Single-Agent and Mean-Field Time-Inconsistent Stopping Problems in Discrete Time

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This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

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Abstract

In this thesis, we first consider single-agent time-inconsistent stopping problems under non-exponential discounting in discrete time with infinite horizon. We extend the iterative approach introduced by Huang and Zhou (2017) to time-inhomogeneous setting and establish the existence of nonstationary subgame perfect Nash equilibria. Under certain continuity assumptions, we further show the existence of a unique optimal equilibrium which dominates any other equilibria pointwisely. Explicit examples of time-homogeneous model with time-inhomogeneous equilibria are also constructed. We then apply the single-agent results to mean field stopping games where each agent plays against other agents as well as against future selves. We construct a single-agent optimal equilibrium for each fixed mean field interaction represented by the proportion of players that have stopped at each time and use this to show the existence of two-layer equilibria in two examples of mean field time-inconsistent stopping games.

Contents

Chapter I

Introduction

Consider a classical optimal stopping problem: given the initial condition of a stochastic process $X_t = x$, we seek a stopping time τ to maximize the following objective function,

$$J(t, x, \tau) := \mathbb{E}^{t,x}[\delta(\tau - t) f(X_\tau)].$$

Under general conditions, there exists an optimal stopping time $\tau^*_{t,x}$ for each (t, x) and there has been abundant and vibrant literature on finding solutions to such problems with the most important one being the Snell Envelop theory. It is well know that when δ is not exponential, i.e., $\delta(i + j) \neq \delta(i)\delta(j)$, the problem is time-inconsistent. That is, stopping strategies that are optimal today may not be optimal from a future's perspective. Mathematically, this is due to the failure of Bellman's principle of optimality. We refer to [24, Section 2.2] and [7, Section 1.3] for more details.

Time-inconsistency is known to arise in optimal stopping problems and more generally, in optimal control. Early studies of time-inconsistency date back to the seminal work by Robert H. Strotz in 1955 [40]. There are many ways time-inconsistency can be introduced in optimal stopping problems. An important case is the use of non-exponential discounting such as the real American option example in [28] and the weighted discounting example in [41]. Another case is when the objective function contains a nonlinear functional of $\mathbb{E}[X]$ as in the mean-variance portfolio selection problem; see [8] for more details.

Under time-inconsistency, Strotz [40] identifies three type of agents, the *naive*, who repeatedly solves the stopping problem for each time and therefore the agent' strategies are always changing and can be considered myopic; the *pre-committed*, who commits to a strategy based on the initial preference throughout the planning horizon and understand that this strategy may no longer be optimal at later stages; the *sophisticated*, who employs the *strategy of consistent planning* which consists of two phases:

(1) Phase I: find all the strategies that the agent has no incentive to deviate from over time. These are the sophisticated strategies;
(2) Phase II: choose the best plan among the sophisticated strategies.

Among them, only the last two types of agents are aware of time inconsistency and only the sophisticated agent takes into account time-inconsistency seriously. For explicit examples on the drawbacks of naive and precommitment strategies in both continuous and discrete time, we refer to [21, Section 1], [24, Section 2.2] and the original work of Strotz [40].

By formulating as *subgame perfect Nash equilibrium* of a game between current and future selves through game theoretic terms, Ekeland and Lazrak [15] gives precise mathematical definition to sophisticated strategies (control) in continuous-time stochastic control problems. Their work has led to vibrant research in time-inconsistent stochastic control. This includes [23, 22, 46, 6, 16] in continuous time and [8] in discrete time. For mean-variance portfolio selection problem, see e.g. [9, 20, 39, 18, 17, 45, 14]. [30] and [19] propose and compare different notions of equilibria.

The development of time-inconsistent stopping problem is more recent. Huang and Nguyen-Huu [24] proposes a general framework for continuous-time models. They characterizes equilibrium stopping regions as fixed-points of an operator and to find equilibria, fixed-point iterations were carried out. Their fixed-point iterative approach are adopted and further developed in [28, 29, 25, 26, 27] as well as [31] which studies a time-inconsistent Dynkin game through an alternating fixed-point iteration. In particular, [28] considers discrete-time models and shows the existence of an optimal equilibrium which dominates any other equilibria anywhere and this is the first time that such dominating subgame perfect Nash equilibrium is shown to exist in the literature of time-inconsistency. A different approach, including [12, 13] is to extend the spike variation technique from [15] in stochastic control to stopping problems. In [5, 3] different definitions of equilibria along these two different paths are collected and precise results are established regarding relations between different notions of equilibria. Let us also mention [4] which studies a mean-standard deviation problem and [41] which studies problems under the so-called the weighted discounting. Finally, the survey article [21] by He and Zhou provides an excellent and broad review on the classical literature and recent developments in both time-inconsistent stochastic control and stopping problems.

In this thesis we study equilibrium strategies of time-inconsistent problems in single-agent time-inhomogeneous models, as well as mean field stopping games. In Chapter II we consider a single-agent time-inconsistent stopping problem in discrete time under infinite horizon where the state process X is a time-inhomogeneous Markov process taking values in a Polish space $\mathbb{X}$. We assume the discount function is log-subadditive. We first establish the existence of equilibria by using a fixed-point iterative approach. In particular, if we start with $S := (\mathbb{X}, \mathbb{X}, \cdots)$ then the limit of the fixed-point iteration converges and is an equilibrium. We show that if the intersection of all equilibria, denoted by S^* is an equilibrium then it is the unique optimal equilibrium, i.e., an equilibrium that generates higher value than all other equilibria. Then assuming the semi-continuity of the payoff function and the Markov transition kernel, we show S^* is an equilibrium and thus an optimal equilibrium. Furthermore, we provide examples of time-homogeneous model with time-inhomogeneous equilibrium.

In Chapter III, we consider mean field stopping games that are time-inconsistent. In this game, players interact in a mean field structure via the stopped proportion process μ_t which represents the proportion of stopped players by time t. Each player chooses a stopping strategy based on μ. In this way, each player plays against other agents and future selves, and thus looks for a two-layer equilibrium. We call such (two-layer) equilibria *mean field equilibria*, which is a pair (μ, S) where S is a stopping region such that the law of the stopping time induced by S coincides with μ. Furthermore, given a mean field equilibrium, if the stopping region S is optimal with respect to μ, we then call this mean field equilibrium *sharp*. Under this setup, we first construct an optimal equilibrium with respect to fixed μ, denoted by Γ^μ as in Chapter II. Then, we apply the construction of Γ^μ to study two examples of mean field stopping games. The first one is motivated by models of bank run and the second has a different structure. We then give different methods in finding sharp mean field equilibria to each of the two examples. In the example of bank run model, we establish the monotonicity of Γ^μ in μ and device a monotone iterative approach to find a sharp mean field equilibrium. More specifically, we start with the zero process $\mu^0 = (0, 0, \cdots)$ and construct the optimal equilibrium Γ^{μ^0} and denote μ^1 the corresponding stopped proportion process induced by Γ^{μ^0} and continue. We show that the iterative approach converges and the limit $(\mu^\infty, \Gamma^{\mu^\infty})$ is a sharp mean field equilibrium. In the second example, under appropriate conditions, we show the continuity of the mapping $\mu \mapsto \Gamma^\mu$. This continuity enable us to apply the fixed-point theorem and establish the existence of a sharp mean field equilibrium.

This thesis contributes to the literature of time-inconsistent stopping in two ways. First, we extend the results of [28] from time-homogeneous models to time-inhomogeneous ones. We also introduce the running payoff in the objective function. We provide an example where a time-homogeneous model can have time-inhomogeneous equilibrium. Therefore, under our formulation, we can consider a larger set of equilibria even when the model is time-homogeneous comparing to [28]. We also show that there must exist an optimal equilibrium that is time-homogeneous for time-homogeneous models. This justifies the selection of equilibria among time-homogeneous ones in time-homogeneous models in [28] (see Corollary 3.2.10). The consideration of time-inhomogeneous models also enables us to look at time-inconsistent mean field games.

Second, we formulate time-inconsistent mean-field stopping problems and propose a (two-layer) equilibrium as a pair of the process μ and stopping region S. To the best of our knowledge, this is the first time mean field stopping games are studied under time-inconsistency. Under appropriate assumptions we show the existence of sharp mean field equilibria for two models with different structure. In the first example, we are able to find a sharp mean field equilibrium which generates higher value than any other sharp mean field equilibrium, see Theorem 3.4.8 for more details. In the second example, we use the continuity of stopping boundary to establish the existence of equilibria and this approach is novel in the literature of mean-field stopping games.

1.1 Outline of the Thesis

The rest of the thesis is organized as follows. Section 2.1 formulates our single-agent time-inconsistent stopping problem with time-inhomogeneous Markov processes and introduces equilibrium stopping regions and the iterative approach. Section 2.2 provides the existence of an equilibrium and optimal equilibrium. Section 2.3 gives examples concerning time-inhomogeneous equilibria for time-homogeneous models. Section 3.1 motivates our study of mean field time-inconsistent stopping games. Section 3.2 constructs an optimal equilibrium with respect to each stopped proportion process μ. Section 3.3 provides precise definition of mean field equilibria. Section 3.4 studies the bank run model and through the iterative approach, constructs a sharp mean field equilibrium. Section 3.5 shows the existence of sharp mean field equilibria under a different model structure by using a fixed-point theorem in infinite dimensional spaces. Appendices A to C contain proofs of lemmas.

Chapter II

Single-Agent Time-Inconsistent Stopping Problems

In this chapter we develop a general theory for time-inconsistent stopping of time-inhomogeneous Markov processes under non-exponential discounting. In the seminal work by Huang and Zhou [28], the following discrete-time time-inconsistent stopping problem,

$$\mathbb{E}^x[\delta(\tau)f(X_\tau)],$$

was studied under infinite horizon with the assumption that δ is log sub-additive and the process X is a time-homogeneous Markov process. We shall extend the results established in [28] to the case where X is time-inhomogeneous and add running payoff to the objective function. Without the assumed homogeneity of the process X, the problem now depends on the calendar time t and accordingly our problem turns to,

$$\mathbb{E}^{t,x}[\delta(\tau - t)f(X_\tau)].$$

2.1 Model Formulation

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that supports a (time-inhomogeneous) Markov process $X = \{X_t\}_{t\in\mathbb{N}}$ taking values in some Polish space $\mathbb{X}$. Let $\mathcal{B}(\mathbb{X})$ be the family of Borel sets in $\mathbb{X}$, and κ_t the transition kernel of X at time $t \in \mathbb{T} := \mathbb{N} = \{0, 1, 2, 3, ...\}$ (we will sometimes use $\mathbb{T}$ and $\mathbb{N}$ interchangeably). Specifically, for any $x \in \mathbb{X}$ and $B \in \mathcal{B}(\mathbb{X})$,

$$\mathbb{P}(X_{t+1} \in B \,|\, X_t = x) = \int_B Q_t(x, dy) \quad \forall t \in \mathbb{T}, \tag{2.1.1}$$

where Q_t is the transition kernel at time t and allowed to be different at each different time. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{N}}$ be the filtration generated by X, and $\mathcal{T}$ be the collection of all $\mathbb{F}$-stopping times. Denote $\mathcal{T}_t := \{\tau \in \mathcal{T} \,|\, \tau \geq t\}$. For each $t \in \mathbb{T}$ and $x \in \mathbb{X}$ we denote $\mathbb{E}^{t,x}[\cdot] = \mathbb{E}[\cdot \,|\, X_t = x]$. Similarly, we write $X^{t,x}$ to specify its initial condition (t, x) and will often suppress (t, x) in the superscript when there are no confusions or with the presence of the associated expectation operator $\mathbb{E}^{t,x}$.

For any $t \in \mathbb{T}$, $x \in \mathbb{X}$ and $\tau \in \mathcal{T}_t$, the agent considers the following objective function,

$$J(t, x, \tau) := \mathbb{E}^{t,x}\left[\sum_{s=t}^{\tau-1} \delta(s-t) g(X_s) + \delta(\tau - t) f(X_\tau)\right]. \tag{2.1.2}$$

In the above objective function, $f, g : \mathbb{X} \mapsto \mathbb{R}^+ := [0, \infty)$ are interpreted as terminal and running payoff respectively and are assumed to be nonnegative, bounded and Borel measureable. We also have a strictly decreasing discount function $\delta : \mathbb{T} \mapsto [0, 1]$ such that $\delta(0) = 1$, $\lim_{t\to\infty} \delta(t) = 0$. For any $\omega \in \Omega$ such that $\tau(\omega) = \infty$, we set $\delta(\tau) f(X_\tau^{t,x})(\omega) := 0$ for all $(t, x) \in \mathbb{T} \times \mathbb{X}$, which is consistent with the fact that f is bounded and $\lim_{t\to\infty} \delta(t) = 0$. Finally, we assume the discount function δ satisfy the following.

Assumption A.

(A.1) either $\sum_{t=0}^{\infty} \delta(t) < \infty$ or $g \equiv 0$;

(A.2) δ is log sub-additive, i.e.,

$$\delta(i)\delta(j) \leq \delta(i + j) \quad \forall i, j = 0, 1, \ldots . \tag{2.1.3}$$

Remark 2.1.1. Assumption (A.1) together with the boundness of f and g ensures that for each δ and any $\tau \in \mathcal{T}_t$, there exists a constant C such that

$$J(t, x, \tau) < C \quad \forall (t, x) \in \mathbb{T} \times \mathbb{X}.$$

Remark 2.1.2. Functions satisfying Equation (2.1.3) are said to be log-subadditive and it encompasses all discount functions which decay slower than the exponential discount function. For example,

1. the hyperbolic discounting $\delta(t) = 1/(1 + \beta t)$ with $\beta > 0$;
2. the generalized hyperbolic discounting $\delta(t) = 1/(1 + \beta t)^k$ with $\beta, k > 0$;
3. the psudoexponential discounting $\delta(t) = \lambda e^{-\rho_1 t} + (1 - \lambda) e^{-\rho_2 t}$ with $\lambda \in (0, 1)$ and $\rho_1, \rho_2 \in (0, 1)$;
4. the quasi-hyperbolic discounting defined by $\delta(0) = 1, \delta(i) = \beta \rho^i$ with $\beta, \rho \in (0, 1)$;

and of course the exponential discounting.

Remark 2.1.3. In the context of behavioural economics, Equation (2.1.3) also captures *decreasing impatience*. A characterization that individual tends to discount more in the near comparing to the distant future. This is a well-known and widely acknowledged feature in the studies of empirical discounting and is substantiated by considerable evidences, see, for example, a series of studies conducted by and collaborated between renowned economists Thaler, Loewenstein and Prelec [42, 35, 34, 38], among them, Richard Thaler were awarded the Nobel Memorial Prize in Economic Sciences for his contributions to behavioral economics in 2017.

Remark 2.1.4. The hyperbolic discounting does not satisfy (A.1) of Assumption A. Its generalization $\delta(t) := \frac{1}{(1+\beta t)^k}$ with $k > 1$ and the quasi-hyperbolic discount function as well as the pseudoexponential does satisfy the convergence assumption.

2.1.1 Equilibrium and the Iterative Approach. Given the current state $x \in \mathbb{X}$ and $t \in \mathbb{T}$, an agent intends to maximize Equation (2.1.2) by choosing an appropriate stopping time $\tau \in \mathcal{T}_t$, i.e.,

$$\sup_{\tau \in \mathcal{T}_t} J(t, x, \tau). \tag{2.1.4}$$

A continuous-time version of this stopping problem with a diffusion process was studied by Huang and Nguyen-Huu [24] by giving precise mathematical definition of Strotz's *consistent planning* of a *sophisticated* agent through equilibrium stopping policies (in continuous-time) and the iterative approach. This was then equivalently formulated as equilibrium stopping regions in [28]. We will take advantage of this reformulation and apply it to the stopping policy defined in [24] for the time-inhomogeneous case, we reiterate their reasoning.

From Chapter I it is known that the stopping problem Equation (2.1.4) is generally time-inconsistent given the form of the discount function (Equation (2.1.3)). As such the agent may re-evaluate and change stopping times at later times. Therefore, the agent's stopping strategy is not a specific stopping time τ, but a stopping region given by the following definition.

Definition 2.1.5. A set $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ is called a stopping region.

Given a current state $x \in \mathbb{X}$ and time $t \in \mathbb{T}$, the stopping region S determines whether the agent should stop, which is the first time $X_s^{t,x}$ enters S with $s > t$.

Taking into account time-inconsistency and time-inhomogeneity, the agent take $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ as the stopping region, in other words $S = (S_0, S_1, S_2, \cdots)$ where $(S)_i := S_i \in \mathcal{B}(\mathbb{X})$ for all $i \in \mathbb{T}$. Now, at any time $t \in \mathbb{T}$ and given the corresponding state of the process $x \in \mathbb{X}$, the agent tries to answer the game-theoretic question:"assuming that all future selves will follow $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$, what is my best stopping strategy for today in response to that?". It is apparent that the agent has two possible choices: either (i) stop today (at time t) and obtain the value $f(x)$ immediately, or (ii) continue and follows the predetermined stopping region $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ and stop at time ρ, defined by,

$$\rho(t, x, S) := \inf\{s \geq t + 1 : X_s^{t,x} \in (S)_s\}. \tag{2.1.5}$$

This will lead to the expected payoff $J(t,x,\rho(t,x,S))$. Therefore, if the agent were to compare the two payoffs $f(x)$ and $J(t,x,\rho(t,x,S))$ from the two possible choices, we obtain the optimal stopping strategy for today in response to S defined by the operator Θ as,

$$\Theta(S) := (\widetilde{S}_0, \widetilde{S}_1, \widetilde{S}_2, \cdots), \tag{2.1.6}$$

and $(\Theta(S))_t := \widetilde{S}_t := \{x \in \mathbb{X} : f(x) \geq J(t,x,\rho(t,x,S))\}$.

Remark 2.1.6. Note that the operator Θ updates S simultaneously, i.e., the agent is playing a *simultaneous game* comparing to a *sequential game* in [12]. Note here, we cannot update S in a backward manner. This is because we are in an infinite-time horizon and there are no terminal position for us to start updating.

Definition 2.1.7. We say $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ an equilibrium stopping region (or simply equilibrium) if $\Theta(S) = S$. We denote the set of all equilibria by $\mathcal{E}$.

Remark 2.1.8. By definition, $S = (S_0, S_1, \cdots) \in \mathcal{E}$ if and only if for all t,

$$\begin{cases} f(x) \geq J(t,x,\rho(t,x,S)), & \forall x \in (S)_t, \\ f(x) < J(t,x,\rho(t,x,S)), & \forall x \in \mathbb{X} \setminus (S)_t. \end{cases} \tag{2.1.7}$$

From Remark 2.1.8, we see that $\mathcal{E}$ in fact corresponds to the set of all stopping strategies that the agent will actually follow over time. This corresponds to Phase I of consistent planning.

It is apparent from Equation (2.1.6) that Θ is an operator mapping $\mathcal{B}(\mathbb{T} \times \mathbb{X})$ to itself and by Definition 2.1.7 any equilibrium $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ is a fixed point of the operator Θ. We will show in Section 2.2 that the iterative approach defined by,

$$S^\infty := \lim_{n \to \infty} \Theta^n(S), \tag{2.1.8}$$

is well defined under certain assumptions. In particular, when starts from $S := (\mathbb{X}, \mathbb{X}, \cdots)$

the above limit converges and the limit S^∞ is an equilibrium. We will also find a candidate for optimal equilibrium, i.e., an equilibrium that dominates any other equilibrium anywhere on $\mathbb{T}\times\mathbb{X}$. This is Phase II of consistent planning.

For any $A, B \in \mathcal{B}(\mathbb{T}\times\mathbb{X})$, we write $A = (\subseteq) B$ if and only if $(A)_i = (\subseteq)(B)_i$ for all $i \in \mathbb{T}$, $A \neq B$ if there exists an $i \in \mathbb{T}$ such that $(A)_i \neq (B)_i$ and similarly $A \cap B = ((A)_0 \cap (B)_0, (A)_1 \cap (B)_1, \cdots)$. We use $(A)_i$ to denote the i-th component of A (the stopping region at time i) instead of A_i because there are dependencies of stopping regions on other parameters later in the thesis and the use of brackets adds clarity.

2.2 Existence of Equilibria and Optimal Equilibria

In this section, we prove the existence of equilibria and introduce the notion of optimal equilibrium. We then propose a candidate for optimal equilibria by taking the intersection of all equilibria and give sufficient conditions for which this candidate is in fact an equilibrium and thus optimal. Theorems 2.2.2, 2.2.9 and 2.2.10 are the main results of this section.

2.2.1 Existence of Equilibria. In this subsection, we will establish the convergence result for our iterative approach given by Equation (2.1.8). We first show that the operator Θ is non-increasing when $\Theta(S) \subseteq S$ for any $S \in \mathcal{B}(\mathbb{T}\times\mathbb{X})$,i.e., $\Theta^2(S) \subseteq \Theta(S)$.

Lemma 2.2.1. *Suppose Assumption A holds. For any nonempty set $S \in \mathcal{B}(\mathbb{T}\times\mathbb{X})$, if $\Theta(S) \subseteq S$, then*

$$J(t,x,\rho(t,x,S)) \leq J(t,x,\rho(t,x,\Theta(S))) \quad \forall (t,x) \in \mathbb{T}\times\mathbb{X}. \tag{2.2.1}$$

In particular, this implies $\Theta^2(S) \subseteq \Theta(S)$.

Proof. The proof is postponed to Appendix A.1. ■

Now, we are in a position to present the main result of this section, which establishes the convergence for the iterative approach.

Theorem 2.2.2. *Suppose Assumption A holds. For any nonempty set $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ such that $\Theta(S) \subseteq S$,*

$$S^\infty := \bigcap_{n=1}^{\infty} \Theta^n(S) \tag{2.2.2}$$

is an equilibrium.

Since $\Theta(\mathbb{X}) \subseteq \mathbb{X}$ trivially, we have an immediate corollary to the above theorem.

Corollary 2.2.3. *Suppose Assumption A holds. $\mathbb{X}^\infty$ defined as*

$$\mathbb{X}^\infty := \bigcap_{n=1}^{\infty} \Theta^n(\mathbb{X})$$

is an equilibrium.

Proof of Theorem 2.2.2. By previous discussion, Θ is an operator mapping $B(\mathbb{T}\times\mathbb{X})$ to itself. Therefore if $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ then $\Theta(S) \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$, which implies that $\Theta^n(S) \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ for all $n \in \mathbb{N}$. Consequently, we must have $S^\infty \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$. We claim that for all $(t, x) \in \mathbb{T} \times \mathbb{X}$ and $\omega \in \Omega$, the following convergence result holds,

$$\rho(t, x, S^\infty)(\omega) = \lim_{n\to\infty} \rho(t, x, \Theta^n(S))(\omega). \tag{2.2.3}$$

Fix $(t, x) \in \mathbb{T} \times \mathbb{X}$ and $\omega \in \Omega$, by Lemma 2.2.1, $\{\Theta^n(S)\}_{n\in\mathbb{N}}$ is a non-increasing sequence of Borel sets and therefore $\{\rho(t, x, \Theta^n(S))(\omega)\}_{n\in\mathbb{N}}$ is a $\mathbb{T}$-valued nondecreasing sequence. As a result, the limit on the RHS of Equation (2.2.3) is then well-defined, and since $S^\infty \subseteq \Theta^n(S)$ for all n, we have

$$\rho(t, x, S^\infty)(\omega) \geq \lim_{n\to\infty} \rho(t, x, \Theta^n(S))(\omega) =: \zeta(\omega).$$

If $\zeta(\omega) = \infty$ then it is clear that Equation (2.2.3) holds trivially. On the other hand if $\zeta(\omega) \in \mathbb{N} < \infty$, then there must exist some $N \in \mathbb{N}$ large enough such that $\rho(t, x, \Theta^n(S))(\omega) = \zeta(\omega)$ for all $n \geq N$. By the definition of $\rho(t, x, \cdot)$, this implies that,

$$X^x_\zeta(\omega) \in (\Theta^n(S))_{\zeta(\omega)}, \quad \forall n \geq N$$

Hence, by the monotone property of $\{\Theta^n(S)\}_{n\in\mathbb{N}}$, we have,

$$X_\zeta^x(\omega) \in \bigcap_{n=1}^{\infty} (\Theta^n(S))_{\zeta(\omega)} = (S^\infty)_{\zeta(\omega)}.$$

This shows that $\rho(t, x, S^\infty(\omega)) \leq \zeta(\omega)$. Combining the two inequalities we prove Equation (2.2.3).

Fix $t \in \mathbb{T}$. For any $x \in (S^\infty)_t$, we have $x \in (\Theta^n(S))_t$ for all $n \in \mathbb{N}$, therefore, by definition,

$$f(x) \geq J(t, x, \rho(t, x, \Theta^{n-1}(S))), \quad \forall n \in \mathbb{N}.$$

Taking $n \to \infty$ on the RHS, we have

$$f(x) \geq \lim_{n\to\infty} J(t, x, \rho(t, x, \Theta^{n-1})) = J(t, x, \rho(t, x, S^\infty)),$$

where we applied the dominated convergence theorem to take the limit under the expectation given by (A.1) of Assumption A. Thus, by Remark 2.1.8, we must have $x \in (\Theta(S^\infty))_t$, and since this is true for any t, we conclude that $S^\infty \subseteq \Theta(S^\infty)$. On the other hand, if $x \notin (S^\infty)_t$, then there must exists some $N \in \mathbb{N}$ such that $x \notin (\Theta^n(S))_t$ for all $n \geq N$, again, by the definition of Θ, this implies that

$$f(x) < J(t, x, \rho(t, x, \Theta^{n-1}(S))), \quad \forall n \geq N.$$

Consequently,

$$\begin{aligned} f(x) &< J(t, x, \rho(t, x, \Theta^{n-1}(S))) \\ &\leq \lim_{m\to\infty} J(t, x, \rho(t, x, \Theta^{m-1}(S))) \\ &= J(t, x, \rho(t, x, S^\infty)), \end{aligned}$$

where we invoked the dominated convergence theorem again. This implies that $x \notin (\Theta(S^\infty))_t$

and since the above works for all t, we must have $\Theta(S^\infty) \subseteq S^\infty$. Combining the two subset inclusions, we have a fixed point S^∞ such that $\Theta(S^\infty) = S^\infty$ hence S^∞ is an equilibrium. ■

Theorem 2.2.2 gives us the existence of an equilibrium. However, as demonstrated in [28] as well as [32, 1], there usually exist multiple equilibria. As we shall see in the next subsection, under rather general conditions, there exists an unique optimal equilibrium.

2.2.2 Optimal Equilibria and Uniqueness. In this subsection, we shall address the problem of selecting an optimal equilibrium, an equilibrium that dominates all other equilibria on $\mathbb{T} \times \mathbb{X}$. Then under appropriate continuity assumptions we prove the existence of an optimal equilibrium. For any $S \in \mathcal{E}$, we define the associated value function by,

$$V(t,x,S) := f(x) \vee J(t,x,\rho(t,x,S)) \quad \forall (t,x) \in \mathbb{T} \times \mathbb{X}. \tag{2.2.4}$$

Then by Definition 2.1.7 and Remark 2.1.8, we see that,

$$V(t,x,S) = \begin{cases} f(x) & \text{if } x \in (S)_t, \\ J(t,x,\rho(t,x,S)) & \text{if } x \in \mathbb{X} \setminus (S)_t. \end{cases} \tag{2.2.5}$$

Therefore, we can express the value function V as,

$$V(t,x,S) = J(t,x,\rho^*(t,x,S)) \quad \text{with} \quad \rho^*(t,x,S) := \inf\{s \geq t : X_s^{t,x} \in (S)_s\}. \tag{2.2.6}$$

Definition 2.2.4 (optimal equilibrium)**.** We say $S^* \in \mathcal{E}$ is an optimal equilibrium if for any $S \in \mathcal{E}$,

$$V(t,x,S^*) \geq V(t,x,S) \quad \forall (t,x) \in \mathbb{T} \times \mathbb{X}.$$

Proposition 2.2.5. *If $S^* \in \mathcal{E}$ is an optimal equilibrium, then $S^* = \bigcap_{S \in \mathcal{S}} S$.*

Proof. It is straightforward to see $\bigcap_{S \in \mathcal{E}} S \subseteq S^*$. To prove the other inclusion, suppose there exists $S \in \mathcal{E}$ such that $S^* \not\subseteq S$, in other words, there exists (at least one) $t \in \mathbb{T}$ such that $(S^*)_t \not\subseteq (S)_t$. Fix such a t, then for any $x \in (S^*)_t \setminus (S)_t$, from Equation (2.2.5) and Remark 2.1.8, we have

$$V(t, x, S^*) = f(x) < J(t, x, \rho(t, x, S)) = V(t, x, S),$$

which contradicts the optimality of S^*. ■

The above Proposition says that should an optimal equilibrium exist, it has to be unique. Next, we find a way to construct a better equilibrium if we have two or more old equilibria. The next Lemma says that for any equilibrium $S, R \in \mathcal{E}$, $\Theta(S \cap R) \subseteq S \cap R$. Then we can apply Theorem 2.2.2 and get equilibrium $\langle S \cap R \rangle^\infty := \Theta^\infty(S \cap R)$ through the iterative approach. Moreover, this new equilibrium dominates S and T simultaneously everywhere on $\mathbb{T} \times \mathbb{X}$, i.e.,

$$V(t, x, \langle S \cap R \rangle^\infty) \geq V(t, x, S) \ \vee \ V(t, x, R) \quad \forall (t, x) \in \mathbb{T} \times \mathbb{X}. \tag{2.2.7}$$

Lemma 2.2.6. *Suppose Assumption A holds. For any $S, R \in \mathcal{E}$,*

$$J(t, x, \rho(t, x, S \cap R)) \geq J(t, x, \rho(t, x, S)) \vee J(t, x, \rho(t, x, R)) \quad \forall (t, x) \in \mathbb{T} \times \mathbb{X}. \tag{2.2.8}$$

In particular, this implies $\Theta(S \cap R) \subset S \cap R$.

Proof. The proof is postponed to Appendix A.2. ■

Using Lemma 2.2.6 we can establish a partial converse to Proposition 2.2.5, we have,

Proposition 2.2.7. *Suppose Assumption A holds. If $S^* := \bigcap_{S \in \mathcal{E}} S$ is an equilibrium, then it is optimal.*

Proof. Let $\hat{S} \in \mathcal{E}$ be arbitrary. Then by Lemma 2.2.6 we have,

$$V(t, x, \hat{S}) = f(x) \ \vee \ J(t, x, \rho(t, x, \hat{S}))$$

$$\leq f(x) \vee J(t,x,\rho(t,x,S^* \cap \hat{S})) = f(x) \vee J(t,x,\rho(t,x,S^*)) = V(t,x,S^*)$$

for all $(t,x) \in \mathbb{T} \times \mathbb{X}$. Hence, S^* is optimal by the arbitrariness of $\hat{S}$. ■

Now, assuming under some general continuity conditions, we are in a position to establish the existence results for the optimal equilibrium. In particular, we show that the candidate choice $S^* := \bigcap_{S \in \mathcal{E}} S$ discussed in Proposition 2.2.5 is in fact the optimal equilibrium.

Recall that a function $f : \mathbb{X} \to \overline{\mathbb{R}}$ is said to be *upper-semicontinuous* ($u.s.c$) on $\mathbb{X}$ if for all $x \in \mathbb{X}$ and every $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$ with $x_n \to x$, $\limsup_{n\to\infty} f(x_n) \leq f(x)$. It is *lower-semicontinuous* ($l.s.c$) on $\mathbb{X}$ if $\liminf_{n\to\infty} f(x_n) \geq f(x)$. It is easy to see that the indicator function of a set $B \subset \mathbb{X}$, $\mathbb{1}_B$ is l.s.c if and only if B is open and u.s.c if and only if B is closed.

Similarly, we say a transition (Markov) kernel λ (e.g. Equation (2.1.1)) is $l.s.c$ under the weak* topology if for all $x \in \mathbb{X}$ and every $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$ with $x_n \to x$ and any bounded Borel measurable function $\varphi : \mathbb{X} \to \mathbb{R}$, we have,

$$\liminf_{n\to\infty} \int_{\mathbb{X}} \varphi(y)\,\lambda(x_n, dy) \geq \int_{\mathbb{X}} \varphi(y)\,\lambda(x, dy). \tag{2.2.9}$$

Remark 2.2.8. Suppose λ admits a probability density function, i.e., $\lambda(x,dy) = \eta(x,y)\,dy$. We claim that if for each $y \in \mathbb{X}$, $x \mapsto \eta(x,y)$ is l.s.c then λ is l.s.c under the weak* topology. Indeed, recall that for every fix $x \in \mathbb{X}$, $\lambda(x,\cdot)$ is a probability measure, then by a variation of Fatou's lemma we have,

$$\liminf_{n\to\infty} \int_{\mathbb{X}} \varphi(y)\eta(x_n,y)\,dy \geq \int_{\mathbb{X}} \liminf_{n\to\infty} \varphi(y)\eta(x_n,y)\,dy \geq \int_{X} \varphi(y)\eta(x,y)\,dy,$$

for any bounded measurable function φ and $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to x$. This shows that,

$$\liminf_{n\to\infty} \int_{\mathbb{X}} \varphi(y)\,\lambda(x_n, dy) \geq \int_{\mathbb{X}} \varphi(y)\,\lambda(x, dy),$$

that is, λ is l.s.c under the weak* topology.

Theorem 2.2.9. *Suppose Assumption A holds, f is u.s.c and the transition (Markov) kernel κ_t in Equation (2.1.1) is l.s.c under the weak* topology for all $t \in \mathbb{T}$. Then, $S^* =: \bigcap_{S\in\mathcal{S}} S$ is the optimal equilibrium.*

Proof. For any $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$, the payoff function $J(t, x, \rho(t, x, S))$ can be written as,

$$J(t, x, \rho(t, x, S)) = \int_{\mathbb{X}} \Phi_t(y, S) Q_t(x, dy),$$

where

$$\Phi_t(y, S) := \mathbb{E}^{t,y} \left[\sum_{s=t}^{\rho^*(t,y,S)} \delta(s - t) g(X_s) + \delta(\rho^*(t, y, S) + 1) f(X_{\rho^*(t,y,S)}) \right].$$

Therefore, by definition of lower semicontinuity and Equation (2.2.9), we see that $x \mapsto J(t, x, \rho(t, x, S))$ is l.s.c. Since $S \in \mathcal{E}$, by the definition of equilibrium and the fact that J is l.s.c in x and f is u.s.c,

$$(S)_t = (\Theta(S))_t = \{x \in \mathbb{X} : f(x) \geq J(t, x, \rho(t, x, S))\}$$

is a closed subset of $\mathbb{X}$ for all $t \in \mathbb{T}$. As an arbitrary intersection of closed set, $(S^*)_t = \bigcap_{S\in\mathcal{E}} (S)_t$ is also closed and therefore Borel measurable for all $t \in \mathbb{T}$.

We know that $\mathbb{1}_{(S)_t}$ is u.s.c given $(S)_t$ is closed for all $S \in \mathcal{E}$. By the virtue of [2, Proposition 4.1] there exists a countable subset $\{S_t^n\}_{n\in\mathbb{N}}$ of $\mathcal{E}$ for each $t \in \mathbb{T}$ such that,

$$\mathbb{1}_{(S^*)_t} = \inf_{S\in\mathcal{E}} \mathbb{1}_{(S)_t} = \inf_{n\in\mathbb{N}} \mathbb{1}_{(S_t^n)_t}, \tag{2.2.10}$$

which implies that $(S^*)_t = \bigcap_{n\in\mathbb{N}} (S_t^n)_t$. By taking the countable union of all such countable subset $\bigcup_{t\in\mathbb{T}} \{S_t^n\}_{n\in\mathbb{N}}$ then we have another countable subset, say $\{S^n\}_{n\in\mathbb{N}}$ of $\mathcal{E}$ such that $S^* = \bigcap_{n\in\mathbb{N}} S^n$. By proceeding as in the discussion below Proposition 2.2.5, we first let $R^1 := S^1$ and $R^2 := \langle R^1 \cap S^2 \rangle^\infty$. Then $R^2 \in \mathcal{E}$ such that $R^2 \subseteq R^1 \cap S^2 \subseteq S^1 \cap S^2$ and $J(t, x, \rho(t, x, R^2)) \geq J(t, x, \rho(t, x, R^1))$ for all $(t, x) \in \mathbb{T} \times \mathbb{X}$. Next, let $R^3 := \langle R^2 \cap S^3 \rangle^\infty$, then similarly, $R^3 \in \mathcal{E}$ such that $R^3 \subseteq R^2 \cap S^3 \subseteq S^1 \cap S^2 \cap S^3$ and $J(t,x,\rho(t,x,R^3)) \geq J(t,x,\rho(t,x,R^2))$ for all $(t,x) \in \mathbb{T} \times \mathbb{X}$. Repeating this procedure recursively for all $n \in \mathbb{N}$, we get a sequence $\{R^n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$ with $R^n \subseteq R^{n-1} \cap S^n \subseteq S^1 \cap \cdots \cap S^n$ and $J(t,x,\rho(t,x,R^n)) \geq J(t,x,\rho(t,x,R^{n-1}))$ for all $n \in \mathbb{N}$ and $(t,x) \in \mathbb{T} \times \mathbb{X}$. Therefore, we have,

$$S^* = \bigcap_{S\in\mathcal{E}} S \subseteq \bigcap_{n\in\mathbb{N}} R^n \subseteq \bigcap_{n\in\mathbb{N}} S^n = S^*, \tag{2.2.11}$$

hence $S^* = \bigcap_{n\in\mathbb{N}} R^n$. By Proposition 2.2.5 it remains to show that S^* is indeed an equilibrium, this can be achieved by replacing $\Theta^n(S)$ with R^n in the proof of Theorem 2.2.2. Therefore S^* is the optimal equilibrium. ■

With the help of [2, Proposition 4.1], the existence of optimal equilibrium still stands if we replace the upper-semicontinuity of f with lower-semicontinuity.

Theorem 2.2.10. *Suppose Assumption A holds, f is l.s.c and the transition (Markov) kernel κ_t in Equation (2.1.1) is l.s.c under the weak* topology for all $t \in \mathbb{T}$. Then, $S^* := \bigcap_{S\in\mathcal{S}} S$ is the optimal equilibrium.*

Proof. As shown before, $x \mapsto J(t,x,\rho(t,x,S))$ is l.s.c for any $S \in \mathcal{B}(\mathbb{T}\times\mathbb{X})$ and any $t \in \mathbb{T}$. Since f is assumed to be l.s.c as well, we must have for all $x \in \mathbb{X}$ and every $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$ with $x_n \to x$,

$$\begin{aligned}
\liminf_{n\to\infty} V(t,x_n,S) &= \liminf_{n\to\infty} f(x_n) \vee J(t,x_n,\rho(t,x_n,S)) \\
&\geq \liminf_{n\to\infty} f(x_n) \vee \liminf_{n\to\infty} J(t,x_n,\rho(t,x_n,S)) \\
&\geq f(x) \vee J(t,x,\rho(t,x,S)),
\end{aligned}$$

thus $x \mapsto V(t,x,S)$ is also l.s.c for all $t \in \mathbb{T}$ and any $S \in \mathcal{B}(\mathbb{T}\times\mathbb{X})$. Again, by the virtue of [2, Proposition 4.1], there exists a sequence $\{S^n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that,

$$\sup_{S\in\mathcal{E}} V(t,x,S) = \sup_{n\in\mathbb{N}} V(t,x,S^n) \quad \forall (t,x) \in \mathbb{T}\times\mathbb{X}. \tag{2.2.12}$$

Now let $R^0 := \mathbb{X}$ and $R^1 := S^1$ and by the same procedure in the previous Theorem we have, $R^2 := \langle R^1 \cap S^2 \rangle^\infty$ is such that $R^2 \in \mathcal{E}$ and $R^2 \subseteq R^1 \cap S^2$ with $J(t,x,\rho(t,x,R^2)) \geq J(t,x,\rho(t,x,R^1))$ for all $(t,x) \in \mathbb{X}$. Next, we define $R^3 := \langle R^2 \cap S^3 \rangle^\infty$, then $R^3 \in \mathcal{E}$ and $R^3 \subseteq R^2 \cap S^3$ with $J(t,x,\rho(t,x,R^3)) \geq J(t,x,\rho(t,x,R^2))$ for all $(t,x) \in \mathbb{T} \times \mathbb{X}$. We repeat this recursively for all $n \in \mathbb{N}$, we get a sequence $\{R^n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $R^n \subseteq R^{n-1} \cap S^n$ with $J(t,x,\rho(t,x,R^n)) \geq J(t,x,\rho(t,x,R^{n-1}))$ for all $n \in \mathbb{N}$. By replacing $\Theta^n(S)$ with R^n in the proof of Theorem 2.2.2, it is easy to see that $R^* := \bigcap_{n\in\mathbb{N}} R^n \in \mathcal{E}$ such that $J(t,x,\rho(t,x,R^*)) \geq J(t,x,\rho(t,x,R^n))$ for all $n \in \mathbb{N}$ and all $(t,x) \in \mathbb{T} \times \mathbb{X}$. By Equation (2.2.8), we have, for all $(t,x) \in \mathbb{T} \times \mathbb{X}$,

$$\begin{aligned} J(t,x,\rho(t,x,R^*)) &\geq J(t,x,\rho(t,x,R^n)) \\ &\geq J(t,x,\rho(t,x,S^n)) \vee J(t,x,\rho(t,x,R^{n-1})) \\ &\geq J(t,x,\rho(t,x,S^n)), \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, $V(t,x,R^*) \geq V(t,x,S^n)$ for all $n \in \mathbb{N}$ and $(t,x) \in \mathbb{T} \times \mathbb{X}$. Combine this with Equation (2.2.12) we have,

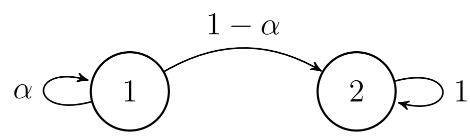
$$\sup_{S\in\mathcal{E}} V(t,x,S) = V(t,x,R^*) \quad \forall (t,x) \in \mathbb{T} \times \mathbb{X}.$$

Therefore, R^* is an optimal equilibrium and by uniqueness, $R^* = \bigcap_{S\in\mathcal{E}} S = S^*$. ■

2.3 Examples

In this section we analyze two examples with the running payoff $g = 0$. In particular, we shall consider some time-homogeneous Markov chains but possibly time-inhomogeneous equilibria. As considered in [28], there exists a unique time-homogeneous optimal equilibrium when the underlying Markov process is time-homogeneous. Moreover, this optimal equilibrium is selected from a set of time-homogeneous equilibria. We will construct an explicit example where there exists a time-inhomogeneous equilibrium for time-homogeneous model and illustrate the fact that using our results, we can consider a larger set of equilibria $\mathcal{E}$ even when the process is time-homogeneous comparing to [28].

2.3.1 General Two-State Markov Chain. We first consider a two state time-homogeneous Markov Chain given by the below transition diagram with state space $\mathbb{X} = \{1, 2\}$. We set $f(1) = 1, f(2) = 2$ and use the quasi-hyperbolic discounting $\delta(0) = 1$ and $\delta(i) = \beta\rho^i$ for $i \geq 1$ and $\beta, \rho \in (0, 1)$, we will choose the values of β, ρ and α later.



We claim that,

$$S = (\{1, 2\}, \{2\}, \{2\}, \{2\}, \{2\}, \cdots) \tag{2.3.1}$$

is an equilibrium. Obviously for all t we have $f(2) = 2 \geq 2\delta(1)$ and hence state 2 is always included in any equilibrium. For $t = 0$, state 1 must satisfy,

$$\begin{aligned} f(1) = 1 &\geq (1-\alpha)2\delta(1) + \alpha(1-\alpha)2\delta(2) + \alpha^2(1-\alpha)2\delta(3) + \cdots \\ &= 2(1-\alpha)\sum_{i=1}^{\infty} \alpha^{i-1}\beta\rho^i \\ &= \frac{2(1-\alpha)\beta\rho}{1-\alpha\rho}. \end{aligned} \tag{2.3.2}$$

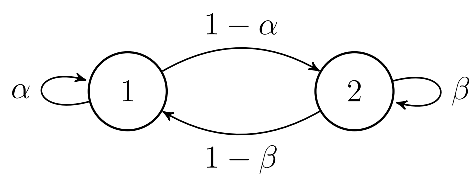
Similarly, when $t \geq 1$, state 1 must also satisfy,

$$\begin{aligned} f(1) = 1 &\leq (1-\alpha)2\delta(1) + \alpha(1-\alpha)2\delta(2) + \alpha^2(1-\alpha)2\delta(3) + \cdots \\ &= 2(1-\alpha)\sum_{i=1}^{\infty} \alpha^{i-1}\beta\rho^i \\ &= \frac{2(1-\alpha)\beta\rho}{1-\alpha\rho}. \end{aligned} \tag{2.3.3}$$

Choosing $\alpha = 0.5$, $\beta = 0.75$ and $\rho = 0.8$, Equation (2.3.2) evaluates to $f(1) = 1 \geq 1$ and Equation (2.3.3) evaluates $f(1) = 1 \leq 1$. In particular, no strict inequality can be achieved for both equations regardless of the choices of the parameters in the hyperbolic discount function and the transition probabilities. Equation (2.3.1) describes a time-inhomogeneous equilibrium, in the sense that not all $(S)_i$ are identical, in this case $(S)_0 = \{1,2\} \neq \{2\} = (S)_n$ for all $n \geq 1$. However, Equation (2.3.1) is somewhat trivial. Ideally, we would like to have an equilibrium where the set $\{1,2\}$ and $\{2\}$ are alternating, i.e.,

$$S = (\{1,2\},\{2\},\{1,2\},\{2\},\{1,2\},\{2\},\cdots) \tag{2.3.4}$$

As we shall see, it is not possible to have Equation (2.3.4) an equilibrium with discount function δ satisfying (A.2) of Assumption A (Equation (2.1.3)) other than the exponential discounting function, i.e., $\delta(i) = \rho^i$ for some $\rho \in (0,1)$. Let us consider a general two state time-homogeneous Markov Chain represented by the following transition diagram,



Without degenerating the example we assume $f(1) = \kappa < f(2) = \nu$ and $\alpha, \beta \in (0,1)$. Under the current setup, $\{2\}$ is always an equilibrium and $\{\emptyset\}$ and $\{1\}$ can never be equilibria by themselves, that leave us the possibility of $\{1,2\}$. We claim that $\{1,2\}$ cannot appear more than once under non-exponential discounting. That is, $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ of the following form,

$$S = (\{1,2\}, \underbrace{\{2\},\{2\},\cdots,\{2\},\{2\}}_{\text{sequence of } n \text{ many } \{2\}}, \{1,2\},\{2\},\{2\},\cdots), \tag{2.3.5}$$

is an equilibrium only under the exponential discounting. We prove this by induction on $n = 1,2,3,...$

Proposition 2.3.1. *Suppose Assumption A holds. If S in Equation (2.3.5) is an equilibrium* *then for any* $n \geq 1$, $\delta(i) = \rho^i$ *for all* $0 \leq i \leq n+1$ *where*

$$0 < \rho = \frac{\kappa}{\alpha\kappa + \nu - \alpha\nu} < 1.$$

Proof. First assume $n = 1$ in Equation (2.3.5), i.e., we restrict our attention to

$$S = (S_0, S_1, S_2, \cdots) = (\{1,2\}, \{2\}, \{1,2\}, \cdots). \tag{2.3.6}$$

At $t = 1$, state 1 must satisfy,

$$f(1) = \kappa \leq (\alpha\kappa + \nu - \alpha\nu) \cdot \delta(1). \tag{2.3.7}$$

At $t = 0$, state 1 must satisfy the following inequality,

$$f(1) = \kappa \geq (1-\alpha)\nu\delta(1) + \alpha(\alpha\kappa + \nu - \alpha\nu)\delta(2) \tag{2.3.8}$$

Now, Equation (2.3.7) gives us that $\delta(1) \geq \kappa/(\alpha\kappa + \nu - \alpha\nu)$ and substituting the lower bound $\delta(1) = \kappa/(\alpha\kappa + \nu - \alpha\nu)$ into Equation (2.3.8) and after some algebraic manipulation we have,

$$\delta(2) \leq \left(\frac{\kappa}{\alpha\kappa + \nu - \alpha\nu}\right)^2 \tag{2.3.9}$$

Suppose that $\delta(1)$ is chosen such that it is strictly larger than $\rho = \kappa/(\alpha\kappa + \nu - \alpha\nu)$ then the above implies that $\delta(2)$ must be strictly smaller than ρ^2 which then violates (A.2) of Assumption A, therefore when $n = 1$, the only choice for δ is the exponential discounting, i.e.,

$$\delta(2) = (\delta(1))^2 = \left(\frac{\kappa}{\alpha\kappa + \nu - \alpha\nu}\right)^2.$$

Next suppose the proposition holds for $n-2$, i.e., $S = (\{1,2\}, \underbrace{\{2\}, \{2\}, \cdots, \{2\}, \{2\}}_{\text{sequence of } n-2 \text{ many } \{2\}}, \{1,2\}, \cdots)$, is only an equilibrium when

$$\delta(i-1) = (\delta(1))^{i-1} = \left(\frac{\kappa}{\alpha\kappa + \nu - \alpha\nu}\right)^{i-1}, \quad \forall i \in \{1, 2, ..., n-2\}. \tag{2.3.10}$$

We prove the case for $n-1$, at $t=1$ state 1 must satisfy the following inequality,

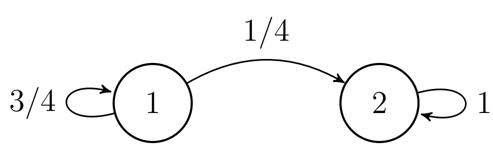
$$f(1) = \kappa \geq \sum_{k=0}^{n-2} \alpha^k (1-\alpha)\nu \cdot \delta(k+1) + \alpha^{n-1}(\alpha\kappa + \nu - \alpha\nu) \cdot \delta(n). \tag{2.3.11}$$

Substituting Equation (2.3.10) into Equation (2.3.11) we have,

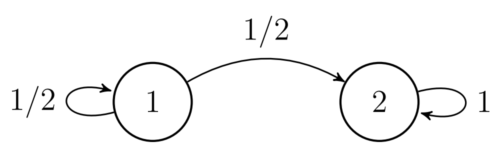
$$\begin{aligned} f(1) = \kappa &\geq \sum_{k=0}^{n-2} \alpha^k (1-\alpha)\nu \cdot \delta(k+1) + \alpha^{n-1}(\alpha\kappa + \nu - \alpha\nu) \cdot \delta(n) \\ &= \kappa - \alpha^n \kappa \left(\frac{\kappa}{\alpha\kappa + \nu - \alpha\nu}\right) - (\alpha - 1)\alpha^{n-1}\nu \left(\frac{\kappa}{\alpha\kappa + \nu - \alpha\nu}\right)^n \\ &\quad + \alpha^{n-1}(\alpha\kappa + \nu - \alpha\nu) \cdot \delta(n). \end{aligned}$$

Solving the above inequality we have, $\delta(n) \leq (\kappa/(\alpha\kappa + \nu - \alpha\nu))^n$. Under (A.2) of Assumption A the only possible choice of $\delta(n)$ is $(\kappa/(\alpha\kappa + \nu - \alpha\nu))^n$. Hence completing the proof. ■

Remark 2.3.2. Recall the time-homogeneous example given by Equation (2.3.1) which admits multiple equilibria including $(\{2\}, \{2\}, \{2\}, \{2\}, \{2\}, \cdots)$ and $(\{1, 2\}, \{2\}, \{2\}, \{2\}, \{2\}, \cdots)$. We now modify the process to be time-inhomogeneous so that the only equilibrium is given by $(\{1, 2\}, \{2\}, \{2\}, \{2\}, \{2\}, \cdots)$. Consider the following transition diagram for $t = 0$,



and when $t \geq 1$ the transition diagram is given by



Similarly, we set $f(1) = 1, f(2) = 2$ and use the quasi-hyperbolic discounting $\delta(0) = 1$ and $\delta(i) = \beta\rho^i$ for $i \geq 1$ and $\beta, \rho \in (0, 1)$. We claim that

$$S = (\{1, 2\}, \{2\}, \{2\}, \{2\}, \{2\}, \cdots) \tag{2.3.12}$$

is the only equilibrium. By choosing $\beta = 0.63$ and $\rho = 0.9$. We have at $t = 0$,

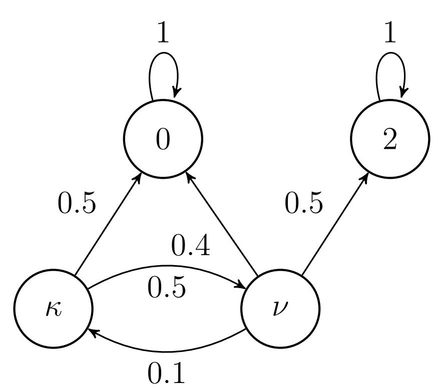
$$\begin{aligned} f(1) = 1 &> \frac{1}{4} \cdot 2\delta(1) + \frac{3}{4} \cdot \frac{1}{2} \cdot 2\delta(2) + \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 2\delta(3) + \cdots \\ &= \frac{1}{4} \cdot 2\delta(1) + \sum_{i=1}^{\infty} \frac{3}{4} \cdot \left(\frac{1}{2}\right)^i \cdot 2\delta(i+1) = 0.97936. \end{aligned}$$

When $t \geq 1$ we have that

$$\begin{aligned} f(1) = 1 &< \frac{1}{2} \cdot 2\delta(1) + \left(\frac{1}{2}\right)^2 \cdot 2\delta(2) + \cdots \\ &= \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \cdot 2\delta(i) = 1.03091. \end{aligned}$$

Here strict inequality has been achieved and this implies $S = (\{1, 2\}, \{2\}, \{2\}, \{2\}, \{2\}, \cdots)$ is the only equilibrium.

2.3.2 A Four-State Example. We consider a model with a four-state time-homogeneous Markov chain model, and show that it has a time-inhomogeneous equilibrium. Consider the following transition diagram,



We use the linear payoff $f(x) = x$ and set $f(\kappa) = \kappa$, $f(\nu) = \nu$, $f(0) = 0$ and $f(2) = 2$. We use the quasi-hyperbolic discount function $\delta(0) = 1$ and $\delta(i) = \beta\rho^i$ for $i \geq 1$ with $\beta, \rho \in (0, 1)$. We will decide the value of κ, ν, β and ρ later. We claim that,

$$S = (\{2\}, \{\nu, 2\}, \{2\}, \{\nu, 2\}, \{2\}, \{\nu, 2\}, \cdots) \tag{2.3.13}$$

is an equilibrium. First observe that state 2 is always an equilibrium as $f(2) = 2 > 2\delta(1)$ and state 0 is always excluded in any equilibrium since $f(0) = 0$. Choose $f(\kappa) = \kappa < \min\{0.5\nu\delta(1), 0.5^2 \cdot 2\delta(2)\}$ then any equilibrium cannot contain κ. To show that Equation (2.3.13) is an equilibrium we consider separately when t is even and odd. When t is even, state ν satisfies,

$$\begin{aligned} f(\nu) = \nu &\leq \frac{1}{2} \cdot 2\delta(1) + \left(\frac{1}{10} \cdot \frac{1}{2}\right) \cdot \frac{1}{2} \cdot 2\delta(3) + \left(\frac{1}{10} \cdot \frac{1}{2}\right)^2 \cdot \frac{1}{2} \cdot 2\delta(5) + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{20}\right)^n \delta(2n+1) = \frac{20\beta\rho}{20 - \rho^2}. \end{aligned} \tag{2.3.14}$$

When t is odd, state ν satisfies,

$$f(\nu) = \nu \geq \frac{1}{2} \cdot 2\delta(1) + \frac{1}{20} \cdot \nu\delta(2) = \beta\rho + \frac{\beta\rho^2}{20} \cdot \nu. \tag{2.3.15}$$

To satisfy Equation (2.3.14) and Equation (2.3.15) simultaneously, one possible choice for the parameters is,

$$\nu = 0.89, \quad \beta = 0.95, \quad \rho = 0.9,$$

giving us $f(\nu) = 0.89 < 0.89109$ in Equation (2.3.14) and $f(\nu) = 0.89 > 0.88924$ in Equation (2.3.15).

Recall the associated value function given by Equation (2.2.4) and define the set,

$$H := (\{2\}, \{2\}, \{2\}, \cdots).$$

We claim that H is an equilibrium such that $V(t,x,H) \geq V(t,x,S)$ for all $(t,x) \in \mathbb{T} \times \mathbb{X}$. We need only to consider state κ and state ν. For state ν, we have

$$\begin{aligned} V(t,\nu,H) &= \frac{1}{2} \cdot 2\delta(1) + \left(\frac{1}{10} \cdot \frac{1}{2}\right) \cdot \frac{1}{2} \cdot 2\delta(3) + \left(\frac{1}{10} \cdot \frac{1}{2}\right)^2 \cdot \frac{1}{2} \cdot 2\delta(5) + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{20}\right)^n \delta(2n+1) = 0.89109. \end{aligned}$$

When t is even, we again have

$$V(t,\nu,S) = \sum_{n=0}^{\infty} \left(\frac{1}{20}\right)^n \delta(2n+1) = 0.89109.$$

When t is odd, we have

$$V(t,\nu,S) = f(\nu) = 0.89.$$

Hence $V(t,\nu,H) \geq V(t,\nu,S)$. Similarly for state κ, when t is even,

$$V(t,\kappa,S) = 0.5\nu\delta(1) = 0.38048,$$

and when t is odd,

$$\begin{aligned} V(t,\kappa,S) &= V(t,\kappa,H) \\ &= 0.5(0.5 \cdot 2)\delta(2) + 0.5(0.5 \cdot 0.1)(0.5 \cdot 2)^2\delta(4) + \cdots \\ &= \sum_{n=1}^{\infty} 0.5(0.5 \cdot 0.1)^{n-1}(0.5 \cdot 2)^n \delta(2n) \\ &= 0.40099. \end{aligned}$$

Therefore, $V(t,\kappa,H) \geq V(t,\kappa,S)$ and we have shown that H is an equilibrium with a value function dominating S anywhere on $\mathbb{X} = \{0,\kappa,\nu,2\}$ and at all times. This is expected, since for a time-homogeneous Markov chain, it always has an optimal equilibrium that is time-homogeneous, see Corollary 3.2.10 in the next chapter.

2.4 References and Remarks

The main reference for this chapter is [28]. The ideas and proofs were borrowed and then extended to the time-inhomogeneous model. The transition diagram for the four state example in Section 2.3.2 is borrowed from [37, Example 5.3], however the computation and the theory behind the two examples are very different.

There are a number of ways we could extend the model given by Equation (2.1.2). Taking the advantage that our setup is time-inhomogeneous, both the running payoff and the final payoff can also depends on time t, i.e., in the form of $g(t, X_t)$ and $f(t, X_t)$ and the results and proofs should still apply. This was more or less considered in a different form in Section 3.2 of the next chapter.

We conclude this chapter by remarking on the case where $X = \{X_t\}_{t\in\mathbb{N}\cup\{0\}}$ is a non Markovian process. In some sense, this would complete the study of time-inconsistent stopping in discrete-time with non-exponential discounting. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that supports a path-dependent process $X = \{X_t\}_{t\in\mathbb{T}}$ taking values in some Polish space $\mathbb{X}$. Let $\mathcal{B}(\mathbb{X})$ be the family of Borel sets in $\mathbb{X}$, and Q_t the transition kernel of X at time $t \in \mathbb{T}$. Denote $\mathbb{X}^t$ the t-fold of $\mathbb{X}$ and $\mathbf{X}^t$ is a $t+1$-random vector taking values in $\mathbb{X}^{t+1}$, i.e., for any $\omega \in \Omega$, $\mathbf{X}^t(\omega) = \mathbf{x}^t$ is a realisation of the path of X starting from 0 and up to time t. For any $\mathbf{x}^t \in \mathbb{X}^{t+1}$ and $B \in \mathcal{B}(\mathbb{X})$,

$$\mathbb{P}(X_{t+1} \in B \,|\, \mathbf{X}^t = \mathbf{x}^t) = \int_B Q_t(\mathbf{x}^t, dy) \quad \forall t = 0, 1, \dots .$$

We then consider the objective function,

$$J(t, \mathbf{x}^t, \tau) := \mathbb{E}^{t,\mathbf{x}^t}\left[\sum_{s=t}^{\tau-1} \delta(s-t) g(X_s) + \delta(\tau - t) f(X_\tau)\right],$$

subjecting to some appropriate conditions as in Section 2.1. We follow the same reasoning as in Section 2.1 but the main difference is the form of stopping regions. Let $S = (S_0, S_1, S_2, \cdots)$ be a stopping region. Instead of $S_n \in \mathcal{B}(\mathbb{X})$ we have $S_n \in \mathcal{B}(\mathbb{X}^{n+1})$ for all $n \in \mathbb{T}$ and then the corresponding stopping policy ρ should be

$$\rho(t, \mathbf{x}^t, S) := \inf\{n \geq t+1 : \mathbf{X}^n \in S_n\} \in \mathcal{T}_t,$$

which is the first time greater than t such that the random vector $\mathbf{X}^n$ enters the stopping region S. Accordingly, for any stopping region S, the operator Θ is defined by,

$$\Theta(S) := (\widetilde{S}_0, \widetilde{S}_1, \widetilde{S}_2, \cdots),$$

where $(\Theta(S))_t := \widetilde{S}_t := \{\mathbf{x}^t \in \mathbb{X}^t : f((\mathbf{x}^t)_t) \geq J(t, \mathbf{x}^t, \rho(t, \mathbf{x}^t, S))\}$. As it stands, the arguments and results in Section 2.2, mainly the iterative procedure, with obvious corresponding adjustment, should be able to cover the above path-dependent case.

Chapter III

Mean Field Time-Inconsistent Stopping Games

In this chapter, we consider mean field time-inconsistent stopping games with a continuum of players where each player chooses a stopping time to a time-inconsistent stopping problem depending on the stopped proportion process μ_t. By introducing μ_t into the objective function we cannot expect our problem to be time-homogeneous even when X is time-homogeneous and as such our theory in Chapter II provides a natural foundation for studying this set of problems.

3.1 Motivation

To motivate our setup, let us first consider the following objective function

$$J_i(t, x, \mu, \tau) := \mathbb{E}^{t,x}\left[\sum_{s=t}^{\tau-1} \delta(s-t) g(X_s^i, \mu_{s,N}) + \delta(\tau - t) f(X_\tau^i, \mu_{\tau,N})\right],$$

where $i \in \{1, 2, ..., N\}$ representing N-numbers of players and $\{X^i\}_{i\in\{1,2,...,N\}}$ a family of independent and identical Markovian processes. For each $i \in \{1, 2, ..., N\}$, X^i corresponds to the private state of player-i and is supported on the probability space $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$ equipped with the filtration $\mathbb{F}^N$. Define τ^i the stopping time chosen by player-i. Each player interacts through the empirical measure μ_N on $\mathbb{T}$ defined by,

$$\mu_{t,N} := \mu_N([0, t]) := \frac{1}{N}\sum_{i=1}^{N} \delta_{\tau^i}[0, t] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{[0,t]}(\tau^i) \quad \forall t \in \mathbb{T}$$

representing the proportion of players that has stopped by time t. Stochastic games of large N-players are notoriously intractable both theoretically and numerically and as such *mean field games* were introduced to study the limiting case when $N \to \infty$. In the limiting regime, each player's decision will have negligible effect on the limit empirical measure $\mu_{t,\infty}$ which we call the *stopped proportion process* and therefore offering great simplification for finding Nash equilibrium to the problem.

By construction, we see that our N-player game extends to an infinite player game as $N \to \infty$. As explained in [11], when N is large, the usual heuristic for mean field games (but without the presence of a common noise in our case), suggests that, the empirical measure μ_N should approaches a probability measure μ on $\mathbb{T}$, i.e., $\mu \in \mathcal{P}(\mathbb{T})$. If we can solve the time-inconsistent stopping problem,

$$J(t, x, \mu, \tau) := \mathbb{E}^{t,x}\left[\sum_{s=t}^{\tau-1} \delta(s-t) g(X_s, \mu_s) + \delta(\tau - t) f(X_\tau, \mu_\tau)\right], \tag{3.1.1}$$

for each fixed measure μ, we can define a map $\mu \to \text{Law}(\tau^*)$ where τ^* is an stopping time induced by a single-agent equilibrium to Equation (3.1.1). Then the final step is to find a fixed-point for this map and any such fixed point is then a *Nash equilibrium*. In Section 3.2, we first solve Equation (3.1.1) for each fixed μ by constructing an (time-inhomogeneous) optimal equilibrium with respect to μ. Then in Section 3.3 we take the stopping time induced by this optimal equilibrium into the above heuristics and make the arguments precise.

3.2 The Optimal Equilibrium Γ^μ

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a Markovian process X taking values in some Borel subset $\mathbb{X} \subseteq \mathbb{R}$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathbb{T}}$ be the filtration generated by X and $\mathcal{T}_t$ the set of $\mathbb{F}$-stopping times τ with $\tau \geq t$ and $\mu = (\mu_0, \mu_1, \cdots) \in [0,1]^{\mathbb{N}}$ a non-decreasing, deterministic process, fixed throughout this section. Consider the following objective function,

$$J(t, x, \mu, \tau) := \mathbb{E}^{t,x}\left[\sum_{s=t}^{\tau-1} \delta(s-t) g(X_s, \mu_s) + \delta(\tau - t) f(X_\tau, \mu_\tau)\right]. \tag{3.2.1}$$

Here, we assume f, g and δ all satisfy the same standing assumptions and Assumption A as in Section 2.1. Given any $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$, define $\Theta^\mu : \mathcal{B}(\mathbb{T} \times \mathbb{X}) \to \mathcal{B}(\mathbb{T} \times \mathbb{X})$ by

$$\Theta^\mu(S) := (\widetilde{S}_0, \widetilde{S}_1, \widetilde{S}_2, \cdots) \tag{3.2.2}$$

where

$$(\Theta^\mu(S))_t := \widetilde{S}_t = \{x \in S_t : f(x, \mu_t) \geq J(t, x, \mu, S)\} \cup \{x \in \mathbb{X} \setminus S_t : f(x, \mu_t) > J(t, x, \mu, S)\}$$

Definition 3.2.1. We say $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ an equilibrium stopping region (or simply equilibrium) if $\Theta(S)^\mu = S$. We denote the set of all equilibria with respect to μ by $\mathcal{E}^\mu$.

In particular, $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ is said to be an equilibrium under Equation (3.2.2), i.e., $\Theta^\mu(S) = S$ if and only if for all $t \in \mathbb{T}$,

$$\begin{cases} f(x, \mu_t) \geq J(t, x, \mu, \rho(t, x, S)), & \forall x \in (S)_t, \\ f(x, \mu_t) \leq J(t, x, \mu, \rho(t, x, S)), & \forall x \in \mathbb{X} \setminus (S)_t. \end{cases} \tag{3.2.3}$$

Remark 3.2.2. This definition of equilibria is borrowed from [31] and is slightly weaker than the definition in Chapter II given by Equation (2.1.6). In particular, it considers a larger set of equilibria and it facilitates our explicit construction of the optimal equilibrium. In the case $f(x, \mu_t) = J(t, x, \mu, \rho(t, x, S))$, under Equation (3.2.2) the agent will not make changes as compared to the previous policy.

Remark 3.2.3. If we are to take Equation (3.2.2) as the definition in Chapter II then it is straightforward to see that all results hold except Theorem 2.2.9 and the uniqueness result given by Proposition 2.2.5.

We first collect a few lemmas that has appeared in previous sections in a similar form. The first one states that the stopping time ρ converges under sequence of monotone stopping region. The proof is exactly the same as in the first part of the proof of Theorem 2.2.2 and therefore omitted.

Lemma 3.2.4. *Let* $\{S^n\}_{n\in\mathbb{N}}$ *be a monotone sequence in* $\mathcal{B}(\mathbb{T}\times\mathbb{X})$. *For any* $(t,x)\in\mathbb{T}\times\mathbb{X}$ *we have* $\rho(t,x,S^n)(\omega)\to\rho(t,x,S^\infty)(\omega)$ *almost surely, where*

$$S^\infty := \begin{cases} \bigcup_{n\in\mathbb{N}} S^n, & \text{if } \{S^n\} \text{ is non-decreasing,} \\ \bigcap_{n\in\mathbb{N}} S^n, & \text{if } \{S^n\} \text{ is non-increasing.} \end{cases}$$

The next result is similar to Lemma 2.2.1, which says that any stopping region containing an equilibrium must be dominated by that equilibrium in terms of the payoff function J.

Lemma 3.2.5. *Suppose Assumption A holds. For any nonempty set* $R,S\in\mathcal{B}(\mathbb{T}\times\mathbb{X})$ *with* $R\subseteq S$ *and* $R\in\mathcal{E}^\mu$,

$$J(t,x,\mu,\rho(t,x,S))\le J(t,x,\mu,\rho(t,x,R)) \quad \forall (t,x)\in\mathbb{T}\times\mathbb{X}.$$

Proof. The proof is postponed to Appendix B.1. ∎

Next, we introduce the operator $\Phi^\mu:\mathcal{B}(\mathbb{T}\times\mathbb{X})\to\mathcal{B}(\mathbb{T}\times\mathbb{X})$ defined by $\Phi^\mu(S):=(\widetilde{S}_0,\widetilde{S}_1,\widetilde{S}_2,\cdots)$ where

$$(\Phi^\mu(S))_t := \widetilde{S}_t = S_t\cup\{x\notin S_t : f(x,\mu_t)>V^*(t,x,\mu,S)\} \tag{3.2.4}$$

for any $S\in\mathcal{B}(\mathbb{T}\times\mathbb{X})$ and

$$V^*(t,x,\mu,S) := \sup_{t<\tau\le\rho(t,x,S)} J(t,x,\mu,\tau). \tag{3.2.5}$$

Observe that for any sequences $S^n\in\mathcal{B}(\mathbb{T}\times\mathbb{X})$ and any $\omega\in\Omega$, if $\rho(t,x,S^n)(\omega)$ converges to some limit, say $\rho(t,x,S^\infty)(\omega)$ then the sequence $\rho(t,x,S^n)(\omega)$ must be eventually constant (when S^n is monotone, this statement is given by Lemma 3.2.4). As such, by the form of

Equation (3.2.5) we expect that $V^*(t,x,\mu,S^n)$ converges as well. This is given by the next lemma.

Lemma 3.2.6. *Let $\{S^n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence in $\mathcal{B}(\mathbb{T}\times\mathbb{X})$. Then, $V^*(t,x,\mu,S^n)\downarrow V^*(t,x,\mu,S^\infty)$ for all $(t,x)\in\mathbb{T}\times\mathbb{X}$, with $S^\infty := \bigcup_{n\in\mathbb{N}} S^n$.*

Proof. The proof is postponed to Appendix B.2. ■

Next, we explicitly construct an equilibrium by using the operator Φ^μ through an iterative procedure.

Proposition 3.2.7. *Let $\{S^n_\mu\}_{n\in\mathbb{N}}$ be a nondecreasing sequence in $\mathcal{B}(\mathbb{T}\times\mathbb{X})$ defined by*

$$S^0_\mu := (\emptyset,\emptyset,\emptyset,\cdots) \quad \textit{and} \quad S^n_\mu := \Phi^\mu(S^{n-1}_\mu) \quad \textit{for } n\geq 1. \tag{3.2.6}$$

Then,

$$\Gamma^\mu := \bigcup_{n\in\mathbb{N}} S^n_\mu \in \mathcal{E}^\mu. \tag{3.2.7}$$

Proof. Fix $(t,x)\in\mathbb{T}\times\mathbb{X}$. Let $x\in(\Gamma^\mu)_t$, by the fact that $\{(S^n_\mu)_t\}_{n\in\mathbb{N}}$ is a non-decreasing sequence, there exists $n\in\mathbb{N}$ such that $x\in(S^{n+1}_\mu)_t\setminus(S^n_\mu)_t$. By the definition of Φ^μ and Lemma 3.2.6, we have

$$f(x,\mu_t) > V^*(t,x,\mu,S^n_\mu) \geq V^*(t,x,\mu,\Gamma^\mu) \geq J(t,x,\mu,\rho(t,x,\Gamma^\mu)),$$

which implies that $x\in(\Theta^\mu(\Gamma^\mu))_t$ and hence $\Gamma^\mu\subseteq\Theta^\mu(\Gamma^\mu)$.

Fix $(t,x)\in\mathbb{T}\times\mathbb{X}$ such that $x\notin(\Gamma^\mu)_t$, we claim that $x\notin(\Theta^\mu(\Gamma^\mu))_t$, i.e.,

$$f(x,\mu_t) \leq J(t,x,\mu,\rho(t,x,\Gamma^\mu)). \tag{3.2.8}$$

We prove the converse inclusion by contradiction. Assume that Equation (3.2.8) fails, define

$\Lambda := \{\Lambda_0, \Lambda_1, \Lambda_2, ...\}$, where

$$\Lambda_t := \{x \notin (\Gamma^\mu)_t : f(x, \mu_t) > J(t, x, \mu, \rho(t, x, \Gamma^\mu))\}. \tag{3.2.9}$$

Consider

$$\nu := \sup_{t \in \mathbb{T}, y \in \Lambda_t} \{f(y, \mu_t) - J(t, y, \mu, \rho(t, y, \Gamma^\mu))\} > 0. \tag{3.2.10}$$

Since $\delta(1) < \delta(0) = 1$, we can choose some $\kappa \in (\Lambda)_t$ such that

$$f(\kappa, \mu_t) - J(t, \kappa, \mu, \rho(t, \kappa, \Gamma^\mu)) > \frac{1 + \delta(1)}{2}\nu. \tag{3.2.11}$$

Since $\kappa \notin (\Gamma^\mu)_t$, we must have $\kappa \notin (S^n_\mu)_t$ for all $n \in \mathbb{N}$, thus $f(\kappa, \mu) \leq V^*(t, \kappa, \mu, S^n_\mu)$ for all $n \in \mathbb{N}$ by Lemma 3.2.6, this implies,

$$f(\kappa, \mu_t) \leq V^*(t, \kappa, \mu, \Gamma^\mu). \tag{3.2.12}$$

Let $\tau^* \in \mathcal{T}_t$ with $t + 1 \leq \tau^* \leq \rho(t, \kappa, \Gamma^\mu)$ be a $\left(\frac{1-\delta(1)}{2}\nu\right)$-optimizer of $V^*(t, \kappa, \mu, \Gamma^\mu)$. Note that for any $\omega \in \Omega$ if $\tau^*(\omega) < \rho(t, \kappa, \Gamma^\mu)(\omega)$ then $X^{t,\kappa}_{\tau^*}(\omega) \notin (\Gamma^\mu)_{\tau^*}$. Hence consider the sets

$$E_1 := \{\omega \in \Omega : \tau^*(\omega) < \rho(t, \kappa, \Gamma^\mu), X^{t,\kappa}_{\tau^*} \in (\Lambda^c \setminus \Gamma^\mu)_{\tau^*}\}$$
$$E_2 := \{\omega \in \Omega : \tau^*(\omega) < \rho(t, \kappa, \Gamma^\mu), X^{t,\kappa}_{\tau^*} \in (\Lambda \setminus \Gamma^\mu)_{\tau^*}\}$$

We have, by Equation (3.2.11) and Equation (3.2.12),

$$\begin{aligned} \frac{1 + \delta(1)}{2}\nu &< f(\kappa, \mu_t) - J(t, \kappa, \mu, \rho(t, \kappa, \Gamma^\mu)) \\ &\leq J(t, \kappa, \mu, \tau^*) - J(t, \kappa, \mu, \rho(t, \kappa, \Gamma^\mu)) + \frac{1 - \delta(1)}{2}\nu \end{aligned} \tag{3.2.13}$$

Writing $\rho_{\Gamma^\mu} = \rho(t, \kappa, \Gamma^\mu)$, by the strong Markov property of X and the same argument as

Equations (A.1.5) and (A.1.4) in the proof of Lemma 2.2.1, we have

$$\begin{aligned} J(t,\kappa,\mu,\tau^*) - J(t,\kappa,\mu,\rho_{\Gamma^\mu}) &\leq \mathbb{E}^{t,\kappa}\left[(\mathbb{1}_{E_1}+\mathbb{1}_{E_2})\delta(\tau^*-t)(f(X_{\tau^*},\mu_{\tau^*}) - J(\tau^*,X_{\tau^*},\mu,\rho_{\Gamma^\mu}))\right] \\ &\leq \mathbb{E}^{t,\kappa}\left[\mathbb{1}_{E_2}\delta(\tau^*-t)(f(X_{\tau^*},\mu_{\tau^*}) - J(\tau^*,X_{\tau^*},\mu,\rho_{\Gamma^\mu})\right] \\ &\leq \delta(1)\nu, \end{aligned}$$

where the second inequality comes from the fact that $X_{\tau^*} \notin (\Lambda)_{\tau^*}$ on E_1 and the definition of Λ and the third equality comes from the fact that $X_{\tau^*} \in (\Lambda)_{\tau^*}$ on E_2 and Equation (3.2.10). Thus continuing from Equation (3.2.13), we have,

$$\frac{1+\delta(1)}{2}\nu < \delta(1)\nu + \frac{1-\delta(1)}{2}\nu = \frac{1+\delta(1)}{2}\nu.$$

The strict inequality above clearly indicates a contradiction, therefore, we conclude Equation (3.2.8) holds true for all $x \notin (\Gamma^\mu)_t$. Hence $(\Theta(\Gamma^\mu))_t \subseteq (\Gamma^\mu)_t$. Then we have $\Gamma^\mu = \Theta^\mu(\Gamma^\mu)$, i.e., $\Gamma^\mu \in \mathcal{E}^\mu$. ■

Lemma 3.2.8. *Fix $T \in \mathcal{E}^\mu$, for any $S \in \mathcal{B}(\mathbb{T}\times\mathbb{X})$ with $S \subseteq T$, $\Phi^\mu(S) \subseteq T$.*

Proof. Fix $t \in \mathbb{T}$, suppose that there exists $x \in (\Phi^\mu(S))_t \setminus (T)_t$ for some $S \in \mathcal{B}(\mathbb{T}\times\mathbb{X})$ with $S \subseteq T$. Now, since $x \notin (S)_t$, the definition of Φ^μ and V^* implies that

$$f(x,\mu_t) > V^*(t,x,\mu,S) \geq V^*(t,x,\mu,T) \geq J(t,x,\mu,\rho(t,x,T)).$$

Therefore, $x \in (\Theta^\mu(T))_t$. However, this contradicts the fact that $T = \Theta^\mu(T)$ as $x \notin (T)_t$ and $T \in \mathcal{E}^\mu$, thus we conclude $\Phi^\mu(S) \subseteq T$. ■

The above lemma states that being a subset of an equilibrium is in fact invariant under Φ^μ. Now, using this lemma, we can deliver the main theorem of this section. We claim that the equilibrium Γ^μ constructed in Proposition 3.2.7 is our optimal equilibrium. Define the associated value function V of an equilibrium $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ similarly as in Section 2.2,

$$V(t, x, \mu, S) = f(x, \mu_t) \vee J(t, x, \mu, \rho(t, x, S)). \tag{3.2.14}$$

Theorem 3.2.9. *Suppose Assumption A holds. Then Γ^μ defined in Equation (3.2.7) is an optimal equilibrium.*

Proof. By Proposition 3.2.7, Γ^μ is an equilibrium. For any $T \in \mathcal{E}^\mu$, $\emptyset \in T$ (trivially) and by Lemma 3.2.8 we have $S_\mu^1 := \Phi^\mu(\emptyset) \in \mathcal{E}^\mu$. Therefore, by applying Lemma 3.2.8 to the sequence $\{S_\mu^n\}_{n\in\mathbb{N}}$ defined recursively in Equation (3.2.6), we have $S_\mu^n = \Phi^\mu(S_\mu^{n-1}) \in T$ for all $n \in \mathbb{N}$, thus $\Gamma^\mu = \bigcup_{n\in\mathbb{N}} S_\mu^n \subseteq T$. Applying Lemma 3.2.5 we have $J(t, x, \mu, \rho(t, x, \Gamma^\mu)) \geq J(t, x, \mu, \rho(t, x, T))$ for all $(t, x) \in \mathbb{T}\times\mathbb{X}$ and hence $V(t, x, \mu, \rho(t, x, \Gamma^\mu)) \geq V(t, x, \mu, \rho(t, x, T))$ for all $(t, x) \in \mathbb{T} \times \mathbb{X}$. Since T is arbitrary we conclude that Γ^μ is an optimal equilibrium. ∎

As a byproduct of our construction of Γ^μ we have the following corollary.

Corollary 3.2.10. *If X is time-homogeneous then there exists a time-homogeneous optimal equilibrium to Equation (2.1.2).*

Proof. By taking $f(x, \mu) = f(x)$ and $g(x, \mu) = g(x)$, Γ^μ is an optimal equilibrium to Equation (2.1.2). Moreover, by construction $(\Gamma^\mu)_i = (\Gamma^\mu)_j$ for any $i \neq j$ since X is time-homogeneous. ∎

3.3 Mean Field Formulation and Objective of Agents

Following the heuristics in Section 3.1, let $\mu \in \mathcal{P}(\mathbb{T})$, we identify each μ as a non-decreasing, deterministic sequence in $[0, 1]^{\mathbb{N}}$ defined by $\mu = (\mu_0, \mu_1, \mu_2, \cdots)$ where $\mu_t = \mu([0, t])$. For each fixed $\mu \in [0, 1]^{\mathbb{N}}$, we solve the time-inconsistent stopping problem given by objective function Equation (3.2.1) by constructing an optimal equilibrium Γ^μ with respect to μ. We now state formally the objective of each players.

We extend the process $\{X_t\}_{t\in\mathbb{T}}$ backwards in time to $t = -1$ and without loss of generality we assume all players start at $X_{-1} = \Delta$ where Δ is some fixed constant in $\mathbb{X}$ where they are not allowed to stop. At $t = 0$, X transition from Δ to some initial condition X_0 according to a known initial distribution independent of the family of transition kernels $\{Q_t\}_{t\in\mathbb{T}}$ of X defined by

$$\mathbb{P}(X_{t+1} \in B \mid X_t = x) = \int_B Q_t(x, dy) \quad \forall t \in \mathbb{T} \quad \text{and} \quad \forall B \in \mathcal{B}(\mathbb{X}).$$

Here $\mathbb{T} = \{0, 1, 2, ...\}$ and does not include $t = -1$. For this process X, players collectively seek an equilibrium strategy to Equation (3.2.1) from the initial starting time $t = -1$ such that when implemented none of the players has incentive to deviate from. We call such strategies soft mean field equilibria.

Definition 3.3.1 (Soft MFE)**.** A pair (μ, S) is a *soft mean field equilibrium* if $\Theta^{\mu}(S) = S$ and $\mu_t = \mathbb{P}(\rho(-1, \Delta, S) \leq t)$ for all $t \in \mathbb{T}$.

This mean field formulation is different from the single-agent formulation as we need to take into account a continuum of players where each player play against other agents and future selves.

Remark 3.3.2. We would like to reiterate again that as we extend X backwards to $t = -1$, the set $\mathbb{T}$ does not include $t = -1$ and μ_{-1} and S_{-1} are not included in our formulation either, or we can simply assume $\mu_{-1} = 0$ and $S_{-1} = \emptyset$.

Given $\mu \in [0, 1]^{\mathbb{N}}$ if (μ, S) is a soft mean field equilibrium such that S is also optimal w.r.t μ, i.e., for any other equilibria $R \in \mathcal{E}^{\mu}$,

$$V(t, x, \mu, S) \geq V(t, x, \mu, R) \quad \forall (t, x) \in \mathbb{T} \times \mathbb{X},$$

then we say (μ, S) is sharp.

Definition 3.3.3 (Sharp MEF)**.** A pair (μ, S) is a *sharp mean field equilibrium* if it is a soft mean field equilibrium and in addition S is optimal w.r.t μ.

In particular, it is easy to see that any fixed-point to the map $\mu \mapsto \{\mathbb{P}(\rho(-1, \Delta, \Gamma^\mu) \le t)\}_{t\in\mathbb{T}}$ must be a sharp mean field equilibrium to Equation (3.2.1).

Remark 3.3.4. We would also like to stress that the purpose of this chapter is to present methods in showing the existence of fixed-point to the limiting game rather than a detailed study concerning the mean field game setup and its asymptotic behaviour as in [11]. However, one thing to note is that even without the sophisticated framework of mean field game we still have the convergence of the empirical measure $\mu_t = \lim_{N\to\infty} \mu_{t,N} = \mathbb{E}[\mathbb{1}_{[0,t]}(\tau^*)]$ simply by the law of large number and the extension to the limiting game is readily justified. For a probabilistic treatment of mean field games, we refer the readers to the excellent two volume textbook/monograph [10] by Carmona and Delarue and the references therein.

3.4 A Bank Run Model

In this section, we analyze an example motivated by bank run which describes the situation where a large number of depositors of a financial institution withdraw their holdings and thus triggering liquidation. Consider the following objective function,

$$\begin{aligned} J(t,x,\mu,\tau) &:= \mathbb{E}^{t,x}\left[\delta(\tau-t) f(X_\tau,\mu_\tau)\right] \\ &:= \mathbb{E}^{t,x}\left[\delta(\tau-t)\{F(X_\tau)\mathbb{1}_{\{\tau<\theta^\mu\}} + \xi\cdot\mathbb{1}_{\{\tau\ge\theta^\mu\}}\}\right] \\ &:= \mathbb{E}^{t,x}\left[\delta(\tau-t)\{F(X_\tau)\mathbb{1}_{\{\mu_\tau\le\zeta\}} + \xi\cdot\mathbb{1}_{\{\mu_\tau>\zeta\}}\}\right], \end{aligned} \tag{3.4.1}$$

where $F : \mathbb{X} \to \mathbb{R}^+$ is strictly positive and $\theta^\mu := \inf\{t : \mu_t > \zeta\}$ with $\zeta \in [0, 1]$ a predetermined constant and is interpreted as the bank's asset to liability ratio. Let ξ be a constant such that $0 < \xi \le \inf f$ representing the recovery payoff in the case of bank liquidation.

Remark 3.4.1. The particular form of the bank run time θ^μ defined above is motivated by [11]. Here for simplicity we do not require ζ to be random as in [11].

Lemma 3.4.2. *Let $S, T \in \mathcal{B}(\mathbb{T}\times\mathbb{X})$ such that $T \subseteq S$ then*

$$\mu_t := \mathbb{P}(\rho(-1,\Delta,S) \le t) \ge \mathbb{P}(\rho(-1,\Delta,T) < t) =: \nu_t, \quad \forall t \in \mathbb{T}.$$

Lemma 3.4.3. *For any (non-decreasing) $\nu, \mu \in [0,1]^{\mathbb{N}}$ such that $\nu \leq \mu$,*

$$J(t,x,\mu,\tau) \leq J(t,x,\nu,\tau), \quad \forall (t,x) \in \mathbb{T} \times \mathbb{X}, \tau \in \mathcal{T}_t. \tag{3.4.2}$$

Hence $V^*(t,x,\mu,S) \leq V^*(t,x,\nu,S)$ for all $x \in \mathbb{X}$, $t \in \mathbb{T}$ and $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$. Moreover, we have

$$\Phi^{\nu}(T) \subseteq \Phi^{\mu}(S)$$

for all $S, T \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ with $T \subseteq S$.

Proof. By definition, if $\nu \leq \mu$ then $\theta^{\mu} \leq \theta^{\nu}$ we have,

$$\begin{aligned}
&\mathbb{E}^{t,x}[\delta(\tau - t) f(X_\tau, \nu_\tau) - \delta(\tau - t) f(X_\tau, \mu_\tau)] \\
&= \mathbb{E}^{t,x}[\delta(\tau - t)(f(X_\tau, \nu_\tau)) - f(X_\tau, \mu_\tau)] \\
&\geq \mathbb{E}^{t,x}[\delta(\tau - t) \cdot 0] = 0,
\end{aligned}$$

where the last inequality comes from the fact θ^{μ} occurs before θ^{ν} and $\xi \leq \inf F$. Therefore, we have $J(t,x,\mu,\tau) \leq J(t,x,\nu,\tau)$ for all $(t,x) \in \mathbb{T} \times \mathbb{X}$ and $\tau \in \mathcal{T}_t$, and hence

$$V^*(t,x,\mu,S) \leq V^*(t,x,\nu,S) \quad \forall S \in \mathcal{B}(\mathbb{T} \times \mathbb{X}) \tag{3.4.3}$$

Next, fix $S, T \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ with $T \subseteq S$. Take $x \in (\Phi^{\nu}(T))_t \setminus (T)_t$. If $x \in (S)_t$ then $x \in (\Phi^{\mu}(S))_t$ by definition of Φ^{μ} (Equation (3.2.4)). Now, assume $x \notin (S)_t$. Consider $t < \theta^{\mu} \leq \theta^{\nu}$. With $x \in (\Phi^{\nu}(T))_t \setminus (T)_t$, Equations (3.2.4) and (3.4.3) give,

$$f(x,\mu_t) = f(x,\nu_t) = F(x) > V^*(t,x,\nu,T) \geq V^*(t,x,\nu,S) \geq V^*(t,x,\mu,S). \tag{3.4.4}$$

This together with $x \notin (S)_t$ yields that $x \in (\Phi^{\mu}(S))_t$ for any $t < \theta^{\mu}$. If $t \geq \theta^{\mu}$ then by

Equation (3.2.4) again,

$$J(t, x, \mu, t) = f(x, \mu_t) = \xi > V^*(t, x, \mu, S) = \delta(1) \cdot \xi \quad \forall x \in \mathbb{X}.$$

Hence $\Phi^\nu(T) \subseteq \mathbb{X} = \Phi^\mu(S)$. In either case, we have $\Phi^\nu(T) \subseteq \Phi^\mu(S)$ ■

Using the above lemma we show that the mapping $\mu \to \Gamma^\mu$ is monotone under our model setup. This monotonicity will play a crucial role in the iterative procedure defined in Theorem 3.4.5.

Corollary 3.4.4. *For any (non-decreasing) $\nu, \mu \in [0,1]^{\mathbb{N}}$ such that $\nu \leq \mu$, we have $\Gamma^\nu \subseteq \Gamma^\mu$.*

Proof. We have $\emptyset \subseteq \emptyset$ therefore by Lemma 3.4.3 and definition of S^n_μ (Equation (3.2.6)), $(S^\nu_0)_t \subseteq (S^\mu_0)_t$ for any $t \in \mathbb{T}$ hence $S^\nu_0 \subseteq S^\mu_0$. Applying this procedure for all $n \in \mathbb{N}$, we have $S^\nu_n \subseteq S^\mu_n$ for any $n \in \mathbb{N}$. Therefore, by definition of Γ^μ (Equation (3.2.7)), we have $\Gamma^\nu \subseteq \Gamma^\mu$. ■

Next, we design an iterative procedure for μ which leads to a sharp mean field equilibrium. To this end, we start with $\mu^0 := (0, 0, 0, \cdots)$. We construct a stopping region induced by the optimal equilibrium w.r.t μ^0 given by Γ^{μ^0}. From Γ^{μ^0} we derive the sequence $\{\mathbb{P}(\rho(-1, \Delta, \Gamma^{\mu^0}) \leq t)\}_{t \in \mathbb{T}} := \mu^1$, by definition, we must have $\mu^1 \geq \mu^0$. Then Corollary 3.4.4 tells us that $\Gamma^{\mu^1} \supseteq \Gamma^{\mu^0}$ where Γ^{μ^1} is the stopping region constructed w.r.t μ^1. Next, we derive $\mu^2 := \{\mathbb{P}(\rho(-1, \Delta, \Gamma^{\mu^1}) \leq t)\}_{t \in \mathbb{T}}$ and by Lemma 3.4.2, $\mu^2 \geq \mu^1$. We apply this iterative procedure for all $n \in \mathbb{N}$, then by the fact that μ^n is non-decreasing and bounded it must converges to some limit in $[0,1]^{\mathbb{N}}$. We show that the limit gives a shape mean field equilibrium.

Theorem 3.4.5. *Let (S^n, μ^n) be a sequence defined by $\mu^0 := (0, 0, 0, \cdots)$,*

$$S^n := \Gamma^{\mu^n} \quad \textit{and} \quad \mu^{n+1} := \{\mathbb{P}(\rho(-1, \Delta, S^n) \leq t)\}_{t \in \mathbb{T}}. \tag{3.4.5}$$

Then S^n and μ^n is non-decreasing. By taking $S^\infty := \cup_n^\infty S^n$ and $\mu^\infty = \lim_{n\to\infty}\mu^n$ we have

$$\Theta^{\mu^\infty}(S^\infty) = S^\infty, \quad S^\infty = \Gamma^{\mu^\infty} \quad \textit{and} \quad \mu^\infty = \{\mathbb{P}(\rho(-1,\Delta,S^\infty)\le t)\}_{t\in\mathbb{T}}. \tag{3.4.6}$$

Proof. The first statement is obvious by the above argument. As S^n is non-decreasing S^∞ is well-defined and as for each t, μ_t^n is bounded and monotone, then the limit $\mu_t^n \to \mu_t^\infty$ exists and μ^n converges to some limit in $[0,1]^{\mathbb{N}}$, denoted by μ^∞.

Next, we show that $\Theta^{\mu^\infty}(S^\infty) = S^\infty$. Fix t and $x \in (S^\infty)_t = \bigcup_n (S^n)_t$. There exists $N \in \mathbb{N}$ such that $x \in (S^{n+1})_t = (\Gamma^{\mu^n})_t$ for all $n > N$. By the fact that $\Gamma^{\mu^n} \in \mathcal{E}^{\mu^n}$, we have,

$$J(t,x,\mu^n,t) = f(x,\mu_t^n) \ge J(t,x,\mu^n,\rho(t,x,S^n)), \quad \forall n \ge N.$$

Letting $n\to\infty$, by each iteration of Equation (3.4.5) μ^n is increasing in n and by definition (Equation (3.4.1)) the function $f(x,\cdot)$ has left limits, therefore we have convergence on the LHS of the above inequality. Again, using this fact and since f is bounded we apply dominated convergence theorem to the RHS then,

$$f(x,\mu_t^\infty) \ge J(t,x,\mu^\infty,\rho(t,x,S^\infty)),$$

which implies $x \in \Theta^{\mu^\infty}(S^\infty)$. Hence $S^\infty \subseteq \Theta^{\mu^\infty}(S^\infty)$. To see the other inclusion, for any $x \notin S^\infty = \bigcup_n S^n$, $x \notin S^{n+1} = \Gamma^{\mu^n}$ for all $n \in \mathbb{N}$. Again due to the fact $\Gamma^{\mu^n} \in \mathcal{E}^{\mu^n}$, $x \notin \Gamma^{\mu^n}$ implies that $f(x,\mu_t^n) \le J(t,x,\mu^n,\rho(t,x,S^n))$. Take $n\to\infty$, $f(x,\mu_t^\infty) \le J(t,x,\mu^\infty,\rho(t,x,S^\infty))$, which implies that $x \notin \Theta^{\mu\infty}(S^\infty)$. Hence $(S^\infty)^c \subseteq (\Theta^{\mu^n}(S^\infty))^c$, or $\Theta^{\mu^\infty} \subseteq S^\infty$. We therefore conclude that $\Theta^{\mu^\infty}(S^\infty) = S^\infty$, i.e., $S^\infty \in \mathcal{E}^{\mu^\infty}$.

Fix t and by Corollary 3.4.4, we have $(\Gamma^{\mu^\infty})_t \supseteq (\Gamma^{\mu^n})_t$ for all n, thus $S^n = \Gamma^{\mu^n} \subseteq \Gamma^{\mu^\infty}$ for all n and thus $S^\infty \subseteq \Gamma^{\mu^\infty}$. On the other have, since $S^\infty \in \mathcal{E}^{\mu^\infty}$, this together with Lemma 3.2.8 implies that $\Gamma^{\mu^\infty} \subseteq S^\infty$ and we conclude that $\Gamma^{\mu^\infty} = S^\infty$. Finally, fix t and as

S^n is non-decreasing, the event $\{\rho(-1, \Delta, S^n) \leq t\}$ is also monotone in n, hence

$$\mathbb{P}(\rho(-1, \Delta, S^\infty) \leq t) = \lim_{n\to\infty} \mathbb{P}(\rho(-1, \Delta, S^n) \leq t) = \lim_{n\to\infty} \mu^n = \mu^\infty.$$

■

By Equation (3.4.6), we have found a fixed-point to the map $\mu \mapsto \{\mathbb{P}(\rho(-1, \Delta, \Gamma^\mu) \leq t)\}_{t\in\mathbb{T}}$ given by μ^∞ through the iterative procedure. This gives us the existence of a nontrivial sharp mean field equilibrium. Since our iterative procedure outlined in Theorem 3.4.5 starts from $\mu^0 = (0, 0, 0, \cdots)$, it is easy to see that the sharp MFE $(\mu^\infty, \Gamma^{\mu^\infty})$ must be the smallest among all sharp MFEs. That is, we have the following result.

Corollary 3.4.6. *Let $(\tilde{\mu}, \Gamma^{\tilde{\mu}})$ be a sharp MFE. Then $\mu^\infty \leq \tilde{\mu}$ and $\Gamma^{\mu^\infty} \subseteq \Gamma^{\tilde{\mu}}$ where μ^∞ is given by Theorem 3.4.5.*

Proof. By definition $\mu^0 = (0, 0, 0, \cdots) \leq \tilde{\mu}$ therefore $\Gamma^{\mu^0} \subseteq \Gamma^{\tilde{\mu}}$ by Corollary 3.4.4. Using the fact that $(\tilde{\mu}, \Gamma^{\tilde{\mu}})$ is a sharp MFE and Lemma 3.4.2 we have $\mu^1 \leq \tilde{\mu}$. Continuing this procedure we see that $\mu^n \leq \tilde{\mu}$ for all $n \in \mathbb{N}$ hence $\mu^\infty \leq \tilde{\mu}$ and $\Gamma^{\mu^\infty} \subseteq \Gamma^{\tilde{\mu}}$. ■

Remark 3.4.7. Our model, similar as other game theoretic papers on bank runs also exhibit the *complementarity* property. Informally, this means that early withdrawal of depositors increase the probability of liquidation of the bank which encourages withdrawal of more depositors. Games with this property is called *supermodular* games (see [44, 36]) and in our model there exists a trivial sharp mean field equilibrium given by $\tilde{\mu} = (1, 1, 1, \cdots)$ and $\Gamma^{\tilde{\mu}} = (\mathbb{X}, \mathbb{X}, \mathbb{X}, \cdots)$. However, this is the worst sharp mean field equilibrium and any other sharp mean field equilibrium such as our construction in Theorem 3.4.5 must be bounded below by this in terms of the value function V.

Recall the associated value function defined by Equation (3.2.14) which gives us the payoff for the agent at time t under fixed $S \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ and $\mu \in [0, 1]^{\mathbb{N}}$.

Theorem 3.4.8. *If (μ, Γ^{μ}) and (ν, Γ^{ν}) are sharp MFEs such that $\nu \leq \mu$ then $V(t,x,\nu,\Gamma^{\nu}) \geq V(t,x,\mu,\Gamma^{\mu})$. In particular, $(\mu^{\infty}, \Gamma^{\mu^{\infty}})$ is an optimal sharp MFE, i.e., for any sharp MFE $(\tilde{\mu}, \Gamma^{\tilde{\mu}})$ we have $V(t,x,\mu^{\infty},\Gamma^{\mu^{\infty}}) \geq V(t,x,\tilde{\mu},\Gamma^{\tilde{\mu}})$ for all $(t,x) \in \mathbb{T} \times \mathbb{X}$.*

Proof. By Corollary 3.4.4 if $\nu \leq \mu$ we have $\Gamma^{\nu} \subseteq \Gamma^{\mu}$. Since $\Gamma^{\nu} \in \mathcal{E}^{\nu}$ and $\Gamma^{\nu} \subseteq \Gamma^{\mu}$, by Lemma 3.2.5 we have that,

$$J(t,x,\nu,\rho(t,x,\Gamma^{\mu})) \leq J(t,x,\nu,\rho(t,x,\Gamma^{\nu})).$$

Moreover, as $\nu \mapsto J(t,x,\nu,\tau)$ is decreasing,

$$J(t,x,\mu,\rho(t,x,\Gamma^{\mu})) \leq J(t,x,\nu,\rho(t,x,\Gamma^{\mu})).$$

Hence, we have

$$J(t,x,\mu,\rho(t,x,\Gamma^{\mu})) \leq J(t,x,\nu,\rho(t,x,\Gamma^{\nu})). \tag{3.4.7}$$

Similarly, as $\nu \mapsto f(x,\nu)$ is decreasing we also have $f(x,\mu_t) \leq f(x,\nu_t)$, combining with Equation (3.4.7),

$$f(x,\mu) \vee J(t,x,\mu,\rho(t,x,\Gamma^{\mu})) \leq f(x,\nu) \vee J(t,x,\nu,\rho(t,x,\Gamma^{\nu})), \tag{3.4.8}$$

that is, $V(t,x,\mu,\Gamma^{\mu}) \leq V(t,x,\nu,\Gamma^{\nu})$. Consequently, by Corollary 3.4.6, the pair $(\mu^{\infty}, \Gamma^{\mu^{\infty}})$ is an optimal sharp mean field equilibrium. ■

3.5 A General Model

In this section, let us consider the following objective function,

$$J(t,x,\mu,\tau) := \mathbb{E}^{t,x}\left[\sum_{k=t}^{\tau-1} \delta(k-t) f(X_k,\mu_k) + \delta(\tau-t) g(\mu_\tau)\right]. \tag{3.5.1}$$

In the above objective function, we make the following assumptions,

Assumption B.

(B.1) $g : [0,1] \to [0, J]$ is bounded and uniformly continuous;

(B.2) $f : \mathbb{R} \times [0,1] \to [0, K]$ is bounded and uniformly continuous in both variables and strictly increasing in the first variable;

(B.3) the discount function δ satisfies $\sum_{t=0}^{\infty} \delta(t) < \infty$ and Equation (2.1.3).

In order for some of our proofs to work, we shall further assume the following,

Assumption C.

(C.1) for each $t \in \mathbb{T}$, given $X_t = x$, $X_{t+1} = h(X_t, \xi_t) = h(x, \xi_t)$, where $h : \mathbb{R}^2 \to \mathbb{R}$ is continuous and strictly increasing in the first variable and $\{\xi_t\}_{t\in\mathbb{T}}$ is a family of independent random variables. Without loss of generality we assume $\{\xi_t\}_{t\in\mathbb{T}}$ generates the filtration $\mathbb{F}$ in this section.

(C.2) given $X_t = x$ the process X_k^x has a probability density function $f_{X_k^x}$ for all $k > t$.

Remark 3.5.1. The process $X_s^{t,x}$ continuously depends on x for any $s \geq t$ and is strictly increasing in x. Indeed for any $x_n \to x$, $X_t^{t,x_n} \to X_t^{t,x}$ by (C.1) of Assumption C and consequently $X_{t+1}^{t,x_n} = h(X_t^{t,x_n}, \xi_t) \to h(X_t^{t,x}, \xi_t) = X_{t+1}^{t,x}$, we then apply this recursively for all $s > t$. Therefore, coupled with (B.2) of Assumption B, we see that f and J is also strictly increasing and continuous in x.

Let us define the topology for the space $[0,1]^{\mathbb{N}}$. For any real-valued sequence v, define the weighted ℓ_1 norm by,

$$\|v\|_1^{\delta} = \sum_{n=1}^{\infty} \delta(n)|v_n|, \tag{3.5.2}$$

where δ is our choice of discount function. We denote ℓ_1^{δ} the space of real-valued sequences having finite $\|\cdot\|_1^\delta$ norm, i.e.,

$$\ell_1^\delta := \left\{ v \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} \delta(n)|v_n| < \infty \right\}. \tag{3.5.3}$$

equipped with the topology induced by $\|\cdot\|_1^\delta$. By (B.3) of Assumption B, for any $\mu \in [0,1]^{\mathbb{N}}$, we have $\|\mu\|_1^\delta \leq \sum_{n=0}^{\infty} \delta(n) < \infty$. Hence, $[0,1]^{\mathbb{N}}$ is a subset of the normed vector space ℓ_1^δ. Moreover, $[0,1]^{\mathbb{N}}$ is convex and compact. Finally, for any sequence $\{\mu^n\}_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$,

$$\mu^n \to \mu \quad \text{if and only if} \quad \|\mu^n - \mu\|_1^\delta = \sum_{k=0}^{\infty} \delta(k) \left|\mu_k^n - \mu_k\right| \to 0, \tag{3.5.4}$$

or equivalently μ^n converges to μ if and only if μ_k^n converges to μ_k for each $k \in \mathbb{N}$.

Remark 3.5.2. At this stage, we do not require each $\mu \in [0,1]^{\mathbb{N}}$ to be non-decreasing. Later we will find μ as a fixed-point for some iteration where non-decreasing is automatically implied.

For $T = (T_0, T_1, \cdots) \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$, $\mu \in [0,1]^{\mathbb{N}}$ and $(t,x) \in \mathbb{T} \times \mathbb{X}$, let us recall the function $V^*(t,x,\mu,T)$, the operator $\Phi^\mu(T)$ and the non-decreasing sequence $\{S_\mu^n\}_{n\in\mathbb{N}}$ defined in Section 3.2 given by Equations (3.2.4) to (3.2.6) respectively. We first establish the strict monotonicity of V^* in x.

Lemma 3.5.3. *Assume Assumptions B and C hold. Suppose $T \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ is such that $(T)_n := (-\infty, C_n) \cap \mathbb{X}$ for some constant C_n, if $x_2 > x_1$ then $\rho(t,x_2,T) \geq \rho(t,x_1,T)$ and $V^*(t,x_2,\mu,T) > V^*(t,x_1,\mu,T)$ for any $t \in \mathbb{T}$ and any $\mu \in [0,1]^{\mathbb{N}}$.*

Proof. The proof is postponed to Appendix C.1. ■

For any sequence $\{T_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{T}\times\mathbb{X})$ such that $(T_n)_k := (-\infty, C_k^n) \cap \mathbb{X}$ for some constant C_k^n, we say,

$$T_n \to T := (-\infty, C) \cap \mathbb{X} \quad \text{if and only if} \quad \left|C_k^n - C_k\right| \to 0 \quad \forall k \in \mathbb{T}. \tag{3.5.5}$$

In this section we shall only consider stopping regions in the above form, the reason for this will be made clear later (Proposition 3.5.10). In the next two lemmas, we establish the continuity of V^* in the stopping region $T \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ and $\mu \in [0,1]^{\mathbb{N}}$.

Lemma 3.5.4. *Assume Assumptions B and C hold. Suppose $T_n \to T$ in the sense of Equation (3.5.5). Then $\rho(t,x,T_n) \to \rho(t,x,T)$ almost surely. This implies that for any fixed $\mu \in [0,1]^{\mathbb{N}}$, $V^*(t,x,\mu,T_n) \to V^*(t,x,\mu,T)$ for all $(t,x) \in \mathbb{T} \times \mathbb{X}$.*

Proof. The proof is postponed to Appendix C.2. ■

Remark 3.5.5. It is evident from first part of the proof of Lemma 3.5.4 above that $\rho(-1,\Delta,T_n) \to \rho(-1,\Delta,T)$ as $T_n \to T$.

Lemma 3.5.6. *Suppose Assumption B holds. Let $\mu^n \to \mu$. Then for any $(t,x) \in \mathbb{T} \times \mathbb{X}$ and $\varepsilon > 0$ there exists N such that*

$$\sup_{T \in \mathcal{B}(\mathbb{T} \times \mathbb{X})} |V^*(t,x,\mu^n,T) - V^*(t,x,\mu,T)| \leq \varepsilon$$

for all $n \geq N$.

Proof. The proof is postponed to Appendix C.3. ■

Next, we combine the previous two lemmas and show the continuity of V^* in $T \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ and $\mu \in [0,1]^{\mathbb{N}}$ which will play a crucial role in our main result.

Lemma 3.5.7. *Suppose Assumptions B and C hold. Let $\mu^n \to \mu$ and $T^n \to T$ in the sense of Equation (3.5.5). Then $V^*(t,x,\mu^n,T^n) \to V^*(t,x,\mu,T)$ for all $(t,x) \in \mathbb{T} \times \mathbb{X}$.*

Proof. We have

$$\begin{aligned}
&\Big|V^*(t,x,\mu,T) - V^*(t,x,\mu^n,T^n)\Big| \\
&\leq \Big|V^*(t,x,\mu,T) - V^*(t,x,\mu,T^n)\Big| + \Big|V^*(t,x,\mu,T^n) - V^*(t,x,\mu^n,T^n)\Big|.
\end{aligned}$$

The convergence in the first term is by Lemma 3.5.4 and the convergence in the second term is by Lemma 3.5.6. Therefore,

$$\lim_{n\to\infty}\Big|V^*(t,x,\mu,T)-V^*(t,x,\mu^n,T^n)\Big|=0.$$

■

In a similar fashion as Lemma 3.5.4 we also establish the continuity of V^* in x.

Lemma 3.5.8. *Suppose Assumptions B and C hold. For any $x\in\mathbb{X}$ and $T\in\mathcal{B}(\mathbb{T}\times\mathbb{X})$ such that $(T)_k=(-\infty,C_k)$ for some constant C_k, if $x_n\to x$ then $\rho(t,x_n,T)\to\rho(t,x,T)$ almost surely.*

Proof. For any $\omega\in\Omega$, let $\zeta(\omega):=\rho(t,x,T)(\omega)$. By Remark 3.5.1, as $x_n\to x$ we have $X^{x_n}_\zeta(\omega)$ converges to $X^x_\zeta(\omega)$ and the rest follows an identical argument as in the proof of the first part of Lemma 3.5.4. ■

Lemma 3.5.9. *Suppose Assumptions B and C hold. For any $T\in\mathcal{B}(\mathbb{T}\times\mathbb{X})$, if $x_n\to x$ then $V^*(t,x^n,\mu,T)\to V^*(t,x,\mu,T)$ for all $t\in\mathbb{T}$ and $\mu\in[0,1]^{\mathbb{N}}$.*

Proof. The proof is postponed to Appendix C.4. ■

The next proposition justifies our consideration of stopping regions satisfying Equation (3.5.5).

Proposition 3.5.10. *Suppose Assumptions B and C hold. For any $\mu\in[0,1]^{\mathbb{N}}$ and $t\in\mathbb{T}$, we have for all $n\in\mathbb{N}$, $(S^n_\mu)_t$ given by Equation (3.2.6) is of the form $(-\infty,C^n_t)\cap\mathbb{X}$ for some constant $C^n_t\in[-\infty,\infty]$.*

Proof. Consider the case $n=0$, then $(S^0_\mu)_t=(\Phi^\mu(\emptyset))_t=\{x\in\mathbb{X}:g(\mu_t)>V^*(t,x,\mu,\emptyset)\}=\{x\in\mathbb{R}:g(\mu_t)>V^*(t,x,\mu,\emptyset)\}\cap\mathbb{X}$. Let $\alpha:=\sup\{x\in\mathbb{R}:g(\mu_t)>V^*(t,x,\mu,\emptyset)\}$. Now for any $y<\alpha$, by Lemma 3.5.3, $V^*(t,y,\mu,\emptyset)<V^*(t,\alpha,\mu,\emptyset)$ therefore $(S^0_\mu)_t=(-\infty,C^0_t)\cap\mathbb{X}$ with $C^0_t=\alpha$ where the openness at C^0_t is the by the strict inequality in the definition of Φ^μ and

the continuity of $V^*(t,x,\mu,T)$ in x by Lemma 3.5.9. If $\{x \in \mathbb{R} : g(\mu_t) > V^*(t,x,\mu,\emptyset)\} = \mathbb{R}$ we take C_t^0 to be ∞ and if $\{x \in \mathbb{R} : g(\mu_t) > V^*(t,x,\mu,\emptyset)\} = \emptyset$ we then take $C_t^0 = -\infty$.

Now assume $(S_\mu^n)_t = (-\infty, C_t^n) \cap \mathbb{X}$ for some constant C_t^n. Then

$$\begin{aligned}(S_\mu^{n+1})_t &= (\Phi(S_\mu^n))_t = (S_\mu^n)_t \cup \{x \notin (S_\mu^n)_t : g(\mu) > V^*(t,x,\mu,S_\mu^n)\} \\ &= (S_\mu^n)_t \cup \{x \in [C_t^n, \infty) : g(\mu) > V^*(t,x,\mu,S_\mu^n)\} \cap \mathbb{X}.\end{aligned}$$

Let $\beta > C_n$ such that $\beta := \sup\{x \in \mathbb{R} : g(\mu_t) > V^*(t,x,\mu,S_\mu^n)\}$. If $\beta > C_t^n$ then by Lemma 3.5.3 $(S_\mu^{n+1})_t = (-\infty, C_t^{n+1}) \cap \mathbb{X}$ with $\beta = C_t^{n+1}$ otherwise $(S_\mu^{n+1})_t = (-\infty, C_t^n) \cap \mathbb{X} = (S_\mu^n)_t$. This concludes the proof. ∎

To prove our main theorem, we need another auxiliary result.

Lemma 3.5.11. *Let $H = H(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be continuous in both and decreasing in the first argument. For any $y_n \in \mathbb{R}$ define $\varphi_n := \sup\{x \in \mathbb{R} : H(x,y_n) > 0\}$. Then if $y_n \to y \in \mathbb{R}$ we have $\varphi_n \to \varphi := \sup\{x \in \mathbb{R} : H(x,y) > 0\}$.*

Proof. The proof is postponed to Appendix C.5. ∎

Using the above auxiliary lemma, we can now prove a key result of this section, which says that S_μ^n defined by Equation (3.2.6) is continuous in μ for all $n \in \mathbb{N}$.

Theorem 3.5.12. *Suppose Assumptions B and C hold. For any $\mu^n, \mu \in [0,1]^{\mathbb{N}}$ such that $\mu^n \to \mu$, we have $S_{\mu^n}^k \to S_\mu^k$ for all $k \in \mathbb{N}$.*

Proof. Fix $(t,x) \in \mathbb{T} \times \mathbb{X}$, by Proposition 3.5.10, each $(\Phi^{\mu^n}(\emptyset))_t$ is in the form of $(-\infty, \varphi^n) \cap \mathbb{X}$ for some constant φ^n where $\varphi^n := \sup\{x \in \mathbb{R} : g(\mu_t^n) > V^*(t,x,\mu^n,\emptyset)\}$. By Lemma 3.5.6, the function $g(\mu_t^n) - V^*(t,x,\mu^n,\emptyset)$ converges to $g(\mu_t) - V^*(t,x,\mu,\emptyset)$ and it is strictly monotone and continuous in x by Lemmas 3.5.3 and 3.5.9. Therefore by Lemma 3.5.11, $\varphi^n = \sup\{x \in \mathbb{R} : g(\mu_t^n) > V^*(t,x,\mu^n,\emptyset)\}$ converges to $\varphi = \sup\{x \in \mathbb{R} : g(\mu_t) > V^*(t,x,\mu,\emptyset)\}$. Hence $(S_{\mu^n}^0)_t := (\Phi^{\mu^n}(\emptyset))_t = (-\infty, \varphi^n) \cap \mathbb{X}$ converges to $(S_\mu^0)_t := (\Phi^\mu(\emptyset))_t = (-\infty, \varphi) \cap \mathbb{X}$ and since this is true for all t and x we have $S_{\mu^n}^0 \to S_\mu^0$.

Next,

$$
\begin{aligned}
(S^1_{\mu^n})_t &:= (S^0_{\mu^n})_t \cup \{x \notin (S^0_{\mu^n})_t : g(\mu^n_t) > V^*(t, x, \mu^n, (S^0_{\mu^n})_t)\} \\
&= (S^0_{\mu^n})_t \cup \{x \in \mathbb{R} : g(\mu^n_t) > V^*(t, x, \mu^n, (S^0_{\mu^n})_t)\} \cap \mathbb{X}.
\end{aligned}
$$

Here the last equality comes from the fact $(S^0_{\mu^n})_t = (-\infty, \varphi^n) \cap \mathbb{X}$ and the set $\{x \in \mathbb{R} : g(\mu^n_t) > V^*(t, x, \mu^n, (S^0_{\mu})_t)\}$ is also in the form of $(-\infty, \psi^n)$ where $\psi^n = \sup\{x \in \mathbb{R} : g(\mu^n_t) > V^*(t, x, \mu^n, (S^0_{\mu^n})_t)\}$ as shown in the proof of Proposition 3.5.10. As $\mu^n \to \mu$, by previous step we have $(S^0_{\mu^n})_t \to (S^0_{\mu})_t$ and by Lemma 3.5.7, $V^*(t, x, \mu^n, S^0_{\mu^n}) \to V^*(t, x, \mu, S^0_{\mu})$. Again by Lemma 3.5.11 we have $\psi^n \to \psi$ and thus $S^1_{\mu^n} \to S^1_{\mu}$. Applying this procedure for all $k \in \mathbb{N}$, we have $S^k_{\mu^n} \to S^k_{\mu}$ as $\mu^n \to \mu$. ∎

Recall the optimal stopping region constructed by Φ w.r.t μ by,

$$
\Gamma^{\mu} := \bigcup_{n \in \mathbb{N}} S^n_{\mu}, \tag{3.5.6}
$$

which is an optimal equilibrium with respect to μ. If Γ^{μ} is also continuous in μ and then we can employ Tikhonov fixed-point theorem to find a fixed-point to the mapping $\mu \mapsto \{\mathbb{P}(\rho(-1, \Delta, \Gamma^{\mu})) \le t\}_{t \in \mathbb{T}}$.

Theorem 3.5.13. *In addition to Theorem 3.5.12, if $\Gamma^{\mu^n} \to \Gamma^{\mu}$ as $\mu^n \to \mu$ then there exists $\mu^* \in [0,1]^{\mathbb{N}}$ such that $\mathbb{P}(\rho(-1, \Delta, \Gamma^{\mu^*}) \le t) = \mu^*_t$ for all $t \in \mathbb{T}$. In particular, (Γ^{μ^*}, μ^*) is a sharp mean field equilibrium.*

Proof. As $\Gamma^{\mu^n} \to \Gamma^{\mu}$, Remark 3.5.5 implies that $\rho(-1, \Delta, \Gamma^{\mu^n}) \to \rho(-1, \Delta, \Gamma^{\mu})$ almost surly, hence $\mathbb{P}(\rho(-1, \Delta, \Gamma^{\mu^n}) \le t)\} \to \mathbb{P}(\rho(-1, \Delta, \Gamma^{\mu}) \le t)\}$ for all $t \in \mathbb{T}$. Therefore the mapping $F(\mu) := \{\mathbb{P}(\rho(-1, \Delta, \Gamma^{\mu}) \le t)\}_{t \in \mathbb{T}}$ is continuous since it is continuous in each component and clearly $F : [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$. As $[0,1]^{\mathbb{N}}$ is a nonempty, convex and compact subset of ℓ^{δ}_1, a locally convex topological vector space then by Tikhonov fixed-point theorem ([43]), we have the existence of a fixed-point μ^*. ∎

Remark 3.5.14. Under the current assumptions, although $S^k_{\mu^n} \to S^k_\mu$ for each $k \in \mathbb{N}$, we cannot conclude that $\Gamma^{\mu^n} = \bigcup_k S^k_{\mu^n}$ converges to $\Gamma^\mu = \bigcup_k S^k_\mu$. To see this, consider the following example. We consider the stopping regions at time $t = 0$ and suppose,

$$\left(S^k_{\mu^n}\right)_0 = \begin{cases} (-\infty, 0) \cap \mathbb{X}, & \forall k \leq n \\ (-\infty, 1) \cap \mathbb{X}, & \forall k > n, \end{cases}$$

and $\left(S^k_\mu\right)_0 = (-\infty, 0) \cap \mathbb{X}$ for all $k \in \mathbb{N}$. Then as $\mu^n \to \mu$, we have (for each fixed $k \in \mathbb{N}$) $(S^k_{\mu^n})_0 \to \left(S^k_\mu\right)_0$ while $(\Gamma^{\mu^n})_0 = (-\infty, 1) \cap \mathbb{X} \not\to (\Gamma^\mu)_0 = (-\infty, 0) \cap \mathbb{X}$.

3.5.1 A Case where $\Gamma^{\mu^n} \to \Gamma^\mu$.

In this subsection, we impose a set of conditions on top of Assumption B and Assumption C such that the convergence $\Gamma^{\mu^n} \to \Gamma^\mu$ is verified for any $\mu^n \to \mu$. For any $(t, x) \in \mathbb{T} \times \mathbb{X}$, $\mu \in [0, 1]^{\mathbb{N}}$ and $\tau \in \mathcal{T}_t$, define the following auxiliary function U, which has appeared in similar forms in the proofs of Proposition 3.5.10 and Theorem 3.5.12,

$$U(t, x, \mu, \tau) = \mathbb{E}^{t,x}\left[\sum_{k=t}^{\tau-1} \delta(k-t) f(X_k, \mu_k) + \delta(\tau - t) g(\mu_\tau)\right] - g(\mu_t). \tag{3.5.7}$$

Assumption D.

(D.1) $X_{t+1} = X_t + \xi_t$ where ξ_t are i.i.d for each $t \in \mathbb{T}$ with a probability density function bounded by some constant L;

(D.2) $\sum_{t=0}^{\infty} t\delta(t) < \infty$;

(D.3) there exists $a, b \in \mathbb{R}$ with $a < b$ such that

(i) for all $\mu \in [0, 1]^{\mathbb{N}}$ and $t \in \mathbb{T}$

$$\sup_{\tau \geq t+1} U(t, a, \mu, \tau) < 0,$$

and $\inf_{\nu \in [0,1]} f(a, \nu) > 0$;

(ii) for all $\mu \in [0,1]^{\mathbb{N}}$ and $t \in \mathbb{T}$ we have

$$f(b, \mu_{t+1}) + \delta(1)g(\mu_{t+1}) > g(\mu_t).$$

(iii) there exists $\alpha > 0$ such that for all $x \in [a,b]$, $\mu \in [0,1]$ and $\Delta x > 0$, we have,

$$\frac{f(x+\Delta x, \mu)}{f(x,\mu)} \geq 1 + \alpha \Delta x.$$

Remark 3.5.15. An example for (D.3) of Assumption D is given by

(1) $f(x,\mu) = F(x)G(\mu)$ where $F(x) \to 0$ as $x \to -\infty$, $\inf G > 0$ and $\sup G < \infty$;

(2) g is decreasing, $\inf g > 0$ and $\sup g < \infty$;

(3) $\lim_{x\to\infty} F(x) \cdot \inf G > \sup g$, F' exists on $[a,b]$ and satisfies $\inf_{x\in[a,b]} F'(x) > 0$.

It is straightforward to see that (2) and (3) implies (ii), and (3) also implies (iii). To see that (1) and (2) implies (i) we first observe that $\sup_{\tau \geq t+1} \mathbb{E}^{t,x}[\delta(\tau - t)g(\mu_\tau)] - g(\mu_t) < 0$ since

$$\sup_{\tau \geq t+1} \mathbb{E}^{t,x}[\delta(\tau-t)g(\mu_\tau)] - g(\mu_t) \leq \delta(1)g(\mu_{t+1}) - g(\mu_t) \leq (\delta(1)-1)g(1) < 0,$$

μ is decreasing and $\inf g > 0$. Denote $\gamma := -(\delta(1)-1)g(1) > 0$ which is independent of t. Next, we can separate the finite sum involving f as in the proof of Lemma 3.5.6,

$$\begin{aligned}
&\sup_{\tau \geq t+1} \mathbb{E}^{t,x}\left[\sum_{k=t}^{\tau-1} \delta(k-t) f(X_k, \mu_k)\right] \leq \sup G \cdot \mathbb{E}^x\left[\sum_{k=0}^{\infty} \delta(k) F(X_k)\right] \\
&\leq \sup G \cdot \left[\mathbb{E}^x\left[\sum_{k=0}^{N-1} \delta(k) F(X_k)\right] + \sup F \sum_{k=N}^{\infty} \delta(k)\right].
\end{aligned}$$

Here we remove the dependency on t in the first line as $F(X_k)$ is now time-homogeneous by (D.1). Choose N large enough so that the second term is smaller than $\frac{\gamma}{4 \sup G}$. By the fact that $\lim_{x\to-\infty} F(x) = 0$ and for each $k \in \{0,1,...,N-1\}$, $F(X_k^x)$ continuously depends on x so we can choose $x = a$ small enough such that the first term is also smaller than $\frac{\gamma}{4 \sup G}$

and hence,

$$\sup_{\tau \geq t+1} U(t, a, \mu, \tau) \leq \sup_{\tau \geq t+1} \mathbb{E}^{t,a}\left[\sum_{k=t}^{\tau-1} \delta(k-t) f(X_k, \mu_k)\right] + \sup_{\tau \geq t+1} \mathbb{E}^t[\delta(\tau - t) g(\mu_\tau)] - g(\mu_t)$$
$$\leq \frac{\gamma}{2} + (\delta(1) - 1) g(1) = -\gamma/2 < 0.$$

By Proposition 3.5.10, for each fixed $\mu \in [0,1]^{\mathbb{N}}$ there exists some constant C_t^n such that $(S_\mu^n)_t = (-\infty, C_t^n)$ for each $t \in \mathbb{T}$ where S_μ^n is defined by Equation (3.2.6). We find a bounded interval for these constants.

Lemma 3.5.16. *Suppose Assumption D holds. We have $C_t^n \in [a, b]$ for all $n \in \mathbb{N}$ and $t \in \mathbb{T}$ where a and b are given in (D.3) of Assumption D.*

Proof. This is a direct consequence of (D.3) of Assumption D. ■

Denote $\Delta C_t^n := C_t^{n+1} - C_t^n \leq b - a$ and $\Delta C^n = \sup_{t \in \mathbb{T}} \Delta C_t^n \leq b - a$.

Lemma 3.5.17. *Suppose Assumption D holds. There exists $\beta \in (0, 1)$ independent of $\mu \in [0,1]^{\mathbb{N}}$ such that $\Delta C_t^{n+1} \leq \beta \Delta C^n$. Hence $\Delta C_t^n \leq \beta^n (b - a)$.*

Proof. The proof is postponed to Appendix C.6. ■

The previous lemma says that the change in the stopping region after each iteration of Φ^μ decrease exponentially and uniformly in μ. This gives us the main result of this subsection.

Proposition 3.5.18. *Suppose Assumption D holds. If $\mu^n \to \mu$ then $\Gamma^{\mu^n} \to \Gamma^\mu$.*

Proof. Fix $t \in \mathbb{T}$ and $\varepsilon > 0$. By Lemma 3.5.17, we have

$$\sum_{k=0}^{\infty} \Delta C_t^k \leq \sum_{k=0}^{\infty} \beta^k (b - a) = (b - a) \sum_{k=0}^{\infty} \beta^k < \infty,$$

as a (convergent) geometric series. Hence there exists N_1 large enough such that for all $k \geq N_1$, $(\Gamma^{\mu^n})_t$ can be approximated by $(S_{\mu^n}^k)_t$ within $\varepsilon/3$ (i.e., $(\Gamma^{\mu^n})_t = (-\infty, C_t^n)$ and $S_{\mu^n}^k = (-\infty, C_t^{n,k})$ where $|C_t^n - C_t^{n,k}| \leq \varepsilon/3$ for all $k \geq N_1$). Note that this is independent of the choice of $\mu^n \in [0,1]^{\mathbb{N}}$ and therefore for the same reason, $(\Gamma^\mu)_t$ can be approximated by $(S^k_\mu)_t$ within $\varepsilon/3$ for all $k \geq N_1$. By Theorem 3.5.12, there exists N_2 large enough such that $(S^k_{\mu^n})_t$ can be approximated by $(S^k_\mu)_t$ within $\varepsilon/3$ for all $n \geq N_2$. Combining this together we have for all $\varepsilon > 0$ there exists N large enough such that for all $n \geq N$, $|C^n_t - C_t| \leq \varepsilon$ where $(\Gamma^{\mu^n})_t = (-\infty, C^n_t)$ and $(\Gamma^\mu)_t = (-\infty, C_t)$, i.e., $(\Gamma^\mu)_t$ can be approximated by $(\Gamma^{\mu^n})_t$ within ε for all $n \geq N$ thus completing the proof. ■

3.6 References and Remarks

The explicit construction of the optimal equilibrium Γ^μ in Section 3.2 is borrowed from [31, Section 3] and the references therein. The formulation of the bank run model is partially inspired by [11] and the iterative procedure for finding a sharp mean field equilibrium is partially borrowed from [31, Section 4].

One generalization of our model is to incorporate heterogeneous players. Where the choice of f, g and the discount function δ depends on the preference of each player, see, e.g. [33]. This may work in both examples.

We can also incorporate the common noise say Y_t and the process X_t is treated as an *idiosyncratic noise* unique to each player. As such μ is now a $\mathcal{P}(\mathbb{T})$-valued random variable, i.e., a random measure depending on the value of Y. With some effort, we should be able to extend the bank run model in this direction.

Appendix A

Proof of Lemmas in Section 2.2

A.1 Proof of Lemma 2.2.1

Proof. We prove by contradiction. Suppose that Equation (2.2.1) does not hold, in other words, there exists some nonnegative constant

$$\nu := \sup_{(t,x)\in(\mathbb{T}\times\mathbb{X})} \left(J(t,x,\rho(t,x,S)) - J(t,x,\rho(t,x,\Theta(S))\right) > 0. \tag{A.1.1}$$

We can choose some $(t,y) \in (\mathbb{T}\times\mathbb{X})$ such that

$$J(t,y,\rho(t,y,S)) - J(t,y,\rho(t,y,\Theta(S))) > \delta(1)\nu. \tag{A.1.2}$$

Next, fix $S \in \mathcal{B}(\mathbb{T}\times\mathbb{X})$ nonempty and consider the event $E := E_1 \cap E_2$, where

$$E_1 := \{\omega\in\Omega : \rho(t,y,S)(\omega) < \infty\} \quad \text{and} \quad E_2 := \left\{\omega\in\Omega : X^{t,y}_{\rho(t,y,S)}(\omega) \in (S\setminus\Theta(S))_{\rho(t,y,S)}\right\}.$$

First consider the complement event $(E_1)^c$, $\Theta(S)\subseteq S$ implies that $\rho(t,y,\Theta(S)) \geq \rho(t,y,S) = \infty$. By the boundness of f we must have $\delta(\rho(t,y,S)-t)f(X_{\rho(t,y,S)}) = \delta(\rho(t,y,\Theta(S))-t) = 0$. On the other hand, on the event $E_1\cap(E_2)^c$, we have $\rho(t,y,S) < \infty$ and $(\Theta(S))_{\rho(t,y,s)} \subseteq (S)_{\rho(t,y,s)}$ implying that $X^y_{\rho(t,y,S)} \in (\Theta(S))_{\rho(t,y,s)}$. Which further implies that $\rho(t,y,\Theta(S)) =$

$\rho(t, y, S)$ and therefore we have

$$\begin{aligned}
&\sum_{s=t}^{\rho(t,y,S)-1} \delta(s-t) g(X_s) + \delta(\rho(t,y,S)-t) f(X_{\rho(t,y,S)}) \\
= &\sum_{s=t}^{\rho(t,y,\Theta(s))-1} \delta(s-t) g(X_s) + \delta(\rho(t,y,\Theta(s))-t) f(X_{\rho(t,y,\Theta(s))})
\end{aligned}$$

almost surely on $(E_1)^c$ and $E_1 \cap (E_2)^c$. Consequently,

$$\begin{aligned}
&J(t,y,\rho(t,y,S)) - J(t,y,\rho(t,y,\Theta(S))) \\
&= \mathbb{E}^{t,y}\Bigg[\mathbb{1}_E \Bigg\{ \sum_{s=t}^{\rho(t,y,S)-1} \delta(s-t) g(X_s) + \delta(\rho(t,y,S)-t) f(X_{\rho(t,y,S)}) \\
&\qquad - \sum_{s=t}^{\rho(t,y,\Theta(S))-1} \delta(s-t) g(X_s) - \delta(\rho(t,y,\Theta(S))-t) f(X_{\rho(t,y,\Theta(S))}) \Bigg\}\Bigg]
\end{aligned}$$

To simplify the notation we write:

1. $\tau_1 := \rho(t, y, S)$;
2. $\kappa_1 := \rho(t, y, \Theta(S))$ and $\kappa_2 := \rho(\tau_1, X_{\tau_1}^{t,y}, \Theta(S))$.

Now,

$$\begin{aligned}
&J(t,y,\tau_1) - J(t,y,\kappa_1) \\
&= \mathbb{E}^{t,y}\left[\mathbb{1}_E \left\{ \sum_{s=t}^{\tau_1-1} \delta(s-t) g(X_s) + \delta(\tau_1-t) f(X_{\tau_1}) - \sum_{s=t}^{\kappa_1-1} \delta(s-t) g(X_s) - \delta(\kappa_1-t) f(X_{\kappa_1}) \right\}\right] \\
&= \mathbb{E}^{t,y}\left[\mathbb{1}_E \left\{ - \sum_{s=\tau_1}^{\kappa_1-1} \delta(s-t) g(X_s) + \delta(\tau_1-t) f(X_{\tau_1}) - \delta(\kappa_1-t) f(X_{\kappa_1}) \right\}\right] \\
&= \mathbb{E}^{t,y}\left[\mathbb{1}_E \delta(\tau_1-t) \mathbb{E}^{t,y}\left[f(X_{\tau_1}) - \sum_{s=\tau_1}^{\kappa_1-1} \frac{\delta(s-t)}{\delta(\tau_1-t)} g(X_s) - \frac{\delta(\kappa_1-t)}{\delta(\tau_1-t)} f(X_{\kappa_1}) \,\middle|\, \mathcal{F}_{\tau_1} \right]\right], \qquad \text{(A.1.3)}
\end{aligned}$$

where the last equality comes from the tower property of conditional expectation. Using (A.2) of Assumption A we have,

$$\frac{\delta(\kappa_1 - t)}{\delta(\tau_1 - t)} \geq \delta(\kappa_1 - \tau_1) \quad \text{and} \quad \sum_{s=\tau_1}^{\kappa_1 - 1} \frac{\delta(s-t)}{\delta(\tau_1 - t)} g(X_s) \geq \sum_{s=\tau_1}^{\kappa_1 - 1} \delta(s - \tau_1) g(X_s). \tag{A.1.4}$$

Substituting Equation (A.1.4) into Equation (A.1.3) we have,

$$\begin{aligned} J(t, y, \tau_1) - J(t, y, \kappa_1) \leq \mathbb{E}^{t,y} \Bigg[\mathbb{1}_E \delta(\tau_1 - t) \Bigg\{ f(X_{\tau_1}) - \mathbb{E}^{t,y} \Bigg[\Bigg\{ \sum_{s=\tau_1}^{\kappa_1 - 1} \delta(s - \tau_1) g(X_s) \\ + \delta(\kappa_1 - \tau_1) f(X_{\kappa_1}) \Bigg\} \Bigg| \mathcal{F}_{\tau_1} \Bigg] \Bigg\} \Bigg] \end{aligned} \tag{A.1.5}$$

Next, by the strong Markov property of X, it holds a.s. that,

$$\begin{aligned} &\mathbb{E}^{t,y} \left[\sum_{s=\tau_1}^{\kappa_1 - 1} \delta(s - \tau_1) g(X_s) + \delta(\kappa_1 - \tau_1) f(X_{\kappa_1}) \middle| \mathcal{F}_{\tau_1} \right] \mathbb{1}_E \\ &= \mathbb{E}^{\tau_1, X^{t,y}_{\tau_1}} \left[\sum_{s=\tau_1}^{\kappa_2 - 1} \delta(s - \tau_1) g(X_s) + \delta(\kappa_2 - \tau_1) f(X_{\kappa_2}) \right] \mathbb{1}_E \\ &= J(\tau_1, X_{\tau_1}, \kappa_2) \mathbb{1}_E. \end{aligned}$$

Taking the above equation into Equation (A.1.5), we have,

$$\begin{aligned} &J(t, y, \rho(t, y, s)) - J(t, y, \rho(t, y, \Theta(S))) \\ &\leq \mathbb{E}^{t,y} \left[\mathbb{1}_E \delta(\tau_1 - t)) \left\{ f(X_{\tau_1}) - J(\tau_1, X_{\tau_1}, \rho(\tau_1, X_{\tau_1}, \Theta(S)) \right\} \right] \\ &\leq \mathbb{E}^{t,y} \left[\mathbb{1}_E \delta(\tau_1 - t) \left\{ J(\tau_1, X_{\tau_1}, \rho(\tau_1, X_{\tau_1}, S)) - J(\tau_1, X_{\tau_1}, \rho(\tau_1, X_{\tau_1}, \Theta(S)) \right\} \right] \\ &\leq \mathbb{E}^{t,y} \left[\mathbb{1}_E \delta(\tau_1 - t) \nu \right] \\ &\leq \delta(1) \nu. \end{aligned}$$

The second inequality in the above equations is by the definition of Θ, $f(x) < J(t, x, \rho(t, x, S))$ for all $x \notin (\Theta(S))_t$. The last inequality comes from Equation (A.1.1). Therefore we have a contradiction and thus establishing Equation (2.2.1).

Fix $t \in \mathbb{T}$. For any $x \notin (\Theta(S))_t$, using the definition of Θ and what we just proved, we have,

$$f(x) < J(t, x, \rho(t, x, S)) \leq J(t, x, \rho(t, x, \Theta(S))),$$

and therefore $x \notin (\Theta^2(S))_t$ which then implies $(\Theta^2(S))_t \subseteq (\Theta(S))_t$. Since the previous argument works for all $t \in \mathbb{T}$, we have $\Theta(S) \subseteq S$. ■

A.2 Proof of Lemma 2.2.6

Proof. Fix some arbitrary initial condition $(t, x) \in \mathbb{T} \times \mathbb{X}$. We shall define, for simplicity, the following notation for all $n = 0, 1, 2, \ldots$,

1. $y_0 := x, \quad y_{2n+1} := X^{\tau_{2n}, y_{2n}}_{\rho(\tau_{2n}, y_{2n}, R)}, \quad y_{2n+2} := X^{\tau_{2n+1}, y_{2n+1}}_{\rho(\tau_{2n+1}, y_{2n+1}, S)}$;
2. $\tau_0 := t, \quad \tau_{2n+1} := \rho(\tau_{2n}, y_{2n}, R), \quad \tau_{2n+2} := \rho(\tau_{2n+1}, y_{2n+1}, S)$;
3. $E_n := \{\omega \in \Omega : \tau_n(\omega) < \infty \text{ and } y_n(\omega) \notin S \cap R\}$;
4. $\mathfrak{E}_n(Y) := \mathbb{E}^{\tau_{n-1}, y_{n-1}}[\mathbb{1}_{E_n} \delta(\tau_n - \tau_{n-1}) Y]$ for any random variable $Y : \Omega \mapsto \mathbb{R}$;
5. $J(t, x, \widehat{S}) := J(t, x, \rho(t, x, \widehat{S}))$ for any $\widehat{S} \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$.
6. $\kappa_{2n+1} := \rho(\tau_{2n}, y_{2n}, S \cap R)$ and $\kappa_{2n+2} := \rho(\tau_{2n+1}, y_{2n+1}, S \cap R)$

By the definition of E_1, we have,

$$\begin{aligned}
&J(t, x, S \cap R) - J(t, x, R) \\
&= \mathbb{E}^{\tau_0, y_0}\Bigg[\mathbb{1}_{E_1}\Bigg\{\sum_{s=t}^{\kappa_1 - 1} \delta(s - t) g(X_s) + \delta(\kappa_1 - t) f(X_{\kappa_1}) \\
&\qquad - \sum_{s=t}^{\tau_1 - 1} \delta(s - t) g(X_s) - \delta(\tau_1 - t) f(y_1)\Bigg\}\Bigg] \\
&= \mathbb{E}^{\tau_0, y_0}\Bigg[\mathbb{1}_{E_1}\Bigg\{\sum_{s=\tau_1}^{\kappa_1 - 1} \delta(s - t) g(X_s) + \delta(\kappa_1 - t) f(X_{\kappa_1}) \\
&\qquad - \delta(\tau_1 - t) f(y_1)\Bigg\}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}^{\tau_0,y_0}\left[\mathbb{1}_{E_1}\delta(\tau_1-t)\mathbb{E}^{\tau_0,y_0}\left[\sum_{s=\tau_1}^{\kappa_1-1}\frac{\delta(s-t)}{\delta(\tau_1-t)}g(X_s)\right.\right.\\
&\qquad\left.\left.+\frac{\delta(\kappa_1-t)}{\delta(\tau_1-t)}f(X_{\kappa_1})-f(y_1)\,\middle|\,\mathcal{F}_{\tau_1}\right]\right]\\
&\geq \mathbb{E}^{\tau_0,y_0}\left[\mathbb{1}_{E_1}\delta(\tau_1-t)\left\{\mathbb{E}^{\tau_0,y_0}\left[\sum_{s=\tau_1}^{\kappa_1-1}\delta(s-\tau_1)g(X_s)\right.\right.\right.\\
&\qquad\left.\left.\left.+\,\delta(\kappa_1-\tau_1)f(X_{\kappa_1})\,\middle|\,\mathcal{F}_{\tau_1}\right]-f(y_1)\right\}\right],
\end{aligned}
\tag{A.2.1}
$$

where we use the tower property of conditional expectation and invoke (A.2) of Assumption A similarly as in the proof of Lemma 2.2.1. Again, by the strong Markov property of X we obtain,

$$
\begin{aligned}
&\mathbb{E}^{\tau_0,y_0}\left[\sum_{s=\tau_1}^{\kappa_1-1}\delta(s-\tau_1)g(X_s)+\delta(\kappa_1-\tau_1)f(X_{\kappa_1})\,\middle|\,\mathcal{F}_{\tau_1}\right]\mathbb{1}_{E_1}\\
&=\mathbb{E}^{\tau_1,y_1}\left[\sum_{s=t}^{\kappa_2-1}\delta(s-\tau_1)g(X_s)+\delta(\kappa_2-\tau_1)f(X_{\kappa_2})\right]\mathbb{1}_{E_1}\\
&=J(\tau_1,y_1,S\cap R)\mathbb{1}_{E_1}.
\end{aligned}
\tag{A.2.2}
$$

Moreover, notice that on the event E_1, we have $y_1 \in (R)_{\tau_1}$ but $y_1 \notin (S\cap R)_{\tau_1}$ this means that $y_1 \notin (S)_{\tau_1}$. Thus by the discussion of Remark 2.1.8 we have,

$$
f(y_1) < J(\tau_1, y_1, S). \tag{A.2.3}
$$

Together Equations (A.2.1), (A.2.2) and (A.2.3) give

$$
\begin{aligned}
J(t,x,S\cap R)-J(t,x,R) &\geq \mathbb{E}^{\tau_0,y_0}\left[\mathbb{1}_{E_1}\delta(\tau_1-\tau_0)\left(J(\tau_1,y_1,S\cap R)-J(\tau_1,y_1,S)\right)\right]\\
&=\mathfrak{E}_1\left(J(\tau_1,y_1,S\cap R)-J(\tau_1,y_1,S)\right).
\end{aligned}
\tag{A.2.4}
$$

Now, we will apply the above argument in a recursive manner. First, repeat the previous procedure for $J(\tau_1, y_1, S \cap R) - J(\tau_1, y_1, S)$, instead of $J(t, x, S \cap R) - J(t, x, R)$ gives us,

$$J(\tau_1, y_1, S \cap R) - J(\tau_1, y_1, S) \geq \mathfrak{E}_2(J(\tau_2, y_2, S \cap R) - J(\tau_2, y_2, R))$$

Combining this with Equation (A.2.4), we have,

$$J(t, x, S \cap R) - J(t, x, R) \geq \mathfrak{E}_1 \circ \mathfrak{E}_2(J(\tau_2, y_2, S \cap R) - J(\tau_2, y_2, R)). \tag{A.2.5}$$

Next, applying the above procedure to $J(\tau_2, y_2, S \cap R) - J(\tau_2, y_2, R)$, instead of $J(t, x, S \cap R) - J(t, x, R)$ and continuing this procedure for $2n$ number of times, we have

$$\begin{aligned} &J(t, x, S \cap R) - J(t, x, R) \\ &\geq \mathfrak{E}_1 \circ \cdots \circ \mathfrak{E}_{2n}(J(\tau_{2n}, y_{2n}, S \cap R) - J(\tau_{2n}, y_{2n}, R)), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{A.2.6}$$

By Assumption A, there exists a constant $C > 0$ such that $|J(s, y, \widehat{S})| \leq C$ for all $(s, y) \in \mathbb{T} \times \mathbb{X}$ and $\widehat{S} \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$. Then,

$$\begin{aligned} &|\mathfrak{E}_1 \circ \cdots \mathfrak{E}_{2n}(J(\tau_{2n}, y_{2n}, S \cap R) - J(\tau_{2n}, y_{2n}, R))| \\ &\quad \leq \mathfrak{E}_1 \circ \cdots \circ \mathfrak{E}_{2n}(|J(\tau_{2n}, y_{2n}, S \cap R) - J(\tau_{2n}, y_{2n}, R)|) \leq \delta(1)^{2n} \cdot 2C \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

as $\delta(1) < 1$. Then by Equation (A.2.6) and the fact that the choice of (t, x) is arbitrary,

$$J(t, x, S \cap R) - J(t, x, R) \geq 0 \quad \forall (t, x) \in \mathbb{T} \times \mathbb{X}. \tag{A.2.7}$$

Similarly, interchanging the role of S and R, we conclude,

$$J(t, x, S \cap R) - J(t, x, S) \geq 0 \quad \forall (t, x) \in \mathbb{T} \times \mathbb{X}. \tag{A.2.8}$$

Combining Equations (A.2.7) and (A.2.8) this completes the proof of Equation (2.2.8). Fix $t \in \mathbb{T}$, for any $x \notin (S \cap R)_t$, if $x \notin (S)_t$, then by definition and Equation (2.2.8) $f(x) < J(t, x, S) \leq J(t, x, S \cap R)$, which implies $x \notin (\Theta(S \cap R))_t$. Similarly, if $x \notin (R)_t$, we have $x \notin (\Theta(S \cap R))_t$. Since the above works for all t, we can conclude $\Theta(S \cap R) \subseteq S \cap R$. ■

Appendix B

Proof of Lemmas in Section 3.2

B.1 Proof of Lemma 3.2.5

Proof. Consider the event $E := E_1 \cap E_2$, where,

$$E_1 := \{\omega \in \Omega : \rho(t, y, S)(\omega) < \infty\} \quad \text{and} \quad E_2 := \left\{\omega \in \Omega : X^{t,x}_{\rho(t,x,S)}(\omega) \in (S \setminus R)_{\rho(t,x,S)}\right\}.$$

For the same argument as in Lemma 2.2.1,

$$\begin{aligned}
&\sum_{s=t}^{\rho(t,y,S)-1} \delta(s-t) g(X_s, \mu_s) + \delta(\rho(t,y,S)-t) f(X_{\rho(t,y,S)}, \mu_{\rho(t,y,S)}) \\
&= \sum_{s=t}^{\rho(t,y,R)-1} \delta(s-t) g(X_s, \mu_s) + \delta(\rho(t,y,R)-t) f(X_{\rho(t,y,R)}, \mu_{\rho(t,y,R)})
\end{aligned}$$

almost surely on the event E^c. To simplify the notation we write:

1. $\tau_1 := \rho(t, x, S)$ and $\tau_2 := \rho(t, x, R)$;

2. $\kappa_1 := \rho(\tau_1, X^{t,x}_{\tau_1}, R)$.

We have,

$$\begin{aligned}
&J(t, x, \mu, \rho(t, x, R)) - J(t, x, \mu, \rho(t, x, S)) \\
&= \mathbb{E}^{t,x}\left[\mathbb{1}_E \left\{\sum_{s=\tau_1}^{\tau_2-1} \delta(s-t) g(X_s, \mu_s) + \delta(\tau_2-t) f(X_{\tau_2}, \mu_{\tau_2}) - \delta(\tau_1-t) f(X_{\tau_1}, \mu_{\tau_1})\right\}\right] \\
&= \mathbb{E}^{t,x}\left[\mathbb{1}_E \delta(\tau_1-t) \mathbb{E}^{t,x}\left[\sum_{s=\tau_1}^{\tau_2-1} \frac{\delta(s-t)}{\delta(\tau_1-t)} g(X_s, \mu_s) + \frac{\delta(\tau_2-t)}{\delta(\tau_1-t)} f(X_{\tau_2}, \mu_{\tau_2}) - f(X_{\tau_1}, \mu_{\tau_1}) \,\middle|\, \mathcal{F}_{\tau_1}\right]\right]
\end{aligned}$$

$$\geq \mathbb{E}^{t,x}\left[\mathbb{1}_E\delta(\tau_1-t)\left\{\mathbb{E}^{t,x}\left[\sum_{s=\tau_1}^{\tau_2-1}\delta(s-\tau_1)g(X_s,\mu_s)+\delta(\tau_2-\tau_1)f(X_{\tau_2},\mu_{\tau_2})\,\middle|\,\mathcal{F}_{\tau_1}\right]-f(X_{\tau_1},\mu_{\tau_1})\right\}\right]$$
$$=\mathbb{E}^{t,x}\left[\mathbb{1}_E\delta(\tau_1-t)\left\{\mathbb{E}^{\tau_1,X^{t,x}_{\tau_1}}\left[\sum_{s=\tau_1}^{\kappa_1-1}\delta(s-\tau_1)g(X_s,\mu_s)+\delta(\kappa_1-\tau_1)f(X_{\kappa_1},\mu_{\kappa_1})\right]-f(X_{\tau_1},\mu_{\tau_1})\right\}\right]$$
$$=\mathbb{E}^{t,x}\left[\mathbb{1}_E\delta(\tau_1-t)\left\{J(\tau_1,X_{\tau_1},\mu,R)-f(X_{\tau_1},\mu_{\tau_1})\right\}\right]\geq 0.$$

Here we use (A.2) of Assumption A in the third line and apply the strong Markov property to the inner expectation in deriving the fourth line. Finally, under the event E and by Equation (3.2.3) we have $f(x,\mu_t)\leq J(t,x,\mu,R)$ for all $x\in\mathbb{X}\setminus(R)_t$, thus the term inside the expectation in the last line must be non-negative and hence establishing the desired result. ■

B.2 Proof of Lemma 3.2.6

Proof. Fix $(t,x)\in\mathbb{X}$. Since $S^n\subseteq S^\infty$, we have $\rho(t,x,S^n)\geq\rho(t,x,S^\infty)$, thus by the definition of V^*, $V^*(t,x,\mu,S^n)\geq V^*(t,x,\mu,S^\infty)$ for all n and

$$\lim_{n\to\infty}V^*(t,x,\mu,S^n)\geq V^*(t,x,\mu,S^\infty). \tag{B.2.1}$$

To prove the other inequality, let $\tau_n\in\mathcal{T}_t$ with $t+1\leq\tau_n\leq\rho(t,x,\mu,S^n)$ be a $\frac{1}{n}$-optimizer of $V^*(t,x,\mu,S^n)$ for each $n\in\mathbb{N}$. Then, writing $\rho_\infty:=\rho(t,x,S^\infty)$ and $\rho_n:=\rho(t,x,S^n)$,

$$\begin{aligned}
&V^*(t,x,\mu,S^n)-V^*(t,x,\mu,S^\infty)\\
&\leq J(t,x,\mu,\tau_n)+\frac{1}{n}-J(t,x,\mu,\tau_n\wedge\rho_\infty)\\
&\leq \mathbb{E}^{t,x}\Bigg[\sum_{s=\tau_n\wedge\rho_\infty}^{\tau_n}\delta(s-t)g(X_s,\mu_s)+(\delta(\tau_n-t)f(X_{\tau_n},\mu_{\tau_n})\\
&\qquad-\delta((\tau_n\wedge\rho_\infty)-t)f(X_{\tau_n\wedge\rho_\infty},\mu_{\tau_n\wedge\rho_\infty})\Bigg]+\frac{1}{n}
\end{aligned} \tag{B.2.2}$$

Now, for any $\omega\in\Omega$, we have, by definition, $\tau_n(\omega)\leq\rho_n(\omega)$ and by Lemma 3.2.4 we also have $\rho_n(\omega)\to\rho_\infty(\omega)$. If $\rho_\infty(\omega)<\infty$, this implies there exists N large enough such that

$\tau_n(\omega) = (\tau_n \wedge \rho_\infty)(\omega)$ for all such $n \geq N$. If $\rho_\infty(\omega) = \infty$ then $\tau_n(\omega) = (\tau_n \wedge \rho_\infty)(\omega)$ holds with triviality. Thus, taking limit on both sides of Equation (B.2.2) and applying the dominated convergence theorem we then have,

$$\lim_{n\to\infty} V^*(t, x, \mu, S^n) \leq V^*(t, x, \mu, S^\infty),$$

proving the other inequality. ■

Appendix C

Proof of Lemmas in Section 3.5

C.1 Proof of Lemma 3.5.3

Proof. For any $\omega \in \Omega$, by Remark 3.5.1, $X^{x_2}(\omega) > X^{x_1}(\omega)$ therefore, by the form of the stopping region T, we must have $\rho(t, x_2, T)(\omega) \geq \rho(t, x_1, T)(\omega)$. Hence, $\rho(t, x_2, T) \geq \rho(t, x_1, T)$ almost surely. Consequently, as J is also strictly increasing in x for any $\tau \geq t$,

$$\begin{aligned}
V^*(t, x_1, \mu, T) &= \sup_{t<\tau\leq\rho(t,x_1,T)} J(t, x_1, \mu, \tau) \\
&< \sup_{t<\tau\leq\rho(t,x_1,T)} J(t, x_2, \mu, \tau) \\
&\leq \sup_{t<\tau\leq\rho(t,x_2,T)} J(t, x_2, \mu, \tau) \\
&= V^*(t, x_2, \mu, T),
\end{aligned}$$

where the strict inequality comes from (B.2) of Assumption B that f is strictly increasing in x. Indeed if $x_1 < x_2$ then

$$\begin{aligned}
V^*(t, x_1, \mu, T) &= \sup_{t<\tau\leq\rho(t,x_1,T)} \mathbb{E}^{t,x_1}\left[\sum_{k=t}^{\tau-1} \delta(k-t) f(X_k, \mu_k) + \delta(\tau - t) g(\mu_\tau)\right] \\
&= f(X_t^{t,x_1}, \mu_t) + \sup_{t<\tau\leq\rho(t,x_1,T)} \mathbb{E}^{t,x_1}\left[\sum_{k=t+1}^{\tau-1} \delta(k-t) f(X_k, \mu_k) + \delta(\tau - t) g(\mu_\tau)\right] \\
&< f(X_t^{t,x_2}, \mu_t) + \sup_{t<\tau\leq\rho(t,x_1,T)} \mathbb{E}^{t,x_2}\left[\sum_{k=t+1}^{\tau-1} \delta(k-t) f(X_k, \mu_k) + \delta(\tau - t) g(\mu_\tau)\right]
\end{aligned}$$

$$\leq f(X_t^{t,x_2}, \mu_t) + \sup_{t<\tau\leq\rho(t,x_2,T)} \mathbb{E}^{t,x_2}\left[\sum_{k=t+1}^{\tau-1} \delta(k-t) f(X_k, \mu_k) + \delta(\tau - t) g(\mu_\tau)\right]$$
$$= V^*(t, x_2, \mu, T),$$

since $f(X_t^{t,x_1}, \mu_t) < f(X_t^{t,x_2}, \mu_t)$ by (B.2) of Assumption B. ■

C.2 Proof of Lemma 3.5.4

Proof. Fix any $\omega \in \Omega$. Let $\zeta(\omega) := \rho(t,x,T)(\omega)$ and assume $X_\zeta^{t,x}(\omega) \in (T)_{\zeta(\omega)}$. Then by the openness of $(T)_{\zeta(\omega)}$ there exists N large enough such that $X_{\zeta(\omega)} \in (T_n)_{\zeta(\omega)}$ for all $n \geq N$ hence we have,

$$\limsup_{n\to\infty} \rho(t,x,T_n) \leq \rho(t,x,T) \quad \text{a.s.} \tag{C.2.1}$$

Next, by (C.2) of Assumption C, the process $X_k^{t,x}$ has a density therefore the probability that $X_k^{t,x}$ sits on the boundary C_k is 0 and $X_k^{t,x} \in (C_k, \infty)$ with probability 1. Hence, consider $\omega \in A := \{\omega : X_k^{t,x}(\omega) \neq C_k \, \forall\, k \geq t\}$ with $\mathbb{P}(A) = 1$ and suppose there exists k such that $t < k < \zeta(\omega)$ with $X_k^{t,x}(\omega) \in (T_n)_k$ for some n but $X_k^{t,x}(\omega) \notin (T)_k = (-\infty, C_k) \cap \mathbb{X}$. Then, for a.e. $\omega \in \Omega$ there exists another N such that $X_k^{t,x}(\omega) \notin (T_n)_k$ for all $n \geq N$ and all such k. Therefore,

$$\liminf_{n} \rho(t,x,T_n) \geq \rho(t,x,T) \quad \text{a.s.} \tag{C.2.2}$$

Note that if $\rho(t,x,T) = \infty$ then the first inequality holds trivially and since $\mathbb{P}(\cup_k X_k^{t,x} = C_k) = 0$ as a countable union of sets with probability zero the second inequality also holds. Combining Equations (C.2.1) and (C.2.2), we have for all $(t,x) \in \mathbb{T} \times \mathbb{X}$

$$\lim_{n\to\infty} \rho(t,x,T_n) = \rho(t,x,T) \quad \text{a.s.} \tag{C.2.3}$$

For $(t, x) \in \mathbb{T} \times \mathbb{X}$ and $\mu \in [0,1]^{\mathbb{N}}$ denote

$$G(\mu, \tau) = \sum_{k=t}^{\tau-1} \delta(k-t) f(X_k, \mu_k) + \delta(\tau - t) g(\mu_\tau).$$

Let τ_n such that $t < \tau_n \le \rho(t, x, T_n)$ be a $\frac{1}{n}$-optimizer of $V^*(t, x, \mu, T_n)$ for $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\limsup_{n\to\infty} [V^*(t, x, \mu, T_n) - V^*(t, x, \mu, T)] \\
&\le \limsup_{n\to\infty} \left[\mathbb{E}^{t,x}[G(\mu, \tau_n)] + \frac{1}{n} - \mathbb{E}^{t,x}[G(\mu, \tau_n \wedge \rho(t, x, T))]\right] \\
&= \limsup_{n\to\infty} \left[\mathbb{E}^{t,x}[G(\mu, \tau_n) - G(\mu, \tau_n \wedge \rho(t, x, T))] + \frac{1}{n}\right] = 0 \qquad \text{(C.2.4)}
\end{aligned}$$

By first part of this proof, for a.e. $\omega \in \Omega$, if $\rho(t, x, T)(\omega) < \infty$ there exists N such that $\tau_n(\omega) \le \rho(t, x, T_n)(\omega) = \rho(t, x, T)(\omega)$ for all $n \ge N$. Otherwise, if $\rho(t, x, T)(\omega) = \infty$ then the equality $\tau_n(\omega) \le \rho(t, x, T_n)(\omega) = \rho(t, x, T)(\omega)$ holds trivially. Hence, taking limit on both sides and applying the dominated convergence theorem we have the convergence to 0.

On the other hand, let τ_n such that $t < \tau_n \le \rho(t, x, T)$ be a $\frac{1}{n}$-optimizer of $V^*(t, x, \mu, T)$ for all $n \in \mathbb{N}$ instead. Then similarly, for each $\omega \in \Omega$, if $\rho(t, x, T)(\omega) < \infty$ then there exists N such that $\tau_n(\omega) \le \rho(t, x, T)(\omega) = \rho(t, x, T_n)(\omega)$ for all $n \ge N$. If $\rho := \rho(t, x, T)(\omega) = \infty$ we denote $\tau_n := \tau_n(\omega)$, $\rho_n := \rho(t, x, T_n)(\omega)$ for simplicity, then

$$\begin{aligned}
&G(\mu, \tau_n) - G(\mu, \tau_n \wedge \rho(t, x, T_n)) \\
&\le \sum_{k=\tau_n\wedge\rho_n}^{\tau_n - 1} \delta(k-t) f(X_k, \mu_k) + \delta(\tau_n) g(\mu_{\tau_n}) - \delta(\tau_n \wedge \rho_n) g(\mu_{\tau_n\wedge\rho_n}) \\
&\le K \sum_{k=\tau_n\wedge\rho_n}^{\tau_n - 1} \delta(k-t) + J \cdot [\delta(\tau_n) - \delta(\tau_n \wedge \rho_n)] \\
&= \left\{ K \sum_{k=\rho_n}^{\tau_n - 1} \delta(k-t) + J \cdot [\delta(\tau_n) - \delta(\tau_n \wedge \rho_n)] \right\} \cdot \mathbb{1}_{\{\tau_n > \rho_n\}} \to 0.
\end{aligned}$$

As $\rho_n \to \rho = \infty$, the first sum converges to 0 and the second term also converges to zero as $\tau_n \wedge \rho_n \to \tau_n$. Therefore, by dominated convergence theorem, we can take the limit under the expectation and hence

$$
\begin{aligned}
&\limsup_{n\to\infty} \left[V^*(t,x,\mu,T) - V^*(t,x,\mu,T_n)\right] \\
&\leq \limsup_{n\to\infty} \left[\mathbb{E}^{t,x}[G(\mu,\tau_n) - G(\mu,\tau_n \wedge \rho(t,x,T_n))] + \frac{1}{n}\right] = 0. \qquad \text{(C.2.5)}
\end{aligned}
$$

Combining Equations (C.2.4) and (C.2.5), we have,

$$
\lim_{n\to\infty} \Big|V^*(t,x,\mu,T_n) - V^*(t,x,\mu,T)\Big| = 0,
$$

whenever, $T_n \to T$ in the sense of Equation (3.5.5). ■

C.3 Proof of Lemma 3.5.6

Proof. Fix any $\mu \in [0,1]^{\mathbb{N}}$ and let $\{\mu^n\}_{n\in\mathbb{N}}$ converges to μ. By (B.3) of Assumption B, for every $\varepsilon_2 > 0$ there exists $N^\dagger$ such that $\sum_{k=N^\dagger}^{\infty} \delta(k-t) < \varepsilon_2$. and there exists δ_1 and δ_2 such that whenever $|\mu_i^n - \mu_i| < \delta_1$ and $|\mu_i^n - \mu_i| < \delta_2$ we have $|f(x,\mu_i) - f(x,\mu_i^n)| < \varepsilon_1$ and $|g(\mu_i) - g(\mu_i^n)| < \varepsilon_1$, let $\delta := \min\{\delta_1, \delta_2\}$. We consider the first $N^\dagger$ terms of each n of the sequence μ^n and the limit μ. We can find $N^* = \max\{N_1, N_2, ..., N_{N^\dagger}\}$ where N_i is such that for all $n \geq N_i$, $|\mu_i^n - \mu_i| \leq \delta$ then for all $n \geq N^\dagger$, $|\mu_i^n - \mu_i| \leq \max_{l\leq N^\dagger} |\mu_l^n - \mu_l| \leq \delta$ for each $i \leq N^\dagger$. Hence, for all $n \geq N^*$,

$$
\begin{aligned}
&|V^*(t,x,\mu,T) - V^*(t,x,\mu^n,T)| \\
&\leq \sup_{t<\tau\leq\rho(t,x,T)} \mathbb{E}^{t,x}\left[\Big|\sum_{k=t}^{\tau-1} \delta(k-t)(f(X_k,\mu_k) - f(X_k,\mu_k^n)) + \delta(\tau-t)(g(\mu_\tau) - g(\mu_\tau^n))\Big|\right] \\
&\leq \sup_{t<\tau\leq\infty} \mathbb{E}^{t,x}\left[\sum_{k=t}^{\infty} \delta(k-t)\Big|f(X_k,\mu_k) - f(X_k,\mu_k^n)\Big| + \delta(\tau-t)\Big|g(\mu_\tau) - g(\mu_\tau^n)\Big|\right] \\
&\leq \mathbb{E}^{t,x}\left[\sum_{k=t}^{N^\dagger-1} \delta(k-t)\Big|f(X_k,\mu_k) - f(X_k,\mu_k^n)\Big| + \sum_{k=N^\dagger}^{\infty} \delta(k-t)\Big|f(X_k,\mu_k) - f(X_k,\mu_k^n)\Big|\right] \\
&\quad + \sup_{t<\tau\leq\infty} \mathbb{E}^{t,x}\left[\delta(\tau-t)\Big|g(\mu_\tau) - g(\mu_\tau^n)\Big|\mathbb{1}_{\{\tau<N^\dagger\}} + \delta(\tau-t)\Big|g(\mu_\tau) - g(\mu_\tau^n)\Big|\mathbb{1}_{\{\tau\geq N^\dagger\}}\right] \\
&\leq M\varepsilon_1 + K\varepsilon_2 + \varepsilon_1 + J\varepsilon_2 = (M+1)\varepsilon_1 + (K+J)\varepsilon_2 =: \varepsilon.
\end{aligned}
$$

Here J, K, M is by (B.1), (B.2) and (B.3) of Assumption B, respectively. Since this is true for all $T \in \mathcal{B}(\mathbb{T} \times \mathbb{X})$ the lemma is then established. ■

C.4 Proof of Lemma 3.5.9

Proof. Let $\rho_n = \rho(t, x_n, T)$ and $\rho = \rho(t, x, T)$ then,

$$\begin{aligned}
&\left|V^*(t,x,\mu,T) - V^*(t,x_n,\mu,T)\right| = \left|\sup_{t<\tau\le\rho} J(t,x,\mu,\tau) - \sup_{t<\tau\le\rho_n} J(t,x_n,\mu,\tau)\right| \\
&\le \left|\sup_{t<\tau\le\rho} J(t,x,\mu,\tau) - \sup_{t<\tau\le\rho} J(t,x_n,\mu,\tau)\right| + \left|\sup_{t<\tau\le\rho} J(t,x_n,\mu,\tau) - \sup_{t<\tau\le\rho_n} J(t,x_n,\mu,\tau)\right|.
\end{aligned}$$

By Remark 3.5.1 X_k^x is continuous in x for each $k > t$, we can control the convergence of the first $N^\dagger$ terms of $X_k^{x_n}$ as in the proof of Lemma 3.5.6, where $N^\dagger$ is such that $\sum_{t=N^\dagger}^{\infty} \delta(t) \le \varepsilon_2$ for some $\varepsilon_2 > 0$, therefore, there exists $N^* > 0$ such that for all $n \ge N^*$,

$$\begin{aligned}
&\left|\sup_{t<\tau\le\rho} J(t,x,\mu,\tau) - \sup_{t<\tau\le\rho} J(t,x_n,\mu,\tau)\right| \\
&\le \sup_{t<\tau\le\rho} \left|J(t,x,\mu,\tau) - J(t,x_n,\mu,\tau)\right| \\
&\le \sup_{t<\tau\le\rho} \mathbb{E}^t\left[\sum_{k=t}^{\tau-1} \delta(k-t)\left|f(X_k^x,\mu_k) - f(X_k^{x_n},\mu_k)\right|\right] \\
&\le \mathbb{E}^t\left[\sum_{k=t}^{\infty} \delta(k-t)\left|f(X_k^x,\mu_k) - f(X_k^{x_n},\mu_k)\right|\right] \\
&\le \mathbb{E}^t\left[\sum_{k=t}^{N^\dagger-1} \delta(k-t)\left|f(X_k^x,\mu_k) - f(X_k^{x_n},\mu_k)\right|\right] \\
&\qquad + \mathbb{E}^t\left[\sum_{k=N^\dagger}^{\infty} \delta(k-t)\left|f(X_k^x,\mu_k) - f(X_k^{x_n},\mu_k)\right|\right] \\
&\le M\varepsilon_1 + K\varepsilon_2,
\end{aligned}$$

where K and M is given by (B.2) and (B.3) of Assumption B respectively. Therefore, we have the convergence of the first term. To see the convergence of the second term, using Lemma 3.5.8 we employ the same argument as in the second part of the proof of Lemma 3.5.4.

We can show that,

$$\lim_{n\to\infty}\left|\sup_{t<\tau\le\rho(t,x,T)} J(t,x,\mu,\tau) - \sup_{t<\tau\le\rho(t,x^n,T)} J(t,x_n,\mu,\tau)\right| = 0.$$

Therefore, collecting the convergence of the two terms, we have the desired results. ■

C.5 Proof of Lemma 3.5.11

Proof. Indeed as H is continuous and strictly decreasing $\varphi_n = \sup\{x\in\mathbb{R} : H(x,y_n) > 0\} = \{x\in\mathbb{R} : H(x,y_n) = 0\} := \{x\in\mathbb{R} : \mathfrak{h}_n(x) = 0\}$ and is unique. By continuity in the second argument we have $\mathfrak{h}_n \to \mathfrak{h} := H(\cdot,y)$. As $\mathfrak{h}$ is continuous and strictly decreasing, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, $\mathfrak{h}(\varphi-\varepsilon) > \delta$ and $\mathfrak{h}(\varphi+\varepsilon) < -\delta$. Therefore by continuity, there exists N large enough such that for all $n \ge N$, $\mathfrak{h}_n(\varphi-\varepsilon) > \delta/2$ and $\mathfrak{h}_n(\varphi+\varepsilon) < \delta/2$. Hence $\varphi^n \in (\varphi-\varepsilon,\varphi+\varepsilon)$ and $\varphi^n \to \varphi$.

Let us denote $\sup\emptyset = -\infty$. If $\sup\{x\in\mathbb{R} : H(x,y) > 0\}$ is empty then $\varphi = -\infty$ and for any $M > 0$ we have $\mathfrak{h}(-M) < -\delta$ for some $\delta > 0$ and again by continuity there exists N large enough such that $\mathfrak{h}_n(-M) < -\delta/2$ for all $n \ge N$. Since M is arbitrary this completes the proof. ■

C.6 Proof of Lemma 3.5.17

Proof. Let $\tau \in \mathcal{T}_t$ be a stopping and denote $\widetilde{\tau}$ the shifted version of τ: if under τ, X stops when $X \le \widetilde{x}$ then under $\widetilde{\tau}$, X stops if $X \le \widetilde{x} + (1-\eta)\Delta C^n$ where $\eta \in (0,1)$ is some constant to be determined. Denote $(S_\mu^{n+1})_t = (-\infty, C_t^{n+1})$ and $(\widehat{S}_\mu^{n+1}) = (-\infty, C_t^n + \Delta C^n) \supset (S_\mu^{n+1})_t$ for all $t \in \mathbb{T}$. Throughout the proof we fix $t \in \mathbb{T}$ and $n \in \mathbb{N}$, for simplicity let us also define,

1. $c = C_t^{n+1} + (1-\eta)\Delta C^n$,
2. $\widetilde{\tau}$ the shifted version of τ with $t+1 \le \tau \le \rho(t, C_n^{t+1}, S_\mu^n)$,
3. $\widehat{\tau} = \widetilde{\tau} \wedge \rho(t, c, \widehat{S}_\mu^{n+1})$.

We wish to show that the following inequalities holds,

$$
\begin{aligned}
&\sup_{t+1\le\sigma\le\rho(t,c,S_\mu^{n+1})} U(t, C_t^{n+1} + (1-\eta)\Delta C^n, \mu, \sigma) \\
&\ge \sup_{t+1\le\sigma\le\rho(t,c,\widehat{S}_\mu^{n+1})} U(t, C_t^{n+1} + (1-\eta)\Delta C^n, \mu, \sigma) \\
&\ge \sup_{t+1\le\sigma\le\rho(t,C_t^{n+1},S_\mu^{n})} U(t, C_t^{n+1}, \mu, \sigma) = 0.
\end{aligned}
$$

If this holds true for any $t \in \mathbb{T}$ then we have $C_t^{n+2} \le C_t^{n+1} + (1-\eta)\Delta C^n$, i.e., $\Delta C_t^{n+1} \le (1-\eta)\Delta C^n$ and therefore $\Delta C_t^n \le \beta^n(b-a)$ for all $t \in \mathbb{T}$ where $\beta := 1-\eta$. In the above inequalities, the second line is straightforward as $\rho(t,c,S_\mu^{n+1}) \ge \rho(t,c,\widehat{S}_\mu^{n+1})$. We now show that for any τ with $t+1 \le \tau \le \rho(t, C_n^{t+1}, S_\mu^n)$ we can construct another stopping time $\widehat{\tau}$ given by 3 above satisfying $t+1 \le \widehat{\tau} \le \rho(t,c,S_\mu^{n+1})$ such that,

$$
U(t, C_t^{n+1} + (1-\eta)\Delta C^n, \mu, \widehat{\tau}) \ge U(t, C_t^{n+1}, \mu, \tau), \tag{C.6.1}
$$

this would establish the third line and complete the proof. We have,

$$
\begin{aligned}
I &:= |U(t,c,\mu,\widetilde{\tau}) - U(t,c,\mu,\widehat{\tau})| \\
&= \left| \mathbb{E}^{t,c}\left[\mathbb{1}_{\{\widehat{\tau}<\widetilde{\tau}\}} \sum_{N=t+1}^{\infty} \mathbb{1}_{\{\widehat{\tau}=N\}} \left\{ \left[\sum_{k=N}^{\widetilde{\tau}-1} \delta(k-t) f(X_k,\mu_k) \right] + \delta(\widetilde{\tau}-t) g(\mu_{\widetilde{\tau}}) - \delta(N-t) g(\mu_N) \right\} \right] \right| \\
&\le \mathbb{E}^{t,c}\left[\sum_{k=t+1}^{\infty} (k-t)\delta(k-t)\, |f(X_k,\mu_k)|\, \mathbb{1}_{\{\widehat{\tau}<\widetilde{\tau},\widehat{\tau}=k\}} \right] \\
&\quad + \mathbb{E}^{t,c}\left[\sum_{k=t+1}^{\infty} |\delta(\widetilde{\tau}-t) g(\mu_{\widetilde{\tau}}) - \delta(k-t) g(\mu_k)|\, \mathbb{1}_{\{\widehat{\tau}<\widetilde{\tau},\widehat{\tau}=k\}} \right] \\
&\le K \sum_{k=t+1}^{\infty} (k-t)\delta(k-t)\mathbb{P}(\widehat{\tau}<\widetilde{\tau},\widehat{\tau}=k) + 2J \sum_{k=t+1}^{\infty} \delta(k-t)\mathbb{P}(\widehat{\tau}<\widetilde{\tau},\widehat{\tau}=k)
\end{aligned}
$$

Note that by (D.1) of Assumption D and definition of $\widehat{\tau}$ and $\widetilde{\tau}$,

$$
\mathbb{P}(\widehat{\tau}<\widetilde{\tau},\widehat{\tau}=k) \le \mathbb{P}\left(X_k^{t,c} \in (C_k^{n+1} + (1-\eta)\Delta C^n, C_k^{n+1} + \Delta C^n) \right) \le L\eta\Delta C^n.
$$

Then by (D.2) of Assumption D we have,

$$I \leq ML\eta\Delta C^n,$$

for some constant M. Then by (D.3) of Assumption D again, we have

$$\begin{aligned}
&U(t, c, \mu, \widehat{\tau}) \geq U(t, c, \mu, \widetilde{\tau}) - ML\eta\Delta C^n \\
&\geq \mathbb{E}^{t,c}\left[\sum_{k=t}^{\tau-1} \delta(k-t) f(X_k, \mu_k) + \delta(\tau - t) g(\mu_\tau)\right] - g(\mu_t) + \alpha(1-\eta)\Delta C^n f(C_t^{n+1}, \mu_t) - ML\eta\Delta C^n \\
&\geq U(t, C_t^{n+1}, \mu, \tau) + \alpha(1-\eta)\Delta C^n f(a, \mu_t) - ML\eta\Delta C^n.
\end{aligned}$$

By (D.3) of Assumption D, $\inf_{\nu \in [0,1]} f(a, \nu) > 0$ therefore choosing $\eta \in (0, 1)$ such that

$$\alpha(1-\eta) f(a, \mu_t) - ML\eta > 0$$

for all $\mu_t \in [0, 1]$, note that η is independent of μ and n (iterations). This establishes Equation (C.6.1) and the proof is complete. ■

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