Separation of Two-Query-Adaptive from Two-Query-Non-adaptive Completeness If NP Is Not Small *

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Abstract. Under the hypothesis that NP does not have p-measure 0 (roughly, that NP contains more than a negligible subset of exponential time), it is shown that there is a language that is $\leq^{\mathrm{P}}_{2-\mathrm{T}}$-complete but not $\leq^{\mathrm{P}}_{2-tt}$-complete for NP. This conclusion, widely believed to be true, is not known to follow from $\mathrm{P} \neq \mathrm{NP}$ or other traditional complexity-theoretic hypotheses. This is a very preliminar version.

1. Introduction

The NP-completeness of decision problems has different formulations. We will deal here with two very close ones, defined respectively by two adaptive and two non-adaptive queries. We will exhibit a reasonable complexity-theoretic hypothesis that implies the distinctness of these two completeness notions.

In general, given a polynomial-time reducibility $\leq^{\mathrm{P}}_{r}$ (e.g., $\leq^{\mathrm{P}}_{T}$ or $\leq^{\mathrm{P}}_{m}$), a language (i.e., decision problem) C is *$\leq^{\mathrm{P}}_{r}$-complete for* NP if $C \in$ NP and, for all $A \in$ NP, $A \leq^{\mathrm{P}}_{r} C$. If A and B are languages, then A is *polynomial-time 2-Turing reducible to* B, and we write $A \leq^{\mathrm{P}}_{2-\mathrm{T}} B$, if A is decided in polynomial time by some oracle Turing machine that consults B as an oracle, making at most two queries per input. On the other hand, A is *polynomial-time 2-truth-table reducible to* B, and we write $A \leq^{\mathrm{P}}_{B-tt}$, if A is decided in polynomial time by some oracle Turing machine that consults B as an oracle, making at most two queries per input in a non-adaptive way, that is, both queries are written by the machine before consulting the oracle. It is clear that $A \leq^{\mathrm{P}}_{2-tt} B$ implies $A \leq^{\mathrm{P}}_{2-\mathrm{T}} B$, and hence that every $\leq^{\mathrm{P}}_{2-tt}$-complete language for NP is $\leq^{\mathrm{P}}_{2-\mathrm{T}}$-complete for NP.

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It is widely conjectured (e.g., [2], [7], [3], [1]) that Turing completeness is more general than many-one completeness:

CvKL Conjecture ("Cook versus Karp-Levin"). There exists a language that is $\leq_T^P$-complete, but not $\leq_m^P$-complete, for NP.

Lutz and Mayordomo prove in [6] that the CvKL Conjecture follows from the hypothesis that "NP does not have p-measure 0". We improve this result here by showing that the same hypothesis implies that there exists a language that is $\leq_{2-T}^P$-complete, but not $\leq_{2-tt}^P$-complete, for NP.

In section 3 below we review the definition of resource-bounded measure that gives the meaning of the hypothesis "NP does not have p-measure 0". In section 4 we prove our Main Theorem.

See [6] for a discussion on the reasonableness of the hypothesis "NP does not have p-measure 0".

2. Preliminaries

In this paper, $[\![\psi]\!]$ denotes the *Boolean value* of the condition ψ.

All *languages* here are sets of binary strings, i.e., sets $A \subseteq \{0,1\}^*$. We identify each language A with its *characteristic sequence* $\chi_A \in \{0,1\}^\infty$ defined by

$$\chi_A = [\![s_0 \in A]\!][\![s_1 \in A]\!][\![s_2 \in A]\!]...,$$

where $s_0 = \lambda$, $s_1 = 0$, $s_2 = 1$, $s_3 = 00, ...$ is the standard enumeration of $\{0,1\}^*$. Relying on this identification, the set $\{0,1\}^\infty$, consisting of all infinite binary sequences, will be regarded as the set of all languages.

If $w \in \{0,1\}^*$ and $x \in \{0,1\}^* \cup \{0,1\}^\infty$, we say that w is a *prefix* of x, and write $w \sqsubseteq x$, if $x = wy$ for some $y \in \{0,1\}^* \cup \{0,1\}^\infty$. The *cylinder generated by* a string $w \in \{0,1\}^*$ is

$$\mathbf{C}_w = \{x \in \{0,1\}^\infty \mid w \sqsubseteq x\} = \{A \subset \{0,1\}^* \mid w \sqsubseteq \chi_A\}.$$

Note that $\mathbf{C}_\lambda = \{0,1\}^\infty$, where λ denotes the empty string.

Let $\mathbf{B}_2$ be the set of two input, one output boolean functions.

As noted in section 1, we work with the exponential time complexity class E =

=DTIME(2^{linear}).
We let $\mathbf{D} = \{m2^{-n} \mid m \in \mathbf{Z}, n \in \mathbf{N}\}$ be the set of ***dyadic rationals***. We also fix a one-to-one pairing function $\langle , \rangle$ from $\{0,1\}^* \times \{0,1\}^*$ onto $\{0,1\}^*$ such that the pairing function and its associated projections, $\langle x, y\rangle \mapsto x$ and $\langle x, y\rangle \mapsto y$, are computable in polynomial time.
Several functions in this paper are of the form $d : \mathbf{N}^k \times \{0,1\}^* \to Y$, where Y is $\mathbf{D}$ or $[0, \infty)$, the set of nonnegative real numbers. Formally, in order to have uniform criteria for their computational complexities, we regard all such functions as having domain $\{0,1\}^*$, and codomain $\{0,1\}^*$ if $Y = \mathbf{D}$. For example, a function $d : \mathbf{N}^2 \times \{0,1\}^* \to \mathbf{D}$ is formally interpreted as a function $\tilde{d} : \{0,1\}^* \to \{0,1\}^*$. Under this interpretation, $d(i,j,w) = r$ means that $\tilde{d}(\langle 0^i, \langle 0^j, w\rangle\rangle) = u$, where u is a suitable binary encoding of the dyadic rational r.
For a function $d : \mathbf{N} \times X \to Y$ and $k \in \mathbf{N}$, we define the function $d_k : X \to Y$ by $d_k(x) = d(k,x) = d(\langle 0^k, x\rangle)$. We then regard d as a "uniform enumeration" of the functions $d_0, d_1, d_2, \ldots$. For a function $d : \mathbf{N}^n \times X \to Y$ $(n \geq 2)$, we write $d_{k,l} = (d_k)_l$, etc.
In general, complexity classes of functions from $\{0,1\}^*$ into $\{0,1\}^*$ will be denoted by appending an 'F' to the notation for the corresponding complexity classes of languages. Thus, for $t : \mathbb{N} \to \mathbb{N}$, DTIMEF($t$) is the set of all functions $f : \{0,1\}^* \to \{0,1\}^*$ such that $f(x)$ is computable in $O(t(|x|))$ time. Similarly, PF $= \bigcup_{k=0}^{\infty}$ DTIMEF(n^k). (For technical reasons [5], when discussing resource bounds for measure, we will deviate from this practice, writing p for PF, etc., as in section 3 below.)

3. Definition of Resource Bounded Measure

In this section we review the definition of measure in complexity classes.
Resource-bounded measure ([4], [5]) is a very general theory whose special cases include classical Lebesgue measure, the measure structure of the class REC of all recursive languages, and measure in various complexity classes. In this paper we are interested only in measure in E, so our discussion of measure is specific to these class. The interested reader may consult section 3 of [4] for more discussion and examples.
Throughout this section, we identify every language $A \subseteq \{0,1\}^*$ with its characteristic sequence $\chi_A \in \{0,1\}^\infty$, defined as in section 2.
Definition 1. The class p consists of functions $f : \{0,1\}^* \to \{0,1\}^*$, such that f is computable in polynomial time
The measure structure of E is developed in terms of the class p.
Definition 2. A *martingale* is a function $d : \{0,1\}^* \to [0, \infty)$ satisfying

$$d(w) = \frac{d(w0) + d(w1)}{2} \tag{3.1}$$

for all $w \in \{0,1\}^*$. A martingale d *succeeds* on a language $A \subset \{0,1\}^*$ if

$$\limsup_{n\to\infty} d(\chi_A[0..n-1]) = \infty.$$

The class of languages on which a martingale d is successful is denoted $S[d]$. A martingale d *is successful* on a set $X \subseteq \{0,1\}^\infty$ if $X \subseteq S[d]$.

Intuitively, a martingale d is a betting strategy that, given a language A, starts with capital (amount of money) $d(\lambda)$ and bets on the membership or nonmembership of the successive strings $s_0, s_1, s_2, \cdots$ (the standard enumeration of $\{0,1\}^*$) in A. Prior to betting on a string s_n, the strategy has capital $d(w)$, where

$$w = [\![s_0 \in A]\!] \cdots [\![s_{n-1} \in A]\!].$$

After betting on the string s_n, the strategy has capital $d(wb)$, where $b = [\![s_n \in A]\!]$. Condition (3.1) ensures that the betting is fair. The strategy succeeds on A if its capital is unbounded as the betting progresses.

More generally, we will be interested in "uniform systems" of martingales that are computable within some resource bound.

Definition 3. An n-dimensional *martingale system* (n-*MS*) is a function

$$d : \mathbb{N}^n \times \{0,1\}^* \to [0, \infty)$$

such that $d_{\vec{k}}$ is a martingale for every $\vec{k} \in \mathbb{N}^n$. It is sometimes convenient to regard a martingale as a 0-MS.

Definition 4. A *computation* of an n-MS d is a function $\hat{d} : \mathbb{N}^{n+1} \times \{0,1\}^* \to \mathrm{D}$ such that

$$\left|\hat{d}_{\vec{k},r}(w) - d_{\vec{k}}(w)\right| \leq 2^{-r}$$

for all $\vec{k} \in \mathbb{N}^n$, $r \in \mathbb{N}$, and $w \in \{0,1\}^*$. A p-*computation* of an n-MS d is a computation $\hat{d}$ of d such that $\hat{d} \in \mathrm{p}$. An n-MS d is p-*computable* if there exists a p-computation $\hat{d}$ of d. If d is an n-MS such that $d : \mathbb{N}^n \times \{0,1\}^* \to \mathrm{D}$ and $d \in \mathrm{p}$, then d is trivially p-computable. This fortunate circumstance, in which there is no need to compute approximations, occurs frequently in practice. (Such applications typically do involve approximations, but these are "hidden" by invoking fundamental theorems whose proofs involve approximations.)

We now come to the key idea of resource-bounded measure theory.

Definition 5. A set X has p-*measure 0*, and we write $\mu_{\mathrm{p}}(X) = 0$, if there exists a p-computable martingale d such that d is successful on X. A set X has p-*measure 1*, and we write $\mu_{\mathrm{p}}(X) = 1$, if $\mu_{\mathrm{p}}(X^c) = 0$.

This definition says that X has p-measure 0 if and only if there is a single p-computable strategy d that succeeds (bets successfully) on every language $A \in X$. The fact that the strategy d is p-computable means that, when betting on the condition "$x \in A$", d requires only $2^{c|x|}$ time for some fixed constant c. (This is because the running time of d for this bet is polynomial in the number of predecessors of x in the standard ordering of $\{0,1\}^*$.)

We now turn to the internal measure structures of the class E.

Definition 6. A set X has *measure 0 in* E, and we write $\mu(X \mid \mathrm{E}) = 0$, if $\mu_{\mathrm{p}}(X \cap \mathrm{E}) = 0$. A set X has *measure 1 in* E, and we write $\mu(X \mid \mathrm{E}) = 1$, if $\mu(X^c \mid \mathrm{E}) = 0$. If $\mu(X \mid \mathrm{E}) = 1$, we say that *almost every* language in E is in X.

We write $\mu(X \mid \mathrm{E}) \neq 0$ to indicate that X does *not* have measure 0 in E. Note that this does *not* assert that "$\mu(X \mid \mathrm{E})$" has some nonzero value.

It is shown in [4] that these definitions endow E with internal measure structure. This structure justifies the intuition that, if $\mu(X \mid \mathrm{E}) = 0$, then $X \cap \mathrm{E}$ is a *negligibly small* subset of E. The next two results state aspects of this structure that are especially relevant to the present work.

Theorem 7. ([4]) For all cylinders $\mathbf{C}_w$, $\mu(\mathbf{C}_w \mid \mathrm{E}) \neq 0$. In particular, $\mu(\mathrm{E} \mid \mathrm{E}) \neq 0$. ∎

The next lemma, which will be used in proving our main results, involves the following computational restriction of the notion of "countable union."

Definition 8. Let $Z, Z_0, Z_1, Z_2, \cdots \subset \{0,1\}^\infty$. Then Z is a p-*union of the p-measure 0 sets* $Z_0, Z_1, Z_2, \cdots$ if $Z = \bigcup_{j=0}^{\infty} Z_j$ and there exists a p-computable 1-MS d such that each d_j is successful on Z_j.

Lemma 9. ([4]) Let $Z, Z_0, Z_1, Z_2, \cdots \subset \{0,1\}^\infty$. If Z is a p-union of the p-measure 0 sets $Z_0, Z_1, Z_2, \cdots$, then Z has p-measure 0. ∎

In particular, an easy consequence of this lemma is that any finite union of p-measure 0 sets has p-measure 0.

4. Separating Completeness Notions in NP

In this section we prove our main result, that is:

Theorem 10. *Main Theorem.* If NP does not have p-measure 0, then there is a language C that is $\leq^{\mathrm{P}}_{2-\mathrm{T}}$-complete, but not $\leq^{\mathrm{P}}_{2-tt}$-complete, for NP.

Our proof of Theorem 10 uses the following definitions and lemma.

Definition 11. The *tagged union* of languages $A_0, \cdots, A_{k-1} \subset \{0,1\}^*$ is the language

$$A_0 \oplus \cdots \oplus A_{k-1} = \left\{ x10^i \,|\, 0 \leq i < k \text{ and } x \in A_i \right\}.$$

Definition 12. For $j \in \mathbb{N}$, the j^{th} *strand* of a language $A \subset \{0,1\}^*$ is

$$A_{(j)} = \left\{ x \,\middle|\, x10^j \in A \right\}.$$

Lemma 13. *Main Lemma.* For any language $S \in \mathrm{E}$, the set

$$X = \left\{ A \subset \{0,1\}^* \,\middle|\, A_{(0)} \leq^{\mathrm{P}}_{2-tt} A_{(1)} \oplus (A_{(1)} \cap S) \oplus (A_{(1)} \cup S) \right\}$$

has p-measure 0.

Before proving the Main Lemma, we use it to prove the Main Theorem.

Proof . *Proof of Theorem 10.* Assume that NP does not have p-measure 0. Let

$$X = \left\{ A \,\middle|\, A_{(0)} \leq^{\mathrm{P}}_{2-tt} A_{(1)} \oplus (A_{(1)} \cap \mathrm{SAT}) \oplus (A_{(1)} \cup \mathrm{SAT}) \right\}.$$

By the Main Lemma, X has p-measure 0, so there exists a language $A \in \mathrm{NP} - X$. Fix such a language A and let

$$C = A_{(1)} \oplus (A_{(1)} \cap \mathrm{SAT}) \oplus (A_{(1)} \cup \mathrm{SAT}).$$

Since $A \in$ NP, we have $A_{(0)}, A_{(1)} \in$ NP. Since $A_{(1)}$, SAT $\in$ NP and NP is closed under $\cap$, $\cup$, and $\oplus$, we have $C \in$ NP. Also, the algorithm

Begin
input x;
if $x1 \in C$
then if $x10 \in C$ **then accept**
else reject
else if $x100 \in C$ **then accept**
else reject
end

clearly decides SAT using just two (adaptive) queries to C, so SAT$\leq^{\rm P}_{2-{\rm T}}C$. Thus C is $\leq^{\rm P}_{2-{\rm T}}$-complete for NP. On the other hand, $A \notin X$, so $A_{(0)} \not\leq^{\rm P}_{2-tt}C$. Since $A_{(0)} \in$ NP, it follows that C is not $\leq^{\rm P}_{2-tt}$-complete for NP. ∎

The Main Lemma is proven in the Appendix.

5. Appendix

This section is devoted to proving the Main Lemma. For this we need the following definitions.

A $\leq^{\rm P}_{2-tt}$*-reduction* is a polynomial time computable function f with domain $\{0,1\}^*$ such that, for all $x \in \{0,1\}^*$,

$$f(x) = (f^1(x), f^2(x), f^3(x)) \in {\{0,1\}^*}^2 \times \mathbf{B}_2.$$

Each $f^i(x)$ is called a *query* of f on input x. $f^3(x)$ is the encoding of a 2-input, 1-output Boolean function. We write $f^3(x)(w)$ for the output of this function on input $w \in \{0,1\}^2$. Let $A, B \subseteq \{0,1\}^*$. A $\leq^{\rm P}_{2-tt}$*-reduction of* A *to* B is a $\leq^{\rm P}_{2-tt}$-reduction f such that, for all $x \in \{0,1\}^*$,

$$[\![x \in A]\!] = f^3(x)([\![f^1(x) \in B]\!], [\![f^2(x) \in B]\!]).$$

(Recall that $[\![\psi]\!]$ denotes the Boolean value of the condition ψ.) In this case we say that $A \leq^{\rm P}_{2-tt} B$ via f. We say that A is $\leq^{\rm P}_{2-tt}$*-reducible to* B, and write $A \leq^{\rm P}_{2-tt} B$, if there exists f such that $A \leq^{\rm P}_{2-tt} B$ via f.

We now sketch the proof the Main Lemma.

Proof. *Sketch of the proof of the Main Lemma.* Our objective is to prove that $\mu_{\rm p}(X) = 0$. For this we will write X as a union of four classes X^1, X^2, X^3 and X^4 and then prove that each X^i has p-measure 0. By the observation following Lemma 9 this is enough to have $\mu_{\rm p}(X) = 0$.

Let us define X^i for $i = 1,2,3,4$. We say that a language A is in X via a reduction f when

$$A_{(0)} \leq^{\rm P}_{2-tt} A_{(1)} \oplus (A_{(1)} \cap S) \oplus (A_{(1)} \cup S) \text{ via } f.$$

For a $\leq^{\rm P}_{2-tt}$-reduction f and for each $x \in \{0,1\}^*$, we denote by $q^1_f(x)$ and $q^2_f(x)$ the following prefixes of $f^1(x)$ and $f^2(x)$

$$f^1(x) = q^1_f(x) 1 0^{a^1_f(x)},$$

$$f^2(x) = q_f^2(x)10^{a_f^2(x)},$$

for $a_f^1(x), a_f^2(x) \in \mathbb{N}$.

$$X^1 = \{A \subset \{0,1\}^* \mid \text{there exists a } \leq_{2-tt}^{\mathrm{P}} \text{ reduction } f \text{ such that } A \in X \text{ via } f$$
$$\text{and there exists infinite } x \in \{0,1\}^* \text{ such that } (q_f^1(x)10 < x1 \text{ and } q_f^2(x)10 < x1)\}$$

$$X^2 = (X^1)^c \cap \{A \subset \{0,1\}^* \mid \text{there exists a } \leq_{2-tt}^{\mathrm{P}} \text{ reduction } f \text{ such that } A \in X \text{ via } f$$
$$\text{and there exists infinite } x \in \{0,1\}^* \text{ such that } q_f^1(x)10 < x1 < q_f^2(x)10\}$$

$$X^3 = (X^1 \cup X^2)^c \cap \{A \subset \{0,1\}^* \mid \text{there exists a } \leq_{2-tt}^{\mathrm{P}} \text{ reduction } f \text{ such that } A \in X \text{ via } f$$
$$\text{and there exists infinite } x \in \{0,1\}^* \text{ such that } q_f^1(x) \neq q_f^2(x)\}$$

$$X^4 = (X^1 \cup X^2 \cap X^3)^c \cap X$$

Let $f \in \mathrm{DTIMEF}(n^{\log n})$ be a function that is universal for PF, in the sense that

$$\mathrm{PF} = \{f_j \mid j \in \mathbb{N}\}.$$

Define the sets

$$X_j^i = \{A \in X^i \text{ via } f_j\}$$

for $i = 1,2,3,4,\ j \in \mathbb{N}$. Note that $X^i = \bigcup_{j=0}^{\infty} X_j^i$.

Now we prove that for each i from 1 to 4, X^i has p-measure 0. By using Lemma 9, it is enough to define for each i a p-computable 1-MS d^i, such that $X_j^i \subseteq S[d_j^i]$, for all $j \in \mathbb{N}$.

X^1 has p-measure 0.

Let $j \in \mathbb{N}, w \in \{0,1\}^*, b \in \{0,1\}$, let A be a language in X_j^1 such that $w \sqsubseteq A$,
$d_j^1(\lambda) = 2^{-j}$

If $s_{|w|} = x1$, for $x \in \{0,1\}^*$, $q_{f_j}^1(x)10 < x1$ and $q_{f_j}^2(x)10 < x1$ then

$$d_j^1(wb) = 2 \cdot d_j^1(w), \text{ if } b = A(x1)$$
$$d_j^1(wb) = 0, \qquad \text{if } b \neq A(x1)$$

Otherwise, $\quad d_j^1(wb) = d_j^1(w)$

X^2 has p-measure 0.

Let $j \in \mathbb{N}$. we first define $\{x_n \mid n \in \mathbb{N}\}$, an infinite sequence of very separated strings that are witnesses of the condition in the definition of X_j^2.

$x_1 = \min\{x \in \{0,1\}^* \mid q_{f_j}^1(x)10 < x1 < q_{f_j}^2(x)10\}$

$x_{n+1} = \min\{x \in \{0,1\}^* \mid x1 > q_{f_j}^2(x_n)10 \text{ and } q_{f_j}^1(x)10 < x1 < q_{f_j}^2(x)10\}$

Let $w \in \{0,1\}^*, b \in \{0,1\}$, let A be a language in X_j^2 such that $w \sqsubseteq A$, let $n \in \mathbb{N}$ such that $x_n 1 \leq s_{|w|} < x_{n+1}1$. Let b_1 be the answer to $f_j^1(x_n)$ according to w and S.
Let g be the function from $\{0,1,2\}$ to $\{0,1\}$ defined as follows

$$g(0) = 1, \quad g(1) = 1, \quad g(2) = 0.$$

$d_j^2(\lambda) = 2^{-j}$
If $s_{|w|} = x_n 1$ thenif $b \neq f_j^3(b_1, g(a_{f_j}^2(x_n)))$ $d_j^2(wb) = \frac{2}{3} \cdot d_j^2(w)$,
if $b = f_j^3(b_1, g(a_{f_j}^2(x_n)))$ $d_j^2(wb) = \frac{4}{3} \cdot d_j^2(w)$,
If $s_{|w|} = q_{f_j}^2(x_n)10$ then
if $A(x_n) = f_j^3(b_1, g(a_{f_j}^2(x_n)))$ $d_j^2(wb) = d_j^2(w)$,
if $A(x_n) \neq f_j^3(b_1, g(a_{f_j}^2(x_n)))$ and $b \neq g(a_{f_j}^2(x_n))$ $d_j^2(wb) = 2 \cdot d_j^2(w)$,
if $b = g(a_{f_j}^2(x_n))$ $d_j^2(w1) = 0$,
Otherwise, $d_j^2(wb) = d_j^2(w)$

X^3 has p-measure 0.

Let $j \in \mathbb{N}$. we first define $\{x_n \mid n \in \mathbb{N}\}$, an infinite sequence of very separated strings that are witnesses of the condition in the definition of X_j^3.
$x_1 = \min\{x \in \{0,1\}^* \mid x1 < q_{f_j}^1(x)10 < q_{f_j}^2(x)10\}$
$x_{n+1} = \min\{x \in \{0,1\}^* \mid x1 > q_{f_j}^2(x_n)10 \text{ and } x1 < q_{f_j}^1(x)10 < q_{f_j}^2(x)10\}$
Let $w \in \{0,1\}^*, b \in \{0,1\}$, let A be a language in X_j^2 such that $w \sqsubseteq A$, let $n \in \mathbb{N}$ such that $x_n 1 \leq s_{|w|} < x_{n+1}$.
Let h be the function from $\{0,1,2\} \times \{0,1,2\}$ to $\{0,1\} \times \{0,1\}$ defined as follows

$$\text{If } (a,b) \notin \{(0,2),(2,0)\} \text{ then } h((a,b)) = (g(a), g(b))$$

$$h((0,2)) = h((2,0)) = (0,0)$$

$d_j^3(\lambda) = 2^{-j}$
If $s_{|w|} = x_n 1$ and if $b \neq f_j^3(h((a_{f_j}^1(x_n), a_{f_j}^2(x_n))))$then $d_j^3(wb) = \frac{6}{7} \cdot d_j^3(w)$,
if $b = f_j^3(h((a_{f_j}^1(x_n), a_{f_j}^2(x_n))))$then $d_j^3(wb) = \frac{8}{7} \cdot d_j^3(w)$,
If $s_{|w|} = q_{f_j}^1(x_n)10$ and if $A(x_n) \neq f_j^3(h((a_{f_j}^1(x_n), a_{f_j}^2(x_n))))$ and
$b = h_1((a_{f_j}^1(x_n), a_{f_j}^2(x_n)))$ then $d_j^3(wb) = \frac{2}{3} \cdot d_j^3(w)$,
$b \neq h_1((a_{f_j}^1(x_n), a_{f_j}^2(x_n)))$ then $d_j^3(wb) = \frac{4}{3} \cdot d_j^3(w)$,
If $s_{|w|} = q_{f_j}^2(x_n)10$, if $A(x_n) = f_j^3(h((a_{f_j}^1(x_n), a_{f_j}^2(x_n))))$,
$A(q_{f_j}^1(x_n)10) = h_1((a_{f_j}^1(x_n), a_{f_j}^2(x_n)))$ and
$b \neq h_2((a_{f_j}^1(x_n), a_{f_j}^2(x_n)))$ then $d_j^3(wb) = 2 \cdot d_j^3(w)$
$b = h_2((a_{f_j}^1(x_n), a_{f_j}^2(x_n)))$ then $d_j^3(wb) = 0$
Otherwise, $d_j^3(wb) = d_j^3(w)$

X^4 has p-measure 0.

Let $j \in \mathbb{N}$. we first define $\{x_n \mid n \in \mathbb{N}\}$, an infinite sequence of very separated strings that are witnesses of the condition in the definition of X_j^4.

$x_1 = \min\{x \in \{0,1\}^* \mid x1 < q^1_{f_j}(x)10 = q^2_{f_j}(x)10\}$
$x_{n+1} = \min\{x \in \{0,1\}^* \mid x1 > q^2_{f_j}(x_n)10 \text{ and } x1 < q^1_{f_j}(x)10 = q^2_{f_j}(x)10\}$
Let $w \in \{0,1\}^*, b \in \{0,1\}$, let A be a language in X^4_j such that $w \sqsubseteq A$, let $n \in \mathbb{N}$ such that $x_n1 \leq s_{|w|} < x_{n+1}$.
$d^4_j(\lambda) = 2^{-j}$
If $(a^1_{f_j}(x_n), a^2_{f_j}(x_n)) \notin \{(1,2),(2,1)\}$ then
if $s_{|w|} = x_n1$ and if $b \neq f^3_j(h((a^1_{f_j}(x_n), a^2_{f_j}(x_n))))$then $d^4_j(wb) = \frac{2}{3} \cdot d^4_j(w)$,
if $b = f^3_j(h((a^1_{f_j}(x_n), a^2_{f_j}(x_n))))$then $d^4_j(wb) = \frac{4}{3} \cdot d^4_j(w)$,
if $s_{|w|} = q^1_{f_j}(x_n)10$ and if $A(x_n) \neq f^3_j(h((a^1_{f_j}(x_n), a^2_{f_j}(x_n))))$ then
if $b \neq h_1((a^1_{f_j}(x_n), a^2_{f_j}(x_n)))$ then $d^4_j(wb) = 2 \cdot d^4_j(w)$
if $b = h_2((a^1_{f_j}(x_n), a^2_{f_j}(x_n)))$ then $d^4_j(wb) = 0$
Otherwise, $d^4_j(wb) = d^3_j(w)$
If $(a^1_{f_j}(x_n), a^2_{f_j}(x_n)) = (1,2)$ then if $f^3_j(x_n)$ is

b_1	b_2	$f^3_j(x_n)$
0	0	0
1	0	1
1	1	0

If if $s_{|w|} = q^1_{f_j}(x_n)10$ then
if $A(x_n1) = 0$ and $S(q^1_{f_j}(x_n)) = 0$ then $d^4_j(w0) = 2 \cdot d^4_j(w)$, $d^4_j(w1) = 0$.
if $A(x_n1) = 0$ and $S(q^1_{f_j}(x_n)) = 1$ then $d^4_j(w1) = 2 \cdot d^4_j(w)$, $d^4_j(w0) = 0$.
if $A(x_n1) = 1$ and $S(q^1_{f_j}(x_n)) = 0$ then $d^4_j(w1) = 2 \cdot d^4_j(w)$, $d^4_j(w0) = 0$.
if $A(x_n1) = 1$ and $S(q^1_{f_j}(x_n)) = 1$ then $d^4_j(w0) = 2 \cdot d^4_j(w)$, $d^4_j(w1) = 0$.
Otherwise, $d^4_j(wb) = d^4_j(w)$.
The other cases of $f^3_j(x_n)$ are analogous.

The proof that this 1-MS are witnesses of $\mu_p(X^i) = 0$ is based on the fact that $x \in A_{(1)}$ implies that $x \in A_{(1)} \cup S$, and $x \notin A_{(1)}$ implies that $x \notin A_{(1)} \cap S$ ∎

6. Conclusion

We have shown that the hypothesis "NP does not have p-measure 0" separates $\leq^P_{2-T}$ and $\leq^P_{2-tt}$-completeness. The most immediate open problem is to generalize this result to separate $\leq^P_{k-T}$ and $\leq^P_{k-tt}$-completeness for other values of k. There is also a large spectrum of completeness notions between $\leq^P_T$ and $\leq^P_{b-tt}$ that deserve further attention.

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