DYNAMIC USER EQUILIBRIUM ON TRAFFIC NETWORKS. AN ANALYSIS AND A DISCRETIZED MODEL

E. Codina, J. Barceló

Dept. of Statistics and Operations Research UPC. Spain
e-mail:esteve@eio.upc.es

Abstract: This paper formulates a discretized version of the Dynamic User Equilibrium problem on traffic networks as a variational inequality problem with special emphasys on the traffic dynamics. A finite difference approximation to the simple continuum model is generalized to the case of a multidestination network, giving to the resulting model a bilivel programming structure and an heuristic algorithm is outlined to solve the model.

Keywords: Road Traffic, Dynamic Models, Mathematical Programming, Continuous Models, Partial Differential Equations.

1. INTRODUCTION

In few words the Dynamic User Equilibrium (DUE) problem on a multidestination traffic network in the presence of a predictable, inelastic but time-varying demand could be defined as determining which routes must be followed by the vehicles along a time horizon so that for a given origin destination and time instant of departure, the total individual cost in time spent in the travel is the minimum possible. If instead, the goal is to minimize the total amount of travel time spent by all travellers along a time horizon the Dynamic System Optimal problem is obtained. Formulations, algorithmic methods, heuristics and even the very problem definitions have been the target of a number of authors. The seminal paper by Merchant and Nemhauser (1978) provided the first mathematical model for the evolution in time of traffic flows approximating the traffic dynamics while trying to optimize an index function. Since then, the complexity of the problem has led to many authors to formulate models in the fields of optimal control (Friesz *et al.*, 1989; Ran *et al.*, 1993; Codina and Barceló, 1995a), variational inequalities on abstract spaces (Friesz *et al.* 1993) and other areas of advanced mathematical programming. They used the concepts of wardropian principles to analyze their models and tried to generalize them. Bernstein *et al.* (1993) and Smith (1993) were the first ones in properly defining the concept of DUE on a traffic network. The increasing diversity of formulations has led to the annalysis of the models in terms of consistency of travel times and the type of traffic dynamics modelization (Codina and Barceló, 1995b).

This paper formulates a discretized version of the DUE model on traffic networks assuming that travel times depend basically on traffic densities and that traffic dynamics follows the Lighthill-Whitham model (Lighthill and Whitham, 1955) or simple continuum model. The DUE model developed has a bilivel programming structure. The outer level of the bilivel program is formulated as a variational inequality problem with two main components. The first one is the path-cost operator which is recursively defined as in Wie (1995) in the second section. The second component are the traffic densities once a given set of routes at each time interval are given. The calculation of the traffic densities on a multidestination network is developed in the third section and can be considered as a generalization of the finite difference approximation to the Lighthill-Whitham model. It results in a nonlinear programming problem that makes up the inner level of the bilevel formulation. Finally in the fourth section, an heuristic algorithm is developed to solve the variational inequality formulation based on the projection methods.

2. THE BILEVEL PROGRAMMING STRUCTURE OF THE PROBLEM

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a graph modelling a traffic network and a time horizon divided in N time subintervals, each of them of length δ_t. By $\mathcal{O} \subseteq \mathcal{N}$ it is denoted the subset of nodes at which traffic flow enters the network (origins) and by $\mathcal{D} \subseteq \mathcal{N}$ it is denoted the subset of nodes at which traffic flow arrives (destinations). Then $\mathcal{I} = \{ (o,d) \mid o \in \mathcal{O},\ d \in \mathcal{D},\ o \textit{ connects } d \textit{ in } \mathcal{G} \}$ is the set of origin-destination pairs and Γ_i, $i = (o,d) \in \mathcal{I}$, is the set of paths connecting origin o with destination d on $\mathcal{G}$. Finally $\Gamma = \cup_{i \in \mathcal{I}} \Gamma_i$ is the set of all possible paths in the network and $\mathcal{C}_t = \Gamma \times \{1, \ldots, N\}$ is the set of pairs (p, ℓ) denoting the path p on $\mathcal{G}$ for cars that enter the network during ℓ-th time slice.

The simple continuum model (SCM) $x_t + (x \cdot \omega(x))_z = 0$ is considered now as a description of traffic behaviour. Let be a link of length L, let be $x(z,0)$, $0 \le z \le L$, an initial distribution of traffic density on the link and let $u(t)$, $v(t)$ input and exit flow functions defined on a time horizon $[0,T]$. It is assumed that traffic propagation is described by means of a decreasing speed-density function $\omega(x)$ on $[0, \hat{x}_a]$ with $\omega(0) = \hat{\omega} > 0$, $\omega(\hat{x}) = \varepsilon > 0$. A way to find an approximate solution of SCM is by means of finite difference methods. By dividing the rectangle $[0,L] \times [0,T]$ forming a grid mesh (δ_z, δ_t), approximations x_k^j to the SCM solutions are made at points (z_k, t_j), $k = 0,1,2,\ldots, M = L/\delta_z$ and $j = 0,1,2,\ldots, N = T/\delta_t$ or, equivalently, the link is subdivided into M sublinks and the time horizon is subdivided into N time slices. It is suposed that $\tilde{\omega} = \delta_z/\delta_t = 1/\nu$ verifies the Courant-Friedrich-Levy (CFL) condition: $\nu \cdot \hat{\omega} \le 1$. This condition, as is known, is equivalent to say that the minimum time to traverse a sublink is greater than or equal to the time slice δ_t. In Codina and Barceló (1995a) it is shown how to split a set of links in a network and determine a maximum time slice length δ_t so that CFL is observed.

Consider now the graph $\mathcal{G}$ after spliting the links or, equivalently, consider the resulting sublinks as links of a more dense graph. Let x_a^ℓ, $\ell = 0, \ldots, N$, be the traffic density on link $a \in \mathcal{A}$ at time slice ℓ-th determining link travel time by means of the function $c_a(x_a^\ell) = \delta_{z_a}/\omega((x_a^\ell)$. Let $x_a = (x_a^1, \ldots, x_a^N)$ be the set of values for link $a \in \mathcal{A}$ in all time slices and $x = (\ldots, x_a, \ldots\ a \in \mathcal{A})$ the set of values for all links along the whole time period. Let now $\phi(\cdot,\cdot)$ an interpolating function for the values in x_a over the mesh of time instants $(t_0, t_1, \ldots, t_N,$. An example of $\phi(\cdot,\cdot)$ can be $\phi(t, x_{a_j}) = \sum_{i=0}^{N} s_i(t) \cdot x_{a_j}^i$, with $s_i(\cdot)$ a cardinal spline on the mesh $(\ldots, t_\ell, \ldots)$, i.e. $s_i(t_j) = \delta_{ij}$ and $\phi(t_\ell, x_{a_j}) = x_{a_j}^\ell$.

Let $\gamma = (p, \ell) \in \mathcal{C}_t$, with $p = (a_1, a_2, \ldots, a_m)$ being a path on the network. If τ_j denotes the (continuous) time instant at which a car would enter link a_j in the path p, following Wie *et al.* (1995), it is possible to define the cost for γ recursively as:

$$
\begin{aligned}
C_\gamma(\ell, x) &= \tau_m - \ell \cdot \delta_t \\
\tau_j &= \tau_{j-1} + c_{a_j}(\phi(\tau_{j-1}, x_{a_j})) \qquad (1) \\
(\tau_0 &\triangleq \ell \cdot \delta_t,\ 1 \le j \le m)
\end{aligned}
$$

Let $r_{i,\ell}$, $i \in (o,d)$, $1 \le \ell \le N$ be the amount of cars entering at node $o \in \mathcal{O}$ during ℓ-th time slice going towards destination $d \in \mathcal{D}$. Let us assume that the $r_{i,\ell}$ cars will split amongst different paths $p \in \Gamma_i$ and if $h_\gamma, \gamma = (p, \ell)$ is the amount following path p then, $\sum_{c \in \Gamma_i} h_\gamma = r_{i,\ell}$. Assume also that in the time slice ℓ-th, the order of input at a given origin of the corresponding $r_{i,\ell}$ is given for distinct $i \in \mathcal{I}$. Accordingly to the paths p being followed and the dispersion phenomenon of traffic there must exist a map $x = x(h)$ between densities x_a^j and the amounts h_γ. Following Smith (1993) and having into account this map, the DUE problem could be stated as the following variational inequality (VI) problem:

DUE as a VI problem. *For each time slice ℓ, $1 \le \ell \le N$, and each O-D pair $i \in \mathcal{I}$, find the amounts of traffic flow, h_γ^*, $\gamma = (p, \ell)$ on paths $p \in \Gamma_i$ such that verify the following VI problem:*

$$\sum_{\ell=1}^{N}\sum_{i\in\mathcal{I}}\sum_{p\in\Gamma_i} C_\gamma(\ell, x(h^*))(h_\gamma - h^*_\gamma) \geq 0,$$
$$\sum_{p\in\Gamma_i} h_\gamma = r_{i,\ell}, \forall i \in \mathcal{I}, (\gamma = (p,\ell)) \quad (2)$$
$$h_\gamma \geq 0, \ell = 1, 2, \ldots, N.$$

The correspondence between the solutions of the previous VI problem and the definition of the DUE problem are shown in Smith (1993), as well as its existence of solutions and it will not be repeated here. In the next section it is going to be shown how the map $x = x(h)$ obbeys to a mathematical program, thus illustrating the bilevel structure of the DUE problem.

3. THE MAP $x = x(h)$ AND A PROGRESSION MODEL

In order to set a continuous map $x = x(h)$ a progression model based on the explicit method to approximate the SCM model on links is developed. Let $\bar{x}_k^{j-1} = \delta_z \cdot x_k^{j-1}$ and $\bar{x}_k^j = \delta_z \cdot x_k^j$ be the number of vehicles at the beginning of time slice j-th and j + 1-th (or at the end of time slice j-th) on sublink k-th and let $\bar{u}_k^j = \delta_t \cdot u_k^j$, $\bar{v}_k^j = \delta_t \cdot v_k^j$ be the number of vehicles entering and leaving k-th sublink on time slice j-th respectively. Let $\bar{\omega}(\bar{x}) = \nu \cdot \omega(\bar{x}/\delta_z)$ a rescaled speed-density function and let $\varphi(\bar{x}) = \bar{x} \cdot \bar{\omega}(\bar{x})$ the rescaled flow density function. Assuming that the initial number of cars $\bar{x}_k^0$ on each sublink and the inputs to the initial sublink $\bar{u}_1^j$ are given, the explicit method can be simply stated as:

- For each time slice j let us denote by φ_k to $\varphi(x_k^{j-1})$. Then $\bar{v}_k^j = \varphi_k$ for $k = 1, 2, \ldots, M$. and:

$$\bar{x}_k^j = \bar{x}_k^{j-1} + \bar{v}_{k-1}^j - \bar{v}_k^j, \quad (k = 1, 2, \ldots, M,\ \bar{v}_0^j \triangleq \bar{u}_1^j) \quad (3)$$

It is assumed that no constraints or interruptions on exit flows $\bar{v}_M^j$ have to be observed. However, this is precisely what happens when the link is part of a traffic network or there exist interruptions due to traffic signals. In this case the relationship between exit flows and densities is weakened to $\bar{v}_k^j \leq \varphi_k$ as in Carey (1986). Being $\bar{v}_k^j = \bar{u}_{k+1}^j$ the number of cars that exit sublink k-th, the maximum number of cars that can enter is given by $x^{0k} + \hat{v}_k^j$, being $x^{0k} = \hat{x} - \bar{v}_k^{j-1}$ the spare capacity of sublink k-th at the beginning of time slice j-th. Therefore $\bar{u}_k^j$ must verify the inequalities $0 \leq \bar{u}_k^j \leq x^{0k} + min\{\bar{u}_{k+1}^j, \varphi_k\}$.

Then the solution of the SCM can be determined as follows:

- Assume that $\bar{x}_k^{j-1}$ are known for $1 \leq k \leq M$ and let $\tilde{v}_M^j$, $\tilde{u}_1^j$ be the maximum number of vehicles outgoing and entering at the link respectively.
- Determine the maximum number of vehicles $\bar{v}_M^j$ that exit in the last sublink: $\bar{v}_M^j = min\{\tilde{v}_M^j, \varphi_M\}$. Let $P(\cdot)$ be a strictly increasing function on $[0, \hat{x}]$. Find the unique inputs $\bar{u}_k^j$ at sublinks $k = 2, \ldots, M$ that solve:

$$\begin{aligned} &Max_{\ \bar{u}_k^j} \sum_{k=1}^{M} P(\bar{u}_k^j) \\ &\bar{u}_k^j \leq x^{0k} + \bar{u}_{k+1}^j \\ &\bar{u}_k^j \leq u^{0k} \\ &\bar{u}_k^j \geq 0, \quad k = 1, \ldots M; \\ &(\bar{u}_{k+1}^j \triangleq \bar{v}_M^j) \end{aligned} \quad (4)$$

  (with $u^{0k} = x^{0k} + \varphi_k$ for $k = 2, \ldots M-1$ and $u^{0\ell} = x^{0\ell} + min\{\varphi_\ell, \tilde{u}_\ell^j\}$ for $\ell = 1$ or $\ell = M$.)
- Calculate $\bar{x}_k^j$ for the next time slice as $\bar{x}_k^j = \bar{x}_k^{j-1} + \bar{u}_k^j - \bar{v}_k^j$ for $k = 1, \ldots, M$ and $\bar{v}_{k-1}^j = \bar{u}_k^j$ for $k = 2, \ldots, M$.

There exists the possibility that certain inputs $\tilde{u}^j$ can not be satisfied as the initial sublink might not have enough capacity or it could be reduced by backward propagation of queues. This would imply that some cars would never enter this link. To overcome this problem inputs to the link should first enter to a fictious 0-sublink with infinite capacity that would store cars, releasing them at a maximum throughput capacity $\bar{x}/\delta_t$ as the first sublink is able to admit them. In this modified model inputs to the system could be $\tilde{u}_0^j$ and if $\bar{x}_0^j$ is the number of cars stored by the 0-sublink then $\bar{x}_0^j = \bar{x}_0^{j-1} + \tilde{u}_0^j - \tilde{u}_1^j$ and $\tilde{u}_1^j \leq \varphi_0(\bar{x}_0^{j-1})$, being φ_0 the exit function of the 0-sublink defined by $\varphi_0(x) = x$ if $x < \tilde{x}$ and $\varphi_0(x) = \tilde{x}$ if $x \geq \tilde{x}$.

The previous program (4) expresses a "maximum instantaneous progression" principle because its solution can be easily found by setting the output $\bar{u}^j_{M+1}$ of the last sublink to its maximum value $min\{ \bar{v}^j_M, \varphi_M \}$, and then the optimum input $\bar{u}^j_k$ to k-th sublink is $\{ u^{0k}, x^{0k} + \bar{u}^j_{k+1} \}$ for $k = M, M-1, \ldots$. Thus if output of the last sublink is maximum then the input to each sublink is also maximum accordingly to the spare capacity $\hat{x} - \bar{x}^{j-1}_k + \bar{v}^j_k$.

The previous concept of maximum instantaneous progresion can be generalized when several links form a traffic network or when, given a subset of paths $\mathcal{C}'_t \subseteq \mathcal{C}_t$ that are to be followed by the cars entering at time instants ℓ, we want to calculate the flows on the links of the previous paths $\gamma = (p, \ell) \in \mathcal{C}'_t$.

For each path $\gamma = (p, \ell) \in \mathcal{C}'_t$ let us build the space-time grid network starting at time slice ℓ and with a number of sublinks $M_\gamma = \sum_{a \in p} M_a$. This grid network results from the balance equations $\bar{x}^{j-1}_k - \bar{x}^j_k + \bar{u}^j_k - \bar{u}^j_{k+1} = 0$, $(k = 1, 2, \ldots, M_\gamma, \ell \leq j \leq N)$. If path p connects origin $o \in \mathcal{O}$ to destination $d \in D$, associated to $\gamma = (p, \ell) \in \mathcal{C}'_t$ it can be considered the set $\tilde{c}_{(p,\ell)}$ of all t-paths or space-time trajectories on its grid mesh. All these space-time trajectories start at node o at time ℓ and end at destination d at time $\ell' \geq \ell$. If $p_{i,\ell}$, $i = (o, d)$, is the number of cars entering at origin o at time ℓ with destination d and $p_{i,\ell} = \sum_{i \in \Gamma_i} h_\gamma$ then h_γ is decomposed as $h_\gamma = \sum_{\kappa \in \tilde{c}_{(p,\ell)}} f_\kappa$. This decomposition is of course *not unique*, even if the traffic difusion on a link obbeys to a model as (4). However, given a solution of (4) it is always possible to find a subset of trajectories in $\tilde{c}_{(p,\ell)}$ for which the FIFO discipline is observed (Codina and Barceló, 1995). At the beginning of a given time slice j-th, the number of cars $\bar{x}^{j-1}_k$ present on sublink k-th of a link a contained in some path $\gamma \in \mathcal{C}'_t$ can be decomposed into an ordered set of traffic packets, $\bar{y}_{q_1}, \ldots, \bar{y}_{q_n}$, $(q_1, \ldots, q_n \in \cup_{\ell \leq j} \cup_{a \in p} \tilde{c}_{(p,\ell)})$. Accordingly to this ordering, the $\bar{y}_{q_1}$ cars of the first packet shall leave sublink k-th "before" than the $\bar{y}_{q_2}$ cars in the second packet if that is possible at all. Into an isolated link the progression of packets on each sublink is determined by the solutions $\bar{u}^j_k$ of (4). If the link a is included in a graph $\mathcal{G}$ then it is necessary to consider other links a' accessing to the same junction or node $\mathcal{J}$ as link a does and the links that emerge from node $\mathcal{J}$.

Let $E(\mathcal{J})$ be the set of emerging links and $I(\mathcal{J})$ the set of incident links at node $\mathcal{J}$ that

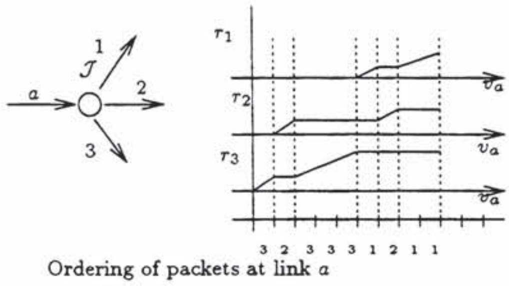

Fig. 1. An example of turning flow functions for a link a accessing to junction $\mathcal{J}$.

are included in some path p contained in $\mathcal{C}'_t$. Assume for simplicity first that all turns are allowed at node $\mathcal{J}$ and that all links consist of only one sublink. Then, accordingly to the emerging links $b \in E(\mathcal{J})$, it is possible to associate to the previously defined packets the corresponding outgoing links $b_1, \ldots, b_n$. As the packets are ordered, the turning flows r_{ab} are continuous piecewise defined functions of the total outgoing flow $\bar{v}_a$ for a link $a \in E(\mathcal{J})$ (see figure 1 for an example). They can be defined as follows:

Let $y_L = \sum_{i=1}^{L} y_{b_i}$, $y_0 = 0$, $0 \leq L \leq M$ and $y_{L,b} = \sum_{b_i = b, i \leq L} y_{b_i}$. If there are n subpackets on link a associated to the sequence of emerging links $b_1 \ldots b_n$, then for $0 \leq L \leq n-1$ and for $y_L \leq v \leq y_{L+1}$:

$$r_{ab}(v) = \begin{cases} y_{L,b} + (v - y_L), b_{L+1} = b \\ y_{L,b}, b_{L+1} \neq b \end{cases} \tag{5}$$

If in the sequence $(b_1 \ldots b_n)$, the emerging links $b \in E(\mathcal{J})$ appear evenly distributed, then an approximation to r_{ab} can be:

$$r_{ab}(v) = \chi_{ab} \cdot v, \quad \left(\chi_{ab} = \frac{\sum_{b_i = b} y_{b_i}}{\sum_{i=1}^{n} y_{b_i}} = \frac{\sum_{b_i = b} y_{b_i}}{\bar{x}^{j-1}} \right) \tag{6}$$

To any origin $o \in \mathcal{O}$ in the network we can associate a connector or 0-sublink for which at a given time slice there will be also a set of ordered subpaquets.

Let ϱ_{oa} be the turning flows entering at link a emerging from origin $o \in \mathcal{O}$ and let $\bar{r}_{ad}(\bar{v}_a)$ be the turning flow functions to a destination $d \in \mathcal{D}$ coming from a link $a \in I(d)$. With this statement of the turning flow functions the traffic flows for time slice j-th must verify:

$$\bar{u}_a = \sum_{b \in I(\mathcal{J})} \rho_{ba}(\bar{v}_b), \quad (\forall a \in E(\mathcal{J}), \forall \mathcal{J} \in \mathcal{N} - \mathcal{O}) \tag{7}$$

$$\bar{u}_a = \sum_{b \in I(o)} \rho_{ba}(\bar{v}_b) + \varrho_{oa}, \quad (8)$$
$$(\forall a \in E(o), \forall o \in \mathcal{O})$$

$$\bar{v}_a = \sum_{b \in E(d)} \rho_{ab}(\bar{v}_a) + \bar{r}_{ad}(\bar{v}_a),$$
$$s_d = \sum_{a \in I(d)} \bar{r}_a(\bar{v}_a), \quad (9)$$
$$(\forall a \in I(d), \forall d \in \mathcal{D})$$

It could be possible to have into account capacity restrictions on junctions expressed as inequalities of the type:

$$\sum_{a \in I(\mathcal{J})} \bar{v}_a \leq \hat{v}_{\mathcal{J}} \quad (10)$$

In the case of limited throughput due to link interactions at junctions, the turning flow functions $r_{ab}; r_{a'b}, \ldots$ at junction $\mathcal{J}$ would verify a relationship as:

$$r_{a'b} \leq \hat{r}_{a'b}\left(1 - \eta_{a'b} \cdot \sum_{ab \in \pi(a'b)} r_{ab}\right) \quad (11)$$

with $\pi(a'b)$ the set of priorized movements over ab on $\mathcal{J}$ and being $\hat{r}_{a'b}$ the maximum throughput for movement $a'b$ and $\eta_{a'b}$ a nonnegative constant.

Let each link $a \in \mathcal{A}$ be divided into M_a sublinks, let $P_a(\cdot)$ strictly increasing functions on $[0, \hat{x}_a]$ and consider the subgraph $\mathcal{G}'$ that contains all paths p in $\mathcal{C}'_t$. For a given time slice j-th form the turning flow functions (5) or their linear approximations (6) at the junctions $\mathcal{J} \in \mathcal{N}$, origins $o \in \mathcal{O}$ and destinations $d \in \mathcal{D}$. Now the maximum instantaneous progression for the set of paths considered as an extension of the previous program (4) appears as:

$$\begin{aligned} Max \sum_{a \in \mathcal{A}} \sum_{k=1}^{M_a} & P_a(\bar{u}^j_{a,k}) + \sum_{d \in \mathcal{D}} P_d(s_d) \\ s.t. \quad & u^j_{a,k} \leq \hat{x}_a - x^{j-1}_{a,k} + u^j_{a,k+1} \\ & u^j_{a,k} \leq \hat{x}_a - x^{j-1}_{a,k} + \varphi_a(x^{j-1}_{a,k}) \qquad (12) \\ & u^j_{a,k} \; subject\ to\ (7),\ (8), \\ & (9),\ (10),\ (11)\ and\ u^j_{a,k} \geq 0 \\ & 1 \leq k \leq M_a \end{aligned}$$

Because of the subpackets structure of traffic on links adopted and the fact that the left hand side of (11) must remain nonnegative although the right hand side may take negative values, the previous program is a nonlinear-integer one difficult to solve. However if in the program (12) the linearized turning flow functions (6) are adopted and, due to design aspects of the traffic network, the right hand side of constraints (11) due to link interactions remains always nonegative, the feasible set of the resulting modified problem (12) is a bounded polytope. If the program (12) has uniqueness of solutions, then the relationship between path flows h_γ, $\gamma \in \mathcal{C}'_t$ and densities on links $x(h)$ is univoquely defined, otherwise the traffic evolution may experiment bifurcation points. Once the solutions of the modified problem (12) in terms of the output flows $\bar{u}^j_{a,M_a+1}$ of each link are known, update the subpackets on each sublink and set $x^j_{a,k} = x^{j-1}_{a,k} + u^j_{a,k} - x^j_{a,k+1}$ for the next time slice. After solving (12) for each time slice the densities $x(h)$ would be obtained on all sublinks of the network and the cost defined in (1) could be evaluated. Its continuity follows from the continuity of $x(h)$ and that of the function $\phi(\cdot,\cdot)$ used to define $C_\gamma(\ell, x)$.

4. AN ALGORITHM FOR THE DUE PROBLEM

In this section an heuristic algorithm to solve problem (2) is developed. This heuristic algorithm is based on the projection methods for VI's. Because of the continuity of cost operator (1) and the finiteness of the feasible set there follows the existence of solutions to VI problem (2) (Smith, 1993) but stronger assumptions others than continuity of the cost $C_\gamma(\ell, x)$ are required to apply a method that succesfully converges to one of them (Harker and Pang, 1990). The heuristic algorithm outlined below can be summarized as follows for a generic VI problem, i.e., find $x^* \in X$ verifying $F(x^*)^\top(x - x^*) \geq 0$, $\forall x \in X$, with $F(\cdot) : \Re^n \mapsto \Re^n$ continuous and X a compact and convex set. Then it is known that, if M is symmetric definite positive real matrix, $\|x\|_M = \frac{1}{2}x^\top M x$ its associated norm and $G(x) = x - M^{-1}F(x)$. Then the unique solution of the following problem (13) defines for any $y \in X$ a function $\varphi(\cdot)$

$$Min_{x \in X} \|G(y) - x\|_M \quad (13)$$

Then any fix point of the function $\varphi(\cdot)$ is a solution of the VI (Kinderlehrer and Stampacchia, 1980). Bearing this in mind the heuristic algorithm to solve the VI problem (2) can be considered as: "perform a fix point iteration of the type $x_{\kappa+1} = \tilde{\varphi}(x_\kappa)$, where $\tilde{\varphi}(x_\kappa)$ is an approximate solution to problem (13) with $y = x_\kappa$."

Heuristic algorithm for VI (2). In order to solve VI (2) a constant definite positive and diagonal matrix $M = diag(..,a_\gamma,..)$ is fixed. At a given iteration κ, a path $\gamma = (p,\ell)$ is said to be "used" if the flow assigned to it, h_γ^κ, is positive. Let $\mathcal{C}_t^{\kappa,+} = \{\gamma \in \mathcal{C}_t \mid h_\gamma^\kappa > 0\}$ the set of all used paths at iteration κ.

1- By solving for each time slice $1 \leq \ell \leq N$ the problem (12), determine the densities $x^{(\kappa} = x(h^\kappa)$ on each sublink and calculate the cost (1) for paths $\gamma \in \mathcal{C}_t^{\kappa,+}$ and let us denote it here by $F_\gamma^\kappa = C_\gamma(\ell, x)$.

2- For each $i \in \mathcal{I}$ calculate the shortest path in time using Kaufman and Smith algorithm (1993). Let $\gamma_i^{*,\kappa}$ the shortest path for o-d pair $i \in \mathcal{I}$.

3- Stop if $\forall i \in \mathcal{I}$, the shortest paths in time satisfy:

$$Max_{\gamma \in \mathcal{C}_t^{\kappa,+}}\{F_\gamma^\kappa\} - F_{\gamma_i^{*,\kappa}} \leq \varepsilon \cdot F_{\gamma_i^{*,\kappa}} \quad (14)$$

4- Let $\hat{\mathcal{C}}_t^{\kappa,+} = \mathcal{C}_t^{\kappa,+} \cup \{\gamma_i^{*,\kappa} \mid i \in \mathcal{I}\}$ and denote by b_γ to $b_\gamma = a_\gamma h_\gamma^\kappa - F_\gamma^\kappa$, $\gamma \in \hat{\mathcal{C}}_t^{\kappa,+}$. Solve the approximate projection:

$$Min \quad \sum_{\gamma \in \hat{\mathcal{C}}_t^{\kappa,+}} (a_\gamma h_\gamma^2 + b_\gamma h_\gamma)$$

$$\sum_{p \in \Gamma_i} h_\gamma = r_{i,\ell},\ h_\gamma \geq 0,\ \forall i \in \mathcal{I}, \quad (15)$$

$$(\gamma = (p,\ell), \ell = 1, 2, \ldots, N.)$$

5- Let $h_\gamma^{*,\kappa}$ be the solution of problem (15). Set $\mathcal{C}_t^{\kappa+1,+} = \hat{\mathcal{C}}_t^{\kappa,+} - \{\gamma \in \mathcal{C}_t \mid h_\gamma^{*,\kappa} = 0\}$. Set $\kappa = \kappa + 1$ and return to step 1.

The previous quadratic problem (15) is an approximation to the projection (13). Also, dimensionality of the problem difficults the use of an approximation to the jacobian of the cost (1) in order to use other methods like linearized Jacobi or quasi-Newton's method (Harker and Pang, 1990).

5. REFERENCES

Bernstein D., T.L. Friesz (1993). A variational control formulation of the simultaneous route and departure-time choice equilibrium problem. In: *Transportation and Traffic Theory*. (C.F. Daganzo, Ed.) Elsevier Science.

Carey M. (1986). A constraint qualification for a dynamic traffic assignment model. *Transp. Sci.*, **20**, pp. 55-58.

Codina E., J. Barceló (1995a). A system optimal dynamic traffic assignment model with distributed parameters. In: *Advanced Methods in Transportation Analysis* (L. Bianco, P. Toth, Eds.), Chap. 13, pp. 299-320. Springer Verlag, Berlin.

Codina E., Barceló J. (1995b). Dynamic Traffic Assignment: Considerations on Some Deterministic Modelling Approaches. *Annals of Operations Research*. **60**, pp.1-58.

Friesz T.L., F.J.Luque, R.L.Tobin and B.W.Wie (1989). Dynamic network traffic assignment considered as a continuous time optimal control problem. *Op. Res.*, **37-6**, pp. 58-69.

Friesz T.L., Bernstein D., Smith T.E., R.L. Tobin and B.W. Wie (1993). A Variational Inequality Formulation of the Dynamic Network User Equilibrium Problem. *Op. Res.*, **41-1**.

Harker T. and J.S. Pang (1990). Finite dimensional variational inequalities and nonlinear complementary problems: a survey of theory, algorithms and applications. *Math. Prog.*, **48**, pp. 161-220.

Kaufman D.E., R.L. Smith (1993). Fastest paths in time-dependent networks for intelligent highway systems application. *IVHS Journal*. **I(I)** pp. 1-11.

Kinderlehrer D. and G. Stampacchia (1980). In: *An introduction to variational inequalities and their applications*. Chap. 1. Academic Press, New York.

Lighthill M.J. and G.B. Whitham, (1955). On Kinematic Waves: a theory of traffic flow on long crowded reads. *Proceedings of the Royal Society*. **229A**, pp. 317-345.

Merchant D.K., G.L.Nemhauser (1978). A model and an algorithm for the dynamic traffic assignment problem. *Transp. Sci.*, **12**, N 3.

Ran B., D.E. Boyce, L.J. Leblanc (1993). A new class of instantaneous dynamic user optimal traffic assignment model. *Op. Res.*, **41**, N 1. pp. 192-202.

Smith M.J. (1993). A new dynamic traffic model and the existence and calculation of dynamic user equilibria on congested capacity-constrained road networks. *Transp. Res.*, **27B**, N 1. pp. 49-63.

Wie B.W. (1995). A Discrete Time, Nested Cost Operator Approach to the Dynamic Network User Equilibrium Problem. *Transp. Sci.*, **29-1**.