \neq j$;

Considering the continuous maps

$$\begin{aligned} \xi_i : \Omega &\to \quad \mathbb{R} \\ Z &\mapsto X_i \cdot Y_i(\mathbf{0}) \end{aligned} \tag{8}$$

it follows that conditions stated in items C1 to C3 are open. Then for each $Z \in \Omega$ satisfying conditions C1 to C3 there exists a neighborhood $\mathcal{V}_Z \subset \Omega$ such that $sgn(\xi_i|_{\mathcal{V}_Z})$ is constant. Moreover, conditions C1 to C3 define a generic set, since its complement in Ω is $\xi_1^{-1}(0) \cup \xi_2^{-1}(0)$. Nevertheless, even if these conditions are open, we will see that vector fields satisfying some of them are not structurally stable. We will analyze each case separately.

Definition 16 Let $\Omega_0^1 \subset \Omega$ be the subset of all $Z \in \Omega$ satisfying condition C1 and therefore, condition A on Theorem 1.

The Proposition 4 gives the local $\Sigma-$equivalence between any $Z \in \Omega_0^1$ and the corresponding normal form $\tilde{Z}$.

Let $Z \in \Omega_0^1$ and $\mathcal{V}_Z \subset \Omega$ be a neighborhood of Z such that $sgn(\xi_i|_{\mathcal{V}_Z}) > 0$, $i = 1, 2$. By Proposition 4, each $Z' \in \mathcal{V}_Z$ is locally $\Sigma-$equivalent to $\tilde{Z}$. By transitivity it follows that Z' is $\Sigma-$equivalent to Z. Then every $Z \in \Omega_0^1$ is locally $\Sigma-$structurally stable.

Proposition 4 *Suppose that $Z \in \Omega_0^1$. Then Z is locally $\Sigma-$equivalent to the piecewise smooth system*

$$\tilde{Z}(p) = \begin{cases} \tilde{X}(p) = (a, b), & \text{if } p \in \mathcal{U}^+ \\ \tilde{Y}(p) = (a, b), & \text{if } p \in \mathcal{U}^- \end{cases}, \tag{9}$$

where $a = sgn(X_1(\mathbf{0}))$ and $b = sgn(X_2(\mathbf{0}))$.
In other words, Z is $\mathcal{C}^0-$equivalent to the continuous vector field $\tilde{Z}(p) = (a, b)$.

Proof We will present the construction for the case $X_1(\mathbf{0}), X_2(\mathbf{0}) > 0$. The other cases can be done analogously. In this case, the vector field $\tilde{Z}$ has the form

$$\tilde{Z}(x, y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$

Since X and Y are both transverse to Σ at the origin, there is a neighborhood $\mathcal{U}$ of the origin such that for each initial condition $p \in \mathcal{U}$ the trajectory of Z through p reaches the discontinuity Σ in finite time.

We are going to construct the homeomorphism piecewisely.
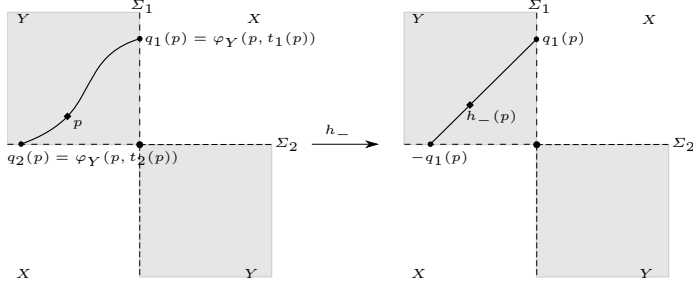
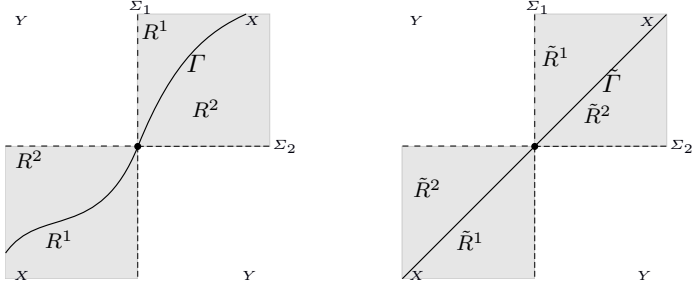**Fig. 6** The map h_- in $\mathcal{U}^-$ for $X_1 \cdot X_2(\mathbf{0}) > 0$.

Fig. 7 The regions R^i for $i = 1, 2$ of $\mathcal{U}$ and the respective cross sections Γ and $\tilde{\Gamma}$ for Z and $\tilde{Z}$ when $X_1 \cdot X_2(\mathbf{0}) > 0$.

Let $p \in \overline{\mathcal{U}^-}$. As $Y_1 \cdot Y_2(\mathbf{0}) > 0$, Z is transient in $\mathcal{U}^-$. Then there exists a unique time $t_i(p) \in \mathbb{R}$ such that $Q_i(p) = \varphi_Y(p; t_i(p)) \in \Sigma_i$ for $i = 1, 2$.

On the other hand, given $Q_1(p) = (0, q_1(p)) \in \Sigma_1$, the point $\tilde{Q}_2(p) = (-q_1(p), 0) = \varphi_{\tilde{Y}}(Q_1(p); -q_1(p))$ belongs to Σ_2.

As illustrated in Figure 6, we define the homeomorphism h^- on $\overline{\mathcal{U}^-}$ as

$$h^-(p) = \varphi_{\tilde{Y}}(Q_1(p), \sigma(p))$$

where $\sigma(p) = -\dfrac{q_1 \cdot t_1}{t_2 - t_1}(p)$.

Observe that $h^-|_{\Sigma_1} = Id$, $h^-(p) = (-q_1(p), 0)$ if $p \in \Sigma_2$ and $h^-(\mathbf{0}) = \mathbf{0}$. Moreover, the map h^- is an homeomorphism in $\mathcal{U}^-$.

Consider now the cross section $\Gamma = \{\varphi_X(0, t), \ t \in \mathbb{R}\} \cap \mathcal{U}$ and define the regions

$$R_1 = \{p \in \mathcal{U}_+^+ : p \text{ is above } \Gamma\} \cup \{p \in \mathcal{U}_-^+ : p \text{ is below } \Gamma\} \cup \Gamma,$$

$$R_2 = \{p \in \mathcal{U}_+^+ : p \text{ is below } \Gamma\} \cup \{p \in \mathcal{U}_-^+ : p \text{ is above } \Gamma\} \cup \Gamma.$$

Since $\Sigma = \Sigma^c$ for each $p \in \overline{R_i}$ there exists a unique $t_i(p) \in \mathbb{R}$ such that $Q_i(p) = \varphi_X(p, t(p)) \in \Sigma_i$, $i = 1, 2$.

Define on each region $\overline{R_i}$ the homeomorphisms

$$h_1^+(p) = \varphi_{\tilde{X}}(Q_1(p); -t_i(p)), \; p \in R_1,$$
$$h_2^+(p) = \varphi_{\tilde{X}}(h_-(Q_2(p)); -t_i(p)), \; p \in R_2.$$

Observe that if $p \in \Gamma$ then $h_1^+(p) = \varphi_{\tilde{X}}(0, -t(p)) = h_2^+(p)$. Therefore, the map h^+, defined as $h^+(p) = h_1^+(p)$, if $p \in R_1$ and $h^+(p) = h_2^+(p)$, if $p \in R_2$ is a homeomorphism in $\mathcal{U}^+$.

Moreover, the maps h^+ and h^- agree on Σ. If $p \in \Sigma_i$, then $Q_i(p) = p$ and $t_i(p) = 0$, thus $h^-(p) = h_{1,2}^+$. Therefore the map h defined as follow is an homeomorphism.

$$h(p) = \begin{cases} h^-(p), & p \in \overline{\mathcal{U}^-}, \\ h_1^+(p), & p \in \overline{R_1}, \\ h_2^+(p), & p \in \overline{R_2} \end{cases} .$$
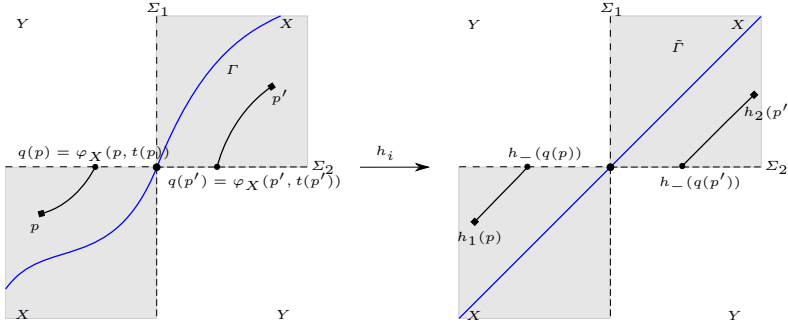

Fig. 8 The homeomorphism h_i define on regions R^i for $X_1 \cdot X_2(\mathbf{0}) > 0$

Regarded the way that h has been constructed it is clear that h carries the trajectories of Z to trajectories of $\tilde{Z}$ preserving the orientation. Moreover, if $p = (p_1, p_2) \in \mathcal{U}$, a straightforward calculation shows that the map

$$g(p) = \begin{cases} \varphi_Y((0, p_2 - p_1)); \tau(p)), & p \in \overline{\tilde{\mathcal{U}}^-}, \\ \varphi_X((0, p_2 - p_1); p_1), & p \in \overline{\tilde{R}_1}, \\ \varphi_X(\varphi_Y((0, p_2 - p_1); -t_2(p_2 - p_1); p_2) & p \in \overline{\tilde{R}_2}, \end{cases}$$

where $\tau(p) = \dfrac{p_1 \cdot t_2(0, p_2 - p_1)}{p_1 - p_2}$ is the inverse of h. Thus h is a homeomorphism.

The case $X_1(\mathbf{0}), X_2(\mathbf{0}) < 0$ is analogous. The only difference is that we must consider the cross section $\Gamma = \{\varphi_Y(t, 0)\} : t \in \mathbb{R}\} \cap \mathcal{U}$, define the regions R_i and then proceed in the same way as in the case $X_1(\mathbf{0}), X_2(\mathbf{0}) > 0$.

Opposed to the case C1, if $Z \in \Omega$ satisfies C2 one can not automatically conclude that Z is locally Σ-structurally stable even if C2 is an open condition.

This occurs because one can have that Z satisfies C2 and $\det Z(\mathbf{0}) = 0$ and then the origin is a pseudo-equilibrium for both Z_i. Lets take for instance

$$Z_\alpha(x,y) = \begin{cases} X_\alpha(x,y) & = (1 - \alpha + x, 1) \\ Y(x,y) & = (-1 + y, -1) \end{cases}. \tag{11}$$

In this case, the sliding vector field is defined in Σ_1 and Σ_2. However, for $\alpha = 0$ the vector field Z_0 has just one pseudo-equilibrium at the origin, while when $\alpha \neq 0$ the vector field Z_α has two pseudo-equilibria in Σ_1 and Σ_2 near the origin. Therefore, one can not establish a local $\Sigma-$equivalence between Z_0 and Z_α for $\alpha \neq 0$.

Since $\det : Z \in \Omega \mapsto \det Z(\mathbf{0}) \in \mathbb{R}$ is a continuous function, once Z satisfies $\det Z(\mathbf{0}) \neq 0$, we obtain a neighborhood $\mathcal{W}_Z \subset \mathcal{V}_Z$ such that $\mathrm{sgn}\,(\det|_{\mathcal{W}_Z})$ is constant. Then $Z' \in \mathcal{W}_Z$ satisfies condition C2 and $\det Z'(\mathbf{0}) \neq 0$.

Definition 17 Let Ω_0^2 be the set of all $Z \in \Omega$ satisfying C2 and $\det Z(\mathbf{0}) \neq 0$ and therefore condition B of Theorem 1.

Then from the argument exposed above and the next proposition we conclude that $Z \in \Omega_0^2$ is local $\Sigma-$structurally stable.

Remark 2 Even if the formula stated for the normal form of $\tilde{Z} \in \Omega_0^2$ in system (12) is cumbersome, it is a good way to write all the normal forms in a concise way. Substituting the values for a, b and c indicated in Proposition 5 the expression of $\tilde{Z}$ becomes very simple.

Proposition 5 *Let $Z \in \Omega_0^2$. Then Z is locally $\Sigma-$equivalent to the piecewise smooth system*

$$\tilde{Z}(p) = \begin{cases} ((\delta_{-1(ab)}\delta_{1c} + 1) \cdot a, -(\delta_{-1(ab)}\delta_{-1c} - ab) \cdot a), & \text{if } p \in \mathcal{U}^+ \\ (-(\delta_{1(ab)}\delta_{1c} + 1) \cdot a, -(\delta_{1(ab)}\delta_{-1c} + ab) \cdot a), & \text{if } p \in \mathcal{U}^- \end{cases}, \tag{12}$$

where $a = sgn(X_1(\mathbf{0}))$, $b = sgn(X_2(\mathbf{0}))$, $c = sgn(\det Z(\mathbf{0}))$ and δ_{rs} is the Kronecker function.

Proof Observe that $\tilde{Z}$ also satisfies condition C2. Moreover, substituting the value of ab in the formula (12), we obtain $\det \tilde{Z}(\mathbf{0}) = \delta_{1c} - \delta_{-1c}$. Thus, $\mathrm{sgn}\left(\det \tilde{Z}(\mathbf{0})\right) = \mathrm{sgn}\,(\det Z(\mathbf{0})) \neq 0$.

As Z_i^s and $\tilde{Z}_i^s$ are one dimensional vector fields with the same sign, the identity is an equivalence between them. Therefore, the same construction of Proposition 4 can be applied in this case.

The last case to be studied is when Z belongs to Ω_0^3, that is, the set of all $Z \in \Omega$ satisfying C3. Observe that in the previous cases there were no meaningful differences on the dynamics of Z depending of $\mathrm{sgn}\,(X_1 \cdot X_2(\mathbf{0}))$, since one case is just the reflection of the other. This is not true when we are considering $Z \in \Omega_0^3$.

For this reason, we define,

Definition 18 Let Ω_0^3 the set of all $Z \in \Omega$ satisfying condition $C3$ and

$$\Omega_0^{3,1} = \{Z \in \Omega_0^3 : X_1 \cdot X_2(\mathbf{0}) > 0\}.$$

A standard argument and Proposition 6 show that $Z \in \Omega_0^{3,1}$ is always local $\Sigma-$structurally stable. Observe that this is not the case when $Z \in \Omega_0^3 \setminus \Omega_0^{3,1}$. In fact, if $X_1 \cdot X_2(\mathbf{0}) < 0$, the vector field X is transient. Moreover, condition C3 gives straightforwardly that the vector field Y is also transient. Consequently, since Z is transient, one needs to analyze the first return map defined in (5) in order to avoid a non hyperbolic fixed point of ϕ_Z at the origin. This situation would lead to a higher codimension bifurcation.

Proposition 6 *Suppose $Z \in \Omega_0^{3,1}$. Then Z is locally $\Sigma-$equivalent around at the origin to*

$$\tilde{Z}(p) = \begin{cases} \tilde{X}(p) = (a, a), & \text{if } p \in \mathcal{U}^+ \\ \tilde{Y}(p) = (b, -b), & \text{if } p \in \mathcal{U}^- \end{cases},$$

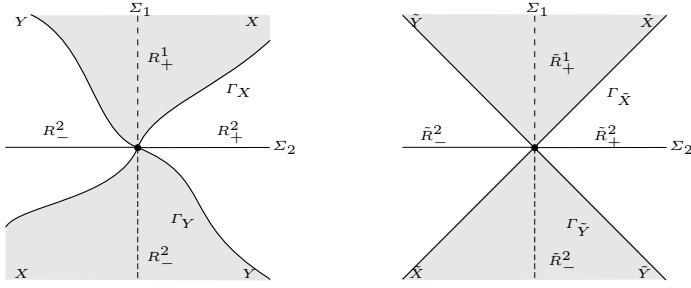*where $a = sgn(X_1(\mathbf{0}))$ and $b = sgn(Y_1(\mathbf{0}))$.*

Fig. 9 The cross sections $\Gamma_{X,Y}$ and regions R^i, $i = 1, 2$ when $a = 1$.

Proof Fix $a = b = 1$. Other cases can be done similarly.

In this case we have $\Sigma_1 = \overline{\Sigma^c}$ and $\Sigma_2 = \overline{\Sigma^e} \cup \overline{\Sigma^e}$, by Proposition 3 the origin is a regular point of Z_2^s and by Proposition 2, Z_2^s is locally conjugated to the constant vector field $\tilde{Z}_2^s(p) = 1$. Moreover, by Proposition 2 there exists a homeomorphism h^* with $h^*(0) = 0$ which gives the $\mathcal{C}^0-$equivalence between these vector fields in a neighborhood of the origin.

The trajectories of X and Y through the origin are both admissible and intersect Σ transversely at this point and the same occurs for $\tilde{Z}$.

Consider the following cross sections of Σ given by $\Gamma_X = \{\varphi_X(t, 0) : t \in \mathbb{R}\} \cap \mathcal{U}$ and $\Gamma_Y = \{\varphi_Y(t, 0) : t \in \mathbb{R}\} \cap \mathcal{U}$. Analogously, we define the cross sections $\Gamma_{\tilde{X}}$ and $\Gamma_{\tilde{Y}}$ for $\tilde{Z}$. See Figure 9.

Let $R_i \subset \mathcal{U}$ be the region between Γ_X and Γ_Y which contains Σ_i for $i = 1, 2$. In addition, decompose R_2 into two regions given by $R_2^{\pm} = R_2 \cap \mathcal{U}^{\pm}$. Proceeding in the same way as above, we define $\tilde{R}_1$ and $\tilde{R}_2^{\pm}$ contained in $\tilde{\mathcal{U}}$.

For each $p \in \overline{R_1}$ there exists a unique $t_1(p) \in \mathbb{R}$ such that the point $q(p) = \varphi_Z(p, t_1(p))$ belongs to Σ_1. Then set

$$h(p) = \varphi_{\tilde{Z}}(\varphi_Z(p, t_1(p)), -t_1(p)).$$

If $p \in \overline{R_2}$ then $t_2(p) \in \mathbb{R}$ is the unique time such that $q(p) = \varphi_X(p, t_2(p)) \in \Sigma_2$ if $p \in \overline{R_2^+}$ and $q(p) = \varphi_Y(p, t_2(p)) \in \Sigma_2$ if $p \in \overline{R_2^-}$. In both cases, $h^*(q(p))$ belongs to $\tilde{\Sigma}_2$ and if $p \in \overline{R_2^+}$ we define $h(p) = \varphi_{\tilde{X}}(h^*(q(p)), -t_2(p))$ and in case $p \in \overline{R_2^-}$ we set $h(p) = \varphi_{\tilde{Y}}(h^*(q(p)), -t_2(p))$.

The three functions defined above are homeomorphisms and due to the way they were constructed they also agree on the intersections, hence the map

$$h(p) = \begin{cases} \varphi_{\tilde{Z}}(\varphi_Z(p, t_1(p)), -t_1(p)), & p \in \overline{R_1}, \\ \varphi_{\tilde{X}}(h^*(\varphi_X(p, t_2(p)))), -t_2(p)), & p \in \overline{R_2^+}, \\ \varphi_{\tilde{Y}}(h^*(\varphi_Y(p, t_2(p)))), -t_2(p)), & p \in \overline{R_2^-}. \end{cases}$$

is a homeomorphism which carries trajectories of Z into trajectories of $\tilde{Z}$ preserving the orientation and so they are locally $\Sigma-$equivalent.

Finally suppose that $Z \in \Omega_0^3 \setminus \Omega_0^{3,1}$, that is, Z satisfies $X_1 \cdot X_2(\mathbf{0}) < 0$. Under these hypothesis, Z is transient. Thus we need to understand what happens with the first return map ϕ_Z defined in (5).
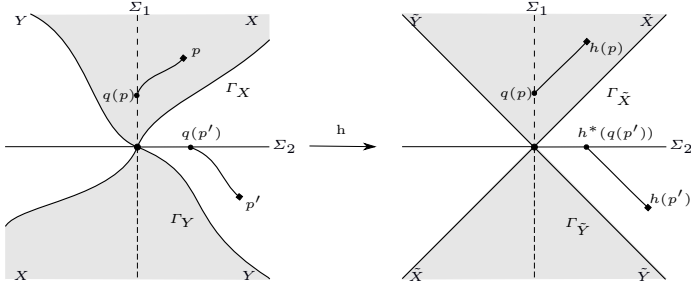

Fig. 10 The homeomorphism h for $a = 1$. In this case h is defined independently in each region R^i in a way to agree on the cross sections.

It is important to notice that since there are sliding and escaping regions in Σ, given $p \in \Sigma_2^-$ its Z trajectory does not reach Σ_2^- again. Thus there are no regular periodic orbits for this case. Even though, one can exist closed contours which are preserved by $\Sigma-$equivalences.

In general, it is not easy to compute explicitly the first return map. The next proposition gives its approximation when Z is transient and transverse to Σ around the origin.

Proposition 7 *Let $Z \in \Omega_0^3$ satisfying $X_1 \cdot X_2(\mathbf{0}) < 0$. Then the first return map is given by*

$$\phi_Z(x) = \alpha_Z^2 x + \mathcal{O}(x^2)$$

with

$$\alpha_Z = \frac{X_1 \cdot Y_2(\mathbf{0})}{X_2 \cdot Y_1(\mathbf{0})}. \tag{13}$$

Proof Near the origin, the trajectories of Y through a point $(x,0) \in \Sigma_2$ can be written as

$$\varphi_Y((x,0),t) = (x,0) + (Y_1(x,0), Y_2(x,0))t + \mathcal{O}(t^2). \qquad (14)$$

From this equation we obtain that the time to arrive Σ_1 is $t = \dfrac{x}{Y_1(\mathbf{0})} + \mathcal{O}(x^2)$. Then, using again (14), we have

$$\phi_Y(x) = -\frac{Y_2(\mathbf{0})}{Y_1(\mathbf{0})}x + \mathcal{O}(x^2).$$

Analogously, for all $(y,0)$ near the origin, we obtain

$$\phi_X(y) = -\frac{X_1(\mathbf{0})}{X_2(\mathbf{0})}y + \mathcal{O}(y^2).$$

By composing twice ϕ_X and ϕ_Y we obtain the desired map.

Observe that the constant α_Z is always negative in this case. Then by Proposition 7, the origin is a hyperbolic fixed point for ϕ_Z if and only if,

$$\gamma_Z = X_1 \cdot Y_2(\mathbf{0}) + X_2 \cdot Y_1(\mathbf{0}) \neq 0 \qquad (15)$$

In addition, the origin will be stable if $\alpha_Z + 1 > 0$ and unstable if $\alpha_Z + 1 < 0$.

Definition 19 Let $\Omega_0^{3,2}$ be the subset of Ω_0^3 such that $X_1 \cdot X_2(\mathbf{0}) < 0$ and $\alpha_Z \neq -1$.

It is clear that given $Z \in \Omega_0^{3,2}$ there exists a neighborhood $\mathcal{W}_Z$ of Z such that $Z' \in \mathcal{W}_Z$ then $Z' \in \Omega_0^{3,2}$ with $\operatorname{sgn}(\alpha_Z + 1) = \operatorname{sgn}(\alpha_{Z'} + 1)$. As an easy consequence of the next proposition we have that Z and Z' are locally $\Sigma-$equivalent and thus Z is locally $\Sigma-$structurally stable.

Remark 3 Observe that the origin is attractive of ϕ_Z if $|\alpha_Z| < 1$, this is equivalent to $\gamma_Z = |X_1 \cdot Y_2(\mathbf{0})| + |X_2 \cdot Y_1(\mathbf{0})| = X_1 \cdot Y_2(\mathbf{0}) + X_2 \cdot Y_1(\mathbf{0}) < 0$. Analogously, the origin is repelling if $\gamma_Z > 0$.

Proposition 8 *Suppose $Z \in \Omega_0^{3,2}$. Then Z is locally Σ-equivalent to*

$$\tilde{Z}(p) = \begin{cases} (a,-a), & \text{if } p \in \mathcal{U}^+, \\ (b(1+\delta_{1c}), b(1+\delta_{-1c})), & \text{if } p \in \mathcal{U}^-, \end{cases}$$

with $a = sgn(X_1(\mathbf{0}))$, $b = sgn(Y_1(\mathbf{0}))$ and $c = sgn(\alpha_Z + 1)$.

Proof We are going to fix $a, b, c = 1$, the other cases can be treated analogously. First of all, observe that $\alpha_{\tilde{Z}} = -\frac{1}{2}$, thus the origin is an stable hyperbolic fixed point to $\phi_{\tilde{Z}}$. By the Grobman-Hartman Theorem there exists a homeomorphism h^* defined in a neighborhood of the origin such that $\phi_Z \circ h^* = h^* \circ \phi_{\tilde{Z}}$.

Moreover, by Proposition 2 and Proposition 3 we have that Z_2^s and $\tilde{Z}_2^s$ are $\mathcal{C}^0-$equivalent in a suitable neighborhood of the origin.

Let consider $R^+ = \{(x,y) \in \mathcal{U} : y > 0\}$ and $R^- = \{(x,y) \in \mathcal{U} : y < 0\}$. Since $a, b = 1$ then $\Sigma_1 = \overline{\Sigma^c}$ and so the trajectories of Z through a point of $\mathcal{U}^+$ or $\mathcal{U}^-$

is an appropriate concatenation of the trajectories of X and Y. The same is valid for $\tilde{Z}$.

For each $(q,0) \in \Sigma_2^-$ there exist unique times $t_1(q), t_2(q) \in \mathbb{R}$ such that $(0, q_1^+) = \varphi_Y((q,0); t_1(q)) \in \Sigma_1^+$ and $(q_2^+, 0) = \varphi_X((0, q_1^+), t_2(q) - t_1(q)) \in \Sigma_2^+$. This defines a continuous curve (see Figure 11) in $\overline{R^+}$

$$\eta_q^+(t) = \begin{cases} \varphi_Y((q,0); t), & t \in I_q^1 = [0, t_1(q)], \\ \varphi_X(\varphi_Y((q,0); t_2(q) - t_1(q)), t), & t \in I_q^2 = [t_1(q), t_2(q)]. \end{cases} \tag{16}$$
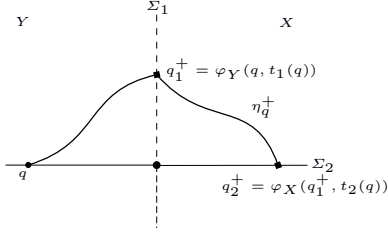

Fig. 11 The trajectory of every point $q \in \Sigma_2^-$ can be seen as a continuous curve η_q^+.

Analogously we define the curve η_q^- in $\overline{R^-}$:

$$\eta_q^-(s) = \begin{cases} \varphi_Y((q,0); s), & s \in J_q^1 = [0, s_1(q)], \\ \varphi_X(\varphi_Y((q;0); s_2(q) - s_1(q)); s), & s \in J_q^2 = [s_1(q), s_2(q)], \end{cases} \tag{17}$$

where $s_1(q), s_2(q) \in \mathbb{R}$ satisfy $(0, q_1^-) = \varphi_Y((q,0); s_1(q)) \in \Sigma_1^-$, and $(q_2^-, 0) = \varphi_Y((0, q_1^-); s_2(q) - s_1(q)) \in \Sigma_2^+$.

For each $p \in \overline{R^+}$ there exists a unique point $(q(p), 0) \in \Sigma_2^-$ and a unique $t(p)$ such that $p = \eta_{q(p)}^+(t(p))$ for some $t(p) \in I_{q(p)} = I_{q(p)}^1 \cup I_{q(p)}^2$.

Consider $h^*(q(p)) \in \tilde{\Sigma}_2^-$ and let

$$\tilde{\eta}_{h^*(q(p))}^+ : \tilde{I}_{h^*(q(p))} \to \overline{\tilde{R}^+}$$

be its $\tilde{Z}$ trajectory. Where

$$\tilde{I}_{h^*(q(p))} = [0, -\tfrac{1}{2} h^*(q(p))] \cup [-\tfrac{1}{2} h^*(q(p)), -h^*(q(p))].$$

In order to get an equivalence that preserves Σ, we will make a reparametrization of time σ_p^+ which preserves the subintervals of $I_{q(p)}$ and $\tilde{I}_{h^*(q(p))}$. Then define the homeomorphism h^+ on $\overline{R^+}$ by

$$h^+(p) = \tilde{\eta}_{h^*(q(p))}^+(\sigma_p^+(t(p))). \tag{18}$$

Let $r(p)$ be the intersection between the trajectory of p in R^- with Σ_2^-. Proceeding in the same way we define the homeomorphism h^- in $\overline{R^-}$ by

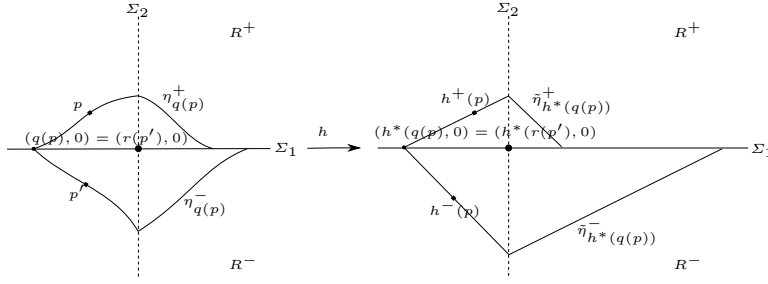$$h^-(p) = \tilde{\eta}_{h^*(r(p))}^-(\sigma_p^-(s(p))). \tag{19}$$

Fig. 12 The homeomorphism which gives the equivalence between $Z \in \Omega_0^{3,2}$ and its normal form for $a, b, c = 1$.

Given $p \in \Sigma_2$, then if $p \in \overline{\Sigma_2^-}$ then $h^+(p) = h^*(p) = h^-(p)$ since both maps coincide with h^* in Σ_2^-. On the other hand, given $p \in \overline{\Sigma_2^+}$ the trajectory of p in $\overline{R^\pm}$ intersects Σ_2^- at the point $q(p)$ and $r(p)$, respectively. Observe that $r(p) = \phi_Z(q(p))$ and since $h^* \circ \phi_Z = \phi_{\tilde{Z}} \circ h^*$ we have that $\phi_{\tilde{Z}}(h^*(q(p))) = h^*(r(p))$, therefore $h^*(q(p))$ and $h^*(r(p))$ belong to the same trajectory, then $h^+(p) = h^-(p)$.

As the maps h^+ and h^- agree in Σ_2, then we can define $h : \mathcal{U} \to \tilde{\mathcal{U}}$ given by

$$
h(p) = \begin{cases} \tilde{\eta}_{h^*(q(p))}^+(\sigma_p^+(t(p))), & p \in \overline{R^+}, \\ \tilde{\eta}_{h^*(r(p))}^-(\sigma_p^-(s(p))), & p \in \overline{R^-}. \end{cases}
$$

Since all the maps involved are continuous and bijective, we conclude that h is also a bijective and continuous function. The inverse of h can be constructed in the same way and is also continuous, then h is a homeomorphism.

Due to the way that h was constructed it is clear that h maps all trajectories of Z in trajectories of $\tilde{Z}$, including the trajectories of the sliding vector field, since h preserves the order on Σ_2.

Let us observe that the set $\Omega_0^{3,1} \cup \Omega_0^{3,2}$ consists of the piecewise vector fields satisfying condition C of Theorem 1.

We are now in conditions to present a characterization of Ω_0, the set of all locally $\Sigma-$structurally stable systems in Ω. Set

$$
\Omega_0' = \Omega_0^1 \cup \Omega_0^2 \cup \Omega_0^{31} \cup \Omega_0^{32} \subset \Omega,
$$

during this section we had shown that $\Omega_0' \subset \Omega_0$. On the other hand, if $Z \in \Omega \setminus \Omega_0'$ then Z satisfies at least one of the following items:

(i) The map $\xi_i(Z) = 0$ for $i = 1$ or 2;
(ii) $Z \in \Omega_0^2$ but $\det Z(\mathbf{0}) = 0$;
(iii) $Z \in \Omega_0^{3,2}$ with $\gamma_Z = 0$ (see (15)), or equivalently, $\alpha_Z = -1$ (see (13)).

If $Z \in \Omega \setminus \Omega_0$ satisfies (i), then Z is tangent to Σ at the origin. In this case, the family

$$
Z_n = \begin{cases} X + (\frac{1}{n}, \frac{1}{n}) & p \in \mathcal{U}^+ \\ Y + (\frac{1}{n}, \frac{1}{n}) & p \in \mathcal{U}^- \end{cases}
$$

converges to Z when $n \to \infty$ and it is transverse to Σ at the origin for all $n \in \mathbb{N}$, then Z and Z_n can not be locally $\Sigma-$equivalent. Therefore, Z is not local $\Sigma-$structurally stable.

In case Z satisfies (ii) or (iii) the family

$$Z_n = \begin{cases} X + (\frac{1}{n}, 0) & p \in \mathcal{U}^+ \\ Y & p \in \mathcal{U}^- \end{cases}$$

also converges to Z. Moreover, observe that $\det Z_n(\mathbf{0}) \neq 0$ and $\gamma_{Z_n} \neq 0$, thus one can not establish a Σ−equivalence between Z and Z_n. Concluding that if $Z \in \Omega$ does not belong to Ω'_0 then Z is not structurally stable, thus $\Omega'_0 = \Omega_0$.

In addition, the above argument also shows that Ω_0 is dense in Ω since for every neighborhood of $Z \in \Omega$ there exists a sequence $Z_n \in \Omega_0$ such that $Z_n \to Z$ when $n \to \infty$. Combining the results we stated by now, we have proved Theorem 1.

4 Codimension one generic bifurcations

Once we have classified the generic behavior of $Z \in \Omega_0$, we will now investigate what happens in the bifurcation set $\Omega_1 = \Omega \setminus \Omega_0$.

The goal of this section is to classify the codimension one generic bifurcation set, which will be called $\Xi_1 \subset \Omega_1$, that is, classify the set of the structurally stable piecewise smooth systems in Ω_1 endowed with the induced topology of Ω. In order to Z belong to Ξ_1 it is necessary to break at most one of the conditions stated in Theorem 1. We must consider the following three groups:

- X or Y has a tangency point at the origin, that is, $X_i(\mathbf{0}) = 0$ or $Y_i(\mathbf{0}) = 0$ for $i = 1, 2$;
- Z satisfying condition C2 and $\det Z(\mathbf{0}) = 0$, that is, the origin is a pseudo-equilibrium for the sliding vector fields Z_i^s, $i = 1, 2$;
- Z satisfying C3, $X_1 \cdot X_2(\mathbf{0}) < 0$ and $\gamma_Z = 0$, or equivalently, $\alpha_Z = -1$, that is, the origin is a non hyperbolic fixed point for the first return map ϕ_Z.

More precisely, we will prove the following theorem:

Theorem 2 (Local Σ−structural stability on Ω_1) *Let $Z \in \Omega_1 = \Omega \setminus \Omega_0$. Then $Z \in \Xi_1$, that is, it has a condimension one singularity at the origin (see definition 15) if, and only if, Z satisfies one of the following conditions:*

A. *$X_i \cdot Y_i(\mathbf{0}) < 0$, $\det Z(\mathbf{0}) = 0$ and $\dfrac{\partial}{\partial x_i} \det Z(\mathbf{0}) \neq 0$ for $i = 1, 2$;*

B. *$X_i \cdot Y_i(\mathbf{0}) > 0$, $X_j \cdot Y_j(\mathbf{0}) < 0$, for $i, j = 1, 2$, $i \neq j$ and $X_1 \cdot X_2(\mathbf{0}) < 0$. In this case, Z is transient and the coefficient α_Z of the first return map (24) satisfies $\alpha_Z = -1$, then we ask the other coefficients to satisfy $\beta_Z \neq 0$ and $\eta_Z \neq 0$.*

C. *the origin is a regular-fold to Z (see Definition 2).*

In addition, the subset Ξ^1 is an open and dense set in Ω_1, therefore local Σ−structural stability is a generic property in Ω_1.

The rest of this section is devoted to prove this theorem.

4.1 The double pseudo-equilibrium bifurcation

We consider now the bifurcation derived from the case where $Z \in \Omega_1$ satisfies condition A. As $\det Z(\mathbf{0}) = 0$, the origin is an pseudo-equilibrium for both sliding vector fields Z_i^s (see (6)). Moreover, as $\frac{\partial}{\partial x_j} \det Z(\mathbf{0}) \neq 0$ for $j = 1, 2$, by proposition 1 we know that the origin is a hyperbolic pseudo-equilibrium for both sliding vector fields.

Definition 20 Let $\Xi_1^1 \subset \Omega_1$ be the subset containing all Z satisfying condition A of Theorem 2, equivalently, for which the origin is a hyperbolic pseudo-equilibrium of the sliding vector fields Z_i^s, $i = 1, 2$.

Proposition 9 *The set Ξ_1^1 is an embedded codimension one submanifold of Ω and it is open in Ω_1.*

Proof Let $Z_0 = (X^0, Y^0) \in \Xi_1^1$. Since conditions $X_i^0 \cdot Y_i^0(\mathbf{0}) < 0$ and $\frac{\partial}{\partial x_i} \det Z_0(\mathbf{0}) \neq 0$ for $i = 1, 2$ of Theorem 2 are open, there is a neighborhood $\mathcal{V}_0 \subset \Omega$ of Z_0 in which these conditions are fulfilled for all $Z \in \mathcal{V}_0$. Moreover, the sign of $X_i \cdot Y_i(\mathbf{0})$ and $\frac{\partial}{\partial x_i} \det Z(\mathbf{0})$ for $i = 1, 2$ are constant in this neighborhood.

Considering the Frechet differentiable map

$$\eta : \mathcal{V}_0 \times \mathcal{D}_0 \quad \to \mathbb{R}^2$$
$$(Z, (x_1, x_2)) \mapsto (\det Z(0, x_1), \det Z(x_2, 0))$$

where $\mathcal{D}_0$ is a neighborhood of the origin in $\mathbb{R}^2$ such that $X_i \cdot Y_i(p) < 0$ for $p \in \Sigma_i$ and $\mathrm{sgn}\left(\frac{\partial}{\partial x_i} \det Z(p)\right)$ is constant for $i = 1, 2$.

By the Implicit Function Theorem applied to η at the point $(Z_0, (0, 0))$, we obtain a Frechet differentiable map

$$g : Z \in \mathcal{W}_0 \subset \mathcal{V}_0 \mapsto (g_1(Z), g_2(Z)) \in \mathcal{U}_0 \subset \mathcal{D}_0, \qquad (20)$$

satisfying

$$\eta(Z, g(Z)) = (\det Z(0, g_1(Z)), \det Z(g_2(Z), 0)) = (0, 0), \ \text{ for all } Z \in \mathcal{W}_0,$$

where $\mathcal{W}_0$ and $\mathcal{U}_0$ are neighborhoods of Z_0 and $\mathbf{0}$, respectively.

The following arguments prove simultaneously that Ξ_1^1 is an embedded codimension one submanifold of Ω and an open set in Ω_1.

Consider the map $g_1 : Z \in \mathcal{W}_0 \mapsto g_1(Z) \in \mathbb{R}$. It is clear that

$$g_1^{-1}(0) = \Xi_1^1 \cap \mathcal{W}_0 = \Omega_1 \cap \mathcal{W}_0 \subset \Xi_1^1.$$

In fact, given $Z \in g_1^{-1}(0)$ then $g_1(Z) = 0$ and therefore $\det Z(\mathbf{0}) = 0$, thus as $Z \in \mathcal{V}_0$, $Z \in \Xi_1^1 \cap \mathcal{W}_0$. On the other hand, if $Z \in \mathcal{W}_0 \cap \Omega_1$ then $\det Z(\mathbf{0}) = 0$, by the exposed above, $g_1(Z)$ is the unique point such that $\det Z(0, x_2) = 0$, then $g_1(Z) = 0$, what proves the desired equality.

To finish the proof one needs to show that $Dg_1(Z_0) \neq 0$. In fact, using the chain rule to the map $\det Z(0, g_1(Z)) = 0$, we obtain

$$Dg_{Z_0} = \frac{D_Z \det Z_0(\mathbf{0})}{\partial_{x_2}(\det) Z_0(\mathbf{0})}$$

which is a non zero linear functional.

Now we are going to study the unfoldings of $Z \in \Xi_1^1$. More precisely, we are going to show that every local unfolding of Z has exactly the same behavior.

Let $Z \in \Xi_1^1$ and $\mathcal{W}_0 \subset \Omega$ be the neighborhood of Z given in (20) where g is defined. One can always suppose that $\mathcal{W}_0$ is connected, therefore the submanifold Ξ_1^1 splits $\mathcal{W}_0$ into two connected open subsets $\mathcal{W}_0^\pm = g_1^{-1}(\mathbb{R}^\pm)$.

Let $\gamma(\delta) = Z_\delta$ a versal unfolding of Z. With no loss of generality, suppose that $\gamma((-\delta_0, 0)) \subset \mathcal{W}_0^-$ and $\gamma((0, \delta_0)) \subset \mathcal{W}_0^+$.

It follows from Proposition 9 that for $\delta \neq 0$ the sliding vector fields $(Z_\delta)_i^s$, $i = 1, 2$ have a unique pseudo-equilibrium $P_i(\delta) = g_i(Z_\delta) \in \Sigma_i$, $i = 1, 2$, which has the same stability as the origin has for Z_i^s. Moreover, if $\delta \neq 0$ then $\det Z_\delta(\mathbf{0}) \neq 0$, therefore the origin is a regular point of the sliding vector fields $(Z_\delta)_i^s$.

To simplify our analysis, fix $X_i(\mathbf{0}) > 0$ for $i = 1, 2$ (and therefore, $Y_i(\mathbf{0}) < 0$ to satisfy condition A of theorem 2), the other case is just the reflection through the $y-$axis. This case leads to four non equivalent configurations, depending on the stability of the origin for the sliding vector fields Z_i^s, $i = 1, 2$. However, we focus only in two, the others can be obtained analogously.

Differentiating (4) and using Proposition 9 it follows that

$$\text{sgn}\left(((Z_\delta)_i^s)'(P_i(\delta))\right) = (-1)^{i-1} c_i, \tag{21}$$

where $c_i = \dfrac{\partial}{\partial x_j} \det Z(\mathbf{0})$, $i, j = 1, 2$ and $i \neq j$.

Moreover, by Proposition 2,

$$\text{sgn}\left((Z_\delta)_i^s(\mathbf{0})\right) = (-1)^{i-1} \det Z_\delta(\mathbf{0}). \tag{22}$$

(a) $\mathrm{x} X_i(\mathbf{0}) > 0$ and $c_i = 1$ for $i = 1, 2$.

(b) Double pseudo-Equilibrium: $X_i(\mathbf{0}) > 0$ for $i = 1, 2$ and $c_1 = 1$ and $c_2 = -1$.
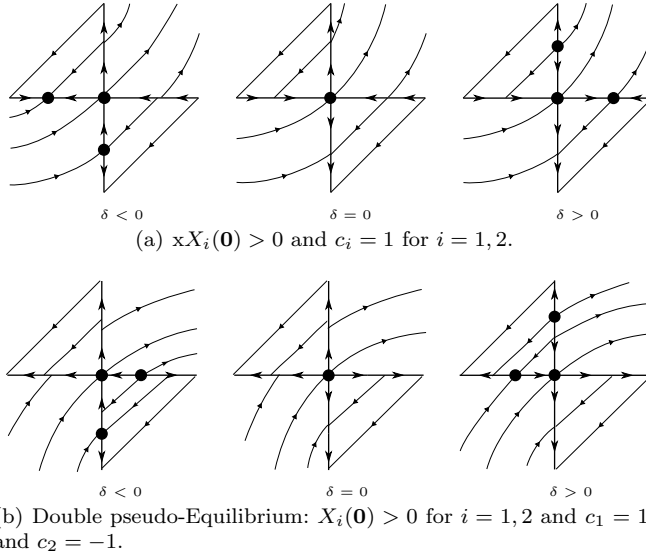
Fig. 13 The unfolding of the double-equilibrium singularity. The big dots represent the pseudo-equilibrium.

One must consider four different cases depending on the signs of c_1 and c_2. We focus in only two, $c_1 = c_2 = 1$ and $c_1 = 1$ and $c_2 = -1$.

• If $c_1 = c_2 = 1$, for $\delta = 0$ the origin is an unstable pseudo-equilibrium for Z_1^s and stable for Z_2^s, as in Fig. 13(a). In one hand, if $\delta < 0$, then $Z_\delta \in \mathcal{W}_0^-$ and thus unstable pseudo-equilibrium $P_1(\delta) = g_1(Z_\delta) \in \Sigma_1^-$. Consequently, $(Z_\delta)_1^s(0) > 0$, therefore $\det Z_\delta(\mathbf{0}) > 0$, thus $(Z_\delta)_2^s(0) < 0$. Then the stable pseudo-equilibrium $P_2(\delta)$ belongs to Σ_2^-. On the other hand, if $\delta > 0$, then $Z_\delta \in \mathcal{W}_0^+$ therefore $P_1(\delta) \in \Sigma_2^+$. It follows that $(Z_\delta)_1^s(0) < 0$ and by the same argument we have $(Z_\delta)_2^s(0) > 0$. Concluding that the attracting pseudo-equilibrium $P_2(\delta)$ belongs to Σ_2^+.

• In the case $c_1 = 1$ and $c_2 = -1$, for $\delta = 0$ the origin is an unstable pseudo-equilibrium for both sliding vector fields, as in Fig. 13(b). Using the same argument as in the case $c_1 = c_2 = 1$ we conclude that for $\delta < 0$ then $P_1(\delta) \in \Sigma_1^-$ and $P_2(\delta) \in \Sigma_2^+$ and for $\delta > 0$ then $P_1(\delta) \in \Sigma_1^+$ and $P_2(\delta) \in \Sigma_2^-$.

As a consequence of the above discussion, we have the following proposition:

Proposition 10 *Let $Z \in \Xi_1^1$. Then any versal unfolding of Z is locally weak equivalent to the one parameter family*

$$
\tilde{Z}_\delta = \begin{cases} \tilde{X}_\delta(x_1, x_2) = \begin{pmatrix} a - bc_2 x_1 \\ b - a\delta \end{pmatrix}, & x_1 \cdot x_2 > 0 \\[2ex] \tilde{Y}(x_1, x_2) = \begin{pmatrix} -a \\ -b + ac_1 x_2 \end{pmatrix}, & x_1 \cdot x_2 < 0 \end{cases} \tag{23}
$$

where $a = sgn(X_1(\mathbf{0}))$, $b = sgn(X_2(\mathbf{0}))$, $c_i = sgn\left(\frac{\partial}{\partial x_j} \det Z(\mathbf{0})\right)$ with $i, j = 1, 2$ and $i \neq j$.

It is easy to see that the family $\tilde{Z}_\delta$ satisfies $\mathrm{sgn}\left(\tilde{X}_1(\mathbf{0})\right) = a$, $\mathrm{sgn}\left(\tilde{X}_2(\mathbf{0})\right) = b$, $\mathrm{sgn}\left(\frac{\partial}{\partial x_j} \det \tilde{Z}(\mathbf{0})\right) = c_i$ with $i, j = 1, 2$ and $i \neq j$. Therefore, the vector field Z and $\tilde{Z}_0$ are locally weak equivalent.

4.2 The pseudo-Hopf bifurcation

In this section we are going to study what happens near a piecewise smooth system $Z \in \Omega_1$ satisfying condition B of theorem 2.

As $X_1 \cdot X_2(\mathbf{0}) < 0$, the vector field Z is transient and we can consider the first return map ϕ_Z, see (5). In Proposition 7 we have computed the first term of the Taylor expansion of ϕ_Z near the origin. Even if one can compute the higher orders terms for ϕ_Z, they have a cumbersome expression and this computation will be omitted. However, suppose that

$$
\phi_X(x) = a_X x + b_X x^2 + c_X x^3 + \mathcal{O}(x^4)
$$
$$
\phi_Y(x) = a_Y x + b_Y x^2 + c_Y x^3 + \mathcal{O}(x^4).
$$

In fact, in Proposition 7 we have seen that $a_X = -\frac{X_1}{X_2}(\mathbf{0})$ and $a_Y = -\frac{Y_2}{Y_1}(\mathbf{0})$. Therefore, we obtain the following expression for ϕ_Z

$$
\phi_Z(x) = \alpha_Z^2 x + (\alpha_Z + \alpha_Z^2)\beta_Z x^2 + \eta_Z x^3 + \mathcal{O}(x^4), \ x \in \Sigma_2^- \tag{24}
$$

where $\alpha_Z = \frac{X_1 Y_2}{X_2 Y_1}(\mathbf{0})$, as given in Proposition 7 and we are assuming that $\alpha_Z = -1$. In this case, the coefficients β_Z and η_Z are given by $\beta_Z = b_X \cdot a_Y^2 + a_X \cdot b_Y$ and

$$\eta_Z = -2 \left((b_X \cdot a_Y)^2 + c_X \cdot a_Y^3 + \left(\frac{b_Y}{a_Y} \right)^2 + \left(\frac{c_Y}{a_Y} \right)^2 \right).$$

It is clear that as $\alpha_Z = -1$ the origin is an unstable fixed point for ϕ_Z if $\eta_Z > 0$ and it is stable if $\eta_Z < 0$.

Definition 21 Let Ξ_1^2 be the set $Z \in \Omega$ satisfying condition B of Theorem 2.

Proposition 11 *The set $\Xi_1^2 \subset \Omega_1$ is an embedded codimension-one submanifold of Ω. Consequently, it is an open set in Ω_1.*

Proof Given $Z_0 \in \Xi_1^2$, our aim is to find a map $g : \mathcal{W}_0 \to \mathbb{R}$ for which 0 is a regular value, $\mathcal{W}_0$ is a neighborhood of Z_0 and $\Xi_1^2 \cap \mathcal{W}_0 = g^{-1}(0)$.

Since conditions $\beta_Z \neq 0$ and $\eta_Z \neq 0$ are open, one can find a neighborhood $\mathcal{V}_0 \subset \Omega$ of Z_0 such that these conditions hold, moreover, the signs of β_Z and η_Z are constant in $\mathcal{V}_0$.

Observe that the origin is a fixed point for ϕ_Z for all $Z \in \mathcal{V}_0$. In this neighborhood ϕ_Z is written as

$$\phi_Z(x) = x \left(\alpha_Z^2 + \left(\alpha_Z + \alpha_Z^2 \right) \beta_Z x + \eta_Z x^2 + \mathcal{O}\left(x^3 \right) \right)$$

Consider the following Frechet differentiable map

$$F : \mathcal{V}_0 \times \mathcal{D}_0 \to \mathbb{R}$$
$$(Z, x) \mapsto \alpha_Z^2 + (\alpha_Z + \alpha_Z^2)\beta_Z x + \eta_Z x^2 + \mathcal{O}(x^3)$$

where $\mathcal{D}_0 \subset \Sigma_2$ is a neighborhood of the origin.

Let $G = \frac{\partial}{\partial x} F$. Since Z_0 belongs to Ξ_1^2, we have $G(Z_0, 0) = (\alpha_{Z_0} + \alpha_{Z_0}^2)\beta_{Z_0} = 0$ and $\frac{\partial}{\partial x} G(Z_0, 0) = 2\eta_{Z_0} \neq 0$. Then by the Implicit Theorem Function there exists a Frechet differentiable map

$$g : Z \in \mathcal{W}_0 \subset \mathcal{V}_0 \mapsto g(Z) \in \mathcal{U}_0 \subset \mathcal{D}_0$$

such that $G(Z, g(Z)) = 0$ for all $Z \in \mathcal{W}_0$, and $\mathcal{W}_0$ and $\mathcal{U}_0$ are neighborhood of Z_0 and $0 \in \Sigma_2$, respectively.

Then for all $Z \in \mathcal{W}_0$ we have

$$\alpha_Z + \alpha_Z^2 = \frac{-2\eta_Z}{\beta_Z} g(Z) + \mathcal{O}(g(Z)^2) \tag{25}$$

Notice that if $Z \in \Xi_1^2 \cap \mathcal{W}_0$ then $\alpha_Z = -1$ and then $G(Z, 0) = 0$ thus $g(Z) = 0$. On the other hand, if $g(Z) = 0$, by Equation (25) we have that $\alpha_Z + \alpha_Z^2 = 0$ therefore $\alpha_Z = -1$. Thus $Z \in \Xi_1^2$ implying that $g^{-1}(0) = \Xi_1^2 \cap \mathcal{W}_0$.

To finish the proof we need to show that Dg_{Z_0} is different from zero. In fact, by the Chain rule we obtain:

$$Dg_{Z_0} = \frac{DG_{(Z_0, 0)}}{2\eta_{Z_0}} = -\frac{\beta_{Z_0}}{2\eta_{Z_0}} \cdot (D\alpha_Z)_{Z_0} \neq 0.$$

Let $Z \in \Xi_1^2$ and consider $\gamma(\delta) = Z_\delta$ a versal unfolding of Z. Suppose that γ is transverse to Ξ_1^2 at $\gamma_0 = Z$ and that $Z_\delta \in \mathcal{W}_0^+$ if $\delta > 0$ and $Z_\delta \in \mathcal{W}_0^-$ if $\delta < 0$, where $\mathcal{W}_0^\pm = g^{-1}(\mathbb{R}^\pm)$.

For each $\delta \in (-\delta_0, \delta_0)$ we associate the first return map

$$\phi_\delta(x) = \alpha_\delta^2 x + (\alpha_\delta + \alpha_\delta^2)\beta_\delta x^2 + \eta_\delta x^3 + \mathcal{O}(x)^4.$$

Observe that $x \neq 0$ is a fixed point for ϕ_δ if, and only if, $F(Z_\delta, x) = 1$. The point $g(Z_\delta) \in \Sigma_2^-$ given by Proposition 11 is a critical point of $F_\delta(x) = F(Z_\delta, x) - 1$ since

$$(F_\delta)'(g(Z_\delta)) = \frac{\partial}{\partial x} F(Z_\delta, g(Z_\delta)) = G(Z_\delta, g(Z_\delta)) = 0.$$

Therefore $g(Z_\delta)$ is a local maximum or minimum of F_δ depending on $\mathrm{sgn}(\eta_\delta) = \mathrm{sgn}(\eta_Z)$ if δ is small enough.

$\delta < 0$ $\qquad\qquad$ $\delta = 0$ $\qquad\qquad$ $\delta > 0$
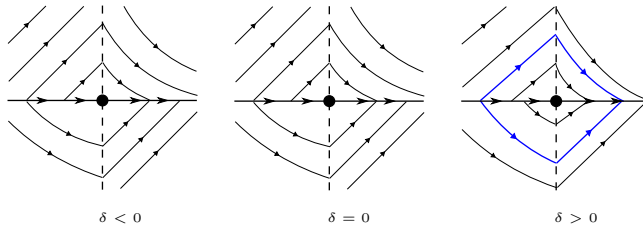
Fig. 14 The pseudo-Hopf bifurcation for Z satisfying $\eta_Z > 0$. The origin is always a fixed point for $\phi_\delta(x)$.

To simplify the analysis, suppose that $X_1(\mathbf{0}) > 0$. Fix η_Z and $\beta_Z > 0$. Therefore, the origin is a repelling fixed point of ϕ_Z and the point $g(Z_\delta)$ is a minimum of $F_\delta(x)$.

Remark 4 The case $\beta_Z < 0$ is just a reparametrization of γ making $\delta \mapsto -\delta$.

If $\delta > 0$, by Equation (25) we have $\alpha_\delta + \alpha_\delta^2 < 0$, thus $-1 < \alpha_\delta < 0$ therefore the origin is an stable fixed point of ϕ_δ.

Moreover using Equation (25), one can see that $F_\delta(g(Z_\delta)) = \alpha_\delta^2 - \eta_\delta g(Z_\delta)^2 + \mathcal{O}(g(Z_\delta)^3) < 0$ and therefore there exist unique points $p_\delta^- < 0 < p_\delta^+$ satisfying $F_\delta(p_\delta^\pm) = 0$ which are unstable fixed points of ϕ_δ since $(\phi_\delta)'(p_\delta^\pm) > 1$. Associated to this fixed point there is a closed contour of Z_δ.

When $\delta < 0$ we obtain $\alpha_\delta + \alpha_\delta^2 > 0$ then the origin is an unstable fixed point of ϕ_δ since $\alpha_\delta < -1$. In this case we have $F_\delta(g(Z_\delta)) > 0$ and therefore there are no fixed points of ϕ_δ around the origin.

The same reasoning can be done when $\eta_Z < 0$, in this case $g(Z_\delta)$ is a maximum. It follows that for $\delta > 0$ the origin becomes unstable. Moreover, $F(g(Z_\delta)) > 0$ and being a maximum, two stable fixed appears, given rise to a closed contour of Z_δ. For $\delta < 0$ the origin remains stable and there are no closed contour around the origin.

Proposition 12 *Let $Z \in \Xi_2^1$. Then any versal unfolding of Z is locally weak equivalent to the one parameter family*

$$\tilde{Z}_\delta = \begin{cases} \tilde{X}_\delta(x_1, x_2) = \begin{pmatrix} ac \\ -(ac + \delta) \end{pmatrix}, & x_1 \cdot x_2 > 0 \\ \tilde{Y}(x_1, x_2) = \begin{pmatrix} a \\ a + x + abx_1^2 \end{pmatrix}, & x_1 \cdot x_2 < 0 \end{cases} \tag{26}$$

where $a = sgn\,(Y_1(\mathbf{0}))$, $b = sgn\,(\eta_Z)$, $c = sgn\,(X_1 \cdot Y_1(\mathbf{0}))$.

Proof It is enough to observe that $\alpha_{Z_\delta} = -1$, $\beta_{Z_\delta} \neq 0$ and $\mathrm{sgn}\,(\eta_{Z_\delta}) = \mathrm{sgn}\,(\eta_Z)$. $\qquad \blacksquare$

4.3 Regular-fold bifurcation

In this section we are going to study what happens near a piecewise smooth system $Z \in \Omega_1$ satisfying condition C of theorem 2.

In this case, we allow the trajectories of X or Y being tangent to Σ at the origin. If the trajectory through the origin of X, is tangent to both Σ_i for $i = 1, 2$ then X has a singularity at the origin and this situation leads to a higher codimension bifurcation. Another situation which raises a higher codimension bifurcation is when the trajectories through zero of both vector fields X and Y are tangent to Σ.

In this section we will see that Z having a regular fold is a codimension one bifurcation.

Definition 22 We call Ξ_1^3 the set of all elements of Ω_1 having a regular fold at the origin.

Without loss of generality, we can fix $X_2(\mathbf{0}) = 0$, $X_1(\mathbf{0}) > 0$ and $\frac{\partial}{\partial x_1} X_2(\mathbf{0}) > 0$. Under these conditions we have four cases to analyze, which are not equivalent, depending on sign of $Y_1(\mathbf{0})$ and $Y_2(\mathbf{0})$. The other ones can be obtained by reversing time and combinations of reflections and rotations of the previous ones.

Remark 5 When $Z \in \Xi_1^3$, provided Z_i^s is defined in $\overline{\Sigma_i}$ for $i = 1, 2$, we have $Z_1^s(0) = \dfrac{(X_1 \cdot Y_2)(\mathbf{0})}{(X_1 - Y_1)(\mathbf{0})}$ and $Z_2^s(0) = X_1(\mathbf{0})$, therefore, the origin is a regular point for the sliding vector fields Z_i^s.

Proposition 13 *The set Ξ_1^3 is an embedded codimension-one submanifold of Ω and an open set of Ω_1.*

Proof Let $Z_0 = (X^0, Y^0) \in \Xi_1^3$. In order to simplify the notation, we will make the proof for the case when the origin is a fold of X^0 at Σ_2. Any other case is analogous.

Let $\mathcal{V}_0 \times \mathcal{D}_0 \subset \Omega \times \mathcal{U}$ a connected neighborhood of $(Z_0, \mathbf{0})$ for which all $Z \in \mathcal{V}_0$ satisfies:

(a) $\mathrm{sgn}\,(\det Z(p)) = \mathrm{sgn}\,(\det Z_0(\mathbf{0}))$ for all $p \in \mathcal{D}_0 \cap \Sigma$;
(b) $\mathrm{sgn}\,(\gamma_Z) = \mathrm{sgn}\,(\gamma_{Z_0})$, where γ_Z is defined in (15);

Table 1 Decomposition of Σ when $X_2(\mathbf{0}) = 0$, $X_1(\mathbf{0}) \cdot \frac{\partial}{\partial x} X_2(\mathbf{0}) > 0$, depending on signs of $Y_1(\mathbf{0})$ and $Y_2(\mathbf{0})$.

$Y_1 \cdot Y_2(\mathbf{0}) > 0$			
	$\delta > 0, g(Z_\delta) > 0$	$\delta = 0, g(Z_\delta) = 0$	$\delta < 0, g(Z_\delta) < 0$
	$\Sigma_1 = \Sigma^c$	$\Sigma_1 = \Sigma^c$	$\Sigma_1 = \Sigma^c$
$Y_1(\mathbf{0}) > 0$	$\Sigma_2^- = \Sigma^e$	$\Sigma_2^- = \Sigma^e$	$\Sigma_2^- = \Sigma^e \cup \Sigma^c$
	$\Sigma_2^+ = \Sigma^c \cup \Sigma^s$	$\Sigma_2^+ = \Sigma^c$	$\Sigma_2^+ = \Sigma^c$
	$\Sigma_1^- = \Sigma^s$	$\Sigma_1^- = \Sigma^s$	$\Sigma_1^- = \Sigma^s$
$Y_1(\mathbf{0}) < 0$	$\Sigma_1^+ = \Sigma^e$	$\Sigma_1^+ = \Sigma^e$	$\Sigma_1^+ = \Sigma^e$
	$\Sigma_2^- = \Sigma^c$	$\Sigma_2^- = \Sigma^c$	$\Sigma_2^- = \Sigma^c \cup \Sigma^s$
	$\Sigma_2^+ = \Sigma^c \cup \Sigma^e$	$\Sigma_2^+ = \Sigma^e$	$\Sigma_2^+ = \Sigma^e$

$Y_1 \cdot Y_2(\mathbf{0}) < 0$			
	$\delta > 0, g(Z_\delta) > 0$	$\delta = 0, g(Z_\delta) = 0$	$\delta < 0, g(Z_\delta) < 0$
	$\Sigma_1 = \Sigma^c$	$\Sigma_1^+ = \Sigma^c$	$\Sigma_1 = \Sigma^c$
$Y_1(\mathbf{0}) > 0$	$\Sigma_2^- = \Sigma^c$	$\Sigma_2^- = \Sigma^c$	$\Sigma_2^- = \Sigma^c \cup \Sigma^s$
	$\Sigma_2^+ = \Sigma^c \cup \Sigma^e$	$\Sigma_2^+ = \Sigma^e$	$\Sigma_2^+ = \Sigma^e$
	$\Sigma_1^- = \Sigma^s$	$\Sigma_1^- = \Sigma^s$	$\Sigma_1^- = \Sigma^s$
$Y_1(\mathbf{0}) < 0$	$\Sigma_1^+ = \Sigma^e$	$\Sigma_1^+ = \Sigma^e$	$\Sigma_1^+ = \Sigma^e$
	$\Sigma_2^- = \Sigma^e$	$\Sigma_2^- = \Sigma^e$	$\Sigma_2^- = \Sigma^e \cup \Sigma^c$
	$\Sigma_2^+ = \Sigma^s \cup \Sigma^c$	$\Sigma_2^+ = \Sigma^c$	$\Sigma_2^+ = \Sigma^c$

(c) The sign of $X_1(x,0)$, $Y_i(x,0)$, $i = 1,2$, and $\frac{\partial}{\partial x} X_2(x,0)$ is constant in $\mathcal{V}_0 \times \mathcal{D}_0$.

Consider the Frechet differentiable application

$$\begin{aligned} \xi : \mathcal{V}_0 \times (\mathcal{D}_0 \cap \Sigma_2) &\to \mathbb{R} \\ (Z, x) &\mapsto X_2(x, 0) \end{aligned}$$

We have $\xi(Z_0, \mathbf{0}) = 0$, since the origin is a fold point to X at Σ_2 and $\frac{\partial}{\partial x} \xi(Z_0, \mathbf{0}) = \frac{\partial}{\partial x_1} X_2^0(0,0) \neq 0$. Then by the Implicit Function Theorem we obtain open sets $\mathcal{W}_0 \subset \mathcal{V}_0$, $\mathcal{U}_0 \subset \mathcal{D}_0 \cap \Sigma_2$ and a Frechet differentiable map $g : \mathcal{W}_0 \to \mathcal{U}_0$ such that $\xi(Z, x) = 0$ if, and only if, $x = g(Z)$. The point $(g(Z), 0)$ is the unique tangency point of Z with Σ and it is a regular-fold of X at Σ_2.

Therefore, $g^{-1}(0) = \mathcal{W}_0 \cap \Xi_1^3$. Using the Chain rule it is easy to see that Dg_{Z_0} is a surjective linear functional. Therefore Ξ_1^3 is a codimension one embedded submanifold of Ω and also an open set of Ω_1.

From now on we are going to present the generic unfoldings of $Z \in \Xi_1^3$. Fixing $X_2(\mathbf{0}) = 0$ and $X_1 \cdot \frac{\partial}{\partial x} X_2(\mathbf{0}) > 0$, the other cases can be done identically.

Let $\gamma(\delta) = Z_\delta$ a versal unfolding of Z. Let $\mathcal{W}_0^\pm = g^{-1}(\mathbb{R}^\pm)$ and without loss of generality, suppose that for $\delta > 0$, $Z_\delta \in \mathcal{W}_0^+$, therefore, X has a fold in Σ_2^+ which is visible. Analogously, for if $\delta < 0$, $Z_\delta \in \mathcal{W}_0^-$, therefore, X has a fold in Σ_2^- which is invisible. Knowing the position of the fold for every δ, it allows us to give the decomposition of Σ. See Figure 16 and also Table 1.

Since $\mathrm{sgn}\,(\det Z_\delta(p))$ does not change for all $(Z, p) \in \mathcal{U} \times (\mathcal{U}_0 \cap \Sigma)$, whenever it is defined, $\mathrm{sgn}\,((Z_\delta)_i^s(p)) = \mathrm{sgn}\,(Z_i^s(0)) > 0$ for all $\delta \in (-\delta_0, \delta_0)$.

(a) $Y_1 \cdot Y_2(\mathbf{0}) > 0$ and $Y_1(\mathbf{0}) > 0$

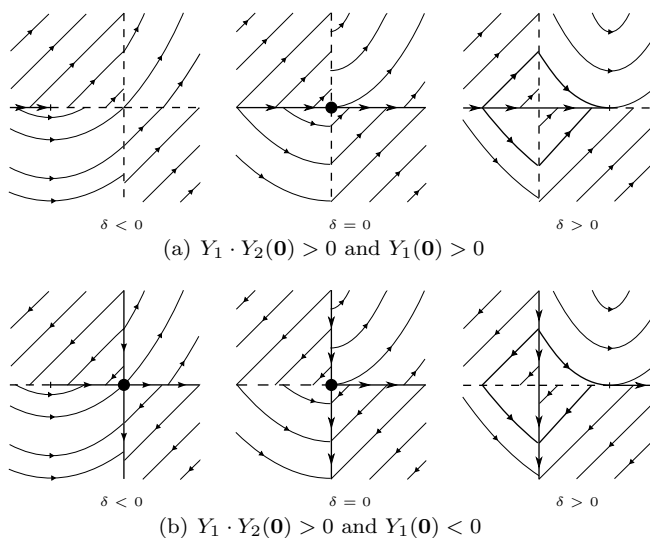(b) $Y_1 \cdot Y_2(\mathbf{0}) > 0$ and $Y_1(\mathbf{0}) < 0$

Fig. 15 The unfoldings of a regular-fold singularity satisfying $Y_1 \cdot Y_2(\mathbf{0}) > 0$. The black dots represent the singularities of the vector fields.
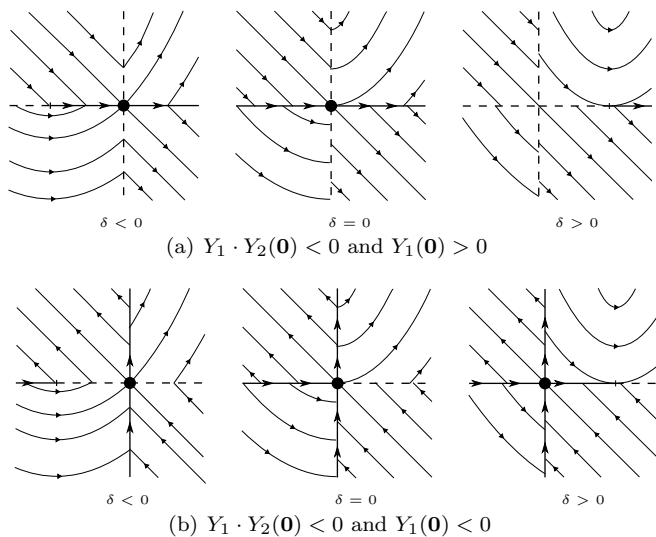

(a) $Y_1 \cdot Y_2(\mathbf{0}) < 0$ and $Y_1(\mathbf{0}) > 0$

(b) $Y_1 \cdot Y_2(\mathbf{0}) < 0$ and $Y_1(\mathbf{0}) < 0$

Fig. 16 The unfoldings for a regular-fold singularity satisfying $Y_1 \cdot Y_2(\mathbf{0}) < 0$. The black dots represent the singularities of the vector fields.

– Suppose that $Y_1 \cdot Y_2(\mathbf{0}) > 0$, thus the Y trajectory trough zero has positive slope.
  – Let $Y_1(\mathbf{0}) > 0$. Fig. 15(a)
    • If $\delta < 0$, $X_2 > 0$ near the origin and therefore Z_δ satisfies hypothesis A of theorem 1.

- When $\delta > 0$, $X_2 < 0$ near the origin. In this case, the vector field Z_δ is transient and therefore, it is necessary to analyze the stability of the origin for the first return map ϕ_δ. By Remark 3, the origin is a hyperbolic fixed point of ϕ_δ if

$$\gamma_\delta = ((X_\delta)_1 \cdot (Y_\delta)_2 + (X_\delta)_2 \cdot (Y_\delta)_1)(\mathbf{0}) \neq 0,$$

but γ_δ is positive for δ sufficiently small because for $\delta = 0$ we have $\gamma_0 = X_1 \cdot Y_2(\mathbf{0}) > 0$. Therefore, Z_δ satisfies the hypothesis C of Theorem 1 and therefore, it is locally Σ-structurally stable.

- Let $Y_1(\mathbf{0}) < 0$, see Fig. 15(b).
  - If $\delta < 0$, $X_2 < 0$ near the origin and therefore the origin satisfies condition B of Theorem 1.
  - The case $\delta > 0$ is analogous to the case $Y_1(\mathbf{0})$.

- Suppose that $Y_1 \cdot Y_2(\mathbf{0}) < 0$, that is the Y trajectory of zero has negative slope. We have different dynamics on the switching manifold depending on $\mathrm{sgn}\,(Y_1(\mathbf{0}))$.
  - Fix $Y_1(\mathbf{0}) > 0$, see figure 16(a).
    - For $\delta < 0$, $X_2 > 0$ near the origin and therefore, Z_δ satisfies the condition C with $X_1 \cdot X_2 > 0$, therefore, Z_δ it is also locally Σ-structurally stable.
    - If $\delta > 0$, then $X_2 < 0$ near the origin and therefore, Z_δ satisfies hypothesis A of theorem 1.
  - Fix $Y_1(\mathbf{0}) < 0$, see figure 16(b).
    - If $\delta < 0$, then $(X_\delta)_2 > 0$ near the origin and therefore Z_δ satisfies hypothesis C of theorem 1 with $(X_\delta)_1 \cdot (X_\delta)_2(\mathbf{0}) > 0$, therefore Z_δ is locally $\Sigma-$structurally stable.
    - If $\delta > 0$ then $(X_\delta)_2 < 0$ near the origin and therefore, as $\det Z_\delta(\mathbf{0}) \neq 0$, Z_δ satisfies hypothesis B of theorem 1, thus Z_δ is locally $\Sigma-$structurally stable.

The following proposition gives a normal form which satisfies all the previous conditions.

Proposition 14 *Let $Z \in \Xi_3^1$ and fix $X_1(\mathbf{0})$ and $\frac{\partial}{\partial x}X_2(\mathbf{0})$ positive. Then any versal unfolding of Z is locally weak equivalent to the one parameter family*

$$\tilde{Z}_\delta = \begin{cases} \tilde{X}_\delta(x_1, x_2) = \begin{pmatrix} 1 \\ x_1 - \delta \end{pmatrix}, & x_1 \cdot x_2 > 0 \\ \tilde{Y}(x_1, x_2) = \begin{pmatrix} a \\ b \end{pmatrix}, & x_1 \cdot x_2 < 0 \end{cases} \tag{27}$$

where $a = \mathrm{sgn}\,(Y_1(\mathbf{0}))$, $b = Y_2(\mathbf{0})$. The other normal forms can be obtained from this proposition by reflections and rotations.

Combining all the results of the last three subsections we prove Theorem 2.

5 Declarations

5.1 Funding

5.2 Conflicts of interest/Competing interests

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