MSc in Photonics

PHOTONICSBCN

Universitat Politècnica de Catalunya (UPC)
Universitat Autònoma de Barcelona (UAB)
Universitat de Barcelona (UB)
Institut de Ciències Fotòniques (ICFO)

http://www.photonicsbcn.eu

Master in Photonics

MASTER THESIS WORK

NETWORK NONLOCALITY WITH CONTINUOUS VARIABLES

José Carlos Guerra Copete

Supervised by Prof. Antonio Acín, (ICFO) and Paolo Abiuso (ICFO)

Presented on date 8th September 2021

Registered at

Escola Tècnica Superior d'Enginyeria de Telecomunicació de Barcelona

Network nonlocality with continuous variables

José Carlos Guerra Copete

ICFO - Institut de Ciències Fotòniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain

E-mail: jcgc97@yahoo.es

August 2021

Abstract. Since the birth of quantum mechanics, quantum information theory has seen a rise due to the contradiction with classical models. Some new concepts have surged up from the application of quantum models to information theory, such as entanglement and Bell nonlocality. Here we sum up the key concepts of these topics, which can be extended to multipartite settings in networks. Later on, we introduce the continuous-variable description of quantum systems of infinite dimension information and its basic formalism. In the end, we prove a theorem stating that Gaussian states (and more in general states with a positive Wigner distribution), which are experimentally easy to reproduce, do not show network nonlocality when measured in phase-space.

Keywords: Quantum nonlocality, causal networks, Gaussian states, entanglement.

1 Introduction

In the early days of Quantum Mechanics, Einstein together with Podolsky and Rosen, noticed in their seminal EPR paper [1] that quantum mechanics entails a "spooky action at a distance" that is incompatible with a description of reality that is at the same time local and realistic. That feature is nowadays known as "nonlocality". In 1964 Bell [2] formalized the same idea showing that the predictions of quantum mechanics are incompatible with local hidden variable (LHV) models trying to explain the emerging uncertainty in quantum measurements results.

In more recent years the theory of Quantum Nonlocality has seen its new renaissance, due to theoretical foundational advances, new numerical methods and experimental progress [3]. Moreover it has found practical importance in connection to practical tasks for quantum cryptography, device-independent detection of entanglement, random number generation etc. [3]. It has been now realized that the standard Bell scenario is no other than a particular instance of a problem of causal model compatibility [4–6], that is, when looking at the statistical output of an experiment:

- We assume a given network of parties (outputting variables publicly) with shared, possibly hidden resources, and given causal relations (normally described by a directed acyclic graph, or DAG) that describe which variables can influence others.

- We then ask the question: are the results of the experiment compatible with such a network?

- There is usually a gap between the probabilistic outputs that can be obtained for a given DAG with quantum sources and classical sources, which allows one to detect the hidden “quantumness” in the network without knowing any specifics of the systems (device-independently).

In this project, we want to study the problem of nonlocality in simple causality networks, such as the bilocality scenario [7,8], with a particular focus on the continuous variable (CV) setting and states and measurements that are closer to photonic implementations, such as combinations of gaussian states, photodetection and phase measurements [9, 10].

2 Classical strategies

A classical description of information theory is based on classical physics and its models. Two of the assumptions, inherited from classical models, are locality (no bit of information can reach instantly to other party because it would contradict relativity theory) and reality (every measurement is made on an entity that is there independently of the measurement). These two assumptions altogether contradict a quantum physical description of nature.

When Einstein, Podolski and Rosen wrote their paper [1], they introduced the mathematical formulation of the locality assumption. While their intention was to prove that quantum theory was incomplete due to the contradiction between locality and reality, in the end, further developments gave out that, in the quantum theory frame, both assumptions cannot coexist. This was presented in Bell's paper [2], which presented a clear framework consisting of experiments evaluating the correlations of measurements performed by distant parties. In a standard Bell bipartite experiment, a source (S) distributes two physical systems to two distant observers. Then, they perform a measurement, given an input (denoted by x and y) and produce an output (denoted a and b), as in the Figure 1.

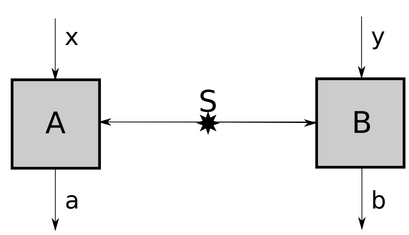

Figure 1: A generic Bell experiment. The source S distributes physical systems to the two distant observers A (Alice) and B (Bob). Labels x and y refer to the inputs and a and b are the outputs of the respective measurements of each part.

2.1 Locality

The local model is formulated in terms of a classical source λ which stores all causal factors from the past that could affect the measurement such that the probability function for the system is given by the probabilities of obtaining x and y taking into account the variable λ:

$$p(ab|xy,\lambda) = p(a|x,\lambda)p(b|y,\lambda) \tag{1}$$

The probability of the system is obtained by integrating the function depending on λ by a probability distribution on that local variable, giving out a result that is in general different from that of independent outcomes. The local set is defined by those probabilities that can be written as [3]:

$$p(ab|xy) = \int g(\lambda)p(a|x,\lambda)p(b|y,\lambda)d\lambda \tag{2}$$

It is known that the local set is convex as its elements can be written as a linear combination of its extremal points. It is also a polytope, which means that it has a finite number of extremal points. The facets of the local set join extremal points and are named Bell inequalities. Any probabilistic element that is outside the previous definition of the local set (Eq. 2) is considered nonlocal.

2.2 An example of a Bell inequality

To simplify the explanation, we consider the specific example where both the inputs and outputs are binary, $x, y \in \{0, 1\}$ and $a, b \in \{1, -1\}$ and derive the Clauser-Horne-Shimony-Holt (CHSH) inequality.

To do so, we first define the expectation value of outputs a and b for some given inputs x and y: $\langle a_x b_y\rangle = \sum_{a,b} ab\, p(ab|xy)$. The inequality is

$$S = \langle a_0b_0\rangle + \langle a_0b_1\rangle + \langle a_1b_0\rangle - \langle a_1b_1\rangle \leq 2. \tag{3}$$

This inequality holds for all local realistic models. To prove this, we substitute the expression for the probability, defined before:

$$\begin{aligned} S &= \langle a_0b_0\rangle + \langle a_0b_1\rangle + \langle a_1b_0\rangle - \langle a_1b_1\rangle \\ &= \int g(\lambda) \left(\sum_{a,b} ab\, p(a|0,\lambda)p(b|0,\lambda) + \sum_{a,b} ab\, p(a|0,\lambda)p(b|1,\lambda) \right. \\ &\quad \left. + \sum_{a,b} ab\, p(a|1,\lambda)p(b|0,\lambda) - \sum_{a,b} ab\, p(a|1,\lambda)p(b|1,\lambda) \right) d\lambda \end{aligned} \tag{4}$$

The expected values of ab can be written as $\langle a_x b_y\rangle = \int g(\lambda) \sum_a a\, p(a|x,\lambda)$ $\sum_b b\, p(b|y,\lambda)d\lambda = \int g(\lambda) \langle a_x\rangle_\lambda \langle b_y\rangle_\lambda d\lambda$ so that the sums inside can be expressed as $D_\lambda = \langle a_0\rangle_\lambda \langle b_0\rangle_\lambda + \langle a_0\rangle_\lambda \langle b_1\rangle_\lambda + \langle a_1\rangle_\lambda \langle b_0\rangle_\lambda - \langle a_1\rangle_\lambda \langle b_1\rangle_\lambda$. The expected values of a and b are defined between -1 and 1, so we can limit the expression as $D_\lambda \leq |\langle b_0\rangle_\lambda + \langle b_1\rangle_\lambda| + |\langle b_0\rangle_\lambda - \langle b_1\rangle_\lambda|$. Then, the expression is limited by the expected value with greater absolute value as $D_\lambda \leq 2\max_y |\langle b_y\rangle_\lambda| \leq 2$, so $S = \int g(\lambda)D_\lambda d\lambda \leq 2$ is the limit on the local set with two inputs and outputs, the so called CHSH inequality.

3 Quantum nonlocality

The quantum description of a system cannot take into account the classical concept of locality at the same time as reality, contrary to a classical description. This follows from the fact that quantum experiments violate Bell inequalities.

On the other hand, if a quantum experiment does not violate any Bell inequality, one may not in principle distinguish if it fulfills a quantum description or a classical one. So to speak, this may be stated as the local set being strictly included in the quantum set ($\mathcal{L} \subset \mathcal{Q}$) of correlations/probabilistic outputs. The same way as $\mathcal{L}$ was defined in (2), $\mathcal{Q}$ can be defined as the set of probabilistic outputs that can be written with local measurement operators acting on bipartite quantum states $p(ab|xy) = Tr[\rho_{AB} M_{a|x} \otimes M_{b|y}]$ [3].

3.1 Violation of the CHSH inequality

To see how a quantum experiment violates the local limits, one can perform measurements on a Bell state $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ with two sets of orthogonal projectors. The key here is that the shared state is entangled, and the two measurement sets are not orthogonal to each other. An entangled state is defined as a density matrix acting on a composite Hilbert space that cannot be described as a distribution of a product of its constituent states. The opposite is a separable state, describable as:

$$\rho_{AB} = \int d\lambda g(\lambda) \rho_A(\lambda) \otimes \rho_B(\lambda) \ . \tag{5}$$

Let's say that Alice measures the x and z axis so that the input values $x = 0$, resp. $x = 1$, correspond to σ_x, resp. σ_z, measurements. Meanwhile, Bob measures a combination between those axes such that he measures $(\sigma_x - \sigma_z)/\sqrt{2}$ for $y = 0$ and $(\sigma_x + \sigma_z)/\sqrt{2}$ for $y = 1$:

$$A_0 = (\sigma_x)_A = |0\rangle_A \langle 1| + |1\rangle_A \langle 0| \tag{6}$$

$$A_1 = (\sigma_z)_A = |0\rangle_A \langle 0| - |1\rangle_A \langle 1| \tag{7}$$

$$B_0 = \left(\frac{\sigma_x - \sigma_z}{\sqrt{2}}\right)_B = \frac{-|0\rangle_B \langle 0| + |0\rangle_B \langle 1| + |1\rangle_B \langle 0| + |1\rangle_B \langle 1|}{\sqrt{2}} \tag{8}$$

$$B_1 = \left(\frac{\sigma_x + \sigma_z}{\sqrt{2}}\right)_B = \frac{|0\rangle_B \langle 0| + |0\rangle_B \langle 1| + |1\rangle_B \langle 0| - |1\rangle_B \langle 1|}{\sqrt{2}} \ . \tag{9}$$

To calculate the obtained quantum correlations, we must average the value of each pair of operators $\langle \Psi^- | A_x B_y | \Psi^- \rangle$:

$$\langle A_0 B_0 \rangle = \left(\frac{\langle 01| + \langle 10|}{\sqrt{2}}\right) (\sigma_x)_A \left(\frac{\sigma_x - \sigma_z}{\sqrt{2}}\right)_B \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \tag{10}$$

$$\langle A_0 B_1 \rangle = \frac{1}{\sqrt{2}};\ \langle A_1 B_0 \rangle = \frac{1}{\sqrt{2}};\ \langle A_1 B_1 \rangle = -\frac{1}{\sqrt{2}} \tag{11}$$

$$S = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle = 2\sqrt{2} > 2, \tag{12}$$

hence violating the CHSH inequality.

3.2 Entanglement and nonlocality

Now we come to talk about the relation between entanglement and nonlocality. They are close concepts as one must have the first to reveal the other, but it is not necessarily the other way around. In principle, as nonlocality is shown depending on the states and the measurements, not all measurements on entangled states would show it.

To prove that entanglement is necessary for nonlocality, we prove the equivalent contraposition, i.e. that separable states lead to local models of correlation. One can see how separability implies locality by the following:

$$P(ab|xy) = Tr[M_{a_A}^{(x)} \otimes M_{b_B}^{(y)} \rho_{AB}] = Tr\left[\int d\lambda g(\lambda) M_{a_A}^{(x)} \rho_A(\lambda) \otimes M_{b_B}^{(y)} \rho_B(\lambda)\right]$$
$$= \int d\lambda g(\lambda) Tr_A\left[M_{a_A}^{(x)} \rho_A(\lambda)\right] Tr_B\left[M_{b_B}^{(y)} \rho_B(\lambda)\right] = \int d\lambda g(\lambda) p(a|x,\lambda) p(b|y,\lambda) \quad (13)$$

which is the local probability expression already seen in (2).

3.3 Visibility and noise

When treating with real experiments, conditions are not ideal. Due to this, we are not always able to prepare a determined state for a channel. This can be modeled by considering a parameter v that represents the visibility of the channel. When transmitting a pure entangled state $|\psi_i\rangle$ the obtained state is:

$$\rho(v) = v\,|\psi_i\rangle\langle\psi_i| + (1-v)\frac{\mathbb{1}}{4} \quad (14)$$

The second term refers to the noise induced by the channel. The noise state is modeled as a source with random (not entangled) outcomes. This is represented as:

$$\frac{\mathbb{1}_{AB}}{4} = \frac{|00\rangle_{AB}\langle 00| + |01\rangle_{AB}\langle 01| + |10\rangle_{AB}\langle 10| + |11\rangle_{AB}\langle 11|}{4} = \frac{\mathbb{1}_A}{2} \otimes \frac{\mathbb{1}_B}{2} \quad (15)$$

As it is seen, this state is not entangled and when it is measured, it does not violate the CHSH inequality. The prepared state may be entangled, so there is a certain value of the visibility above which CHSH inequality is violated.

For the bipartite local setup, the average of the operators that give the CHSH inequality can be calculated by tracing the state described in (14). To keep the previous demonstrations, we will consider the state $|\Psi^-\rangle$:

$$Tr(A_x B_y \rho) = v\,\langle\Psi^-|A_x B_y|\Psi^-\rangle + \frac{1-v}{4}\cancelto{0}{Tr(\mathbb{1}_A A_x)Tr(\mathbb{1}_B B_y)} \quad (16)$$

It can be easily seen that for $v > 1/\sqrt{2} \approx 0.707 \to S > 2$, the contribution of the prepared entangled state violates the CHSH inequality.

4 Beyond the standard Bell test

Now we come to analyze more complex systems that just a bipartite Bell test. First, it is better to stick to a system with three parties and later extend the conclusions to an arbitrary network. Starting with a tripartite system, it can be considered local if it can be modeled with a similar equation as in the bipartite case. Having one more party in the experiment implies that the local model is more constrained and thus the set of quantum states allowing a violation of locality is larger.

4.1 Bilocality

The bilocality problem is the simplest case of a quantum nontrivial network. It is based on a central node connected to two other parties and it can be thought of as two Bell experiments like the one explained before, with one common party.

In this case, the model can be expressed as:

$$p(abc|xyz) = \int\int g(\lambda_1,\lambda_2) p(a|x,\lambda_1) p(b|y,\lambda_1,\lambda_2) p(c|z,\lambda_2) d\lambda_1 d\lambda_2 \quad (17)$$

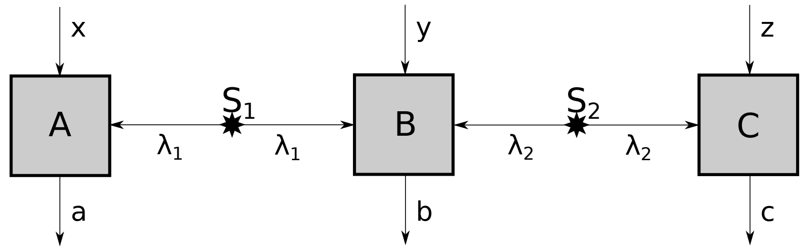

Figure 2: Bell bilocality experiment. The sources emit a state to each of neighbouring parties. Bob is a common party to both sources.

and it can be considered bilocal if the local variables are independent, e.g. $g(\lambda_1, \lambda_2) = g(\lambda_1)g(\lambda_2)$.

We consider here the case in which Bob performs a full Bell state measurement, so he gets four possible outcomes from a single measurement. In this case, we can say that Bob has two binary outputs, $b^0b^1 = 00, 01, 10, 11$, which correspond to $|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle$ and $|\Psi^-\rangle$ respectively. The probability corresponding to each value of the inputs and outputs (with measurements A_x and C_z defined both as A_x in Eq. (3)) can be calculated as:

$$P_Q(a, b^0b^1, c|x, z) = Tr\left[\hat{A}_x \otimes |b^0b^1\rangle\langle b^0b^1| \otimes \hat{C}_z |\Psi^-\rangle_{AB^0}\langle\Psi^-| \, |\Psi^-\rangle_{B^1C}\langle\Psi^-|\right] \tag{18}$$

whose results give out a compact form [8]:

$$P_Q(a, b^0b^1, c|x, z) = \frac{1}{16}\left[1 + (-1)^{a+c}\frac{(-1)^{b^0} + (-1)^{x+z+b^1}}{2}\right] \tag{19}$$

It is worth noticing that, here, the values $a, b^0, b^1, c, x, z \in \{0, 1\}$ are used instead of $a, b^0, b^1, c \in \{-1, 1\}$ used above, so the expression for our previous notation would substitute the exponential functions by products, e.g. $(-1)^{a+c} \to ac$, for a correspondence $0 \to 1, 1 \to -1$, but it's just a matter of notation. What is important from that expression is the structure of the probabilistic output.

We introduce the visibility as before, with the only exception that it is a property of a shared state so in the bilocality Bell experiments we would have 2 different visibilities:

$$\rho_i(v_i) = v_i |\psi_i\rangle\langle\psi_i| + (1 - v_i)\frac{\mathbb{1}}{4}; \qquad i = 1, 2 \tag{20}$$

If we look at the total probability, in the end it depends on the product of visibilities $V = v_1v_2$:

$$\begin{aligned} P(v) &= Tr\Big[\hat{A}_x \otimes |b^0b^1\rangle\langle b^0b^1| \otimes \hat{C}_z \\ &\left[v_1|\Psi_1\rangle\langle\Psi_1| + (1 - v_1)\frac{\mathbb{1}}{4}\right]_{AB^0}\left[v_2|\Psi_2\rangle\langle\Psi_2| + (1 - v_2)\frac{\mathbb{1}}{4}\right]_{B^1C}\Big] \\ &= Tr\Bigg[\hat{A}_x \otimes |b^0b^1\rangle\langle b^0b^1| \otimes \hat{C}_z\Bigg[v_1v_2\underbrace{|\Psi_1\rangle\langle\Psi_1| \otimes |\Psi_2\rangle\langle\Psi_2|}_{P_Q} + v_1(1 - v_2) \\ &|\Psi_1\rangle\langle\Psi_1| \otimes \frac{\mathbb{1}}{4} + (1 - v_1)v_2\frac{\mathbb{1}}{4} \otimes |\Psi_2\rangle\langle\Psi_2| + (1 - v_1)(1 - v_2)\frac{\mathbb{1}}{16}\Bigg]\Bigg] \end{aligned} \tag{21}$$

As we have chosen before, the states on both sides are maximally entangled (e.g. $|\Psi_1\rangle = |\Psi_2\rangle = |\Psi^-\rangle$) and their marginal distributions are random, meaning that a prepared state on one side and a random state on the other will have a random probability, as if both were random:

$$P(v) = VP_Q + (1 - V)P_R \tag{22}$$

where P_Q is the quantum distribution described in (19) and P_R is the random noisy distribution, which corresponds to $1/16$ for each output combination.

4.2 Multipartite networks

Until now, we have seen the simplest case of a network, that is a three-party network with a central party sharing sources with the other two. The results about the boundaries of the correlation sets give us a hint of how sharing states between nodes can show locality properties less restrictively, e.g. with lesser visibility. That is, there are states that for a standard Bell experiment do not show nonlocality, but they do for a network. The essence of these more general scenarios is to activate nonlocality by means of adding parties or preparing several copies of the state to perform multiple measurements. For example, a bipartite entangled state can exhibit nonlocal behaviour in a star-shaped network in which the state is distributed as copies from the central node, even if a single copy of the state itself is not nonlocal for that number of parties [6].

The power of networks in revealing nonlocality of a multipartite system has been explored [6, 11, 12] taking into account a more general set of operations, allowing us to perform joint measurements and post-selection in spacelike separated measurements.

It remains an open question the complete relation between entanglement and nonlocality. The article [6] shows that higher dimensions can lower the thresholds for nonlocal correlations, but the question is if for any entangled state one could build a network to demonstrate its nonlocality.

5 Continuous-variable quantum systems

Quantum information does not only consist in transmitting quantum bits by preparing a state and performing discrete operations. That can be assimilated as a digital approach to quantum information, encoded in a discrete degrees of freedom of particles, such as spin, momentum or energy levels. A continuous description of quantum information takes advantage of the operators used in quantum mechanics to describe, e.g., bosonic systems, such as the quadrature operators. Instead of the common logical gates used to manipulate (quantum) bits such as `CNOT` or Hadamard gates, one uses other unitary operators such as squeezing and displacement (see below). Continuous variable (CV) representation is widely used as some kinds of state are much easier to reproduce experimentally, for example in atomic ensembles.

The CV treatment starts from the description of a harmonic oscillator mode Hamiltonian $\hat{H} = \sum_k \hat{H}_k$, where $\hat{H}_k = \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + 1/2\right)$. The $\hat{a}$ and $\hat{a}^\dagger$ operators obey the commutator(anticommutator) relations for bosonic(fermionic) systems. Normally an equivalent basis is used, defined as:

$$\hat{q}_k = \frac{\hat{a}_k + \hat{a}_k^\dagger}{\sqrt{2}}; \quad \hat{p}_k = \frac{\hat{a}_k - \hat{a}_k^\dagger}{i\sqrt{2}}; \quad \hat{\boldsymbol{R}} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \ldots)^T \tag{23}$$

obeying the conmutation relation $[\hat{q}_k, \hat{p}_l] = i\delta_{kl}$. These are the position and momentum operators and are especially relevant because they are Hermitian, corresponding to observables with a continuous spectrum.

CV states can be described in the (discrete) Fock basis, or by the continuous wavefunction representation, e.g. in the $\hat{q}$ basis [9]. Another complete description, much more handy to use, is from quasi-probability distributions W_ρ^s, defined in phase space. They act as a complex Fourier transform of a family of characteristic functions [10]. They are called quasi-probabilities because, although they are normalized to 1, they can have negative values, and are defined as:

$$W_\rho^s = \frac{1}{\pi^2} \int_{\mathbb{R}^{2N}} \chi_p^s(\boldsymbol{\xi}) e^{i\boldsymbol{\kappa}^T \boldsymbol{\Omega} \boldsymbol{\xi}} d^{2N} \boldsymbol{\kappa} \tag{24}$$

where $\chi_p^s(\boldsymbol{\xi}) = \mathrm{Tr}[\rho \hat{D}(\boldsymbol{\xi})] e^{s||\boldsymbol{\xi}||^2/2}$ is called characteristic function, where $\hat{D}(\boldsymbol{\xi}) = e^{i\hat{\boldsymbol{R}}^T \boldsymbol{\Omega} \boldsymbol{\xi}}$ is the displacement operator, defined in terms of the canonical operators $\hat{\boldsymbol{R}}$, $\boldsymbol{\Omega} = \bigoplus_{k=1}^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\boldsymbol{\xi} = \sqrt{2}\,(\Re(\alpha_1), \Im(\alpha_1), \ldots)$.

The so-called Wigner function is the quasi-probability function with $s = 0$. Expressed in terms of position and momentum operators its expression is:

$$W_\rho^0(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{\pi^N} \int \langle \boldsymbol{q} + \boldsymbol{x} | \rho | \boldsymbol{q} - \boldsymbol{x} \rangle \, e^{2i\boldsymbol{x}\cdot\boldsymbol{p}} d^N \boldsymbol{x} \,, \tag{25}$$

where the state $|\boldsymbol{x}\rangle$ is an eigenvector of the quadrature operators [10].

This quasi-probability distribution has the property of getting the probability distribution (non-negative) of position or momentum by marginalizing on the other, for example:

$$\begin{aligned} \int W_\rho^0(q,p) dp &= \frac{1}{\pi} \iint \langle q + x | \rho | q - x \rangle \, e^{2ix\cdot p} dx dp \\ &= \int \langle q + x | \rho | q - x \rangle \, \delta(x) dx = \langle q | \rho | q \rangle = P(q) \end{aligned} \tag{26}$$

Even if the previous demonstration is done in one mode, as parties are independent, one can marginalize any variable on each mode in a N-mode system ($2N$-dimensional Wigner function) to give the distribution of any set of N commuting observables.

5.1 Wigner distributions and nonlocality

Now let's consider a bipartite system, whose inputs are discrete ($\{0, 1\}$) and determine if $\hat{q}$ or $\hat{p}$ is measured. The probability to have a determined output on each party is:

$$P(a, b|x, y) = \int dq_1 dq_2 dp_1 dp_2 W_\rho^0(q_1, p_1, q_2, p_2) \delta_x(a) \delta_y(b) \tag{27}$$

where $\delta_x(a)$ should be noted as $\delta(a - q_1)$ if $x = 0$ and $\delta(a - p_1)$ if $x = 1$ and similarly with $\delta_y(b)$. The δ in previous equation should be thought of as a deterministic output to marginalize that variable. For these measures to be local, one should be able to write down the probability as in (2). We can therefore consider $\{q_1, q_2, p_1, p_2\} \rightarrow \vec{\lambda}$ as local variables. To complete the LHV description, we need to consider the Wigner distribution as a distribution $g(\vec{\lambda})$ of these local variables. This is automatically fulfilled if W_ρ^0 is non-negative at all points, given that it is already normalized by definition. This leads to the following:

Theorem: Any system that counts with a non-negative Wigner function will always exhibit local behaviour in Bell tests where the parties measure $\hat{q}$ and $\hat{p}$.

This theorem can be extended to more general scenarios, for which we now give few examples.

First, we give an extension to continuous inputs for the measurement setting: we know that measuring q and p while having a positive Wigner distribution leads to local behaviour. W_ρ^0 can be defined in p basis instead of q basis as seen in Eq. (25), or even a combination of both, using canonical transformations [13]. We can define a variable that is a combination of both quadratures and is still an Hermitian variable to perform measurements: $\hat{q}_\theta = (e^{i\theta}\hat{a} + e^{-i\theta}\hat{a}^\dagger)/\sqrt{2}$. By substituting a and $a^\dagger$ by the expressions of the quadrature operators, one gets $\hat{q}_\theta = \cos(\theta)\hat{q} + \sin(\theta)\hat{p}$, thus $\hat{q}_0 = \hat{q}$ and $\hat{q}_{\pi/2} = \hat{p}$. To extend the theorem to continuous inputs, we consider a Wigner distribution with $|q - x\rangle \to |q_\theta - x_\theta\rangle$, now $|x_\theta\rangle$ being an eigenvector of $\hat{q}_\theta$. The locality of the output distribution follows in the same way as in Eq. (27), by just substituting the discrete subindex of the δ for this variable, i.e. $\delta_\theta(a) = \delta(a - q_\theta)$. It naturally follows the same conclusion as before, but in this case the system measures a continuous variable $\hat{q}_\theta$, proving the validity of this extension.

For the second generalization, we may consider a multipartite Bell state, shared among an arbitrary number of parties as in the Figure 3.

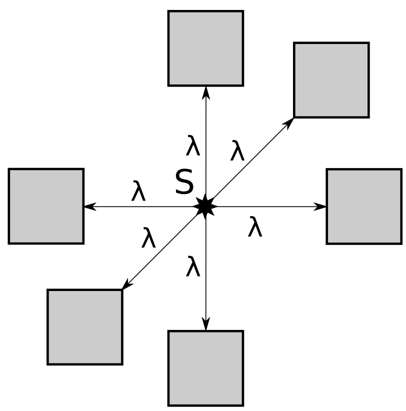

Figure 3: A generic multipartite single state Bell experiment

A natural extension of the bipartite probability stated in (27) comes when considering more than two parties, written as:

$$P(\boldsymbol{a}|\boldsymbol{x}) = \int d\boldsymbol{q}d\boldsymbol{p}W_\rho^0(\boldsymbol{q},\boldsymbol{p})\delta_{\boldsymbol{x}}(\boldsymbol{a}), \text{ where } \delta_{\boldsymbol{x}}(\boldsymbol{a}) = \prod_i \delta_{x_i}(a_i) \tag{28}$$

The delta function marginalizes on each party, and if the Wigner function is non-negative, the previous theorem holds true.

Finally, another extension in this sense is a bilocal setup (cf. Sec. 4.1), whose probability is given by:

$$\begin{aligned} P(a,b,c|x,y,z) = &\int dq_1 dq_2^1 dq_2^2 dq_3 dp_1 dp_2^1 dp_2^2 dp_3 W_\rho^0(q_1,p_1,q_2^1,p_2^1) \\ &\times W_\rho^0(q_2^2,p_2^2,q_3,p_3)\delta_x(a)\delta_y(b^1b^2)\delta_z(c) \end{aligned} \tag{29}$$

To assimilate this expression to a bilocal probability distribution (Eq. 17), we may consider $\{q_1, q_2^1, p_1, p_2^1\} \to \vec{\lambda}_1$ and $\{q_2^2, q_3, p_2^2, p_3\} \to \vec{\lambda}_2$ as the local variables. Bob receives two modes, one from Alice and other from Charlie, so y has two values, e.g.

$\delta_y(b^1b^2) \Rightarrow \delta(b^1 - q_2^1)\delta(b^2 - q_2^2)$ if $y = 00$; $\delta(b^1 - q_2^1)\delta(b^2 - p_2^2)$ if $y = 01\ldots$ Both modes b^1 and b^2 are independent because they come from different sources, so if both Wigner functions are non-negative, they can represent local variables distributions $g(\lambda_1)g(\lambda_2)$.

What we showed above, i.e. extending this theorem to continuous variables, an arbitrary number of parties and to non-trivial networks, act as tools for a more general theorem. The idea is that when constructing a quantum network, if the condition of non-negativity for each Wigner function is fulfilled, when measuring canonical observables of the form $\hat{q}_\theta$, it will always exhibit a local behaviour.

5.2 Gaussian states

Within the CV framework, there is a type of states that is especially useful in practice, which are the Gaussian states [10]. Some of the most feasible states in a laboratory, such as coherent states from a laser or thermal states, are Gaussian. They are defined as the ones whose characteristic function and quasi-probability functions are Gaussian. As Gaussian functions are always positive, we can conclude that Gaussian states, even if entangled, when measuring $\hat{q}$ and $\hat{p}$, can never show nonlocality in a network.

6 Conclusions

We have seen the differences between classical and quantum strategies in information theory, in particular in Bell experiments, where we have defined local models.

In quantum systems, we have seen how classical locality boundaries can be violated and how this nonlocality is related to entanglement. Later, we introduced bilocality in a tripartite experiment and multipartite networks.

We have summarised the continuous variables framework for quantum information, defining the Wigner quasi-probability distribution. We showed that states with positive Wigner distribution (including Gaussian states), even if entangled, can never be used to show nonlocality in networks where the parties only measure phase-space quadratures.

References

[1] Einstein A, Podolsky B and Rosen N 1935 *Physical Review* **47** 777–780
[2] Bell J S 1964 *Physics Physique Fizika* **1** 195–200
[3] Brunner N, Cavalcanti D, Pironio S, Scarani V and Wehner S 2014 *Reviews of Modern Physics* **86** 419–478
[4] Fritz T 2012 *New Journal of Physics* **14** 103001
[5] Henson J, Lal R and Pusey M 2014 *New Journal of Physics* **16** 113043
[6] Cavalcanti D, Almeida M, Scarani V and Acín A 2011 *Nature Communications* **2**
[7] Branciard C, Gisin N and Pironio S 2010 *Physical Review Letters* **104**
[8] Branciard C, Rosset D, Gisin N and Pironio S 2012 *Physical Review A* **85**
[9] Braunstein S and van Loock P 2005 *Reviews of Modern Physics* **77** 513–577
[10] Adesso G, Ragy S and Lee A 2014 *Open Systems & Information Dynamics* **21** 1440001
[11] Ciampini M A, Vigliar C, Cimini V, Paesani S, Sciarrino F, Crespi A, Corrielli G, Osellame R, Mataloni P, Paternostro M and Barbieri M 2017 *New Journal of Physics*
[12] Pozas-Kerstjens A, Rabelo R, Rudnicki L, Chaves R, Cavalcanti D, Navascués M and Acín A 2019 *Physical Review Letters* **123**
[13] Garcia-Calderon G; Moshinsky M 1980 *Journal of Physics A: Mathematical and General Physics* **13**(6) L185–L188 ISSN 0305-4470,0022-3689