Sum of a random number of independent random variables

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Let N and $X_1, X_2, \ldots, X_n, \ldots$ be independent random variables such that:

- N takes nonnegative integer values.
- The variables X_i, $i \geqslant 1$, are identically distributed.

We are interested in the sum

$$S = X_1 + X_2 + \cdots + X_N$$

in which the number N of terms is random. (We take $S = 0$ if $N = 0$ happens.)

Sum of a random number of independent random variables

The characteristic function of the random variable S can be obtained as follows:

$$\begin{aligned} M_S(\omega) &= \mathbb{E}\left(e^{i\omega S}\right) \\ &= \sum_{k \geqslant 0} \mathbb{E}\left(e^{i\omega S} \mid N = k\right) \mathbb{P}(N = k) \end{aligned}$$

Sum of a random number of independent random variables

Observe that $\mathbb{E}\left(e^{i\omega S} \mid N=0\right) = \mathbb{E}\left(e^{i\omega 0}\right) = 1$.

If $N = k$, $k \geqslant 1$, then

$$\begin{aligned}
\mathbb{E}\left(e^{i\omega S} \mid N=k\right) &= \mathbb{E}\left(e^{i\omega(X_1+X_2+\cdots+X_N)} \mid N=k\right) \\
&= \mathbb{E}\left(e^{i\omega(X_1+X_2+\cdots+X_k)} \mid N=k\right) \\
&= \mathbb{E}\left(e^{i\omega(X_1+X_2+\cdots+X_k)}\right) \\
&= \mathbb{E}\left(e^{i\omega X_1}\right) \mathbb{E}\left(e^{i\omega X_2}\right) \cdots \mathbb{E}\left(e^{i\omega X_k}\right) \\
&= \left(M_X(\omega)\right)^k
\end{aligned}$$

Sum of a random number of independent random variables

Let $G_N(z) = \sum_{k\geqslant 0} \mathbb{P}(N=k)\, z^k$ be the probability generating function of N (considered here as a function of a complex variable z).

Then

$$M_S(\omega) = \sum_{k\geqslant 0} \left(M_X(\omega)\right)^k \mathbb{P}(N=k) = G_N\left(M_X(\omega)\right)$$

- Notice that the composition $G_N\left(M_X(\omega)\right)$ is well-defined, because $|M_X(\omega)| \leqslant 1$ for all $\omega \in \mathbb{R}$ and $G_N(z)$ converges for all $z \in \mathbb{C}$ such that $|z| \leqslant 1$.

Sum of a random number of independent random variables

Remark. A shortest way to formulate the previous calculation is as follows:

$$M_S(\omega) = \mathbb{E}\left(e^{i\omega S}\right) = \mathbb{E}\left(\mathbb{E}\left(e^{i\omega S}|N\right)\right),$$

where

$$\mathbb{E}\left(e^{i\omega S}\middle| N\right) = \left(M_X(\omega)\right)^N$$

Therefore

$$M_S(\omega) = \mathbb{E}\left(\left(M_X(\omega)\right)^N\right) = G_N(M_X(\omega))$$

Example 1

- Let the number N of costumers arriving at a service point be a $\mathrm{Po}(\lambda)$-distributed random variable.
- Each arrival is randomly and independently served with probability p.
  (Hence an arrival is not served with probability $q = 1 - p$.)

Let us determine the probability distribution of the number S of served customers.

Example 1

We have

$$S = X_1 + X_2 + \cdots + X_N,$$

where X_i is the indicator random variable of the event "the i-th arrival is served". So $X_i \sim \text{Be}(p)$.

Moreover,

$$M_{X_i}(\omega) = q + pe^{i\omega}, \quad \omega \in \mathbb{R}$$

$$G_N(z) = e^{\lambda(z-1)}, \quad z \in \mathbb{C},$$

Example 1

Therefore

$$M_S(\omega) = G_N\left(M_X(\omega)\right) = e^{\lambda(z-1)}\Big|_{z=q+pe^{i\omega}}$$

$$= e^{\lambda(q+pe^{i\omega}-1)} = e^{\lambda p(e^{i\omega}-1)}$$

This is the characteristic function of a Poisson random variable with parameter λp. Thus

$$S \sim \text{Po}(\lambda p)$$

Example 1

Analogously, if R denotes the number of non-served customers, then $R \sim \text{Po}(\lambda q)$.

- It can be proved that S and R are independent variables.
- Notice how the convolution theorem applies. Indeed,

$$M_S(\omega)\, M_R(\omega) = e^{\lambda p(e^{i\omega}-1)}\, e^{\lambda q(e^{i\omega}-1)}$$
$$= e^{\lambda(p+q)(e^{i\omega}-1)} = e^{\lambda(e^{i\omega}-1)} = M_N(\omega),$$

in accordance with the fact that $S + R = N$.

Example 2

As a second example consider the following scenario.

- The number N of costumers arriving at a service point is a $\text{Ge}(p)$-distributed.
- The service times X_i, $i \geqslant 1$, are independent and $\text{Exp}(\mu)$-distributed random variables.

Let S be total time of occupancy of the service point,

$$S = X_1 + X_2 + \cdots + X_N$$

Example 2

Now

$$M_X(\omega) = \frac{\mu}{\mu - i\omega}, \quad G_N(z) = \frac{pz}{1 - qz}$$

Therefore

$$M_S(\omega) = G_N\left(M_X(\omega)\right) = \left.\frac{pz}{1 - qz}\right|_{z=\frac{\mu}{\mu - i\omega}}$$

$$= \frac{p\,\frac{\mu}{\mu - i\omega}}{1 - q\,\frac{\mu}{\mu - i\omega}} = \frac{\mu p}{\mu p - i\omega}$$

We conclude that

$$S \sim \mathrm{Exp}(\mu p)$$

Expected values

We can obtain the expected value of $S = X_1 + X_2 + \cdots + X_N$ by conditioning on N, that is to say, $\mathbb{E}(S) = \mathbb{E}\left(\mathbb{E}(S \mid N)\right)$.

We have

$$\begin{aligned} \mathbb{E}(S \mid N = k) &= \mathbb{E}(X_1 + X_2 + \cdots + X_k \mid N = k) \\ &= \mathbb{E}(X_1 + X_2 + \cdots + X_k) = k\, m, \end{aligned}$$

where $m = \mathbb{E}(X)$ is the common expected value of the variables X_i.

Therefore $\mathbb{E}(S \mid N) = mN$ and so

$$E(S) = \mathbb{E}\left(\mathbb{E}(S \mid N)\right) = \mathbb{E}(m\,N) = m\,\mathbb{E}(N) = \mathbb{E}(N)\,\mathbb{E}(X).$$

Expected values

- In Example 1 we have $X \sim \mathrm{Be}(p)$, $N \sim \mathrm{Po}(\lambda)$, and $S \sim \mathrm{Po}(\lambda p)$. Therefore

$$\mathbb{E}(S) = \lambda p = \mathbb{E}(N)\,\mathbb{E}(X)$$

- In Example 2 we have $X \sim \mathrm{Exp}(\mu)$, $N \sim \mathrm{Ge}(p)$, and $S \sim \mathrm{Exp}(\mu p)$. Hence

$$\mathbb{E}(S) = \frac{1}{\mu p} = \frac{1}{p} \cdot \frac{1}{\mu} = \mathbb{E}(N)\,\mathbb{E}(X)$$