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Energy Decay in Thermoelastic Bodies with Radial Symmetry

Noelia Bazarra[1] · José R. Fernández[1] · Ramón Quintanilla[2]

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Abstract
In this paper, we consider the energy decay of some problems involving domains with radial symmetry. Three different settings are studied: a strong porous dissipation and heat conduction, a weak porous dissipation and heat conduction and poro-thermoelasticity with microtemperatures. In all the three problems, the exponential energy decay is shown. Moreover, for each of them some finite element simulations are presented to numerically demonstrate this behavior.

Keywords Heat conduction · Dissipation · Porous-thermoelasticity · Energy decay · Numerical experiments

Mathematics Subject Classification (2010) 35Q74 · 74H10 · 80A20 · 74F05

1 Introduction

The study of elastic solids with voids started with the contributions of Cowin and Nunziato [1–3]. This kind of materials has deserved much attention in the last forty years [4–14]. Interesting physical applications of the thermoelasticity with voids are directed to the study of solids with small distributed porous. Rocks, soils, woods, ceramics or even biological materials as bones are natural examples. In fact, in recent years multiporosity structure has been considered [15–18]. This kind of solids can be seen as a particular kind of elastic

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✉ J.R. Fernández
jose.fernandez@uvigo.es

N. Bazarra
nbazarra@uvigo.es

R. Quintanilla
ramon.quintanilla@upc.edu

1 Universidade de Vigo, Departamento de Matemática Aplicada I, 36310 Vigo, Spain

2 Departament de Matemàtiques, Universitat Politècnica de Catalunya, C. Colom 11, 08222 Terrassa, Barcelona, Spain

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materials with microstructure. Indeed, an increasing interest has been developed to consider the theories with microstructure along the last and current centuries [19–25]. A relevant sub-class of these solids corresponds to the case that the microstructure [26] is determined by microtemperatures [27]. The wide applicability of this kind of structures made that many scientists are interested in the study of elastic materials with microtemperatures [28–35].

Since Quintanilla [36] showed the slow decay of the solutions when only one mechanism of porous dissipation is considered, much studies have been done to analyze the decay of the solutions when different types of dissipative mechanisms are introduced in the system [4–7, 9–12, 28, 33, 37]; however, we can say that all the contributions correspond to the one-dimensional problem. To the knowledge of the authors the only exception to this statement corresponds to the article Nicaise and Valein [38]. In that contribution, the authors showed the polynomial decay to a certain kind of thermo-porous elastic problem. We can say that it is generally accepted that for different problems porous-thermo-elasticity is not natural to expect exponential decay of the solutions for dimension greater than one. Therefore, one can ask if it is possible to identify solutions in such a way that they decay in an exponential way. In this sense, we believe that the *geometry of the body* plays a relevant role in this kind of study. The main aim of the present contribution is to identify solutions for different porous-thermo-elastic problems in such a way that they decay in this fast way. For this reason, in this paper we restrict our attention to geometries as which are *radially symmetric*. Examples of radially symmetric domains are the balls centered at the origin as well as the circular crowns. In this case we can see that the *radially symmetric* solutions decay in an exponential way. To this end, we will follow the energy arguments, which are naturally applied because they are usual in the study of decay of solutions within several thermoelastic theories (see for instance [39, 40]).

In the next section we describe the basic equations corresponding to the theories we are going to work with. Section 3 is devoted to the study of the time decay of solutions for porous-thermoelasticity with strong porous dissipation. Later, we consider the case when the porous dissipation is weak, but in this case we need to change the boundary conditions. Last section is devoted to the case of porous thermoelasticity with microtemperatures.

2 Several Theories

In this section we recall the basic equations we will work in this paper.

First problem we consider corresponds to the porous-thermo-elasticity with strong porous dissipation. In this case, the problem is determined by the evolution equations [41]:

$$\rho \ddot{u}_i = t_{ij,j}, \tag{1}$$

$$J\ddot{\phi} = h_{k,k} + g, \tag{2}$$

$$\rho T_0 \dot{\eta} = q_{j,j}, \tag{3}$$

and the constitutive equations:

$$\begin{aligned} t_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \mu_0\phi\delta_{ij} - \beta\theta\delta_{ij}, \\ h_i &= a\phi_{,i} + a^*\dot{\phi}_{,i} + k_1\theta_{,i}, \\ g &= -\mu_0 e_{ii} - \eta\phi + \beta_1\theta, \end{aligned}$$

$$\rho\eta = \beta e_{ii} + \beta_1\phi + c\theta,$$
$$q_i = k\theta_{,i} + k_2\dot{\phi}_{,i}.$$

In this system of equations we have that ρ is the mass density, u_i is the displacement vector, t_{ij} is the stress tensor, h_i is the equilibrated stress tensor, g is the equilibrated body force, η is the entropy, q_i is the heat flux vector, J is the equilibrated inertia, T_0 is the reference temperature at the equilibrium state (assumed uniform and equal to one to simplify the calculations), e_{ij} is the strain tensor which is related to the displacement by the relation

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

ϕ is the volume fraction, θ is the temperature. Here λ and μ are the usual Lamé parameters, β determines the coupling between the displacement and the temperature, μ_0 is the coupling between the displacement and the porosity, k is the thermal conductivity and c the thermal capacity. Other parameters are usual in the study of the present theory.

We will consider this system with homogeneous Dirichlet boundary conditions. Therefore it is suitable to recall that some certain geometries there exist undamped solutions. The main idea is to consider domains where certain spectral conditions are satisfied. Here, we will assume that $u_i = 0$ on the boundary and homogeneous (Dirichlet or Neumann) boundary conditions for the domain for the other variables. We consider domains that satisfy the condition:

Condition D. There exists a nonzero field $\psi_i \in \mathbf{H}_0^1$ such that $\psi_{i,jj} = \gamma_2\psi_i$ and $\psi_{i,i} = 0$, where $\gamma_2 \neq 0$.

We note that in dimension one there are no functions satisfying this condition; however, in dimension greater than one this condition is satisfied for several symmetric domains. For instance, if the domain is a ball there are an infinity of eigenvalues satisfying this condition (see [42]).

When the domain is such that condition (D) is satisfied, we look for solutions of the form $(u_i, \phi, \theta) = (\exp(\omega t)\psi_i, 0, 0)$ whenever $\omega^2\rho + \gamma^2\mu = 0$. It is clear that our solutions are undamped and therefore they do not decay; however, the aim of this paper is different and we are looking for solutions that decay in an exponential way. For this reason, we will study the radial solutions which are totally different from the ones considered previously.

Second problem we will study corresponds to the porous-thermo-elasticity with weak porous dissipation. Again we will work with the evolution equations (1)-(3), but we need to change the constitutive equations for the equilibrated stress tensor and the equilibrated body force. They should be

$$h_i = a\phi_{,i}, \quad g = -\mu_0 e_{ii} - \eta\phi - \eta^*\dot{\phi} + \beta_1\theta,$$

meanwhile the other constitutive equations continue in the same form, but $k_2 = 0$. It is worth noting that the examples of undamped solutions can be considered also in this problem.

Third system we are going to study in this note corresponds to thermoelasticity with voids and microtemperatures. In this case to the system (1)-(3) we have to add the equation:

$$\rho\dot{\epsilon}_i = q_{ij,j} + q_i - Q_i,$$

where ϵ_i is the first heat flux moment tensor, Q_i is the microheat flux average, q_{ij} is the first heat flux moment tensor. The constitutive equations are ([41]):

$$
\begin{aligned}
t_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \mu_0\phi\delta_{ij} - \beta\theta\delta_{ij},\\
h_i &= a\phi_{,i} - \mu_2 T_i,\\
g &= -\mu_0 e_{ii} - \xi\phi + \beta_1\theta,\\
\rho\eta &= \beta_0 e_{ii} + \beta_1\phi + c\theta,\\
\rho\epsilon_i &= -\mu_2\phi_{,i} - bT_i,\\
q_i &= k\theta_{,i} + \kappa_1 T_i,\\
q_{ij} &= -\kappa_4 T_{r,r}\delta_{ij} - \kappa_5 T_{j,i} - \kappa_6 T_{i,j},\\
Q_i &= (k-\kappa_3)\theta_{,i} + (\kappa_1-\kappa_2)T_i,
\end{aligned}
$$

where the new constitutive parameters are the usual ones in the theories with microtemperatures. Again, we can obtain undamped solutions of the form $(u_i, \phi, \theta, T_i) = (\exp(\omega t)\psi_i, 0, 0, 0)$.

Though we have seen examples of undamped solutions for certain domains, our main aim in this paper is to prove the exponential decay of solutions for the problems defined previously whenever we consider radially symmetric solutions. It is relevant to recall that this kind of solutions are only possible in the case that the domain would be radially symmetric as in the case of balls or crowns. For this reason, we want to emphasize how the geometry of the domain allow the existence of exponentially stable solutions which is the main aim of the paper.

3 First Case: Strong Porous-Dissipation and Heat Conduction

The aim of this section is to study the radial solutions of the problem determined by the system obtained after the substitution of the constitutive equations into the evolution equations:

$$\rho\ddot{u}_i = \mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \mu_0\phi_{,i} - \beta\theta_{,i}, \tag{4}$$

$$J\ddot{\phi} = a\phi_{,jj} + a^*\dot{\phi}_{,jj} + k_1\theta_{,jj} - \mu_0 u_{r,r} - \eta\phi + \beta_1\theta, \tag{5}$$

$$c\dot{\theta} = k\theta_{,jj} - \beta\dot{u}_{r,r} - \beta_1\dot{\phi} + k_2\dot{\phi}_{,jj}. \tag{6}$$

This system corresponds to the case of the porous-thermoelasticity where the dissipative mechanisms are given by the heat conduction and the *strong porous viscosity* given by the term $a^*\dot{\phi}_{,jj}$. As we said before, (u_i) is the displacement vector, ϕ is the volume fraction, θ is the temperature, ρ is the mass density, J is the equilibrated inertia, λ and μ are the Lamé's parameters, k is the thermal conductivity and μ_0, β, a, a^*, η and β_1 are constitutive coefficients for this theory. We will study the problem determined by this system with the Dirichlet boundary conditions:

$$u_i(\boldsymbol{x},t) = \phi(\boldsymbol{x},t) = \theta(\boldsymbol{x},t) = 0, \quad \boldsymbol{x}\in\partial B, \tag{7}$$

and the initial conditions:

$$
\begin{aligned}
&u_i(\boldsymbol{x},0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x},0) = v_i^0(\boldsymbol{x}), \quad \phi(\boldsymbol{x},0) = \phi^0(\boldsymbol{x}), \quad \dot{\phi}(\boldsymbol{x},0) = \psi^0(\boldsymbol{x}),\\
&\theta(\boldsymbol{x},0) = \theta^0(\boldsymbol{x}).
\end{aligned} \tag{8}
$$

It is clear that in the case we assume that the initial conditions have radial symmetry we obtain that the solutions are always of radial type. That is, the solutions can be given in the following way:

$$u_i(\boldsymbol{x},t) = x_i U(r,t), \quad \phi(\boldsymbol{x},t) = \phi(r,t), \quad \theta(\boldsymbol{x},t) = \theta(r,t), \tag{9}$$

where $r^2 = x_1^2 + x_2^2$ for the two dimensional case and $r^2 = x_1^2 + x_2^2 + x_3^2$ in the three-dimensional case. It is worth noting that in this situation we have that $u_{i,j}(\boldsymbol{x},t) = u_{j,i}(\boldsymbol{x},t)$. Therefore, the first equation of our system can be written as

$$\rho \ddot{u}_i = (\lambda + 2\mu) u_{j,ji} + \mu_0 \phi_{,i} - \beta \theta_{,i}. \tag{10}$$

Our aim in this section is to prove that the radial solutions decay in a exponential decay. Therefore we should recall the assumptions that we need to work. In this section we assume that $\mu > 0$, $\lambda + \mu > 0$, $\eta\mu > \mu_0^2$, $c > 0$, $a > 0$, $a^* > 0$, $4a^*k > (k_1 + k_2)^2$, $\rho > 0$ and $J > 0$. The sign of the other constitutive coefficients is not a restriction, however we need that β must be different from zero.

The problem described previously has solutions (for instance by means of the semigroup arguments) in the Hilbert space $\mathcal{H} = \mathbf{H}_0^1 \times \mathbf{L}^2 \times H_0^1 \times L^2 \times L^2$. Existence and uniqueness of solutions are guaranteed. Even more, the stability of solutions can be obtained; however, we cannot expect (in the general case) its exponential decay.

Now, our aim here is to obtain the exponential decay of the radial solutions to the problem proposed previously. We have the following.

Theorem 1 *Let us introduce the functions E_1, E_2 and E_3 given by*

$$E_1(t) = \frac{1}{2}\int_B \big(\rho \dot{u}_i \dot{u}_i + J|\dot{\phi}|^2 + c\theta^2 + (\lambda + 2\mu) u_{i,j} u_{i,j} + 2\mu_0 u_{i,i}\phi + \eta\phi^2 + a\phi_{,i}\phi_{,i}\big) dv,$$

$$E_2(t) = \frac{1}{2}\int_B \big(\rho \ddot{u}_i \ddot{u}_i + J|\ddot{\phi}|^2 + c|\dot{\theta}|^2 + (\lambda + 2\mu) \dot{u}_{i,j} \dot{u}_{i,j} + 2\mu_0 \dot{u}_{i,i}\dot{\phi} + \eta|\dot{\phi}|^2 + a\dot{\phi}_{,i}\dot{\phi}_{,i}\big) dv,$$

$$E_3(t) = \frac{1}{2}\int_B \big(\rho \dot{u}_{i,j} \dot{u}_{i,j} + J\dot{\phi}_{,i}\dot{\phi}_{,i} + c\theta_{,i}\theta_{,i} + (\lambda + 2\mu) u_{i,jj} u_{i,kk} + 2\mu_0 u_{i,jj}\phi_{,i}$$
$$+ \eta\phi_{,i}\phi_{,i} + a\phi_{,ii}\phi_{,jj}\big) dv,$$

and define the energy of the system as

$$E(t) = E_1(t) + E_2(t) + E_3(t).$$

Then, this energy decays exponentially, that is, there exist two positive constants M and ω such that

$$E(t) \le M E(0) \exp(-\omega t).$$

Proof First, we have that

$$\dot{E}_1(t) = -\int_B \big(k\theta_{,i}\theta_{,i} + a^*\dot{\phi}_{,i}\dot{\phi}_{,i} + (k_1 + k_2)\theta_{,i}\dot{\phi}_{,i}\big) dv,$$

$$\dot{E}_2(t) = -\int_B \big(k\dot{\theta}_{,i}\dot{\theta}_{,i} + a^*\ddot{\phi}_{,i}\ddot{\phi}_{,i} + (k_1 + k_2)\dot{\theta}_{,i}\ddot{\phi}_{,i}\big) dv,$$

$$\dot{E}_3(t) = -\int_B \left(k\theta_{,ii}\theta_{,jj} + a^*\dot{\phi}_{,jj}\dot{\phi}_{,ii} + (k_1+k_2)\theta_{,ii}\dot{\phi}_{,jj}\right)dv$$
$$-\int_{\partial B}\left(\beta\dot{u}_{i,i}\frac{\partial\theta}{\partial\mathbf{n}} + \mu_0 u_{i,i}\frac{\partial\dot{\phi}}{\partial\mathbf{n}}\right)da,$$

where $\mathbf{n}$ is the normal vector to the surface ∂B.

We note that

$$\left|\int_{\partial B}\beta\dot{u}_{i,i}\frac{\partial\theta}{\partial\mathbf{n}}da\right| \le \epsilon(\lambda+2\mu)\int_{\partial B}\dot{u}_{i,i}\dot{u}_{j,j}da + \frac{C}{\epsilon}\int_{\partial B}\left(\frac{\partial\theta}{\partial\mathbf{n}}\right)^2 da,$$

where ϵ is as small as we want and C is a calculable positive constant. If we recall Lemma 4.1 of the book of Jiang and Racke [40], we obtain

$$\epsilon(\lambda+2\mu)\int_{\partial B}\dot{u}_{i,i}\dot{u}_{j,j}da \le \frac{2\epsilon(\lambda+2\mu)}{\lambda+\mu}\frac{d}{dt}\int_B \ddot{u}_i\sigma_k\dot{u}_{i,k}dv + C\epsilon\int \Pi_1 dv$$
$$-\frac{\epsilon\mu(\lambda+2\mu)}{\lambda+\mu}\int_{\partial B}\frac{\partial\dot{u}_i}{\partial\mathbf{n}}\frac{\partial\dot{u}_i}{\partial\mathbf{n}}da,$$

where σ_k is a smooth field defined on B such that $\sigma_i = n_i$ on the boundary of B and

$$\Pi_1 = \ddot{u}_i\ddot{u}_i + \dot{u}_{i,i}\dot{u}_{j,j} + \dot{\phi}_{,i}\dot{\phi}_{,i} + \dot{\theta}_{,i}\dot{\theta}_{,i}.$$

Recalling again Lemma 4.1 of the book of Jiang and Racke [40] we find that

$$\frac{C}{\epsilon}\int_{\partial B}(\frac{\partial\theta}{\partial\mathbf{n}})^2 da \le \epsilon\int_B(\dot{u}_{i,i}\dot{u}_{j,j} + \dot{\phi}_{,jj}\dot{\phi}_{,ii})dv + \frac{C}{\epsilon^2}\int_B(\theta_{,i}\theta_{,i} + \dot{\theta}_{,i}\dot{\theta}_{,i} + |\dot{\phi}|^2)dv,$$
$$\left|\int_{\partial B}\mu_0 u_{i,i}\frac{\partial\dot{\phi}}{\partial\mathbf{n}}da\right| \le \epsilon(\lambda+2\mu)\int_{\partial B}u_{i,i}u_{j,j}da + \frac{C}{\epsilon}\int_{\partial B}\left(\frac{\partial\dot{\phi}}{\partial\mathbf{n}}\right)^2 da.$$

The first integral can be bounded in a similar way as previously to obtain

$$\epsilon(\lambda+2\mu)\int_{\partial B}u_{i,i}u_{j,j}da \le \frac{2\epsilon(\lambda+2\mu)}{\lambda+\mu}\frac{d}{dt}\int_B \dot{u}_i\sigma_k u_{i,k}dv + C\epsilon\int \Pi_2 dv$$
$$-\frac{\epsilon\mu(\lambda+2\mu)}{\lambda+\mu}\int_{\partial B}\frac{\partial u_i}{\partial\mathbf{n}}\frac{\partial u_i}{\partial\mathbf{n}}da,$$

where

$$\Pi_2 = \dot{u}_i\dot{u}_i + u_{i,i}u_{j,j} + \phi_{,i}\phi_{,i} + \theta_{,i}\theta_{,i}.$$

The other integral can be bounded by

$$\frac{C}{\epsilon}\int_{\partial B}\left(\frac{\partial\dot{\phi}}{\partial\mathbf{n}}\right)^2 da \le \frac{C^*}{\epsilon}\left(\int_B\dot{\phi}_{,i}\dot{\phi}_{,i}\right)^{1/2}\left(\int_B\dot{\phi}_{,ii}\dot{\phi}_{,jj}\right)^{1/2}$$

If we select a parameter χ large enough we can show that

$$\chi(\dot{E}_1+\dot{E}_2) + \dot{E}_3 \le -C_1\int_B(\theta_{,i}\theta_{,i} + \dot{\theta}_{,i}\dot{\theta}_{,i} + \theta_{,ii}\theta_{,jj} + \dot{\phi}_{,i}\dot{\phi}_{,i} + \ddot{\phi}_{,i}\ddot{\phi}_{,i} + \dot{\phi}_{,jj}\dot{\phi}_{,ii})dv$$

$$
\begin{aligned}
&-\frac{2\epsilon(\lambda+2\mu)}{\lambda+\mu}\frac{d}{dt}\int_B(\dot{u}_i\sigma_k u_{i,k}+\ddot{u}_i\sigma_k\dot{u}_{i,k})dv\\
&-\frac{\epsilon\mu(\lambda+2\mu)}{\lambda+\mu}\int_{\partial B}\left(\frac{\partial u_i}{\partial \mathbf{n}}\frac{\partial u_i}{\partial \mathbf{n}}+\frac{\partial \dot{u}_i}{\partial \mathbf{n}}\frac{\partial \dot{u}_i}{\partial \mathbf{n}}\right)da+C\epsilon\int_B \Pi_3 dv,
\end{aligned}
$$

where

$$
\Pi_3=\ddot{u}_i\ddot{u}_i+\dot{u}_{i,i}\dot{u}_{j,j}+u_{i,i}u_{j,j}+\phi_{,i}\phi_{,i}.
$$

As β is different from zero, we can obtain the estimate

$$
\int_B \dot{u}_{i,i}\dot{u}_{j,j}dv\le C\int_B(|\dot{\theta}|^2+\theta_{,ii}\theta_{,jj}+|\dot{\phi}|^2+\dot{\phi}_{,jj}\dot{\phi}_{,ii})dv, \tag{11}
$$

and therefore the next inequality can be found:

$$
\begin{aligned}
&\chi(\dot{E}_1+\dot{E}_2)+\dot{E}_3\\
&\le -C_1\int_B(\theta_{,i}\theta_{,i}+\dot{\theta}_{,i}\dot{\theta}_{,i}+\theta_{,ii}\theta_{,jj}+\dot{\phi}_{,i}\dot{\phi}_{,i}+\ddot{\phi}_{,i}\ddot{\phi}_{,i}+\dot{\phi}_{,jj}\dot{\phi}_{,ii}+\dot{u}_{i,i}\dot{u}_{j,j})dv\\
&\quad-\frac{2\epsilon(\lambda+2\mu)}{\lambda+\mu}\frac{d}{dt}\int_B(\dot{u}_i\sigma_k u_{i,k}+\ddot{u}_i\sigma_k\dot{u}_{i,k})dv\\
&\quad-\frac{\epsilon\mu(\lambda+2\mu)}{\lambda+\mu}\int_{\partial B}\left(\frac{\partial u_i}{\partial \mathbf{n}}\frac{\partial u_i}{\partial \mathbf{n}}+\frac{\partial \dot{u}_i}{\partial \mathbf{n}}\frac{\partial \dot{u}_i}{\partial \mathbf{n}}\right)da+C\epsilon\int_B \Pi_4 dv,
\end{aligned}
$$

where

$$
\Pi_4=\ddot{u}_i\ddot{u}_i+u_{i,i}u_{j,j}+\phi_{,i}\phi_{,i}.
$$

We now consider the functions:

$$
\begin{aligned}
F_1(t)&=\int_B\left(\rho u_i\dot{u}_i+J\phi\dot{\phi}+\frac{a^*}{2}\phi_{,j}\phi_{,j}\right)dv,\\
F_2(t)&=\int_B\left(\rho u_{i,i}\dot{u}_{j,j}+J\phi_{,i}\dot{\phi}_{,i}+\frac{a^*}{2}\phi_{,ii}\phi_{,jj}\right)dv.
\end{aligned}
$$

We have

$$
\begin{aligned}
\dot{F}_1(t)=&-\int_B\Big((\lambda+2\mu)u_{i,j}u_{i,j}+2\mu_0 u_{i,i}\phi+a\phi_{,j}\phi_{,j}+\eta\phi^2+k_1\theta_{,j}\phi_{,j}\Big)dv\\
&+\int_B\big(\beta_1\theta\phi-\beta\theta_{,i}u_i\big)dv+\int_B(\rho\dot{u}_i\dot{u}_i+J|\dot{\phi}|^2)dv\\
\le&-C_2\int_B(u_{i,j}u_{i,j}+\phi_{,j}\phi_{,j}+\phi^2)dv+C\int_B(\theta_{,i}\theta_{,i}+\dot{u}_i\dot{u}_i+|\dot{\phi}|^2)dv.
\end{aligned}
$$

On the other hand, since

$$
F_2(t)=-\int_B\big(\rho\dot{u}_i u_{i,jj}+J\dot{\phi}\phi_{,jj})dv+\int_B\frac{a^*}{2}\phi_{,ii}\phi_{,jj}\,dv, \tag{12}
$$

we see that

$$\dot{F}_2(t) = -\int_B \Big((\lambda+2\mu)u_{i,jj}u_{i,kk} + 2\mu_0 u_{i,jj}\phi_{,i} + a\phi_{,jj}\phi_{,ii} + \eta\phi_{,i}\phi_{,i} + k\theta_{,ii}\phi_{,jj}\Big)dv$$
$$-\int_B \big(\beta_1\theta\phi_{,ii} - \beta\theta_{,i}u_{i,jj}\big)dv + \int_B (\rho\dot{u}_{i,j}\dot{u}_{i,j} + J\dot{\phi}_{,i}\dot{\phi}_{,i})dv - \int_{\partial B} u_{i,i}\frac{\partial\phi}{\partial\mathbf{n}}da.$$

It follows that

$$\begin{aligned}\dot{F}_2(t) \le & -C_2\int_B \Big(u_{i,jj}u_{i,kk} + \phi_{,jj}\phi_{,ii} + \phi_{,i}\phi_{,i}\Big)dv \\ & + C\int_B \big(\theta_{,ii}\theta_{,jj} + \theta_{,i}\theta_{,i} + \dot{u}_{i,i}\dot{u}_{j,j} + \dot{\phi}_{,j}\dot{\phi}_{,j}\big)dv \\ & + C_3\Bigg[\Big(\int_B u_{i,i}u_{j,j}dv\Big)^{1/2}\Big(\int_B u_{i,jj}u_{i,kk}dv\Big)^{1/2} \\ & + \Big(\int_B \phi_{,i}\phi_{,i}dv\Big)^{1/2}\Big(\int_B \phi_{,jj}\phi_{,kk}dv\Big)^{1/2}\Bigg].\end{aligned}$$

If we select χ^* large enough we get

$$\begin{aligned}\chi^*\dot{F}_1(t) + \dot{F}_2(t) \le & -C_2\int_B (u_{i,jj}u_{i,kk} + \phi_{,jj}\phi_{,ii} + \phi_{,i}\phi_{,i} + u_{i,i}u_{i,i} + \phi^2)dv \\ & + C\int_B \big(\theta_{,ii}\theta_{,jj} + \theta_{,i}\theta_{,i} + \dot{u}_{i,i}\dot{u}_{j,j} + \dot{\phi}_{,j}\dot{\phi}_{,j}\big)dv.\end{aligned}$$

At the same time, we also note that there exists a positive K such that

$$\int_B \ddot{u}_i\ddot{u}_i dv \le K\int_B (u_{i,jj}u_{i,kk} + \phi_{,i}\phi_{,i} + \theta_{,i}\theta_{,i})dv. \tag{13}$$

We now define the function

$$\begin{aligned}E^*(t) = & \chi(E_1 + E_2) + E_3 + \epsilon^{1/2}(\chi^* F_1(t) + F_2(t)) \\ & + \frac{2\epsilon(\lambda+2\mu)}{\lambda+\mu}\frac{d}{dt}\int_B (\dot{u}_i\sigma_k u_{i,k} + \ddot{u}_i\sigma_k\dot{u}_{i,k})dv.\end{aligned}$$

In view of the previous estimates we see that

$$\begin{aligned}\dot{E}^*(t) \le & -C_1\int_B (\theta_{,i}\theta_{,i} + \dot{\theta}_{,i}\dot{\theta}_{,i} + \theta_{,ii}\theta_{,jj} + \dot{\phi}_{,i}\dot{\phi}_{,i} + \ddot{\phi}_{,i}\ddot{\phi}_{,i} + \dot{\phi}_{,jj}\dot{\phi}_{,ii} + \dot{u}_{i,i}\dot{u}_{j,j})dv \\ & -C_1\epsilon^{1/2}\int_B (u_{i,i}u_{j,j} + u_{i,jj}u_{i,kk} + \phi_{,i}\phi_{,i} + \phi_{,ii}\phi_{,jj} + \ddot{u}_i\ddot{u}_i)dv \\ & +C\epsilon^{1/2}\int_B \big(\theta_{,ii}\theta_{,jj} + \theta_{,i}\theta_{,i} + \dot{u}_{i,i}\dot{u}_{j,j} + \dot{\phi}_{,j}\dot{\phi}_{,j}\big)dv + C\epsilon\int_B \Pi_4 dv.\end{aligned}$$

We note that, for ϵ small enough, we can obtain that

$$\dot{E}^*(t) + CE(t) \le 0,$$

where we recall that

$$E(t) = E_1(t) + E_2(t) + E_3(t)$$

In view that $E^*(t)$ is equivalent to $E(t)$ we obtain the exponential decay of solutions. $\square$

3.1 Numerical Simulations

In this section, we want to present numerical simulations for the solutions to our problem in the case that we have a two-dimensional ball with radius 1. We also assume that the time interval is fixed as $[0, T]$, where T denotes the final time, and we note that, in order to simplify the writing, we do not indicate the dependence on the spatial variable (x_i).

Taking into account the boundary conditions, defining the velocity $v_i = \dot{u}_i$ and the porosity speed $e = \dot{\phi}$ system (4)-(6) leads to the following variational formulation, for a.e. $t \in [0, T]$,

$$\rho(\dot{v}_i(t), w_i) + \mu(u_{i,j}(t), w_{i,j}) + (\lambda + \mu)(u_{j,j}(t), w_{i,i}) = \mu_0(\phi_{,i}(t), w_i) \\ -\beta(\theta_{,i}(t), w_i) \quad \forall (w_i) \in V, \tag{14}$$

$$J(\dot{e}(t), z) + a(\phi_{,j}(t), z_{,j}) + a^*(e_{,j}(t), z_{,j}) + \eta(\phi(t), z) = -\mu_0(u_{r,j}(t), z) \\ +\beta_1(\theta(t), z) \quad \forall z \in E, \tag{15}$$

$$c(\dot{\theta}(t), r) + k(\theta_{,j}(t), r_{,j}) = -\beta(v_{j,j}(t), r) - \beta_1(e(t), r) \quad \forall r \in E, \tag{16}$$

where $v_i(0) = v_i^0$ and $e(0) = \psi^0$, the variational spaces E and V are given by

$$E = \{z \in H^1(B) \quad ; \quad z = 0 \quad \text{on} \quad \partial B\},$$
$$V = \{(w_i) \in [H^1(B)]^2 \quad ; \quad w_i = 0 \quad \text{on} \quad \partial B \quad \text{for } i = 1, 2\},$$

and we denote by $(\cdot, \cdot)$ the inner product in the space $L^2(B)$. Here, the displacements and the porosity are defined as

$$u_i(t) = \int_0^t v_i(s)\, ds + u_i^0, \quad \phi(t) = \int_0^t e(s)\, ds + \phi^0. \tag{17}$$

Now, we consider a fully discrete approximation of problem (14)-(17). This is done in two steps. First, we assume that the domain $\overline{B}$ is polyhedral and we denote by $\mathcal{T}^h$ a regular triangulation in the sense of [43]. Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$E^h = \{r^h \in C(\overline{B})\, ; \, r^h_{|Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad r^h = 0 \quad \text{on} \quad \partial B\},$$
$$V^h = \{w_i^h \in C(\overline{B})\, ; \, w^h_{i_{|Tr}} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad w_i^h = 0 \text{ on } \partial B, \text{ for } i = 1, 2\},$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr, i.e. the finite element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by u_i^{0h}, v_i^{0h}, ϕ^{0h}, e^{0h} and θ^{0h}, are given by

$$u_i^{0h} = \mathcal{P}^h u_i^0, \quad v_i^{0h} = \mathcal{P}^h v_i^0, \quad \phi^{0h} = \mathcal{P}^h \phi^0, \quad e^{0h} = \mathcal{P}^h \psi^0, \quad \theta^{0h} = \mathcal{P}^h \theta^0,$$

where $\mathcal{P}^h$ is the classical finite element interpolation operator over E^h (see, e.g., [43]).

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $\tau = T/N$ and nodes $t_n = n\,\tau$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/\tau$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

$$
\begin{aligned}
&\rho(\delta(v_n^{h\tau})_i, w_i^h) + \mu((u_n^{h\tau})_{i,j}, w_{i,j}^h) + (\lambda+\mu)((u_n^{h\tau})_{j,j}, w_{i,i}^h) = \mu_0((\phi_n^{h\tau})_{,i}, w_i^h) \\
&\qquad -\beta((\theta_n^{h\tau})_{,i}, w_i^h) \quad \forall (w_i^h) \in V^h, \\
&J(\delta e_n^{h\tau}, z^h) + a((\phi_n^{h\tau})_{,j}, z_{,j}^h) + a^*((e_n^{h\tau})_{,j}, z_{,j}^h) + \eta(\phi_n^{h\tau}, z^h) = -\mu_0((u_n^{h\tau})_{r,j}, z^h) \\
&\qquad +\beta_1(\theta_n^{h\tau}, z^h) \quad \forall z^h \in E^h, \\
&c(\delta\theta_n^{h\tau}, r^h) + k((\theta_n^{h\tau})_{,j}, r_{,j}^h) = -\beta((v_n^{h\tau})_{j,j}, r^h) - \beta_1(e_n^{h\tau}, r^h) \quad \forall r^h \in E^h,
\end{aligned}
$$

where the discrete displacements and the discrete porosity are obtained from the relations:

$$
(u_n^{h\tau})_i = \tau(v_n^{h\tau})_i + (u_{n-1}^{h\tau})_i, \quad \phi_n^{h\tau} = \tau e_n^{h\tau} + \phi_{n-1}^{h\tau}.
$$

The above fully discrete problem leads to a linear system written in terms of a product variable which has a unique solution due to the conditions imposed on the constitutive coefficients.

Our aim now is to show numerically the asymptotic energy behavior. Therefore, we use the following data:

$$
\begin{aligned}
&T = 30, \quad \rho = 1, \quad \mu = 5, \quad \lambda = 2, \quad \mu_0 = 2, \quad \beta = 2 \quad J = 5, \quad a = 2, \\
&a^* = 5, \quad \eta = 5, \quad \beta_1 = 3, \quad c = 1, \quad k = 0.1,
\end{aligned}
$$

and the initial conditions:

$$
\phi^0 = \psi^0 = \theta^0 = 0, \quad u_i^0(x, y) = v_i^0(x, y) = x^2 + y^2 - 1 \quad \text{for } (x, y) \in \overline{B}.
$$

Taking the discretization parameter $\tau = 0.001$ and using the finite element mesh shown in Fig. 1, in Fig. 2 we plot the evolution in time, in both natural and semi-log scales, of the discrete energy $E_n^{h\tau}$ defined as:

$$
\begin{aligned}
E_n^{h\tau} = \frac{1}{2}\int_B \Big(&\rho(v_n^{h\tau})_i(v_n^{h\tau})_i + J|e_n^{h\tau}|^2 + c(\theta_n^{h\tau})^2 + (\lambda+2\mu)(u_n^{h\tau})_{i,j}(u_n^{h\tau})_{i,j} \\
&+2\mu_0(u_n^{h\tau})_{i,i}\phi_n^{h\tau} + \eta(\phi_n^{h\tau})^2 + a(\phi_n^{h\tau})_{,i}(\phi_n^{h\tau})_{,i}\Big)dv.
\end{aligned}
$$

As can be seen, the discrete energy tends to zero and the theoretical exponential asymptotic behavior seems to be achieved.

4 Second Case: Weak Porous-Dissipation and Heat Conduction

In this section, we consider a similar problem as in the previous one, but in the case that we assume weak porous dissipation. That is, we replace equation (5) by

$$
J\ddot{\phi} = a\phi_{,jj} - \mu_0 u_{r,r} - \eta\phi - \eta^*\dot{\phi} + \beta_1\theta, \tag{18}
$$

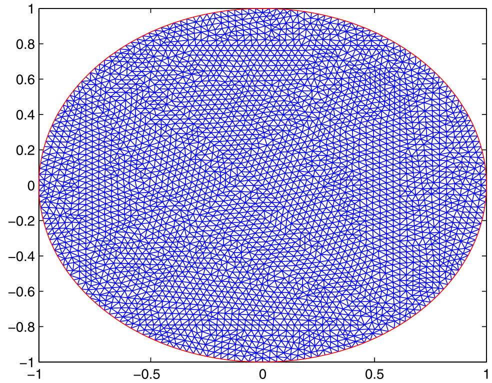

Fig. 1 First system: Finite element mesh

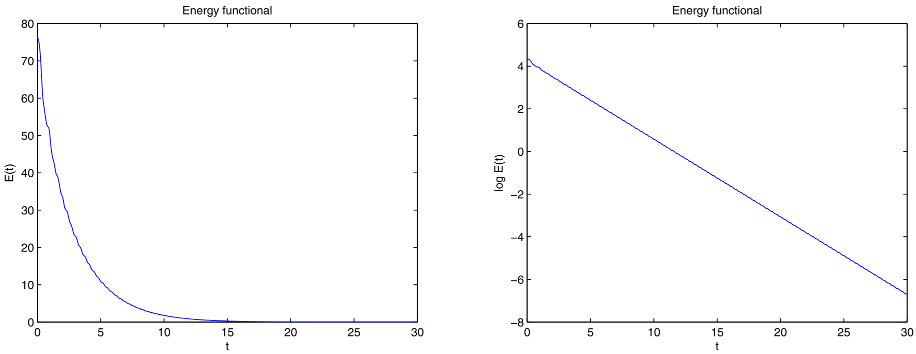

Fig. 2 First system: Evolution in time of the discrete energy in natural (left) and semi-log (right) scales

where η^* is a positive constant and we assume that $k_2 = 0$. The analysis in this section follows the lines proposed before; however, as we have seen in the previous section, the boundary conditions proposed there impose that the manipulations are very cumbersome and it is not clear that we can directly extend to the present situation. In order to overcome this difficulty, we change a little bit the boundary conditions to assume that

$$u_i(\boldsymbol{x}, t) = \frac{\partial \phi(\boldsymbol{x}, t)}{\partial \mathbf{n}} = \frac{\partial \theta(\boldsymbol{x}, t)}{\partial \mathbf{n}} = 0, \quad \boldsymbol{x} \in \partial B. \tag{19}$$

To define a well-posed problem we need to consider the initial conditions proposed in (8). Again, we assume radial solutions of the form proposed in the previous section and equation (4) becomes (10). This problem can be solved in the Hilbert space $\mathcal{H} = \mathbf{H}_0^1 \times \mathbf{L}^2 \times H_*^1 \times L_*^2 \times L_*^2$, where

$$L_*^2(B) = \left\{ f \in L^2, \int_B f dv = 0 \right\},$$

and $H_*^1(B) = H_0^1(B) \cap L_*^2(B)$.

Now, our aim is again to obtain the exponential decay of the radial solutions to the problem proposed in this section. So, we have the following.

Theorem 2 *Let us consider the functions E_i, $i=1,2,3$, defined in Theorem 1 and the energy function E given by*

$$E(t)=E_1(t)+E_2(t)+E_3(t).$$

Then, this energy decays exponentially; i.e., there exist two positive constants M and ω such that

$$E(t)\leq ME(0)\exp(-\omega t).$$

Proof We have

$$\dot{E}_1(t)=-\int_B\big(k\theta_{,i}\theta_{,i}+\eta^*|\dot{\phi}|^2\big)dv,$$
$$\dot{E}_2(t)=-\int_B\big(k\dot{\theta}_{,i}\dot{\theta}_{,i}+\eta^*|\ddot{\phi}|^2\big)dv,$$
$$\dot{E}_3(t)=-\int_B\big(k\theta_{,ii}\theta_{,jj}+\eta^*\dot{\phi}_{,i}\dot{\phi}_{,i}\big)dv.$$

We note that, in view of the boundary conditions, the expression of the time derivative of E_3 is easier to manipulate. Using similar arguments to the ones used in the previous section we see

$$\dot{E}_1+\dot{E}_2+\dot{E}_3\leq -C_1\int_B\big(\theta_{,i}\theta_{,i}+\dot{\theta}_{,i}\dot{\theta}_{,i}+\theta_{,ii}\theta_{,jj}+|\dot{\phi}|^2+|\ddot{\phi}|^2+\dot{\phi}_{,i}\dot{\phi}_{,i}+\dot{u}_{i,i}\dot{u}_{j,j}\big)dv.$$

Proceeding as in the proof of Theorem 1 we also need to introduce a couple of functions, but being a little bit different:

$$F_1(t)=\int_B\left(\rho u_i\dot{u}_i+J\phi\dot{\phi}+\frac{1}{2}\eta^*|\phi|^2\right)dv,$$
$$F_2(t)=\int_B\left(\rho u_{i,i}\dot{u}_{j,j}+J\phi_{,i}\dot{\phi}_{,i}+\frac{1}{2}\eta^*\dot{\phi}_{,i}\dot{\phi}_{,i}\right)dv.$$

It follows that

$$\dot{F}_1(t)\leq -C_2\int_B(u_{i,j}u_{i,j}+\phi_{,j}\phi_{,j}+\phi^2)dv+C\int_B(\theta_{,i}\theta_{,i}+\dot{u}_i\dot{u}_i+|\dot{\phi}|^2)dv,$$
$$\dot{F}_2(t)\leq -C_2\int_B\big(u_{i,jj}u_{i,kk}+\phi_{,jj}\phi_{,ii}+\phi_{,i}\phi_{,i}\big)dv+C\int_B\big(\theta_{,i}\theta_{,i}+\dot{u}_{i,i}\dot{u}_{j,j}+\dot{\phi}_{,j}\dot{\phi}_{,j}\big)dv.$$

Again, the expression of the time derivative of $F_2(t)$ is easier due to the boundary conditions we assume. We also note that inequality (13) also holds here. Therefore, we find that

$$\dot{F}_2(t)\leq -C_2^*\int_B\big(u_{i,jj}u_{i,kk}+\ddot{u}_i\ddot{u}_i+\phi_{,jj}\phi_{,ii}+\phi_{,i}\phi_{,i}\big)dv+C^*\int_B\big(\theta_{,i}\theta_{,i}+\dot{u}_{i,i}\dot{u}_{j,j}+\dot{\phi}_{,j}\dot{\phi}_{,j}\big)dv.$$

The function

$$E^*(t)=E_1(t)+E_2(t)+E_3(t)+\epsilon(F_1(t)+F_2(t))$$

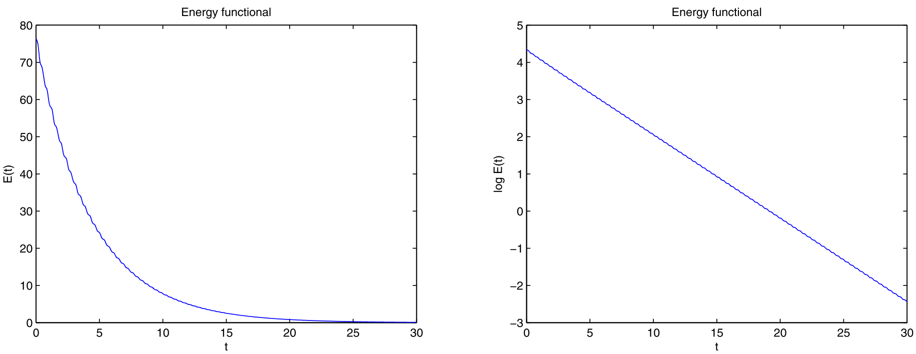

Fig. 3 Second system: Evolution in time of the discrete energy in natural (left) and semi-log (right) scales

is equivalent to the function $E(t) = E_1(t) + E_2(t) + E_3(t)$ whenever ϵ is small enough. At the same time, we get

$$\dot{E}^*(t) \leq C_1 \int_B (\theta_{,i}\theta_{,i} + \dot{\theta}_{,i}\dot{\theta}_{,i} + \theta_{,ii}\theta_{,jj} + |\dot{\phi}|^2 + |\ddot{\phi}|^2 + \dot{\phi}_{,i}\dot{\phi}_{,i} + \dot{u}_{i,i}\dot{u}_{j,j})dv$$

$$-\epsilon C_2 \int_B (u_{i,j}u_{i,j} + \phi_{,j}\phi_{,j} + u_{i,jj}u_{i,kk} + \ddot{u}_i\ddot{u}_i)dv + \epsilon C^* \int_B (\theta_{,i}\theta_{,i} + \dot{u}_{i,i}\dot{u}_{j,j} + \dot{\phi}_{,j}\dot{\phi}_{,j})dv.$$

Obviously, if we select ϵ small enough it follows that

$$\dot{E}^*(t) + CE(t) \leq 0,$$

and, therefore, the exponential decay can be obtained. □

4.1 Numerical Simulations

In this section, the system obtained replacing equation (5) by (18) is numerically solved. Since the differences with respect to the numerical algorithm shown in Sect. 3.1 are minor, we skip the details for the sake of reading.

Therefore, we use the following data:

$$T = 30, \quad \rho = 1, \quad \mu = 5, \quad \lambda = 2, \quad \mu_0 = 2, \quad \beta = 2 \quad J = 5, \quad a = 2,$$
$$\eta^* = 5, \quad \eta = 5, \quad \beta_1 = 3, \quad c = 1, \quad k = 4,$$

and the same initial conditions than in Sect. 3.1.

Taking now the time discretization parameter $\tau = 0.001$ and the finite element mesh shown in Fig. 1, the evolution in time of the discrete energy is shown in Fig. 3 in both natural and semi-log scales. Again, we can observe that the discrete energy tends to zero and the expected asymptotic exponential decay.

5 Third Case: Heat Conduction and Microheat Conduction

Third case we want to analyze in this paper corresponds to the study of porous-thermoelasticity with microtemperatures. The system of equations is:

$$\rho\ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \mu_0\phi_{,i} - \beta\theta_{,i}, \tag{20}$$

$$J\ddot{\phi} = a\phi_{,jj} - \mu_0 u_{r,r} - dw_{i,i} - \eta\phi + \beta_1\theta, \tag{21}$$

$$c\dot{\theta} = k\theta_{,jj} - \beta\dot{u}_{r,r} + \kappa_1 w_{i,i} - \beta_1\dot{\phi}, \tag{22}$$

$$\zeta\dot{w}_i = \kappa_6 w_{i,jj} + (\kappa_4 + \kappa_5) w_{j,ji} - \kappa_3\theta_{,i} - \kappa_2 w_i - d\dot{\phi}_{,i}. \tag{23}$$

We pay attention that this system does not impose any kind of porous dissipation and the damping is proposed by the temperature and the microtemperatures. To study this system, we need to impose the initial conditions proposed in (8), but we also need initial conditions for the microtemperatures. That is,

$$w_i(\boldsymbol{x}, 0) = w_i^0(\boldsymbol{x}).$$

We also impose the boundary conditions proposed in (19) as well as Dirichlet homogeneous boundary conditions for the microtemperatures:

$$w_i(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial B.$$

Apart of the assumptions proposed previously for the different constitutive coefficients in the case that the microtemperatures are present, it is usual to assume that

$$\kappa_6 + \kappa_5 + 3\kappa_4 > 0, \quad \kappa_6 + \kappa_5 > 0, \quad \kappa_6 - \kappa_5 > 0, \quad (\kappa_1 + \kappa_3)^2 < 4k\kappa_2.$$

Under these assumptions we can determine the existence and uniqueness of solutions in the Hilbert space:

$$\mathcal{H} = \mathbf{H}_0^1 \times \mathbf{L}^2 \times H_*^1 \times L_*^2 \times L_*^2 \times \mathbf{L}^2.$$

However, we here concentrate our attention to radial solutions. That is, solutions of the form (9) and such that $w_i(\boldsymbol{x}, t) = x_i W(r, t)$. As we have seen for the displacement, the microtemperatures satisfy $w_{i,j} = w_{j,i}$ and, therefore, equation (23) can be written as

$$\zeta\dot{w}_i = \kappa^* w_{i,jj} - \kappa_3\theta_{,i} - \kappa_2 w_i - d\dot{\phi}_{,i}, \tag{24}$$

where $\kappa^* = \kappa_4 + \kappa_5 + \kappa_6$ and we will assume, from now on, positive. As we have imposed in the previous sections, we assume that β is different from zero, but in this section we also assume that d is different from zero.

Theorem 3 *Let us introduce the functions E_1, E_2 and E_3 given by*

$$E_1(t) = \frac{1}{2}\int_B \big(\rho\dot{u}_i\dot{u}_i + J|\dot{\phi}|^2 + c\theta^2 + (\lambda + 2\mu)u_{i,j}u_{i,j} + 2\mu_0 u_{i,i}\phi + \eta\phi^2 + a\phi_{,i}\phi_{,i} + \zeta w_i w_i\big)\,dv,$$

$$E_2(t) = \frac{1}{2}\int_B \big(\rho\ddot{u}_i\ddot{u}_i + J|\ddot{\phi}|^2 + c|\dot{\theta}|^2 + (\lambda + 2\mu)\dot{u}_{i,j}\dot{u}_{i,j} + 2\mu_0\dot{u}_{i,i}\dot{\phi} + \eta|\dot{\phi}|^2 + a\dot{\phi}_{,i}\dot{\phi}_{,i} + \zeta\dot{w}_i\dot{w}_i\big)\,dv,$$

$$E_3(t) = \frac{1}{2}\int_B \big(\rho\dot{u}_{i,j}\dot{u}_{i,j} + J\dot{\phi}_{,i}\dot{\phi}_{,i} + c\theta_{,i}\theta_{,i} + (\lambda + 2\mu)u_{i,jj}u_{i,kk} + 2\mu_0 u_{i,jj}\phi_{,i} + \eta\phi_{,i}\phi_{,i} + a\phi_{,ii}\phi_{,jj} + \zeta w_{i,j}w_{i,j}\big)\,dv,$$

and define the energy of the system as

$$E(t) = E_1(t) + E_2(t) + E_3(t).$$

Then, this energy decays exponentially, that is, there exist two positive constants M and ω such that

$$E(t) \leq ME(0)\exp(-\omega t).$$

Proof We have

$$\dot{E}_1(t) = -\int_B \big(k\theta_{,i}\theta_{,i} + (\kappa_1+\kappa_3)\theta_{,i}w_i + \kappa_2 w_i w_i + \kappa^* w_{i,j}w_{i,j}\big)dv,$$
$$\dot{E}_2(t) = -\int_B \big(k\dot{\theta}_{,i}\dot{\theta}_{,i} + (\kappa_1+\kappa_3)\dot{\theta}_{,i}\dot{w}_i + \kappa_2 \dot{w}_i \dot{w}_i + \kappa^* \dot{w}_{i,j}\dot{w}_{i,j}\big)dv,$$
$$\dot{E}_3(t) = -\int_B \big(k\theta_{,ii}\theta_{,jj} + (\kappa_1+\kappa_3)\theta_{,jj}w_{i,i} + \kappa_2 w_{i,i} w_{j,j} + \kappa^* w_{i,jj}w_{i,kk}\big)dv.$$

We note that (11) holds and, in a similar way, since d is different from zero, we find that

$$\int_B \dot{\phi}_{,i}\dot{\phi}_{,i}dv \leq C\int_B (w_{i,jj}w_{i,kk} + \dot{w}_i\dot{w}_i + \theta_{,i}\theta_{,i})dv.$$

Therefore, it follows that

$$\dot{E}_1 + \dot{E}_2 + \dot{E}_3$$
$$\leq -C_1\int_B (\theta_{,i}\theta_{,i} + \dot{\theta}_{,i}\dot{\theta}_{,i} + \theta_{,ii}\theta_{,jj} + w_{i,i}w_{j,j} + \dot{w}_i\dot{w}_i + w_{j,ii}w_{j,kk} + \dot{\phi}_{,i}\dot{\phi}_{,i} + \dot{u}_{i,i}\dot{u}_{j,j})dv.$$

We also consider the functions:

$$F_1(t) = \int_B (\rho u_i\dot{u}_i + J\phi\dot{\phi})dv,$$
$$F_2(t) = \int_B (\rho u_{i,i}\dot{u}_{j,j} + J\phi_{,i}\dot{\phi}_{,i})dv,$$

and we obtain that

$$\dot{F}_1(t) = -C_2\int_B (u_{i,j}u_{i,j} + \phi_{,j}\phi_{,j} + \phi^2)dv + C\int_B (\theta_{,i}\theta_{,i} + w_{i,i}w_{j,j} + \dot{u}_i\dot{u}_i + |\dot{\phi}|^2)dv,$$
$$\dot{F}_2(t) \leq -C_2\int_B \big(u_{i,jj}u_{i,kk} + \phi_{,jj}\phi_{,ii} + \phi_{,i}\phi_{,i}\big)dv$$
$$+ C\int_B \big(\theta_{,i}\theta_{,i} + w_{i,i}w_{j,j} + \dot{u}_{i,i}\dot{u}_{j,j} + \dot{\phi}_{,j}\dot{\phi}_{,j}\big)dv.$$

According to inequality (13) we can show that

$$\int_B |\ddot{\phi}|^2 dv \leq K\int_B (u_{i,j}u_{i,j} + \phi_{,ii}\phi_{,jj} + \theta_{,i}\theta_{,i} + w_{i,i}w_{j,j})dv. \tag{25}$$

In view of (13) and (25) it follows that

$$\dot{F}_2(t) \leq -C_2 \int_B \Big(u_{i,jj}u_{i,kk} + \phi_{,jj}\phi_{,ii} + \phi_{,i}\phi_{,i} + \ddot{u}_i\ddot{u}_i + |\ddot{\phi}|^2\Big)dv$$

$$+ C \int_B \big(\theta_{,i}\theta_{,i} + w_{i,i}w_{j,j} + \dot{u}_{i,i}\dot{u}_{j,j} + \dot{\phi}_{,j}\dot{\phi}_{,j}\big)dv.$$

We can define the function

$$E^*(t) = E_1(t) + E_2(t) + E_3(t) + \epsilon(F_1(t) + F_2(t)),$$

which is also equivalent to the function $E(t) = E_1(t) + E_2(t) + E_3(t)$, whenever ϵ is small enough. At the same time, we have

$$\begin{aligned}\dot{E}^*(t) \leq C_1 \int_B &(\theta_{,i}\theta_{,i} + \dot{\theta}_{,i}\dot{\theta}_{,i} + \theta_{,ii}\theta_{,jj} + |\dot{\phi}|^2 + \dot{\phi}_{,i}\dot{\phi}_{,i} + \dot{u}_{i,i}\dot{u}_{j,j} + w_{i,j}w_{i,j}\\ &+\dot{w}_{i,j}\dot{w}_{i,j} + w_{i,jj}w_{i,kk})dv + \epsilon C^* \int_B (\theta_{,i}\theta_{,i} + \dot{u}_{i,i}\dot{u}_{j,j} + \dot{\phi}_{,j}\dot{\phi}_{,j})dv\\ &-\epsilon C_2 \int_B (u_{i,j}u_{i,j} + \phi_{,j}\phi_{,j} + u_{i,jj}u_{i,kk} + \ddot{u}_i\ddot{u}_i + \ddot{\phi}|^2)dv.\end{aligned}$$

It is clear that if we choose ϵ small enough we obtain that

$$\dot{E}^*(t) + CE(t) \leq 0,$$

and, therefore, the exponential decay can be concluded. □

5.1 Numerical Simulations

Finally, we solve numerically system (20)-(23). Since we could obtain straightforwardly the fully discrete approximations following the procedure described in Sect. 3.1, we omit again the details for the sake of reading.

Therefore, we use the following data:

$$\begin{aligned}&T = 40, \quad \rho = 1, \quad \mu = 5, \quad \lambda = 2, \quad \mu_0 = 2, \quad \beta = 2 \quad J = 5, \quad a = 2,\\ &\eta = 5, \quad \beta_1 = 3, \quad c = 1, \quad k = 4, \quad d = 1, \quad \zeta = 1, \quad k_1 = 3, \quad k_2 = 1,\\ &k_3 = 2, \quad k_4 = 1, \quad k_5 = 1, \quad k_6 = 2,\end{aligned}$$

and the same initial conditions than in Sect. 3.1, by adding the following one for the microtemperatures:

$$w_i^0 = 0.$$

Using again the time discretization parameter $\tau = 0.001$ and the finite element mesh shown in Fig. 1, the evolution in time of the discrete energy $E_n^{h\tau}$ given now as

$$\begin{aligned}E_n^{h\tau} = \frac{1}{2}\int_B \Big(&\rho(v_n^{h\tau})_i(v_n^{h\tau})_i + J|e_n^{h\tau}|^2 + c(\theta_n^{h\tau})^2 + (\lambda + 2\mu)(u_n^{h\tau})_{i,j}(u_n^{h\tau})_{i,j}\\ &+2\mu_0(u_n^{h\tau})_{i,i}\phi_n^{h\tau} + \eta(\phi_n^{h\tau})^2 + a(\phi_n^{h\tau})_{,i}(\phi_n^{h\tau})_{,i} + \zeta(w_n^{h\tau})_i(w_n^{h\tau})_i\Big)dv,\end{aligned}$$

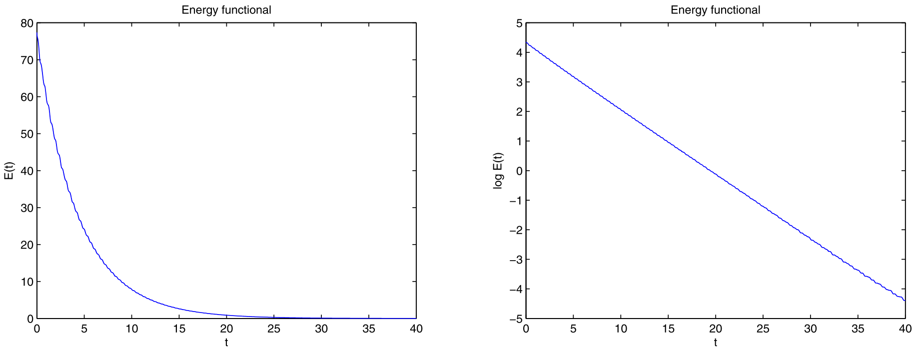

Fig. 4 Third system: Evolution in time of the discrete energy in natural (left) and semi-log (right) scales

is plotted in Fig. 4 in both natural and semi-log scales. Again, we can observe that the discrete energy tends to zero and the expected asymptotic exponential decay.

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