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Some product graphs with power dominating number at most 2

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ABSTRACT

Let S be a set of vertices of a graph G. Let $M[S]$ be the set of vertices built from the closed neighborhood $N[S]$ of S, by iteratively applying the following propagation rule: if a vertex and all but exactly one of its neighbors are in $M[S]$, then the remaining neighbor is also in $M[S]$. A set S is called a power dominating set of G if $M[S] = V(G)$. The power domination number $\gamma_P(G)$ of G is the minimum cardinality of a power dominating set. In this paper, we present some necessary conditions for two graphs G and H to satisfy $1 \le \gamma_P(G * H) \le 2$ for product graphs.

KEYWORDS

Domination; power domination; electric power monitoring; zero forcing

AMS SUBJECT CLASSIFICATION

05C38; 05C76; 05C90

1. Introduction

This paper is devoted to the study of the power domination number of connected graphs introduced in [8]. The notion of power domination in graphs is a dynamic version of domination where a set of vertices (power) dominates larger and larger portions of a graph and eventually dominates the whole graph. The introduction of this parameter was mainly inspired by a problem in the electric power system industry [2]. Electric power networks must be continuously monitored. One usual and efficient way of accomplish this monitoring, consist in placing phase measurement units (PMUs), called PMUs, at selected network locations.

Due to the high cost of the PMUs, their number must be minimized, while maintaining the ability to monitor (i.e. to observe) the entire network. The *power domination problem* consists thus of finding the minimum number of PMUs needed to monitor a given electric power system. In other words, a power dominating set of a graph is a set of vertices that observes every vertex in the graph, following the set of rules for power system monitoring described in [8].

Since it was formally introduced in [17], the power domination number and its variations have generated considerable interest (see, for example, [3, 5, 8–10, 15, 20, 22]).

In this paper, we present a variety of graph families such that all their products have power dominating sets of cardinality at most 2.

All the graphs considered are undirected, simple, finite and (unless otherwise stated) connected. Let v be a vertex of a graph G. The *open neighborhood* of v is $N_G(v) = \{w \in V(G) : vw \in E\}$, and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$ (we will write $N(v)$ and $N[v]$ if the graph G is clear from the context). The *degree* of v is $\deg(v) = |N(v)|$. The minimum degree (resp. maximum degree) of G is $\delta(G) = \min\{\deg(u) : u \in V(G)\}$ (resp. $\Delta(G) = \max\{\deg(u) : u \in V(G)\}$). If $\deg(v) = 1$, then v is said to be a *leaf* of G.

The distance between vertices $v, w \in V(G)$ is denoted by $d_G(v, w)$, or $d(v, w)$ if the graph G is clear from the context. The diameter of G is $\mathrm{diam}(G) = \max\{d(v, w) : v, w \in V(G)\}$. The distance between a vertex $v \in V(G)$ and a set of vertices $S \subseteq V(G)$, denoted by $d(v, S)$, is the minimum of the distances between v and the vertices of S, that is, $d(v, S) = \min\{d(v, w) : w \in S\}$.

Let $W \subseteq V(G)$ be a subset of vertices of G. The *open neighborhood* of W is $N(W) = \cup_{v \in W} N(v)$ and the *closed neighborhood* of W is $N[W] = \cup_{v \in W} N[v]$.

Let $u, v \in V(G)$ be a pair of vertices such that $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$, i.e., such that either $N(u) = N(v)$ or $N[u] = N[v]$. In both cases, u and v are said to be *twins*.

Let H and G be a pair of graphs. The graph H is a *subgraph* of G if it can be obtained from G by removing edges and vertices. The graph H is an *induced subgraph* of G if it can be obtained from G by removing vertices. The subgraph of G induced by a subset of vertices W, denoted by $G[W]$, has W as vertex set and $E(G[W]) = \{vw \in E(G) : v \in W, w \in W\}$. The graph H is a *minor* of G if it can be obtained from G by removing vertices and by removing and contracting edges.

A set D of vertices of a graph G is a *dominating set* if $N[D] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set.

Let K_n, $K_{r,n-r}$, $S_n \cong K_{1,n-1}$, P_n, W_n, C_n and F_n denote, respectively, the complete graph, complete bipartite graph, spider, path, wheel, cycle and friendship graph(It is a planar undirected graph with $2n+1$ vertices and $3n$ edges) of order n. For undefined terminology and notation, we refer the reader to [4].

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The concept of *zero forcing* can be described via the following coloring game on the vertices of a given graph $G = (V, E)$. Let U be a proper subset of V. The elements of U are colored black, meanwhile the vertices of $W = V \setminus U$ are colored white. The color change rule is:

If $u \in U$ and exactly one neighbor w of u is white, then change the color of w to black.

In such a case, we denote this by $u \to w$, and we say, equivalentely, that u forces w, that u is a forcing vertex of w and also that $u \to w$ is a force. The *closure* of U, denoted $cl(U)$, is the set of black vertices obtained after the color change rule is applied until no new vertex can be forced; it can be shown that $cl(U)$ is uniquely determined by U (see [1]).

Definition 1 ([1]). A subset of vertices U of a graph G is called a *zero forcing set* of G if $cl(U) = V(G)$.

A *minimum zero forcing set*, a *ZF-set* for short, is a zero forcing set of minimum cardinality. The *zero forcing number* of G, denoted by $Z(G)$, is the cardinality of a ZF-set.

Proposition 2 ([7]). Let G be a graph of order n. Then, $Z(G) = 1$ if and only if G is the path P_n.

A *chronological list of forces* $\mathcal{F}_U$ associated with a set U is a sequence of forces applied to obtain $cl(B)$ in the order they are applied. A *forcing chain* for the chronological list of forces $\mathcal{F}_U$ is a maximal sequence of vertices $(v_1, ..., v_k)$ such that the force $v_i \to v_{i+1}$ is in $\mathcal{F}_U$ for $1 \le i \le k-1$. Each forcing chain induces a distinct path in G, one of whose endpoints is in U; the other is called a terminal. Notice that a zero forcing chain can consist of a single vertex (v_1), and this happens if $v_1 \in U$ and v_1 does not perform a force. Observe also that any two forcing chains are disjoint.

Zero forcing is closely related to power domination, because power domination can be described as a domination step followed by the zero forcing process or, equivalentely, zero forcing can be described as power domination without the domination step. In other words, the power domination process on a graph G can be described as choosing a set $S \subset V(G)$ and applying the zero forcing process to the closed neighbourhood $N[S]$ of S. The set S is thus a power dominating set of G if and only if $N[S]$ is a zero forcing set for G

Definition 3 ([8]). A subset of vertices S of a graph G is called a *power dominating set* of G if $cl(N[S]) = V(G)$.

A *minimum power dominating set*, a *PD-set* for short, is a power dominating set of minimum cardinality. The *power dominating number* of G, denoted by $\gamma_P(G)$, is the cardinality of a PD-set.

Definition 4 ([14]). If G is a graph and $S \subseteq V(G)$, then the sets $(\mathcal{P}^i_{G,1}(S))_{i \ge 0}$ of vertices monitored by S at step i are as follows:

$\mathcal{P}^0_{G,1}(S) = N_G[S]$ (domination step), and $\mathcal{P}^{i+1}_{G,1}(S) = \cup\{N_G[v] : v \in \mathcal{P}^i_{G,1}(S)$ *such that* $|N_G[v] \setminus \mathcal{P}^i_{G,1}(S)| \le 1$ (propagation steps).

Here, we use the simplified definition below instead of the above definition.

Definition 5. If G is a graph and $S_0 = S \subseteq V(G)$, then the sets $S_i (i > 0)$ of vertices monitored by S_0 at step i are as follows:

$S_1 = N_G[S_0]$ (domination step), and

$S_{i+1} = \cup\{N_G[v] : v \in S_i$ *such that* $|N_G[v] \setminus S_i| \le 1$ (propagation steps).

As a straight consequence of these definitions, it is derived both that $\gamma_P(G) \le Z(G)$ and $\gamma_P(G) \le \gamma(G)$.

The vertex set of graph products constructed from graphs G and H is $V(G) \times V(H)$. Let $u = (g, h)$ and $v = (g', h')$ be a pair of vertices of $V(G) \times V(H)$.

- Vertices u and v are adjacent in the *Cartesian product* $G \Box H$ if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$.
- Vertices u and v are adjacent in the *direct product* $G \times H$ if $gg' \in E(G)$ and $hh' \in E(H)$.
- The edge set of the *strong product* $G \boxtimes H$ is $E(G \Box H) \cup E(G \times H)$.

Let $G \star H$ be any of the three graph products. Then, the subgraph of $G \star H$ induced by $g \times V(H)$ is called an H-fiber and denoted by ${}^gH$, meanwhile the subgraph induced by $V(G) \times h$ is called a G-fiber and denoted by G^h. All of these products are associative and all are also commutative.

2. Cartesian product

While a complete classification of graphs G for which $\gamma_p(G) = 1$ is not known yet and it is certainly far for being simple, several authors were able to solve this problem for the Cartesian product of two graphs.

Theorem 6 ([9, 12, 14]). Let G and H be two nontrivial graphs such that $\gamma(G) \le \gamma(H)$. Then, $\gamma_p(G \Box H) = 1$ if and only if either

1. G and H each has order at least four, $\gamma(G) = 1$ and H is a path, or
2. G is either P_2 or P_3 and H can be obtained by amalgamating any vertex of a graph, say D, with $\gamma(D) = 1$ and an end vertex of P_n with $n \ge 1$, or
3. $G \cong C_3$ and the H is a path.

Theorem 7 ([6]). Let $1 \le m \le n$. Then, $\gamma_P(P_m \Box P_n)$

$$= \begin{cases} \lceil \frac{m+1}{4} \rceil, & \text{if} \quad m \equiv 4 (\text{mod } 8) \\ \lceil \frac{m}{4} \rceil, & \text{otherwise.} \end{cases}$$

Theorem 8 ([14]). Let G and H be two graphs.

1. *If* $\gamma(H) = 1$, *then* $\gamma_p(G \Box H) \le Z(G)$.
2. *If* $H \cong P_n$, *then* $\gamma_P(G \Box P_n) \le \gamma(G)$.

Theorem 9 ([9]). Let G and H be two graphs.

1. $max\{\gamma_p(G), \gamma_p(H)\} \le \gamma_p(G \Box H)$.
2. *If H is a tree T, then* $\gamma_p(G) \cdot \gamma_p(T) \le \gamma_p(G \Box T)$.

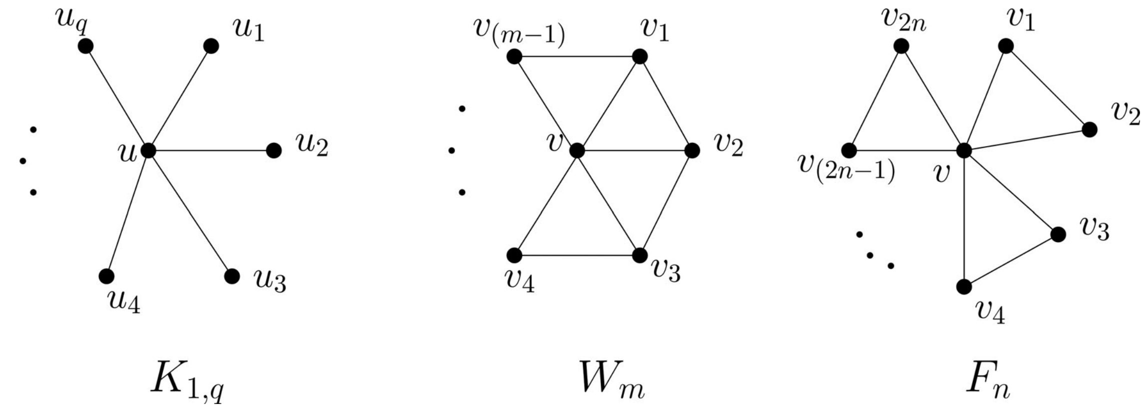

Figure 1. Star, wheel and friendship graph.

Corollary 10 ([14]). *For any graph G, $\gamma_p(G) \leq \gamma_p(G\Box P_2) \leq \min\{\gamma(G), Z(G)\}$.*

Now, we present the theorems that we have obtained.

Theorem 11. *Let G and H be two graphs of order at least four and $\gamma(G) = 1$. $Z(H) = 2$ if and only if $\gamma_p(G\Box H) = 2$.*

Proof. ($\Rightarrow$) According to theorem 8, $\gamma_p(G\Box H) \leq 2$ and according to theorem 6, $\gamma_p(G\Box H) \neq 1$. So $\gamma_p(G\Box H) = 2$. Let $S = \{v\}$ is a γ-set for G and $\{u_1, u_2\}$ is a zero forcing set for H. So it is clear $\{(v, u_1), (v, u_2)\}$ is a γ_p-set for $G\Box H$.

($\Leftarrow$) Let $C = \{g_1, ..., g_r\}$ is the universal vertices set of G. If $\{(g, h), (g', h')\}$ be a γ_p-set for $G\Box H$, then it is clear $g, g' \in C$ and $h \neq h'$. Therefore the propagation occurs from a G-fiber to another G-fiber. This means that the propagation is possible if $Z(H) \leq 2$. In other hand, according to theorem 6, H can not be a path because $\gamma_p(G\Box H) = 2$. According to proposition 2, $Z(H) \neq 1$. So $Z(H) = 2$. □

In [11], the necessary and sufficient condition for $Z(H) = 2$ is expressed.

Theorem 12. *Let G and H graphs such that $\gamma_p(G) = \gamma(G) = 2$ and H is isomorphic to a path, then $\gamma_p(G\Box H) = 2$.*

Proof. Let $\gamma_p(G) = \gamma(G) = 2$ and $\{u_1, u_2\}$ is a $\gamma-$set of G, and also H be isomorphic to a path and $v \in V(H)$ and that $\deg(v) = 1$. So $\{(u_1, v), (u_2, v)\}$ is a γ_p-set for $G\Box H$ and according to theorem 6, $\gamma_p(G\Box H) \neq 1$ then $\gamma_p(G\Box H) = 2$. □

We have kept the following conjecture as a open problem.

Problem 13. Let G and H graphs such that $\gamma_p(G) = \gamma(G) = 2$. If $\gamma_p(G\Box H) = 2$, then H is isomorphic to a path.

3. Direct product

Notice that $K_3 \times K_2 \cong C_6$. Thus, $\gamma_p(K_3 \times K_2) = 1$. Other results are shown below.

Theorem 14 ([13]). *Let K_m and K_n such that $m, n \geq 2$ and $m + n \neq 5$. Then, $\gamma_p(K_m \times K_n) = 2$.*

Theorem 15 ([14]). *For any nontrivial graph G, $\gamma_p(G \times K_2) \leq 2\gamma_p(G)$. The equality holds if G is a bipartite graph.*

Theorem 16 ([13]). *If G and H are graph having each at least 2 universal vertices, then $\gamma_p(G \times H) \leq 2$*

Theorem 17 ([13]). *Let $G = P_n \times C_m$, where n is even and $m \geq 3$. Then*

$$\gamma_P(G) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil, & \text{if } m \text{ is even} \\ \left\lceil \frac{n}{3} \right\rceil, & \text{if } m \text{ is odd} \end{cases}$$

Now, pay attention to the following theorem.

Theorem 18. Let G and H are two connected graphs. If $\gamma_p(G \times H) = 1$, then G or H is P_2.

Proof. Let none of them is not P_2. Without losing generality, let G has 3 vertices $\{v_1, v_2, v_3\}$ and $H \neq P_2$ is an arbitrary graph with vertices set $\{u_1, ..., u_n\}$. So G is $P_3 = (v_1, v_2, v_3)$ or $C_3 = (v_1, v_2, v_3)$. Suppose that $G = C_3$ and $\{(v_1, u_i)\}$ is a power dominating set of $G \times H$. If $N(u_i) = \{u_{i_1}, ..., u_{i_k}\}$, then $S_1 = \{v_2, v_3\} \times N(u_i)$. Now propagation can not continue, because every vertex for example (v_3, u_{i_j}) of S_1 has at least two neighbor $\{v_2\} \times N(u_{i_j})$ and $\{v_1\} \times N(u_{i_j})$. So if $N(u_{i_j}) \subseteq N(u_i)$, then (v_3, u_{i_j}) has no neighbors in $V(G \times H) \setminus S_1$ and if $N(u_{i_j}) \nsubseteq N(u_i)$, then (v_3, u_{i_j}) has at least two neighbor $\{v_2\} \times N(u_{i_j})$ and $\{v_1\} \times N(u_{i_j})$ in $V(G \times H) \setminus S_1$ or has no neighbors in $V(G \times H) \setminus S_1$. So $G \neq C_3$, because if $G = C_3$, then $\gamma_p(G \times H) > 1$.

Now, let $G = P_3 = (v_1, v_2, v_3)$. So power domination set of $G \times H$ is either as $\{(v_1, u_i)\}$ or as $\{(v_2, u_i)\}$. Let $\{(v_1, u_i)\}$ is power domination set of $G \times H$. hence $S_1 = \{v_2\} \times N(u_i)$. If there is $u_j \in N(u_i)$ such that u_j is adjacent with u_k, then (v_2, u_j) is adjacent with $(v_1, u_i), (v_3, u_i), (v_1, u_k)$ and (v_1, u_k) so propagation can not continue. In other hand, if every member of $N(u_i)$ be adjacent only with u_i, then all members of S_1 are adjacent with (v_1, u_i) and (v_3, u_i). So $S_2 = S_1 \cup (v_3, u_i)$ and propagation can not continue because all members of S_2 are not adjacent with none of the members of $V(G) \setminus S_2$. Now let $\{(v_2, u_i)\}$ is power domination set of $G \times H$. hence $S_1 = \{v_1, v_3\} \times N(u_i)$. If every member of $N(u_i)$ be adjacent only with u_i, then none of the members of S_1 are not adjacent with none of the members of $V(G) \setminus S_1$. Otherwise, $S_2 = S_1 \cup (\{v_2\} \times N(N(u_i)))$ but propagation can not continue because every vertex of S_2 either is adjacent with two vertices of $V(G) \setminus S_2$ or is not adjacent with none of the vertices of $V(G) \setminus S_2$. So $\gamma_p(G \times H) \neq 1$. Therefore $G \neq P_3$. □

The converse of this Theorem is not true in general. For example, $\gamma_p(C_4 \times P_2) = 2$ because $C_4 \times P_2 = C_4 \cup C_4$ and we know $\gamma_p(C_4 \times C_4) = 2$.

Look at the three families of graphs in the below;

Theorem 19. *Let n be an arbitrary positive integer and m be an odd number. If $G \in \{C_m, F_n\}$, then $\gamma_p(G \times P_2) = 1$.*

Proof. According to Theorem 17, If m be an odd number, then $\gamma_p(C_m \times P_2) = 1$. Now let $V(P_2) = \{v_1, v_2\}$ and F_n is like Figure 1 with $V(F_n) = \{u, u_1, u_2, ..., u_{2n}\}$.

So $\{(u, v_1)\}$ is a γ_p-set for $F_n \times P_2$ because $S_1 = (V(F_n) \setminus \{u\}) \times \{v_2\}$ and $S_2 = V(F_n \times P_2) \setminus (u, v_2)$ and then $S_3 = V(F_n \times P_2)$. □

We continue this section with a research problem.

Problem 20. Is there another graph G that $G \notin \{C_m, F_n\}$ and $\gamma_p(G \times P_2) = 1$.

We have obtained the following theorems for $\gamma_p = 2$.

Theorem 21. *If $G \in \{W_m, F_n\}$ and m is even, then $\gamma_p(K_{1,q} \times G) = 2$ for $q \geq 2$.*

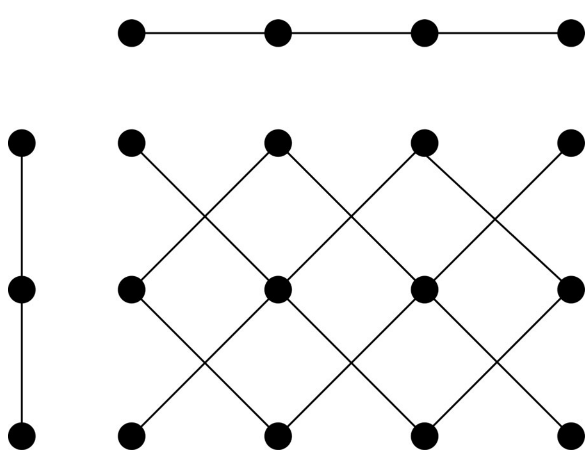

Figure 2. $P_4 \times P_3$.

Proof. According to Theorem 18, If $G \in \{W_m, F_n\}$, then $\gamma_p(K_{1,q} \times G) \neq 1$. According to Figure 1, we claim $\{(u, v), (u, v_1)\}$ is a γ_p- set for $K_{1,q} \times G$.

The first $K_{1,q} \times W_m$, $S_1 = S_0 \cup (V(K_{1,q}) \setminus \{u\}) \times (V(W_m) \setminus \{v\}) \cup (V(K_{1,q}) \setminus \{u\}) \times \{v, v_2, v_{(m-1)}\}$. In other hand, every vertex of $N(u) \times \{v_2\}$ is adjacent with (u, v), (u, v_1) and (u, v_3). So $S_2 = S_1 \cup \{(u, v_3)\}$. $N(u) \times \{v_4\}$ is adjacent with (u, v), (u, v_3) and (u, v_5) and then $S_3 = S_2 \cup \{(u, v_5)\}$. As the same way, $S_{(m-3)} = S_{(m-4)} \cup \{(u, v_{(m-1)})\}$ and $S_{(m-2)} = S_{(m-3)} \cup \{(u, v_2), (u, v_{(m-2)})\}$ and then there is k such that $S_k = V(K_{1,q} \times G)$.

The second $K_{1,q} \times F_n$, $S_1 = S_0 \cup (V(K_{1,q}) \setminus \{u\}) \times (V(F_n) \setminus \{v\}) \cup (V(K_{1,q}) \setminus \{u\}) \times \{v, v_2\}$. every vertex of $(V(K_{1,q}) \setminus \{u\}) \times (V(F_n) \setminus \{v\})$ is adjacent with only one vertex of $V(K_{1,q} \times F_n) \setminus S_1$, for example vertices of $N(u) \times \{v_1\}$ are adjacent with (u, v) and (u, v_2) that $(u, v) \in S_0$. So $S_2 = V(K_{1,q} \times F_n)$. □

Theorem 22. $\gamma_p(F_m \times F_n) = 2$

Proof. According to Theorem 18, $\gamma_p(F_m \times F_n) \neq 1$. We claim $\{(u, v), (u, v_1)\}$ is a γ_p- set for $F_m \times F_n$ that $V(F_m) = \{u, u_1, u_2, ..., u_{2m}\}$ and $V(F_n) = \{v, v_1, v_2, ..., v_{2n}\}$.

$S_1 = S_0 \cup (V(F_m) \setminus \{u\}) \times (V(F_n) \setminus \{v\}) \cup (V(F_m) \setminus \{u\}) \times \{v, v_2\}$. In other hand, every vertex of $(V(F_m) \setminus \{u\}) \times (V(F_n) \setminus \{v\})$ is adjacent with only one vertex of $V(F_m \times F_n) \setminus S_1$, for example vertex $\{u_1, v_1\}$ is adjacent with (u, v), (u_2, v), (u_2, v_2) and (u, v_2) that (u, v), (u_2, v), $(u_2, v_2) \in S_1$. So $S_2 = V(F_m \times F_n)$. □

Note that for $\gamma_p = 2$, having a universal vertex for both graphs can not be a sufficient condition. For example $\gamma_p(K_{1,q} \times K_{1,q}) = 3 \neq 2$ for $q \geq 3$ because $K_{1,q} \times K_{1,q}$ is disjoint union of a star and $K_{q,q}$. Also this condition can not be a necessary condition because according to the figure below $\gamma_p(P_4 \times P_3) = 2$.

4. Strong product

First, we recall the definition of strong product. The vertex set of strong products constructed from graphs G and H is $V(G) \times V(H)$ and the edge set of the strong product $G \boxtimes H$ is $E(G \square H) \cup E(G \times H)$.

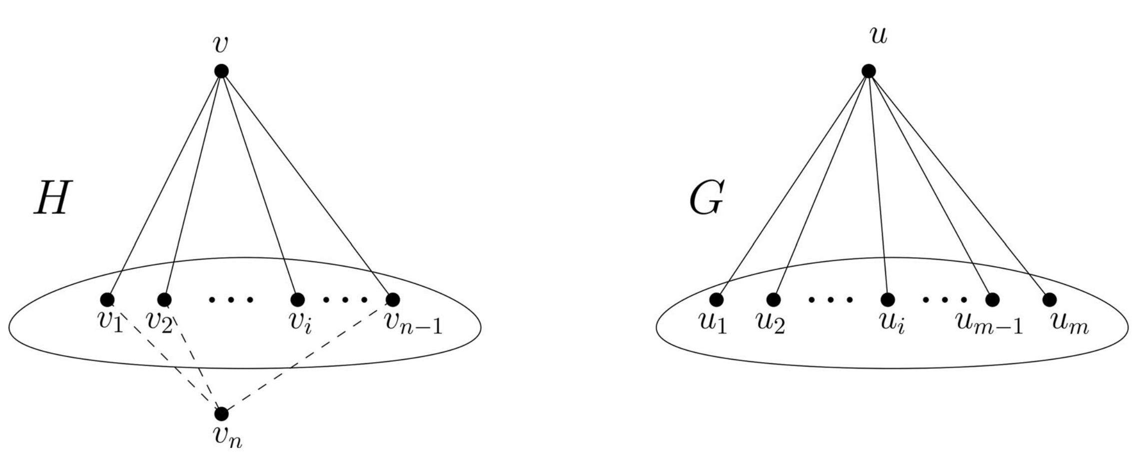

Figure 3. G and H.

Now we present the theorems we have obtained for this product.

Theorem 23. *Let G and H are connected graphs. G and H have at least one universal vertex if and only if* $\gamma_p(G \boxtimes H) = 1$.

Proof. ($\Rightarrow$) Let $\{u_1, u_2, ..., u_n\}$ is vertices set and u_1 is a universal vertex of G and $\{v_1, v_2, ..., v_m\}$ is vertices set and v_1 is a universal vertex of H. We claim $\{(u_1, v_1)\}$ is a γ_p- set of $G \boxtimes H$.

Because of" vertices (u_i, v_j) and (u_r, v_s) are adjacent in $G \boxtimes H$ if either $u_i = u_r$ and $v_j v_s \in E(H)$, or $v_j = v_s$ and $u_i u_r \in E(G)$." hence $\{u_1\} \times (V(H) \setminus \{v_1\}) \cup (V(G) \setminus \{u_1\}) \times \{v_1\} \subseteq S_1$.

In other hand, because of" vertices (u_i, v_j) and (u_r, v_s) are adjacent in $G \boxtimes H$ if $u_i u_r \in E(G)$ and $v_j v_s \in E(H)$." hence $(V(G) \setminus \{u_1\}) \times (V(H) \setminus \{v_1\}) \subseteq S_1$. So $S_1 = V(G \boxtimes H)$.

($\Leftarrow$) Let $\gamma_p(G \boxtimes H) = 1$. (proof by contradiction): suppose that G has a universal vertex u but H does not have a universal vertex. Let such that $N[v] = V(H) \setminus \{v_n\}$. Notice to Figure 3.we have eight states for $\gamma_p(G \boxtimes H)$:

1. $S = \{(u, v)\}$ **5.** $S = \{(u_i, v)\}$
2. $S = \{(u, v_n)\}$ **6.** $S = \{(u_i, v_n)\}$
3. $S = \{u, v_1)\}$ **7.** $S = \{(u_i, v_1)\}$
4. $S = \{(u, v_i)\}$ **8.** $S = \{(u_i, v_i)\}$

That v_i is a vertex that is not adjacent to v_n and u_i is a vertex that is not universal.

State 1. Let $S = \{(u, v)\}$ is a γ_p-set for $G \boxtimes H$. So $S_2 = V(G) \times (V(H) \setminus \{v_n\})$. but every member of $V(G) \times \{v_1, ..., v_s\}$ at least adjacent to two vertices of $V(G) \times \{v_n\}$. For example (u_1, v_1) adjacent to both (u, v_n) and (u_1, v_n). Therefore we can not countinue.

Another states have the same proof. Hence none of the states can not be a γ_p-set for $G \boxtimes H$. □

Theorem 24. *Let* G *and H are connected graphs. If* $\gamma(G) = 2$ *and* $\gamma(H) = 1$, *then* $\gamma_p(G \boxtimes H) = 2$.

Proof. ($\Rightarrow$) Let $\{u_1, u_2, ..., u_n\}$ is vertices set and $\{u_i, u_j\}$ is a dominayion set of G and $\{v_1, v_2, ..., v_m\}$ is vertices set and v_1 is a universal vertex of H. We claim $\{(u_i, v_1), (u_j, v_1)\}$ is a γ_p-set of $G \boxtimes H$.

Because of" vertices (u_i, v_j) and (u_r, v_s) are adjacent in $G \boxtimes H$ if either $u_i = u_r$ and $v_j v_s \in E(H)$, or $v_j = v_s$ and $u_i u_r \in E(G)$." hence $\{u_i, u_j\} \times (V(H) \setminus \{v_1\}) \cup (V(G) \times \{v_1\}) \subseteq S_1$.

In other hand, because of" vertices (u_i, v_j) and (u_r, v_s) are adjacent in $G \boxtimes H$ if $u_i u_r \in E(G)$ and $v_j v_s \in E(H)$." hence $(N(u_i) \cup N(u_j)) \times (V(H) \setminus \{v_1\}) \subseteq S_1$. So $S_1 = V(G \boxtimes H)$. □

The converse of this Theorem is not true in general. For example, $\gamma_p(C_4 \boxtimes C4) = 2$ but $\gamma(C_4) = 2$.

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