Managing Inventory and Financing Decisions Under Ambiguity

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Micro, small and medium-sized enterprises (MSMEs) face persistent challenges in raising capitals, and one of the practical reasons could be the high level of ambiguity in this sector. As many not-for-profit organizations or governmental agencies strengthen financial supports to MSMEs, the important issue of stimulating growth while protecting fund providers under ambiguity arises. We propose a robust optimization framework to jointly determine the firm's production planning and financing decisions in a principal-agent model with the presence of distributional ambiguity. We apply the notion of absolute robustness to derive a financing agreement that is both feasibility-robust and performance-robust. We assume that both the firm and the investor base their decisions on two fundamental descriptive statistics: the mean and the variance of the demand. The firm jointly determines the production quantity and financial agreement to maximize the worst-case expected profit, while the investor approves the financial agreement if the worst case expected return can cover the cost of capital. We show that equity financing is one of the robust optimal financing agreements. We also consider loan financing as an alternative. We derive the firm's robust optimal interest rate and production quantity in closed forms. Notably, the robust optimal interest rate depends on the demand variability and the asset recovery ratio, which comprehensively considers the value of collateral, initial capital, and production quantity.

Key words: Newsvendor, interface between operations and finance, principal–agent model, robust optimization, mean-variance ambiguity

History: Updated on 29 August, 2022

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1. Introduction

Micro, small and medium-sized enterprises (MSMEs) are important engines of innovation, growth, job creation, and social cohesion in high-income, emerging, and low-income developing economies. The World Bank estimates that 600 million jobs will be needed by 2030 to absorb the growing global workforce, making MSME development a high priority for many governments worldwide. However, access to financial capital remains a key obstacle hampering the growth of MSMEs, which are less likely to obtain bank loans than large firms. The International Finance Corporation (IFC) estimates that 65 million firms, or 40% of formal MSMEs in developing countries, have an unmet financing need of $5.2 trillion every year.[1] This figure is equivalent to 1.4 times the current level of the global MSME lending.

Many international organizations and governments have been working together to improve MSMEs' access to financing and to reduce their financing costs. For instance, a key area of the World Bank Group's work is to improve MSMEs' access to financial capital and to find innovative solutions to unlock sources of capital. Since 2015, the G20 has committed to advancing financial inclusion worldwide. A special focus was given to facilitating financial services among vulnerable groups and small and medium-sized enterprises (SMEs) to advance and encourage SMEs' participation in sustainable global value chains.[2] The G20/OECD High-Level Principles on SME Financing (developed by the OECD at the request of the G20 Finance Ministers and Central Banks' Governors) call for strengthening of SMEs' access to traditional bank financing and promoting financial inclusion for SMEs. The principles also call for designing regulation that supports a range of financing instruments for SMEs while ensuring financial stability and investor protection and that enhances SMEs' financial skills and strategic vision.

The issues of supporting growth and protecting investors are exacerbated by the ambiguity in the MSME sector caused by numerous factors, such as loosely established hierarchies or unplanned and rapid changes. Additionally, the political, social or economic instability in many countries and regions has caused contraction in consumer markets and limited expansion of business, posing a major challenge to MSME financing. Given the worldwide impact of the COVID-19 pandemic, MSMEs, especially those in the informal sector, are acutely vulnerable to ambiguity.

To address the persistent challenge caused by ambiguous internal and external factors, we propose a robust optimization framework to study the interaction between a firm and an investor. For simplicity of exposition, we use the feminine pronoun for the investor and the masculine pronoun for the firm. Conceivably, if the firm does not know the demand distribution, then neither does the less informed external investor. We recognize that both decision makers face distributional ambiguity. Demand forecasting could be difficult for MSMEs that do not maintain a large data set. According to a global SME banking survey conducted by IFC in 2018, an overwhelming majority of MSMEs do not have formal record-keeping processes even for financial accounts or audited financial statements. Despite the lack of resources, many practitioners are still able to obtain the two fundamental descriptive statistics–the mean and the variance of the demand (Natarajan et al. 2011, 2018). Therefore, we employ the mean-variance ambiguity model, which is prevalent in the operations and supply chain management literature.

We develop a principal-agent model, in which the firm drafts the financing agreement and determines the production quantity, while the investor decides whether to approve the financing agreement. In the baseline model, the (newsvendor) firm has zero initial capital and zero collateral. He offers a financing agreement to the investor. After the investor accepts the agreement, the firm uses the borrowed money to produce the inventories. Any money that is unused must be refunded to the investor prior to making any repayment. The realized demand arrives, and the firm uses the available inventories to satisfy the demand. Any demand in excess of the available inventories is lost and any any unsold inventories are salvaged. The financing agreement specifies a repayment function such that the firm uses the available cash generated from selling or salvaging the inventories to repay the investor. Consistent with the finance literature, the repayment to be made to the investor is monotonic with respect to the verifiable sales quantity (but not the realized demand due to lost sales). The repayment is also limited to the available cash such that the firm is not required to raise additional money to repay the investor if the realized demand is low. Neither decision maker knows the precise demand distribution except for the values of the mean and the variance.

We assume that both decision makers employ the robust max-min decision rule, enabling us to attain *absolute robustness*. According to Kouvelis and Yu (1997, page 15), absolute robustness involves two requirements: i) ensuring robust feasibility; and ii) optimizing the worst case objective. Because our model involves two decision makers–the firm and the investor– the second requirement of absolute robustness must imply that both decision makers use the robust max-min decision rule to maximize their own worst case objective. Notably, when the second requirement is met, the first requirement is also met. For example, if the investor accepts the financing agreement even under her most unfavorable distribution, she must accept the financing agreement under any other feasible distributions. Conversely, if the firm's worst case distribution is not realized, he obtains a weakly higher expected profit. Thus, the robust financing agreement is both feasibility-robust and performance-robust. Under the max-min decision rule, the investor aims at ensuring that her worst case expected return covers her cost of contributed capital whereas the firm jointly determines his production quantity and repayment function to maximize his worst case expected profit (which we refer to as his utility).

We show that equity financing is one of the firm's robust optimal agreements (or contracts). The lack of collateral or initial capital creates a bailout mentality such that the firm is willing to produce as much inventory as possible. The investor then responds by limiting the cash that can be borrowed by the firm. Interestingly, the worst case distributions of the investor, the firm, and the central planner are identical under equity financing. As such, equity financing attains the centralized robust outcome. We also show that the equity contract is improvable by an ironing procedure, which produces an improved repayment function that has a quadratic component and a jump in the event of stockout. The complex nature of this improved repayment function increases the difficulty in practical implementation. Additionally, the improved repayment function does not strictly improve the firm's utility. Hence, for ease of implementation, we advocate for equity financing.

Although equity financing is robustly optimal, it requires substantial administrative work to implement. For instance, the investor and the firm might need to form a share-holding company, which cannot be frequently terminated or renewed. In addition, some countries restrict banks in offering equity financing. For example, the Chinese government (although actively promoting credit expansions to MSMEs) tightened the use of equity financing offered by banks. Therefore, we also consider loan financing as an easy-to-implement alternative.[3]

We show that the firm's robust optimal interest rate consists of three components. Specifically, it equals the investor's cost of capital plus the sensitivity coefficient multiplied by the asset recovery ratio. In the baseline model, the asset recovery ratio equals the salvage value divided by the production cost. In a more general situation in which the firm provides collateral and has some initial capital, the asset recovery ratio equals the sum of the salvage value of all the inventories and the value of the collateral divided by the principal amount of the loan. This comprehensive asset recovery ratio is intuitive and simple for practical implementation. The higher that the asset recovery ratio is, the lower that the optimal interest rate is. Conversely, the sensitivity coefficient equals the square of the coefficient of variation of the demand such that higher demand variability increases the optimal interest rate. The simple functional form of the optimal interest rate and the benefits of absolute robustness make our robust solution appealing for practitioners. In contrast, a distribution-dependent interest rate can easily become infeasible or perform unsatisfactorily under ambiguity.

1.1 Literature Review

Our paper is at the intersection of inventory-based financing strategy and robust inventory management. We review the relevant literature on these two topics as follows.

The literature on the interface between operations and finance studies the inventory management of capital-constrained firms (see Zhao and Huchzermeier 2015, Babich and Kouvelis 2018, for literature reviews). There exist several financing options that the capital-constrained firm can choose to fund its production activities. For instance, the short-term financing options include trade credit, bank credit, and asset-based financing (Fu et al. 2021), while long-term financing options include equity and long-term debt (Xu and Birge 2004) or a mix of equity and debt (Li et al. 2013). Shortages of cash can also occur in both the downstream and upstream of the supply chain. For instance, Kouvelis and Zhao (2012) considered a supply chain in which both the supplier and the newsvendor-like retailer are capital constrained. The supplier designs a trade credit contract by recognizing that the retailer can choose between trade credit and bank credit. In a similar setting, Kouvelis and Zhao (2016) studied contract design and supply chain coordination with bankruptcy costs. Yang and Birge (2018) examined the supplier's optimal trade credit contract under default costs when the newsvendor-like retailer is financially constrained and can use a portfolio of trade credit and bank loans.

Several more closely related papers have studied asset-based financing. Buzacott and Zhang (2004) developed a multi-period newsvendor framework to study the optimal inventory decisions of a firm when it has access to asset-based loans. Dada and Hu (2008) examined a single-period newsvendor model between a capital-constrained retailer and a bank by assuming that the bank chooses an optimal interest rate to maximize its expected profit. Fu et al. (2021) considered a multi-period stochastic inventory management problem in the context of inventory-based loans. While the majority of research in this area (including the aforementioned articles) has not considered asymmetric information, some scholars have examined joint inventory and financing decisions under information asymmetry. For example, de Véricourt and Gromb (2018) considered a single-period interaction between a capital-constrained firm and a bank in the context of assent-based loans and studied the bank's contract design in the presence of moral hazard. In a similar setting, Alan and Gaur (2018) examined the optimal financial contract under adverse selection when the capital-constrained newsvendor possesses private information regarding its demand distribution.

There is a growing literature studying robust operations management (for Lu and Shen 2021, a recent review). Among various decision criteria under ambiguity, the max-min decision rule (i.e., maximizing the worst-case objective) is the most widely used decision rule (Kouvelis and Yu 1997). Our paper is closely related to research on robust inventory management (e.g., Mamani et al.

2017, Fu et al. 2018, and references therein). The pioneering work by Scarf (1958) used the max-min decision criterion to study a newsvendor problem under distribution ambiguity when only the mean and standard deviation of the demand distribution are known. Gallego and Moon (1993) provided a new compact proof of Scarf's ordering rule and extended the analysis in several directions, such as the recourse case, fixed ordering cost case, random yields case, and multi-item case. Han et al. (2014) extended Scarf's ordering rule by considering simultaneously the newsvendor's risk aversion and ambiguity about demand distribution. Mamani et al. (2017) studied a multi-period newsvendor problem utilizing uncertainty sets motivated by the central limit theorem. Using game theory, Wagner (2015) studied a Stackelberg game between a supplier and a newsvendor retailer in which the supplier first chooses an optimal wholesale price, and then the retailer makes an ordering decision, where one firm knows the probabilistic distribution of demand, and the other only knows the mean and variance. By considering a Stackelberg game, Fu et al. (2018) also studied the retailer's robust max-min inventory decision and profit sharing ratio when both demand and price are random. Chen and Xie (2021) considered an unknown joint distribution for demand and yield. To our knowledge, we make the first attempt here to study the robust joint decisions of inventory and financing.

2. The Model

We consider a supply chain financing problem between a firm and an external investor. The investor can be a bank or a supply chain entity with adequate cash to lend. The firm operates in a newsvendor manner with lost sales. For instance, companies selling influenza vaccine (Chick et al. 2008) or holiday products (Raz and Porteus 2006) are the representative empirical parallels. The random demand is represented by a non-negative random variable $\tilde{\theta}$, and we use θ to indicate the realized demand. The cumulative distribution function of the demand is $F(\theta)$ and is unknown to the firm. If the firm does not know the demand distribution, then neither does the less informed external investor. We assume that both decision makers face the same ambiguity in relation to the demand. Although the exact demand distribution could be difficult to specify, many managers are able to obtain reliable estimates of the mean and the variance (e.g. Natarajan et al. 2011, 2018). Therefore, we assume that both decision makers know the mean μ and standard deviation σ of the demand but not the exact distribution of F. We let $\rho = \frac{\sigma}{\mu}$ be the coefficient of variation. We use σ or $\rho\mu$ to represent the standard deviation whenever convenient. Let Ω be the probability space such that any feasible distribution drawn from space Ω satisfies the constraints on the mean and variance. In other words,

$$\Omega = \left\{ F(\theta) \mid \int_0^\infty dF(\theta) = 1, \int_0^\infty \theta dF(\theta) = \mu, \int_0^\infty \theta^2 dF(\theta) = \mu^2 + \sigma^2 \right\}. \tag{2.1}$$

The number of feasible distributions in space Ω is infinite, and these distributions can be discrete, continuous, or mixed. Many of these feasible distributions violate the assumption of an increasing failure rate, which is important to preserve concavity in the firm's objective function (e.g., Buzacott and Zhang 2004, Alan and Gaur 2018). However, our ambiguous model does not require the assumption of an increasing failure rate. Hence, we broaden the application of our results. When facing demand ambiguity, both decision makers employ the max-min robust decision rule.

The investor's cost of capital (or minimal required rate of return) is $r_0 \geq 0$, which is a publicly known constant. We assume that the financial market is so competitive that the firm possesses the decision right to draft the financing agreement. An alternative interpretation is that the investor is a not-for-profit organization or a government, which is dedicated to assisting SMEs that lack cash. If the money borrowed by the firm is L_0, the investor requires that her worst case expected profit be at least $r_0 L_0$. Otherwise, the investor rejects the financing agreement.

The firm has zero initial capital to finance his operation and borrows $L_0 \geq 0$ from the investor. The production cost, selling price, and salvage value of the firm's product are, respectively, c, p, and s per unit (where $0 \leq s < c < p$). We assume that the cost parameters c, p, and s are publicly known. In practice, the investor must request and verify the information about cost parameters before approving any financing agreement. The firm agrees to use the borrowed money to produce q units of inventory such that $L_0 = cq$. If the firm spends less than he borrows, the firm must refund any unused cash to the investor prior to observing the realized demand. In Section 5, we discuss the use of valuable collateral and the possibility that the firm has some (but inadequate) initial capital. Let $(\cdot) = \max(0, \cdot)$.

At the end of the planning horizon, the demand is realized, and the system's cash position changes to $K(\theta, q) = p \min(\theta, q) + s(q - \theta)^+$, where the first term represents the cash generated from selling the inventories to external customers, and the second term represents the cash generated from salvaging the unsold inventories. Let $w(\theta)$ be the repayment made by the firm to the investor. We require that $0 \leq w(\theta) \leq K(\theta, q)$ and $w(\theta)$ be weakly increasing in θ. These two assumptions on $w(\theta)$ are standard in the finance literature. Under a feasible agreement w, the ex post cash position of the firm becomes $Z(\theta|w,q) = K(\theta,q) - w(\theta)$, while that of the investor becomes $U(\theta|w,q) = w(\theta) - cq$. It holds that $Z(\theta|w,q) + U(\theta|w,q) = K(\theta,q) - cq$, representing the total profit of the system.

2.1 Centralized Benchmark

In the centralized benchmark, a central planner chooses a production quantity to maximize the worst case expected profit of the system. We also assume that the central planner has zero initial capital and faces an interest rate $r \geq 0$. We use an arbitrary interest rate r, rather than r_0, to render the intermediate results more general. The total profit of the system equals $K(\theta,q) - (1+r)cq = (p-s)\min(\theta,q) - (c+rc-s)q$, suggesting that $(p-c-rc)$ is the understock cost, and $(c+rc-s)$ is the overstock cost of the system. The term rc represents the interest expense for one unit of the inventory. The central planner solves the following robust optimization model:

$$T = \max_{q\geq 0} \inf_{F\in\Omega} \left\{ \int_0^\infty (p-s)\min(\theta,q) - (c+rc-s)\, q dF(\theta) \right\}. \tag{2.2}$$

The robust solution of Equation (2.2) has been extensively studied, and we refer readers to Appendix A for the relevant preliminary lemmas. To avoid an uninteresting result with zero production quantity, hereafter we assume that $(p-c-rc) \geq (c+rc-s)\rho^2$.

Lemma 1 *The optimal production quantity for Equation (2.2) satisfies*

$$\bar{q}(r) = \mu \left[1 + \frac{\rho}{2} \left(\sqrt{\frac{p-c-rc}{c+rc-s}} - \sqrt{\frac{c+rc-s}{p-c-rc}} \right) \right]. \tag{2.3}$$

Consequently, the optimized worst-case expected profit equals

$$\bar{T}(r) = \mu \left[(p-c-rc) - \rho\sqrt{(p-c-rc)(c+rc-s)} \right]. \tag{2.4}$$

Equation (2.3) is the well-known Scarf's rule.

2.2 Decentralized Formulation

When both decision makers employ the max-min robust decision rule, the firm drafts the financing agreement and chooses the production quantity by solving the following principal-agent model:

$$Z = \max_{w(\cdot),q} \left\{ \inf_{F\in\Omega} \int_0^\infty K(\theta,q) - w(\theta)\, dF(\theta) \right\} \tag{2.5}$$

$$\text{s.t.} \quad q = \arg\max_{Q\geq 0} \left\{ \inf_{F\in\Omega} \int_0^\infty K(\theta,Q) - w(\theta)\, dF(\theta) \right\}, \tag{2.6}$$

$$U = \inf_{F\in\Omega} \int_0^\infty (w(\theta) - cq)\, dF(\theta) \geq r_0 cq, \tag{2.7}$$

Equation (2.5) is the firm's objective function. Constraint (2.6) is the firm's incentive compatibility (IC) constraint. Constraint (2.7) is the investor's individual rationality (IR) constraint, in which the investor's worst case expected profit must exceed the cost of the contributed capital r_0cq. Conceptually, the investor uses the firm's IC constraint to infer the firm's production quantity q, while the firm uses the investor's IR constraint to predict whether the investor will accept the financing agreement. Firm-led models are common in the literature but are dominated by unambiguous models in which the demand distribution is precisely and publicly known (e.g. Innes 1990, Alan and Gaur 2018, de Véricourt and Gromb 2018, Fu et al. 2021).

2.3 Congruent Expectation

We refer to the worst case expected profit as the utility of the decision maker. We use subscript I to indicate the investor, N to indicate the newsvendor firm, and C to indicate the central planner. Let $Z = \inf_{F\in\Omega}\int_0^\infty [K(\theta,q) - w(\theta|q)]\,dF(\theta)$ be the firm's utility, $U = \inf_{F\in\Omega}\int_0^\infty [w(\theta|q) - cq]\,dF(\theta)$ be the investor's utility, and $T = \inf_{F\in\Omega}\int_0^\infty [K(\theta,q) - cq]\,dF(\theta)$ be the central planner's utility. Let F_N, F_I, and F_C be the worst case distributions, respectively, for the firm, the investor, and the central planner.

Lemma 2 *It holds that $Z + U \leq T$, where the equals sign holds if and only if $F_N = F_I = F_C$.*

When the two decision makers choose the same distribution as their own worst case, we say that *congruent expectation* occurs. Notably, their worst case distribution is also identical to that of the central planner. Hence, in the robust principal-agent model formulated in Equations (2.5) to (2.7), congruent expectation is a necessary condition to achieve the centralized outcome.

When the two decision makers choose different distributions as their own worst cases, we say that *incongruent expectation* occurs. Absolute robustness and incongruent expectation are closely related. When $F_N \neq F_I$, we can verify that

$$Z + U = \int_0^\infty [K(\theta,q) - cq]\,dF_N(\theta) - \left[\int_0^\infty w(\theta|q)\,dF_N(\theta) - \int_0^\infty w(\theta|q)\,dF_I(\theta)\right]. \quad (2.8)$$

We define $\Pi = \int_0^\infty w(\theta|q)\,dF_N(\theta) - \int_0^\infty w(\theta|q)\,dF_I(\theta)$ as the distributional surplus. Conceivably, the firm's worst case distribution F_N could maximize the expected payment to the investor, while the investor's worst case distribution F_I could minimize the expected payment to the investor. The distributional surplus is positive if $F_N \neq F_I$ and represents the money that both decision makers "leave on the table" when ambiguity creates incongruent expectations. However, this distributional surplus will be divided between the two decision makers depending on the realized distribution so that each party will experience a weak increase in their actual expected profits. The distributional surplus enables both decision makers to secure a satisfactory performance while ensuring that both decision makers accept the financing agreement. These properties are important to the absolutely robust contracting outcome.

3. Equity Financing

An important benchmark is equity financing, under which the repayment to the investor equals $w_b(\theta) = bK(\theta, q)$. In other words, the firm gives away $b \times 100\%$ of the firm's ownership in exchange for a capital injection L. We refer to b as the equity ratio of the investor. Under an equity ratio b, the firm's cash flow equals $Z_b(\theta, q) = (1-b) K(\theta, q)$, whereas the investor's cash flow equals $U_b(\theta, q) = bK(\theta, q) - cq$. The equity agreement differs from the profit sharing agreement in Fu et al. (2018). The equity agreement divides the ex post cash position $K(\theta, q)$, while the profit sharing agreement divides the total profit, which equals $K(\theta, q) - cq$.

We use the subscript wst to indicate the worst case objective. Let $K_{wst}(q) = \inf_{F\in\Omega} \left\{\int_0^\infty K(\theta, q)\, dF(\theta)\right\}$ be the system's worst case expected cash position, which is affected by the production quantity q but not by the financing agreement $w(\cdot)$. By following the well-known result of Scarf (1958), we obtain

$$K_{wst}(q) = \begin{cases} (p-s)\left(\frac{q}{1+\rho^2}\right) + sq & \text{if } q < \frac{\mu\left(1+\rho^2\right)}{2}, \\ (p-s)\left(\frac{q+a-\sqrt{(a-q)^2+a^2\rho^2}}{2}\right) + sq & \text{if } q \geq \frac{\mu\left(1+\rho^2\right)}{2}. \end{cases} \tag{3.1}$$

It is known that: i) $K_{wst}(q)$ is continuous, increasing, and concave in q; and ii) the first derivative of $\frac{\partial K_{wst}(q)}{\partial q}$ is continuous and decreasing in q (for example, see Lemmas 8 and 11 in Li and Kirshner 2021, for details).

Under equity financing, the firm solves the following principal-agent model:

$$\begin{aligned} &\max_{q\geq 0, 1\geq b\geq 0} \left\{Z_{wst}(q|b) = (1-b) K_{wst}(q)\right\} \\ \text{s.t.} \quad & U_{wst}(q|b) = bK_{wst}(q) - cq \geq r_0 cq. \end{aligned}$$

Because $K_{wst}(q)$ appears in both worst case objectives, the firm and investor choose the same distribution F_{wst} as their common worst case, indicating the occurrence of concurrent expectations. For any given q, the common worst case distribution F_{wst} satisfies either Equation (A-3) or Equation (A-4) depending on q. The firm maximizes his own worst case objective while satisfying the investor's IR constraint.

Proposition 1 *Assume that only equity financing can be used. In the robust contracting equilibrium, the firm produces* $q^* = \bar{q}(r_0)$ *units of inventories by borrowing* cq^* *from the investor and setting the investor's equity ratio at* $b^* = \frac{(1+r_0)cq^*}{K_{wst}(q^*)}$*. The investor breaks even from her worst case perspective while the firm makes an expected profit of* $\bar{T}(r_0)$ *in his own worst case.*

Proposition 1 predicts that equity financing attains the centralized outcome despite ambiguity and hence is one of the optimal contracts for the robust principal-agent model formulated in Equations (2.5) to (2.7). We explain the intuition of this result as follows. Recall that the firm's worst case objective function is $Z_{wst}(q|b) = (1-b) K_{wst}(q)$. For any given b, the firm's worst case expected cash flow $Z_{wst}(q|b)$ is increasing in q, suggesting that the firm prefers to spend all the borrowed money on production. For any given distribution, $Z(q|F) + U(q|F) = K(q|F) - cq$ holds. Due to congruent expectations, both decision makers refer to the same distribution F_{wst} as their common worst case such that $Z_{wst}(q) + U_{wst}(q) = K_{wst}(q) - cq$. The common worst case distribution satisfies Equation (A-5). In the robust contracting equilibrium, the firm attempts to lower the interest rate as much as possible such that the investor's IR constraint is binding. With $U_{wst}(q) = r_0 cq$, the firm's utility equals $Z_{wst}(q) = K_{wst}(q) - (1 + r_0) cq$, suggesting that the firm benefits from producing the centralized optimal production quantity $\bar{q}(r_0)$.

Proposition 1 can be generalized to the unambiguous model in which the demand distribution is precisely and publicly known to be F. We can first compute the distribution-dependent optimal quantity q_F to maximize the system profit $K(q|F) - cq$ and then use $b_F = \frac{(1+r_0)cq_F}{K(q_F|F)}$ to determine the equity ratio. A shortcoming is that the presence of distributional ambiguity can easily render the distribution-dependent equity ratio b_F infeasible. In contrast, the robust equity financing agreement in Proposition 1 ensures that both decision makers experience an increase in their expected profits if their common worst case is not realized. To illustrate, consider that the demand distribution is $F \neq F_{wst}$. By definition, it holds that $K(\bar{q}|F) \geq K_{wst}(\bar{q})$. Hence, $Z(\bar{q}|F) = (1-b) K(\bar{q}|F) \geq (1-b) K_{wst}(\bar{q})$ and $U(\bar{q}|F) = bK(\bar{q}|F) \geq bK_{wst}(\bar{q})$, indicating that both decision makers experience a higher expected profit than their own worst case. The increased profit is due to the realization of a demand distribution that is more favorable than F_{wst}.

There is a noteworthy result regarding the off-equilibrium behavior. Suppose that the firm is able to privately observe the demand distribution F after signing the financing agreement. If the firm keeps his production quantity unchanged at $\bar{q}$, we have demonstrated that $Z(\bar{q}|F) \geq (1-b) K_{wst}(\bar{q})$ and $U(\bar{q}|F) \geq bK_{wst}(\bar{q})$. Hence, keeping the production quantity unchanged is a feasible option for the firm despite the updated private information about the demand distribution. If the firm decides to deviate from the equilibrium production quantity $\bar{q}$, the financing agreement requires the firm to spend no more than he initially borrows. Thus, the firm's off-equilibrium quantity q must satisfy $q < \bar{q}$. Note that $Z(q|F) = (1-b)\, K(q|F)$, where $K(q|F)$ is known to be concave increasing (see the proof of Proposition 1 in Appendix B). We observe that, by playing an off equilibrium strategy $q < \bar{q}$, the firm is worse off despite his updated private information about the demand distribution. We conclude that the robust contracting equilibrium in Proposition 1 is stable against the subsequent private information about the demand distribution.

In summary, the robust equity agreement in Proposition 1 is feasibility-robust and performance-robust, attaining absolute robustness and the centralized outcome. However, equity agreement requires substantial administrative workloads. For example, the investor and firm might need to form a shareholding company. Once she becomes a shareholder, the investor might find that exiting the shareholding company is not easy. Therefore, equity financing is more suitable for repeated and long term transactions. Conversely, loan agreements are prevalent for short term arrangements and can be easily terminated or renewed, especially if the investor is a government aiming to assist small enterprises. In the subsequent Section 4, we focus on loan agreements.

4. The Robust Loan Agreement

Under a loan agreement, the firm is obligated to repay the investor up to $L = (1+r)\, cq$ using the available cash generated from the operation. The positive constant r is the investor's interest rate, cq is the principal of the loan, and L is the sum of the interest and principal of the loan. We refer to L as the total balance of the loan. Under the loan agreement, the firm's cash flow equals

$$Z(\theta|r,q) = K(\theta,q) - \min\{K(\theta,q), (1+r)\, cq\} = [K(\theta,q) - (1+r)\, cq]^{+}. \tag{4.1}$$

In Equation (4.1), $K(\theta,q)$ is the cash available for repaying the loan, and $(1+r)\, cq$ is the total balance of the loan. We also find that the investor's cash flows equals

$$U(\theta|r,q) = \min\{K(\theta,q), (1+r)\, cq\} - cq,$$

where the first term is the repayment made by the firm to the investor under the loan agreement, and the second term is the investor's contributed capital. The firm can fully repay the total balance of the loan if $K(\theta, q) \geq (1+r)\, cq$; otherwise, he defaults. Solving the equation $K(\theta, q) = (1+r)\, cq$, we obtain a critical threshold on the realized demand as follows:

$$\delta = \left(\frac{c + rc - s}{p - s}\right) q, \tag{4.2}$$

which is increasing in the firm's production quantity and interest rate.

We expand $Z(\theta | r, q)$ in Equation (4.1) as follows:

$$Z(\theta | r, q) = \begin{cases} 0 & \text{if } \theta \leq \delta; \\ (p-s)\,\theta - (c + rc - s)\, q & \text{if } \delta \leq \theta \leq q; \\ pq - (1+r)\, cq & \text{if } \theta \geq q. \end{cases} \tag{4.3}$$

Equation (4.3) shows that the firm's ex post cash flow is piece wise and has a total of three pieces– one more than the standard model in Equation (2.2). Therefore, we show that, in robust equilibrium, a new worst case distribution could emerge depending on various cost parameters.

Similarly, we expand $U(\theta | r, q)$ as follows:

$$U(\theta | r, q) = \begin{cases} K(\theta, q) - cq & \text{if } \theta \leq \delta; \\ rcq & \text{if } \theta \geq \delta. \end{cases} \tag{4.4}$$

Both Equations (4.3) and (4.4) appear in the firm's and the investor's robust optimization models.

4.1 Robust Production Plan

Using Equations (4.3) and (4.4), we formulate the firm's principal-agent model under a loan agreement as follows:

$$Z_f = \max_{r,q} \left\{ \inf_{F \in \Omega} \int_0^\infty \left[K(\theta, q) - (1+r)\, cq\right]^+ dF(\theta) \right\} \tag{4.5}$$

$$\text{s.t.} \quad q = \arg\max_{Q \geq 0} \left\{ \inf_{F \in \Omega} \int_0^\infty \left[K(\theta, q) - (1+r)\, cq\right]^+ dF(\theta) \right\}, \tag{4.6}$$

$$U_f = \inf_{F \in \Omega} \int_0^\infty \min\left\{K(\theta, q) - cq, rcq\right\} dF(\theta) \geq 0, \tag{4.7}$$

where we use the subscript f to indicate the firm-led model with a loan financing agreement.

We first examine the firm's production plan before optimizing the interest rate. With mean-variance ambiguity, we apply strong duality to formulate the firm's robust optimization problem as the following semi-infinite programming (SIP) model:

$$P_f(q) = \max_{y_i} \left\{ y_0 + y_1 \mu + y_2 \left(\mu^2 + \sigma^2\right) \right\}$$

$$
\begin{aligned}
\text{s.t.} \quad & y_0 + y_1\theta + y_2\theta^2 \le 0,\ \forall 0 \le \theta \le \delta, \\
& y_0 + y_1\theta + y_2\theta^2 \le (p-s)\,\theta - (c + rc - s)\,q,\ \forall \delta \le \theta \le q, \\
& y_0 + y_1\theta + y_2\theta^2 \le pq - (1+r)\,cq,\ \forall \theta \ge q.
\end{aligned} \tag{4.8}
$$

Figure 1 visualizes the SIP model in Equation (4.8). The left-hand-side (LHS) of the SIP constraints is a quadratic function of the exogenous state variable θ, while the right-hand-side (RHS) is the firm's ex post cash flow $Z(\theta|r,q)$ shown in Equation (4.3). If $\delta \ge \mu$, it is easy to verify that the firm's worst case expected profit is zero.

We focus on the case with $\delta < \mu$. To determine the conditions on the cost parameters associated with each possible worst case distribution, we define the following quadratic function:

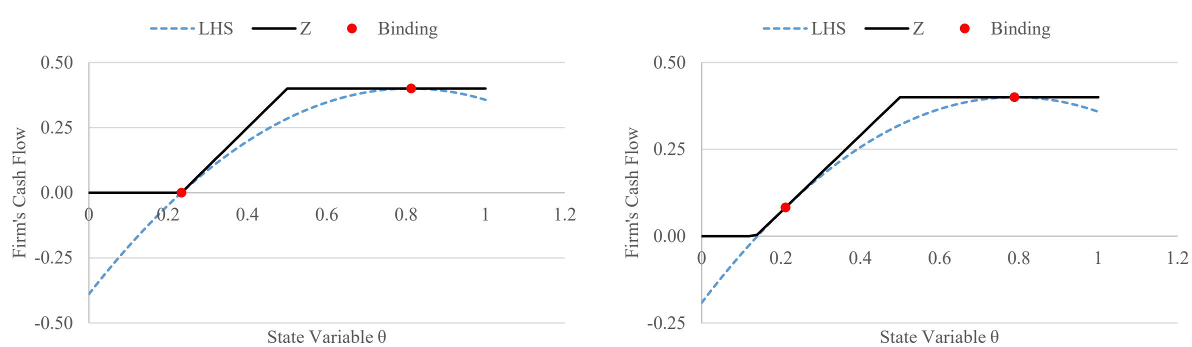
$$
h(q) = q^2\left(1 - \left(\frac{p-c-rc}{p-s}\right)^2\right) - 2\mu q + \left(\mu^2 + \sigma^2\right). \tag{4.9}
$$

If $h(q) \ge 0$, $\theta = \delta$ is a binding SIP constraint, the firm's worst case distribution has the following details:

$$
\begin{cases}
\Pr\left(\tilde{\theta} = \delta\right) = \frac{\sigma^2}{\sigma^2 + (\mu-\delta)^2}, \\
\Pr\left(\tilde{\theta} = \mu + \frac{\sigma^2}{\mu-\delta}\right) = \frac{(\mu-\delta)^2}{\sigma^2+(\mu-\delta)^2},
\end{cases}. \tag{4.10}
$$

We illustrate this case in Figure 1a). If $h(q) < 0$, $\theta = \delta$ is not a binding constraint, and the firm's worst case distribution follows Equation (A-4). We illustrate this case in Figure 1b).

Figure 1 Binding SIP Constraints (Notes: $p = 2$, $c = 1$, $r = 0.2$, and $q = 0.5$)



(a) When $s = 0.2$ (b) When $s = 0.9$

Define the determinant of the quadratic function $h(q)$ in Equation (4.9) as

$$
\Delta(r) = 4\left[\mu^2 - \left(\mu^2 + \sigma^2\right)\left(1 - \left(\frac{p-c-rc}{p-s}\right)^2\right)\right]. \tag{4.11}
$$

Let $\hat{q}_1$ and $\hat{q}_2$ be the two roots of $h(q) = 0$ if $\Delta(r) > 0$. We obtain the following results regarding the firm's worst case objective function under a loan agreement.

Proposition 2 *Under a given loan agreement with interest rate $r \geq 0$, one of the following two cases occurs. i) Suppose that the determinant in Equation (4.11) satisfies $\Delta(r) \leq 0$. The firm's worst case expected profit equals*

$$P_f(q) = \begin{cases} \frac{(\mu-\delta)^2(p-c-rc)q}{\sigma^2+(\mu-\delta)^2}, & 0 \leq q \leq \frac{\mu(p-s)}{p-c-rc} \\ 0, & q \geq \frac{\mu(p-s)}{c+rc-s} \end{cases}. \tag{4.12}$$

ii) Suppose that the determinant satisfies $\Delta(r) > 0$. The firm's worst case expected profit equals

$$P_f(q) = \begin{cases} \frac{(\mu-\delta)^2(p-c-rc)q}{\sigma^2+(\mu-\delta)^2}, & 0 \leq q \leq \hat{q}_1, \\ \frac{(p-s)}{2}\left(\mu + q - \sqrt{(\mu-q)^2+\sigma^2}\right) - (c+rc-s)\,q, & \hat{q}_1 \leq q \leq \hat{q}_2, \\ \frac{(\mu-\delta)^2(p-c-rc)q}{\sigma^2+(\mu-\delta)^2}, & \hat{q}_2 \leq q \leq \frac{\mu(p-s)}{c+rc-s}, \\ 0, & q \geq \frac{\mu(p-s)}{c+rc-s}, \end{cases}. \tag{4.13}$$

In either case i) or case ii), the firm's worst case expected profit $P_f(q)$ is continuous in q.

In case i) of Proposition 2, we verify that the firm's worst case expected profit $P_f(q)$ in Equation (4.12) is quasi-concave in q; hence, we can avoid the assumption of increasing failure rates. In case ii) of Proposition 2, we verify that the first derivative $\frac{\partial P_f(q)}{\partial q}$ is also continuous in q; hence, we establish overall quasi-concavity despite the piece wise nature of $P_f(q)$ in Equation (4.13).

Proposition 3 *Suppose that the firm and investor sign a loan agreement with interest rate $r \geq 0$. i) If $\Delta(r) \leq 0$, then the firm's optimal production quantity q_r is*

$$q_r = \frac{W^2 + \mu(p-s)W - \sigma^2(p-s)^2}{c_o W}, \tag{4.14}$$

where $W = (p-s)\left[\sigma^2\left(\sqrt{\mu^2+\sigma^2} - \mu\right)\right]^{1/3}$. ii) If $\Delta(r) > 0$. If $\hat{q}_1 \leq \bar{q}(r) \leq \hat{q}_2$, where $\bar{q}(r)$ is central planner's optimal production quantity that follows Equation (2.3), then the firm's optimal production quantity is $q_r = \bar{q}(r)$; otherwise, q_r satisfies Equation (4.14).

For any given interest rate $r \geq 0$, Proposition 3 enables us to determine the firm's robust production quantity based on the sign of determinant $\Delta(r)$.

4.2 Robust Interest Rate

The firm must reduce the interest rate as much as possible while satisfying the investor's IR constraint. Let r_f be the firm's robust optimal interest rate and q_f be the firm's robust optimal production quantity.

Proposition 4 *The firm's robust optimal interest rate equals*

$$r_f = r_0 + \rho^2 \left(1 + r_0 - \frac{s}{c}\right). \tag{4.15}$$

Consequently, if $\Delta(r_f) \leq 0$, the firm's optimal production quantity q^* follows case i) of Proposition 3; if $\Delta(r_f) > 0$, the firm's optimal production quantity q^* follows case ii) of Proposition 3.

We refer to $\frac{s}{c}$ as the asset recovery ratio. When the asset recovery ratio increases, the investor can recover more money in the event of defaulting. The firm's optimal interest rate in Proposition 4 is remarkably simple and depends on the coefficient of variation, the asset recovery ratio, and the investor's cost of capital. In the robust contracting equilibrium under a loan agreement, the firm and investor do not share a common worst case distribution, indicating the occurrence of incongruent expectation. For example, if case i) of Proposition 4 occurs, the firm's worst case distribution is based on Equation (A-4) while the investor's worst case distribution is based on Equation (A-3) in Appendix A. Incongruent expectation creates a positive distributional surplus.

Proposition 4 also mitigates moral hazard (e.g. de Véricourt and Gromb 2018) if the coefficient of variation is unaffected by the firm's effort. Specifically, consider that the firm can exert an effort to increase the mean of the demand to μ_1 (where $\mu_1 > \mu$), while the coefficient of variation remains unchanged. This demand model is a multiplicative model that is common in statistics to address heteroscedasticity. The firm sets the interest rate according to Proposition 4 to ensure that the investor participates irrespective of the firm's subsequent choice of effort. Subsequently, the firm chooses the optimal effort and production quantity.

4.3 Performance of Robust Loan Agreement

We conduct a numerical study to evaluate the performance of the robust loan agreement. We fix the mean at $\mu = 0.5$ and the coefficient of variation at $\rho = \frac{1}{\sqrt{3}}$ (so that the standard uniform distribution is one of the feasible distributions). We also fix the production cost at $c = 1$ and then change the salvage value s and selling price p. To mitigate a double margin effect, we lower the investor's cost of capital r_0 to zero. We solve the unambiguous model using the standard uniform distribution to determine the firm's optimal expected profit $Z^*(q_F, r_F|F)$, where (q_F, r_F) denote the firm's unambiguous solution. We then solve the robust solution (q_f, r_f) and compute the firm's expected profit $Z(q_f, r_f|F)$. Finally, we compute the performance gap as

$$Gap = \frac{Z^*(q_F, r_F|F) - Z(q_f, r_f|F)}{Z^*(q_F, r_F|F)} \times 100\%.$$

We report the data of Gap in the following table.

Table 1 Performance Gap

	$s=0.4$	$s=0.5$	$s=0.6$	$s=0.7$	$s=0.8$	$s=0.9$
$p=2$	20.7%	12.9%	12.0%	11.2%	10.5%	9.9%
$p=2.5$	12.0%	11.5%	11.0%	10.5%	10.1%	9.7%
$p=3$	11.2%	10.8%	10.5%	10.2%	9.9%	9.6%
$p=3.5$	10.8%	10.5%	10.2%	10.0%	9.8%	9.6%

Case i) of Proposition 4 occurs only when $p=2$ and $s=0.4$, and we solve the robust solution using the cubic Equation (4.14). For all the other pairs of (p, s), case ii) of Proposition 4 occurs and we use Scarf's rule to determine the firm's robust production quantity. The performance gap is large (at least 9.6%) for two major reasons.

- First, we are considering a two-dimensional model with interest rate and production quantity as the firm's twin decision variables. In the standard Scarf's models, the production quantity is the only decision variable, rendering the models one-dimensional. Therefore, when contrasting the robust solution with the unambiguous solution in our problem contexts, we encounter a performance gap that is compounded by the differences in two dimensions.
- Second, as Equation (2.2) suggested, the performance gap is attributable to the distributional surplus, which plays the important role of ensuring robust feasibility and delivering satisfactory performance. To illustrate, we choose the case with $p=2$ and $s=0.8$ as an example. It can be verified that the robust solution is given by $r_f=0.067$ and $q_f=0.693$. In the robust equilibrium, the firm's expected worst case is $Z_f=0.323$ and the investor's worst case expected profit is $U_f=0$. Due to lost sales, the system expected profit T_F is also distribution-dependent. When the robust solution is implemented under the uniform distribution, the system expected profit equals $T_F=0.405$, and the distribution surplus equals $\Pi_F=T_F-Z_f=0.082$. The distributional surplus will be divided between the firm and the investor so that they experience a higher expected

profit. For instance, when the underline demand distribution is uniform, the firm's expected profit equals $Z(q_f, r_f|F) = 0.373$ (which is 15.5% higher than the firm's worst case), and the investor's expected profit equals $U(q_f, r_f|F) = 0.032$.

A noteworthy advantage of the robust loan agreement is that the issue of infeasibility is effectively mitigated. The unambiguous solution renders the investor's IR constraint binding under the uniform distribution. A binding IR constraint, however, leaves no room for ambiguity. For example, uniform distribution is not the most unfavorable distribution for the investor. With the presence of ambiguity, the investor rejects the loan agreement over many feasible distributions that are less favorable than the uniform distribution. Recall that the objective of many loan assistance programs is to support MSMEs. An outcome with no agreement would be less preferred.

5. Discussion

We discuss several extensions to the baseline model presented in the previous sections.

5.1 Collateral and Initial Capital

We assume that the firm provides an asset with a liquidation value of $I_0 \geq 0$ as the collateral. When the firm defaults, the investor liquidates the collateral to generate an ex post income I_0. We also assume that the firm has an initial capital $I_1 \geq 0$ to initiate the production activities. Therefore, the firm needs to borrow $(cq - I_1)^+$ from the investor. If $I_0 + K(\theta, q) \geq (1 + r_0)(cq - I_1)^+$, the investor refunds the excessive cash to the firm; otherwise, no refund is made to the firm. We observe that the threshold for defaulting equals

$$\delta(I_0, I_1) = \left(\frac{(1+r)(cq - I_1)^+ - I_0 - sq}{p - s} \right)^+.$$

Using a similar method, we obtain the firm's robust solution (r_f, q^*). Before showing the results, we define several identities as follows: $A_1 = \mu(p - s)$, $A_2 = I_0 + (1 + r) I_1$, $A_3 = \sigma^2 (p - s)^2$, and

$$Y^3 = A_3 \left[\sqrt{(A_1 + A_2)^2 + A_3} - (A_1 + A_2) \right].$$

The quadratic function in Equation (4.9) changes to the following:

$$h(q|I_0, I_1) = q^2 \left(1 - \left(\frac{p - c - rc}{p - s} \right)^2 \right) - 2q \left[\mu + \frac{A_2 (p - c - rc)}{(p - s)^2} \right] + (\mu^2 + \sigma^2) - \left(\frac{A_2}{p - s} \right)^2.$$

If $h(q|I_0,I_1)\geq 0$, the $\theta=\delta(I_0,I_1)$ is a binding SIP constraint. The determinant in Equation (4.11) also changes to the following:

$$\Delta(r|I_0,I_1)=4\left[\mu+\frac{A_2(p-c-rc)}{(p-s)^2}\right]^2-4\left[1-\frac{(p-c-rc)^2}{(p-s)^2}\right]\left[\left(\mu^2+\sigma^2\right)-\left(\frac{A_2}{p-c}\right)^2\right].$$

If the determinant $\Delta(r|I_0,I_1)\leq 0$, the firm's best response satisfies

$$q(r)=\frac{Y^2+(A_1+A_2)Y-A_3}{(c+rc-s)Y}. \tag{5.1}$$

If the determinant $\Delta(r|I_0,I_1)>0$, we let $\hat{q}_1(I_0,I_1)$ and $\hat{q}_2(I_0,I_1)$ be the two roots of Equation $h(q|I_0,I_1)=0$. If the central planner's solution $\bar{q}_(r)$ is between these two roots, then the firm's best response is $\bar{q}_r$. Otherwise, Equation (5.1) holds. In a special case with $I_0=I_1=0$, it holds that $A_2=0$ and $Y=W$, making Equations (4.14) and (5.1) identical.

Proposition 5 *If the firm's collateral has a value of $I_0\geq 0$ and initial capital is $I_1\geq 0$, for any given $q\geq 0$, the firm's optimal interest rate $r_f(q|I_0,I_1)$ equals*

$$r_f(q|I_0,I_1)=r_0+\rho^2\left(1+r_0-\frac{sq+I_0}{(cq-I_1)^+}\right)^+. \tag{5.2}$$

The collateral and initial capital change the asset recovery ratio to the following:

$$\text{Asset Recovery Ratio}=\frac{sq+I_0}{(cq-I_1)^+},$$

which is intuitive and simple for practical implementation. In the special case with $I_0=I_1=0$, the asset recovery ratio is unaffected by the production quantity.

Equation (5.2) is the reminiscent of the well-known Capital Asset Pricing Model (CAPM), which states that $R_i=R_f+\beta_i(R_m-R_f)$. The following Table 2 contrasts the CAPM formula and Equation (5.2). In particular, the term $\left(1+r_0-\frac{sq+I_0}{(cq-I_1)^+}\right)^+$ resembles to the market risk premium in the CAPM model. Because we are considering loan agreements, the market risk premium is now replaced by the difference between $(1+r_0)$ and the asset recovery ratio.

In this paper, for ease of formulation, we assume that the firm drafts the financing agreement and chooses the production quantity. In practice, the loan agreement is negotiated between the firm and the investor. The closed form of the robust optimal interest rate in Equation (5.2) can enable the investor to quickly determine whether a loan should be approved.

Table 2 Contrasting CAPM and Equation (5.2)

Equation (5.2)	Managerial Interpretation	Counterpart in CAPM	CAPM Interpretation
r_f	Optimal interest rate	R_i	Return of asset i
r_0	Cost of capital	R_f	Risk-free rate of return
ρ^2	Squared Coefficient of Variation	β_i	Sensitivity factor
$\left(1+r_0-\frac{sq^*+I_0}{(cq^*-I_1)^+}\right)$	Interest rate premium	(R_m-R_f)	Market risk premium

Next, we explain how to compute the firm's robust optimal production quantity q^* and interest rate r_f^*. The difference between equations (4.15) and (5.2) is that the former is unaffected by the firm's production quantity q. With non-zero (I_0, I_1), we need to solve two simultaneous equations to determine $\left(r_f^*, q^*\right)$. Similar to Proposition 3, the firm's best response $q(r)$ has two cases, depending on the sign of the determinant $\Delta(r|I_0, I_1)$. Observe that the interest rate affects A_2 and $(c+rc-s)$ but not A_1 and A_3. Because the firm chooses the interest rate based on Equation (5.2) and her production quantity q, we derive the first candidate solution (q_1, r_1) by solving the following simultaneous two equations:

$$q=\bar{q}(r) \text{ and } r=r_0+\rho^2\left(1+r_0-\frac{sq+I_0}{(cq-I_1)^+}\right)^+.$$

We need to retrospectively verify the sign of the determinant $\Delta(r_1|I_0, I_1)$. If $\Delta(r_1|I_0, I_1)>0$ and $\bar{q}(r_1)\in\left[\hat{q}_1(I_0, I_1), \hat{q}_2(I_0, I_1)\right]$, then the first candidate solution indeed emerges as the firm's robust optimal solution, in which the production quantity follows Scarf's rule and is the central planner's production quantity. Otherwise, we proceed to derive the second candidate solution (q_2, r_2) by solving the following two simultaneous equations:

$$q=\frac{Y^2+(A_1+A_2)Y-A_3}{(c+rc-s)Y} \text{ and } r=r_0+\rho^2\left(1+r_0-\frac{sq+I_0}{(cq-I_1)^+}\right)^+.$$

Whenever the first candidate solution is not optimal, the second candidate solution must be optimal.

5.2 Quadratic Agreement

The analysis so far focuses on two easy-to-implement financing agreements: equity and loan. These two agreements are common in practice, and we have derived the robust version for each of them in closed forms. Although equity is one of the optimal contracts for the principal-agent model formulated in Equations (2.5) to (2.7), we can still weakly improve it using the "ironing" procedure that Li and Kirshner (2021) proposed.[4] However, the ironing procedure does not strictly increase the firm's utility.

Proposition 6 *For any given equity agreement with (q, b), we can construct a quadratic payment function $w_d(\theta)$ that dominates the equity financing agreement. The quadratic payment function $w_d(\theta)$ satisfies*

$$w_d(\theta) = \begin{cases} y_0^* + y_1^*\theta + y_2^*\theta^2, & 0 \le \theta < q \\ bpq, & \theta \ge q \end{cases} \tag{5.3}$$

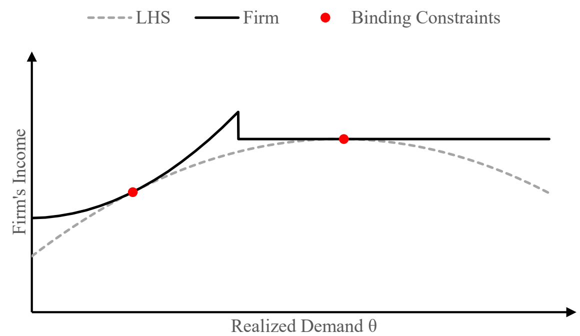
where $y_0^* = bpq + y_2^*\theta_h^2$, $y_1^* = -2y_2^*\theta_h$, and $y_2^* = -\frac{b(p-s)}{2(\theta_h-\theta_l)}$, where $\theta_l = q - \sqrt{(\mu-q)^2+\sigma^2}$ and $\theta_h = q + \sqrt{(\mu-q)^2+\sigma^2}$. Under this quadratic payment function, the firm's utility satisfies

$$Z(w_d) = \inf_{F \in \Omega} \int_0^\infty [K(\theta, q) - w_d(\theta)]\, dF(\theta) = (1-b)\, K_{wst}(q).$$

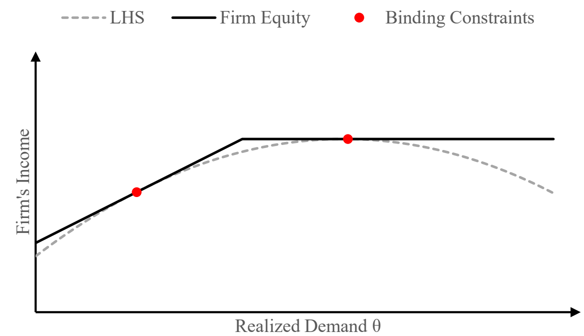
Proposition 6 offers two valuable insights. First, for any given agreement w (not just the equity agreement), the ironing procedure can produce an improved payment function. If the improved payment function produced by the ironing procedure is identical to the initial agreement, we say that the initial agreement is non-improvable by the ironing procedure. The improved payment function $w_d(\theta)$ in Equation (5.3) is complex and involves a jump at point $\theta = q$ and a quadratic component for $\theta \in [0, q)$. The quadratic component and the jump increase the practical difficulty in implementing this complex agreement. Second, although the equity agreement is improvable, from the worst case perspective, the improved payment function does not strictly increase the firm's utility because the firm's worst case distribution remains unchanged.

In Figure 2a), we depict the firm's ex post income $K(\theta, q) - w_d$, where $w_d(\theta)$ is based on Equation (5.3). The jump in the repayment to the investor at point $\theta = q$ creates a drop in the firm's ex post income. The binding constraints remain at points θ_l and point θ_h. For contrasting purposes, we depict the firm's SIP model under the given equity agreement in Figure 2b). We observe that the binding constraints are unaffected by the improved agreement w_d. Therefore, equity agreement remains one of the robust optimal financing agreements despite the equity agreement being weakly improvable by the ironing procedure.

Figure 2 Equity Agreement and Its Improved Agreement



(a) Firm's SIP Model under Improved Agreement



(b) Equity Payment Function

5.3 Investor-Led Model

We now formulate the investor-led model, in which we use the subscript I to indicate that the investor drafts the financing agreement to maximize her utility. The investor solves the following principal-agent model:

$$U_I = \max_{w(\theta)\geq 0, q\geq 0} \inf_{F\in\Omega} \left\{ \int_0^\infty [w(\theta|q) - cq] dF(\theta) \right\} \tag{5.4}$$

$$\text{s.t.} \quad q = \arg\max_{Q\geq 0} \left\{ \inf_{F\in\Omega} \int_0^\infty K(\theta, Q) - w(\theta|Q)\, dF(\theta) \right\}, \tag{5.5}$$

$$Z_I = \inf_{F\in\Omega} \int_0^\infty K(\theta, Q) - w(\theta|Q)\, dF(\theta) \geq 0. \tag{5.6}$$

Equation (5.4) is the investor's objective function. Constraint (5.5) is the firm's IC constraint. Constraint (5.6) is the firm's IR constraint. Conceptually, the investor chooses the interest rate r, and the firm chooses the production quantity q. The investor uses the IC/IR constraints to infer the firm's production quantity q.

Proposition 3 enables us to determine the firm's robust production quantity such that we can simplify the IC constraints (5.5) based on the occurrence of either case i) or ii) of Proposition 3. However, case i) of Proposition 3 involves a cubic function in the firm's first order condition. Therefore, we cannot obtain a closed form solution but leave the investor-led model for the future research.

6. Conclusions

We study the supply chain financing problem between an investor and a firm. By applying the concept of absolute robustness, we characterize robust equity and robust loan financing agreements.

The most interesting result is the asset recovery ratio, which comprehensively considers the firm's initial capital, production quantity, and collateral. Using this asset recovery ratio, we can conveniently determine the firm's optimal interest rate. Our firm-led model is suitable for practice if the investor is a not-for-profit organization or a government that aims to assist firms while financially breaking even. In contrast, if equity financing is permitted by regulators or central banks, it is robust optimal and attains the robust centralized outcome and absolute robustness.

Notes

[1]We refer readers to the following link: https://www.ifc.org/wps/wcm/connect/03522e90-a13d-4a02-87cd-9ee9a297b311/121264-WP-PUBLIC-MSMEReportFINAL.pdf?MOD=AJPERES&CVID=m5SwAQA.

[2]We refer readers to the following links: https://www.worldbank.org/en/topic/smefinance and https://www.oecd.org/finance/sme-financing.htm.

[3]In China, inclusive financial lending to MSMEs by financial institutions increased by 27.3 percent in 2021. This figure represents an increase of more than 40 percent over the previous year. As a result, overall financing costs of businesses dropped steadily. As part of the Chinese government's stepped-up support for the real economy, large state-owned banks will add more than 1.6 trillion yuan (equivalent to $241.7 billion) of inclusive loans for MSMEs in 2022. (https://www.chinadaily.com.cn/a/202205/19/WS62860a65a310fd2b29e5dd00.html) However, to improve risk tolerance and mitigation, the Chinese government also restricts commercial banks from offering equity contracts to any company (including MSMEs). For instance, Article 43 of Law of the P.R. China on Commercial Banks states, "Commercial banks are not allowed to make trust investments, trade in shares or make investments in fixed assets of non-self-use within China." Technically, regular loans are the only option that Chinese commercial banks can offer.

[4]Li and Kirshner (2021) did not use the term "ironing procedure" but the term "monotonic non-negative upper envelope." A key step is to reduce the repayment based on the LHS of the investor's SIP constraints. Specifically, we use the LHS of SIP constraints as the ironing board and then iron (or press) the initial repayment function downward against this ironing board. We believe that ironing is a better illustration of this procedure.

Appendix A: Preliminary Lemmas

In Appendix A, we recap several intermediate results, including strong duality, Scarf's rule and its associated worst case distributions, and bounds on any given percentile.

Strong Duality

First, we clarify what we mean by strong duality, which is frequently applied in this paper. Consider that the decision maker faces an exogenous payoff function $h(\theta)$, which can be discontinuous in θ. The decision maker knows only the values of the mean and variance associated with the random variable θ. The decision maker's worst-case expected payoff is determined by the following robust optimization model:

$$D = \inf_{F \in \Omega} \left\{ \int_0^\infty h(\theta)\, dF(\theta) \right\}$$

$$
\text{s.t.} \begin{cases} \int_0^\infty dF(\theta) = 1, \\ \int_0^\infty \theta dF(\theta) = \mu, \\ \int_0^\infty \theta^2 dF(\theta) = \mu^2 + \sigma^2. \end{cases}
$$

Consistent with the extant literature (e.g., Hettich and Kortanek 1993), we formulate the following SIP model:

$$
\begin{aligned} P = &\max_{y_0, y_1, y_2} \left\{ y_0 + y_1 \mu + y_2 \left(\mu^2 + \sigma^2 \right) \right\} \\ \text{s.t.} \quad & y_0 + y_1 \theta + y_2 \theta^2 \leq h(\theta), \ \forall \theta \geq 0. \end{aligned} \tag{A-1}
$$

where the decision variables y_i have unrestricted signs. The decision variables y_0, y_1, and y_2 are, respectively, the shadow prices of the total probability constraint, the mean constraint, and the variance constraint in model D. Because θ is the exogenous state variable, the SIP model P encompasses an infinite number of constraints but only three decision variables. Weak duality implies that $P \leq D$.

Next, we construct another dual model D' using general finite sequence $\lambda(\theta)$, which is non-negative for any $\theta \geq 0$, but only a finite number of $\lambda(\theta)$ can be strictly positive. The dual model D' is the following:

$$
D' = \inf_{\lambda(\theta) \geq 0} \sum_{\theta \geq 0} h(\theta) \lambda(\theta)
$$

$$
\text{s.t.} \begin{cases} \sum_{\theta \geq 0} \lambda(\theta) = 1, \\ \sum_{\theta \geq 0} \theta \lambda(\theta) = \mu, \\ \sum_{\theta \geq 0} \theta^2 \lambda(\theta) = \mu^2 + \sigma^2. \end{cases}
$$

The difference between D and D' is that model D uses all the distributions (continuous, mixed, or discrete) in the full ambiguity set Ω to construct the decision maker's worst distribution, while model D' uses only the discrete distributions. Because only a finite number of $\lambda(\theta)$ can be strictly positive, model D' also excludes discrete distributions such as Poisson distribution, of which the number of realized values is infinite. It must hold that $P \leq D \leq D'$.

Interestingly, the primal model corresponding to model D' is the same SIP model P shown in Equation (A-1) because both dual models D and D' share the same exogenous state variable θ. Because P is also the primal model of D', we apply Theorem 6.5 of Hettich and Kortanek (1993, page 399) to prove that $P = D'$. Theorem 6.5 of Hettich and Kortanek (1993) requires two conditions: i) model D' is feasible (meaning that the full ambiguity set Ω contains at least one feasible discrete distribution), and ii) model D' is finite (which automatically holds under the liability constraints). We can claim that $P = D = D'$.

Scarf's Rule

Second, we recap the classic result of Scarf (1958), in which Scarf identified the robust optimal production quantity by solving the model formulated in Equation (2.2). The challenge is to derive the lowest expected sales quantity, which is given by $S(q) = \inf_{F\in\Omega}\left\{\int_0^\infty \min(\theta, q)\, dF(\theta)\right\}$. There are two methods to compute $S(q)$: the first method is to apply strong duality (see Lemma 11 of Li and Kirshner 2021, for a complete proof), and the second method is to apply Cauchy-Schwarz inequality (see Gallego and Moon 1993). With debt financing, Cauchy-Schwarz inequality becomes less effective, and we apply SIP tools.

Lemma 3 *(Scarf 1958) When the non-negative demand θ satisfies $E(\theta) = \mu$ and $Var(\theta) = \sigma^2 = \rho^2\mu^2$ (where $\rho \geq 0$ is a constant representing the coefficient of variation of θ), the lowest expected sales quantity satisfies*

$$S(q) = \begin{cases} \frac{q}{1+\rho^2} & \text{if } q < \frac{\mu\left(1+\rho^2\right)}{2}; \\ \frac{q+\mu-\sqrt{(\mu-q)^2+\sigma^2}}{2} & \text{if } q \geq \frac{\mu\left(1+\rho^2\right)}{2}. \end{cases} \tag{A-2}$$

Equation (A-2) leads us to Equation (3.1) in Section 3 and Equation (B-6) in Appendix B. The corresponding worst-case distribution is noteworthy and is one of the following two distributions. i) If $q < \frac{\mu\left(1+\rho^2\right)}{2}$, the decision maker's worst-case distribution satisfies

$$\begin{cases} \Pr\left(\tilde{\theta} = 0\right) = \frac{\rho^2}{1+\rho^2} \\ \Pr\left(\tilde{\theta} = \mu\left(1+\rho^2\right)\right) = \frac{1}{1+\rho^2} \end{cases}. \tag{A-3}$$

ii) If $q \geq \frac{\mu\left(1+\rho^2\right)}{2}$, the decision maker's worst-case distribution satisfies

$$\begin{cases} \Pr\left(\tilde{\theta} = q - \sqrt{(\mu-q)^2+\sigma^2}\right) = \frac{1}{2} + \frac{q-\mu}{2\sqrt{(\mu-q)^2+\sigma^2}}, \\ \Pr\left(\tilde{\theta} = q + \sqrt{(\mu-q)^2+\sigma^2}\right) = \frac{1}{2} - \frac{q-\mu}{2\sqrt{(\mu-q)^2+\sigma^2}}, \end{cases}. \tag{A-4}$$

Third, using Equation (A-2), we can obtain the robust production quantity when the understock cost is c_u, and the overstock cost is c_o. The results are shown in Lemma 1. We omit the proof of Lemma 1 and refer readers to Lemma 12 of Li and Kirshner (2021) for a complete proof.

As stated in Section 2.1, we focus on the case with $c_u > c_o\rho^2$. Otherwise, the worst-case scenario yields an uninteresting result with zero inventory. Substituting the robust production quantity in Equation (2.3) into Equation (A-4), we obtain that the firm's most unfavorable distribution is

$$\begin{cases} \Pr\left(\tilde{\theta} = \mu\left(1-\rho\sqrt{\frac{c_o}{c_u}}\right) \overset{def}{=} \theta_L\right) = \frac{c_u}{c_u+c_o}, \\ \Pr\left(\tilde{\theta} = \mu\left(1+\rho\sqrt{\frac{c_u}{c_o}}\right) \overset{def}{=} \theta_H\right) = \frac{c_o}{c_u+c_o}, \end{cases}. \tag{A-5}$$

The two-point distribution in Equation (A-5) yields two noteworthy implications. First, if the firm knows that the demand follows the two-point distribution in Equation (A-5), any production quantity q within the closed interval $[\theta_L, \theta_H]$ yields the same amount of expected profit to the firm. Second, the robust production quantity in Equation (2.3) happens to be the average of θ_L and θ_H. The pre-condition $c_u > c_o\rho^2$ ensures that $\theta_L > 0$.

The fourth intermediate result is relatively unknown in the literature on robust inventory control but determines the lower and upper bounds of any percentile.

Lemma 4 *For any $F \in \Omega$ such that $\mathbb{E}_F(\theta) = \mu$ and $Var_F(\theta) = \sigma^2$, let q_F be the newsvendor solution based on F satisfying $F(q_F) = \frac{c_u}{c_u+c_o}$. It holds that*

$$\inf_{F\in\Omega}\{q_F\} = \theta_L \textit{ and } \sup_{F\in\Omega}\{q_F\} = \theta_H. \tag{A-6}$$

Proof of Lemma 4:

By contradiction, we hypothesize that there exists a $q > \theta_H$ such that $F(q) = \frac{c_u}{c_u+c_o}$. According to Cantelli's inequality, when $q > \theta_H \geq \mu$, it holds that

$$\Pr(\theta \geq q) = 1 - F(q) \leq \frac{\sigma^2}{(\mu-q)^2 + \sigma^2}, \tag{A-7}$$

which implies that

$$\frac{(\mu-q)^2}{(\mu-q)^2+\sigma^2} \leq F(q),$$

where the left-hand-side (LHS) monotonically increases from zero to 1 when q changes from μ to $+\infty$. When $q = \theta_H$, it holds that

$$\frac{(\mu-\theta_H)^2}{(\mu-_H)^2+\sigma^2} = \frac{\frac{c_u}{c_o}\sigma^2}{\frac{c_u}{c_o}\sigma^2+\sigma^2} = \frac{c_u}{c_u+c_o}.$$

We now obtain that, for any $q > \theta_H \geq \mu$, it holds that

$$\frac{c_u}{c_u+c_p} < \frac{(\mu-q)^2}{(\mu-q)^2+\sigma^2} \leq F(q),$$

which contradicts the initial hypothesis that $F(q) = \frac{c_u}{c_u+c_o}$. Hence, any $q > \theta_H$ cannot be a percentile satisfying the condition $F(q) = \frac{c_u}{c_u+c_o}$.

Similarly, we can prove that any $q < \theta_L$ cannot be a percentile satisfying the condition $F(q) = \frac{c_u}{c_u+c_o}$. Finally, it is easy to verify that the two-point distribution shown in Equation (A-5) causes Equation (A-6) to hold. Q.E.D.

Lemma 4 characterizes the lower and upper bounds of the $p \times 100\%$ percentile (i.e., let $c_u = p$ and $c_o = 1 - p$) of a random variable θ based on the mean and variance. In a special case with $p = 0.5$, we obtain that the median is within the closed interval $[\mu - \sigma, \mu + \sigma]$, which is a well-known result in statistics. Lemma 4 enables us to judge whether the firm's inventory level is reasonable. If the firm orders either less than θ_L or more than θ_H, we can assert that the firm's inventory decision is suboptimal (i.e., off the equilibrium). Notably, Scarf's robust inventory level happens to be the mid-point of the interval $[\theta_L, \theta_H]$.

Under a loan agreement, the firm's understock cost is $c_u(r) = p - c - rc$, and overstock cost is $c_o(r) = c + rc - s$. Consequently, the lower threshold is

$$\theta_L(r) = \mu\left(1 - \frac{\rho}{2}\sqrt{\frac{c + rc - s}{p - c - rc}}\right).$$

The identities of $\theta_L(r)$ and $\bar{q}(r)$ play important roles in proving Propositions 2 and 4.

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Appendix B: Technical Proofs

Proof of Lemma 2:

Using the well-known inequality that $\min A + \min B \leq \min(A+B)$, we obtain

$$\begin{aligned} Z+U &= \inf_{F\in\Omega}\int_0^\infty [K(\theta,q)-w(\theta|q)]\,dF(\theta)+\inf_{F\in\Omega}\int_0^\infty [w(\theta|q)-cq]\,dF(\theta) \\ &\leq \inf_{F\in\Omega}\int_0^\infty [K(\theta,q)-w(\theta|q)+w(\theta|q)-cq]\,dF(\theta) \\ &= \inf_{F\in\Omega}\int_0^\infty [K(\theta,q)-cq]\,dF(\theta)=T. \end{aligned}$$

When the equals sign in the inequality $\min A + \min B \leq \min(A+B)$ holds, both A and B are simultaneously minimized. In our context, the same distribution F minimizes both Z and U, indicating that $F_N = F_I = F_C$.

If $F_N \neq F_I$,

$$\begin{aligned} Z+U &= \int_0^\infty [K(\theta,q)-w(\theta|q)]\,dF_N(\theta)+\int_0^\infty [w(\theta|q)-cq]\,dF_I(\theta) \\ &= \int_0^\infty [K(\theta,q)-cq]\,dF_N(\theta)-\left[\int_0^\infty w(\theta|q)\,dF_N(\theta)-\int_0^\infty w(\theta|q)\,dF_I(\theta)\right], \end{aligned}$$

which proves Equation (2.8). Q.E.D.

Proof of Proposition 1:

Observe that $K(\theta,q) = p\min(\theta,q) + s(q-\theta)^+$ and $K(q|F) = \int_0^\infty K(\theta,q)\,dF(\theta)$. It is well known that: i) the first derivative of $K(q|F)$ satisfies

$$\frac{\partial K(q|F)}{\partial q} = p(1-F(q)) + sF(q) > 0$$

; ii) the second derivative satisfies

$$\frac{\partial^2 K(q|F)}{\partial q^2} = -(p-s)\,f(q) < 0.$$

Because $K(q|F)$ is concave increasing in q for any given F (including the worst case distribution), the firm prefers to produce as much inventory as possible after using up his initial capital. The firm's IC constraints can be relaxed. With the investor's IR constraint binding, the Lagrangian is

$$\Lambda = (1-b)\,K(q|F) - \lambda\,(bK(q|F) - (1+r_0)\,cq).$$

Karush–Kuhn–Tucker (KKT) conditions require

$$\begin{cases} \frac{\partial \Lambda}{\partial q} = (1-b)\frac{\partial K(q|F)}{\partial q} - \lambda b \frac{\partial K(q|F)}{\partial q} + \lambda (1+r_0)\, c = 0 \\ bK(q|F) - (1+r_0)\, cq = 0 \end{cases}$$

When

$$\frac{\partial K(q|F)}{\partial q} = (1+r_0)\, c,\ \lambda = -1, \text{ and } b = \frac{(1+r_0)\, c\bar{q}_F}{K(\bar{q}_F|F)},$$

we can verify that the KKT conditions hold. With the presence of ambiguity, we replace $K(q|F)$ with $K_{wst}(q)$, which has been shown in Equation (3.1), thus proving Proposition 1. Q.E.D.

Proof of Proposition 2:

We apply relaxation to solve the SIP model in Equation (4.8). Figure 1 provides useful references for us to conjecture about the binding constraints.

First, based on Figure 1b), we conjecture that the binding constraints occur at points $\theta = \theta_l$ and $\theta = \theta_h$, where $\delta < \theta_l < q$ and $\theta_h > q$ are to be determined later. The corresponding Lagrangian multipliers are, respectively, λ_l and λ_h. For now, we assume that $\delta \leq \theta_l$ holds. Otherwise, we consider case i) of Proposition 2. The Lagrangian equals

$$\begin{aligned} \Gamma = {} & y_0 + y_1\mu + y_2\left(\mu^2+\sigma^2\right) - \lambda_l\left[y_0 + y_1\theta_l + y_2\theta_l^2 - (p-s)\,\theta_l + (c+rc-s)\, q\right] \\ & -\lambda_h\left[y_0 + y_1\theta_h + y_2\theta_h^2 - (p-c-rc)\, q\right]. \end{aligned} \tag{B-1}$$

The KKT conditions include: $\frac{\partial \Gamma}{\partial y_0} = 1 - \lambda_l - \lambda_h = 0$, $\frac{\partial \Gamma}{\partial y_1} = \mu - \lambda_l\theta_l - \lambda_h\theta_h = 0$, $\frac{\partial \Gamma}{\partial y_2} = \mu^2(1+\rho^2) - \lambda_l\theta_l^2 - \lambda_h\theta_h^2 = 0$, $y_0 + y_1\theta_l + y_2\theta_l^2 = (p-s)\,\theta_l - (c+rc-s)\, q$, and $y_0 + y_1\theta_h + y_2\theta_h^2 = (p-c-rc)\, q$. To ensure that $\theta = \theta_l$ is a locally binding SIP constraint, the so-called "tangent" condition $y_1 + 2y_2\theta_l = (p-s)$ must hold. The tangent condition was first proposed by Scarf (1958) and has become a standard technical device in the extant literature. Similarly, the tangent condition for $\theta = \theta_h$ is $y_1 + 2y_2\theta_h = 0$. We obtain a system with 7 unknowns and 7 equations. We find a solution with the following details: $\theta_l = q - \sqrt{(\mu-q)^2 + \sigma^2}$, $\theta_h = q + \sqrt{(\mu-q)^2+\sigma^2}$, $\lambda_l^* = \frac{1}{2} + \frac{q-\mu}{2\sqrt{(\mu-q)^2+\sigma^2}}$, $\lambda_h^* = \frac{1}{2} - \frac{q-\mu}{2\sqrt{(\mu-q)^2+\sigma^2}}$, $y_0^* = (p-c-rc)\, q + y_2^*(\theta_h)^2$, $y_1^* = -2y_2^*\theta_h$, and $y_2^* = \frac{-(p-s)}{2(\theta_h-\theta_l)}$. The binding constraints and multipliers indicate that the worst-case distribution is identical to Equation (A-4).

By substituting y_i^* into the objective function in Equation (4.8), we find

$$P_f(q) = y_0^* + y_1^*\mu + y_2^*\left(\mu^2+\sigma^2\right)$$

$$
\begin{aligned}
&= (p-c-rc)\,q + y_2^*(\theta_h)^2 + (-2y_2^*\theta_h)\,\mu + \left[\frac{-(p-s)}{2(\theta_h-\theta_l)}\right]\left(\mu^2+\sigma^2\right) \\
&= \frac{(p-s)}{2}\left(\mu + q - \sqrt{(\mu-q)^2+\sigma^2}\right) - (c+rc-s)\,q.
\end{aligned}
$$

To ensure that the above solution is optimal for the SIP model in Equation (4.8), the following critical condition must hold: $\theta_l = q - \sqrt{(\mu-q)^2+\sigma^2} > \delta$. Otherwise, $\theta = \theta_l$ is not a binding SIP constraint. The detailed reasons are the following. Under the loan agreement, the firm's ex post cash position is 0 when $\theta \le \delta$. However, the above solution y_i^* yields $y_0^* + y_1^*\theta_l + y_2\theta_l^2 = (p-s)\,\theta_l - (c+rc-s)\,q < 0$, contradicting the presumption that $\theta = \theta_l$ is a binding constraint. Because $\delta = \left(\frac{c+rc-s}{p-s}\right)q$, the critical condition becomes the following:

$$
q - \sqrt{(\mu-q)^2+\sigma^2} > \left(\frac{c+rc-s}{p-s}\right)q.
$$

After reorganizing the terms and performing some algebra, we find that the above inequality reduces to $h(q) < 0$, where $h(q)$ is already defined by Equation (4.9).

Second, based on Figure 1a), we conjecture that $\theta = \delta$ and $\theta = g$ (where g is to be determined ex-post) are the two binding constraints. The Lagrangian becomes

$$
\Gamma = y_0 + y_1\mu + y_2\mu^2\left(1+\rho^2\right) - \lambda_\delta\left(y_0 + y_1\delta + y_2\delta^2\right) - \lambda_g\left[y_0 + y_1 g + y_2 g^2 - (p-c-rc)\,q\right].
$$

Similar to the early analyses, we obtain a system with 6 unknowns and 6 equations with the following details: $\frac{\partial\Gamma}{\partial y_0} = 1 - \lambda_\delta - \lambda_g = 0$, $\frac{\partial\Gamma}{\partial y_1} = \mu - \lambda_\delta\delta - \lambda_g g = 0$, $\frac{\partial\Gamma}{\partial y_2} = (\mu^2+\sigma^2) - \lambda_\delta\delta^2 - \lambda_g g^2 = 0$, $y_0 + y_1\delta + y_2\delta^2 = 0$, $y_0 + y_1 g + y_2 g^2 = (p-c-rc)\,q$, and $y_1 + 2y_2 g = 0$ (which is the tangent condition for $\theta = g$). Solving this system of 6 unknowns and 6 equations, we obtain the following solution: $y_0^* = y_2^* g^2 + (p-c-rc)\,q$, $y_1^* = -2y_2^* g$, and $y_2^* = \frac{-(p-c-rc)q}{(g-\delta)^2}$, where $g = \frac{\mu^2\left(1+\rho^2\right)-\mu\delta}{\mu-\delta}$, $\lambda_\delta = 1-\lambda_g$, and $\lambda_g = \frac{(\mu-\delta)^2}{(\mu-\delta)^2+\sigma^2}$. It is then easy to verify that

$$
\begin{aligned}
P_f(q) &= y_0^* + y_1^*\mu + y_2^*\left(\mu^2+\sigma^2\right) \\
&= (p-c-rc)\,q + \left(\frac{-(p-c-rc)\,q}{(g-\delta)^2}\right)g^2 - 2\left(\frac{-(p-c-rc)\,q}{(g-\delta)^2}\right)g\mu - \frac{(p-c-rc)\,q\,(\mu^2+\sigma^2)}{(g-\delta)^2} \\
&= (p-c-rc)\,q\left[1 - \frac{\sigma^2\left[\sigma^2+(\mu-\delta)^2\right]}{\left[\mu^2\rho^2+(\mu-\delta)^2\right]^2}\right] = \frac{(\mu-\delta)^2(p-c-rc)\,q}{(\mu-\delta)^2+\sigma^2}.
\end{aligned}
$$

The multipliers and binding constraints imply that the firm's worst case distribution satisfies Equation (4.10). The critical condition is that $q - \sqrt{(\mu-q)^2+\sigma^2} \le \delta$, which is equivalent to $h(q) \ge 0$.

Recall that $h(q)$ in Equation (4.9) is a quadratic function of q. If the determinant of $h(q)$ satisfies that $\Delta(r) \leq 0$, then $h(q) \geq 0$ for any q. We conclude that $P_f(q) = \frac{(\mu-\delta)^2(p-c-rc)q}{(\mu-\delta)^2+\sigma^2}$. When δ approaches μ, $P_f(q)$ approaches zero. When $\delta > \mu$, it is easy to construct a worst case distribution that yields the firm zero profit. Hence, $P_f(q) = 0$ if $\delta = \left(\frac{c+rc-s}{p-s}\right) q > \mu$. We conclude that $P_f(q)$ is continuous in q if $\Delta(r) \leq 0$.

If the determinant of $h(q)$ satisfies that $\Delta(r) > 0$, then $h(q) = 0$ has two positive roots (because $h(0) > 0$). Let these two roots be $\hat{q}_1$ and $\hat{q}_2$. It holds that $h(q) \geq 0$ if $q \in [0, \hat{q}_1] \cup \left[\hat{q}_2, \frac{\mu(p-s)}{c+rc-s}\right]$ such that $P_f(q) = \frac{(\mu-\delta)^2(p-c-rc)q}{(\mu-\delta)^2+\sigma^2}$. If $\hat{q}_1 \leq q \leq \hat{q}_2$, $h(q) < 0$; hence,

$$P_f(q) = \frac{(p-s)}{2}\left(\mu + q - \sqrt{(\mu-q)^2 + \sigma^2}\right) - (c + rc - s)\,q,$$

which is well known to be concave in q.

Our remaining task is to prove that $P_f(q)$ is continuous in q when $\Delta(r) > 0$. We use $c_u = p - c - rc$ and $c_o = c + rc - s$ to streamline the exposition. We obtain a linear relationship between δ and q as follows: $\delta(c_u + c_o) = c_o q$. In other words, we can use δ and q interchangeably as the decision variable. At point $q = \hat{q}_1$, we find that $\hat{q}_1 - \sqrt{(\mu-\hat{q}_1)^2 + \sigma^2} = \hat{\delta}_1$, which implies that

$$\mu^2 - 2\mu\hat{q}_1 + \sigma^2 = -2\hat{q}_1\hat{\delta}_1 + \left(\hat{\delta}_1\right)^2.$$

By reorganizing the terms in the above equation, we obtain that

$$\mu^2 - 2\mu\delta + \delta^2 + \sigma^2 = -2\hat{q}_1\delta + 2\delta^2 + 2\mu\hat{q}_1 - 2\mu\delta.$$

In the above equation, the LHS equals $\left(\mu - \hat{\delta}_1\right)^2 + \sigma^2$, while the RHS is

$$2\left(-\frac{c_u + c_o}{c_o}\left(\hat{\delta}_1\right)^2 + \left(\hat{\delta}_1\right)^2 + \mu\frac{c_u + c_o}{c_o}\hat{\delta}_1 - \mu\hat{\delta}_1\right) = 2\hat{\delta}_1\left(\mu - \hat{\delta}_1\right)\frac{c_u}{c_o}.$$

We find that

$$\left(\mu - \hat{\delta}_1\right)^2 + \sigma^2 = 2\hat{\delta}_1\left(\mu - \hat{\delta}_1\right)\frac{c_u}{c_o}. \tag{B-2}$$

Using Equation (4.13) with $\hat{q}_1 \leq q \leq \hat{q}_2$, we find that, at point $q = \hat{q}_1$, the right limit of $P_f(\hat{q}_1)$ equals

$$P_f(\hat{q}_1) = \frac{(c_u + c_o)}{2}(\mu + \delta - 2\delta) = \frac{1}{2}(c_u + c_o)(\mu - \delta);$$

while the left limit equals

$$P_f(\hat{q}_1) = \frac{(\mu-\delta)^2 (p-c-rc)\hat{q}_1}{(\mu-\delta)^2+\sigma^2} = \frac{(\mu-\delta)^2 c_u}{(\mu-\delta)^2+\sigma^2}\frac{c_u+c_o}{c_o}\delta$$
$$= \frac{(\mu-\delta)^2 c_u \frac{c_u+c_o}{c_o}\delta}{2\delta(\mu-\delta)\frac{c_u}{c_o}} = \frac{1}{2}(c_u+c_o)(\mu-\delta).$$

We conclude that the left limit and right limit are equal at the point $q=\hat{q}_1$. Due to symmetry, we can also verify the continuity at point $q=\hat{q}_2$. This proves that $P_f(q)$ is continuous in q when $\Delta(r)>0$. Q.E.D.

Proof of Proposition 3:

i) We first consider that $\Delta(r)\leq 0$ such that $P_f(q)$ follows Equation (4.12). We let $x=(\mu-\delta)^2$ so that $q=\frac{p-s}{c+rc-s}(\mu-\sqrt{x})$. We rewrite Equation (4.12) as the following:

$$P_f(q) = \frac{x(\mu-\sqrt{x})}{x+\sigma^2}\frac{(p-c-rc)(p-s)}{c+rc-s}, \tag{B-3}$$

of which $\frac{(p-c-rc)(p-s)}{c+rc-s}$ is a constant. Theorem 6.9 of Avriel (1976) states that function $\frac{a(x)}{b(x)}$ is quasi-concave in x if the numerator $a(x)$ is concave in x, and the denominator $b(x)$ is linear and positive in x. In Equation (B-3), the numerator $x(\mu-\sqrt{x})$ is concave in x, and the denominator $x+\sigma^2$ is linear and positive in x. As we have shown in the proof of Proposition 2, at the firm's optimum, $\delta\in[0,\mu)$. We observe that $(\mu-\delta)^2$ is a monotonic convex function of δ when $\delta\in[0,\mu)$. It is easy to verify that the quasi-concavity of $f(g(\cdot))$ is preserved if $f(\cdot)$ is quasi-concave, and $g(\cdot)$ is monotonic and convex. We conclude that $P_f(q)$ is quasi-concave in q.

We solve the firm's optimal production quantity using the first order condition. Recall that q and δ form a perfect linear relationship. We can use δ and q interchangeably as the decision variable. Taking the first derivative with respect to δ, we obtain

$$\frac{\partial P_f(\delta)}{\partial\delta} = \frac{c_u(c_u+c_o)}{c_o\left[(\mu-\delta)^2+\sigma^2\right]^2}(\mu-\delta)\left[\sigma^2(\mu-3\delta)+(\mu-\delta)^3\right]. \tag{B-4}$$

When δ approaches μ, $\frac{\partial P_f(\delta)}{\partial\delta}$ approaches zero. Therefore, both $P_f(\delta)$ and $\frac{\partial P_f(\delta)}{\partial\delta}$ are continuous in δ. By letting Equation (B-4) be zero, we obtain $\sigma^2(\mu-3\delta)+(\mu-\delta)^3=0$. When $\delta<\frac{\mu}{3}$, the first order condition does not hold. There exists an optimal δ^* satisfying the first order condition and the condition $\frac{\mu}{3}<\delta^*<\mu$. After re-organizing terms, we obtain

$$(\mu-\delta)^3+3\sigma^2(\mu-\delta)=2\sigma^2\mu,$$

in which δ follows Equation (4.2). We observe that the above equation is a cubic equation of q. Solving this cubic equation, we confirm Equation (4.14).

Although we have proved quasi-concavity, it is still useful to verify the second order condition. The second derivative is

$$P_f''(\delta) = -\frac{2\sigma^2 c_u (c_u + c_o)}{c_o} \frac{\left[\delta^3 + (\mu^2 + \sigma^2)(2\mu - 3\delta)\right]}{\left[(\mu - \delta)^2 + \sigma^2\right]^3}. \tag{B-5}$$

At the point $\delta = \delta^*$ where $\sigma^2 (\mu - 3\delta^*) + (\mu - \delta^*)^3 = 0$, we find that

$$\begin{aligned}
&(\delta^*)^3 + \left(\mu^2 + \sigma^2\right)(2\mu - 3\delta^*) = (\delta^*)^3 + \left(\mu^2 - \frac{(\mu - \delta^*)^3}{(\mu - 3\delta^*)}\right)(2\mu - 3\delta^*) \\
&= \frac{(\delta^*)^3 (\mu - 3\delta^*) + \left[\mu^2 (\mu - 3\delta^*) - \mu^3 + 3\mu^2\delta^* - 3\mu (\delta^*)^2 + (\delta^*)^3\right](2\mu - 3\delta^*)}{\mu - 3\delta^*} \\
&= \frac{(\delta^*)^3 (\mu - 3\delta^*) + \left[-3\mu (\delta^*)^2 + (\delta^*)^3\right](2\mu - 3\delta^*)}{\mu - 3\delta^*} \\
&= \frac{(\delta^*)^3 \mu - 3(\delta^*)^4 - 6\mu^2 (\delta^*)^2 + (\delta^*)^3 2\mu + 9\mu (\delta^*)^3 - 3(\delta^*)^4}{\mu - 3\delta^*} \\
&= \frac{-6\mu^2 (\delta^*)^2 + 12\mu (\delta^*)^3 - 6(\delta^*)^4}{\mu - 3\delta^*} = \frac{6(\delta^*)^2 (\delta^* - \mu)^2}{3\delta^* - \mu} > 0.
\end{aligned}$$

We conclude that $P_f''(\delta^*)$ in Equation (B-5) is negative.

ii) We consider that $\Delta(r) > 0$ such that $P_f(q)$ is piece-wise and follows Equation (4.13). When $q \in [0, \hat{q}_1] \cup \left[\hat{q}_2, \frac{\mu(p-s)}{c+rc-s}\right]$, we can apply the result related to the case $\Delta(r) \le 0$ to claim that $P_f(q)$ is quasi-concave over these two intervals. When $\hat{q}_1 \le q \le \hat{q}_2$, we also know that $P_f(q)$ is concave (and hence quasi-concave). The first order condition based on $\Delta(r) > 0$ and $\hat{q}_1 \le q \le \hat{q}_2$ yields

$$q_r = \bar{q}(r) = \mu \left[1 + \frac{\rho}{2}\left(\sqrt{\frac{p - c - rc}{c + rc - s}} - \sqrt{\frac{c + rc - s}{p - c - rc}}\right)\right].$$

According to the proof of case i), the first order condition based on $\Delta(r) \le 0$ yields Equation (4.14).

A difference between case i) and case ii) is that Equation (4.13) is piece wise. To establish that the entire $P_f(q)$ function is quasi-concave, we must verify that the first derivative $\frac{\partial P_f(q)}{\partial q}$ is also continuous in q. When $\hat{q}_1 \le q \le \hat{q}_2$, the first derivative with respect to q is

$$\frac{\partial P_f(q)}{\partial q} = \frac{c_u + c_o}{2}\left[1 + \frac{\mu - q}{\sqrt{(\mu - q)^2 + \sigma^2}}\right] - c_o.$$

When $q=\hat{q}_1$, $\hat{\delta}_1=\frac{c_o}{c_o+c_u}\hat{q}_1$ and $\hat{q}_1-\hat{\delta}_1=\sqrt{(\mu-\hat{q}_1)^2+\sigma^2}$ hold. When q approaches $\hat{q}_1$ from the right, we obtain that

$$\begin{aligned}\frac{\partial P_f(q)}{\partial q}|_{q=\hat{q}_1} &= \frac{c_u+c_o}{2}\left[1+\frac{\mu-\frac{c_u+c_o}{c_o}\hat{\delta}_1}{\hat{q}_1-\hat{\delta}_1}\right]-c_o=\frac{c_u+c_o}{2}\left[1+\frac{\mu-\frac{c_u+c_o}{c_o}\hat{\delta}_1}{\frac{c_u}{c_o}\hat{\delta}_1}\right]-c_o\\ &= -\frac{c_o\left[c_o\left(\hat{\delta}_1-\mu\right)+c_u\left(3\hat{\delta}_1-\mu\right)\right]}{2c_u\hat{\delta}_1}.\end{aligned}$$

When q approaches $\hat{q}_1$ from the left, we obtain

$$\frac{\partial P_f(q)}{\partial q}|_{q=\hat{q}_1}=-\frac{2c_uc_o\hat{q}_1\left(\mu-\hat{\delta}_1\right)}{\left[\sigma^2+\left(\mu-\hat{\delta}_1\right)^2\right](c_u+c_o)}+\frac{c_u\left(\mu-\hat{\delta}_1\right)^2}{\sigma^2+\left(\mu-\hat{\delta}_1\right)^2}+\frac{2c_uc_o\hat{q}_1\left(\mu-\hat{\delta}_1\right)^3}{\left[\sigma^2+\left(\mu-\hat{\delta}_1\right)^2\right]^2(c_u+c_o)}.$$

We use Equation (B-2) to simplify the above equation as follows:

$$\begin{aligned}\frac{\partial P_f(q)}{\partial q}|_{q=\hat{q}_1} &= \frac{-2c_uc_o\hat{q}_1\left(\mu-\hat{\delta}_1\right)}{2\hat{\delta}_1\left(\mu-\hat{\delta}_1\right)\frac{c_u}{c_o}(c_u+c_o)}+\frac{c_u\left(\mu-\hat{\delta}_1\right)^2}{2\hat{\delta}_1\left(\mu-\hat{\delta}_1\right)\frac{c_u}{c_o}}+\frac{2c_uc_o\hat{q}_1\left(\mu-\hat{\delta}_1\right)^3}{4\left(\hat{\delta}_1\right)^2\left(\mu-\hat{\delta}_1\right)^2\left(\frac{c_u}{c_o}\right)^2(c_u+c_o)}\\ &= \frac{-c_o^2\hat{q}_1}{\hat{\delta}_1(c_u+c_o)}+\frac{c_o\left(\mu-\hat{\delta}_1\right)}{2\hat{\delta}_1}+\frac{c_o^3\hat{q}_1\left(\mu-\hat{\delta}_1\right)}{2\left(\hat{\delta}_1\right)^2c_u(c_u+c_o)}\\ &= \frac{-c_o^2\hat{\delta}_1\frac{c_u+c_o}{c_o}}{\hat{\delta}_1(c_u+c_o)}+\frac{c_o\left(\mu-\hat{\delta}_1\right)}{2\hat{\delta}_1}+\frac{c_o^3\left(\mu-\hat{\delta}_1\right)\hat{\delta}_1\frac{c_u+c_o}{c_o}}{2\left(\hat{\delta}_1\right)^2c_u(c_u+c_o)}\\ &= -c_o+\frac{c_o\left(\mu-\hat{\delta}_1\right)}{2\hat{\delta}_1}+\frac{c_o^2\left(\mu-\hat{\delta}_1\right)}{2\left(\hat{\delta}_1\right)c_u}=-\frac{c_o\left[c_o\left(\hat{\delta}_1-\mu\right)+c_u\left(3\hat{\delta}_1-\mu\right)\right]}{2c_u\hat{\delta}_1}.\end{aligned}$$

Thus, the first derivative $\frac{\partial P_f(q)}{\partial q}$ is continuous at point $q=\hat{q}_1$. Similarly, we can verify that the first derivative $\frac{\partial P_f(q)}{\partial q}$ is continuous at $q=\hat{q}_2$ due to symmetry.

With a continuous $\frac{\partial P_f(q)}{\partial q}$, we conclude that $P_f(q)$ is quasi-concave in q when $\Delta(r)>0$. Q.E.D.

Proof of Proposition 4:

Let $U_{wst}(r,q)=\inf_{F\in\Omega}\int_0^\infty U(\theta|r,q)\,dF(\theta)$ be the investor's worst case expected cash flow. Using Equation (4.4), we obtain

$$\begin{aligned}U(\theta|r,q) &= (p-s)\min\left(\theta,\frac{c+rc-s}{p-s}q\right)-(c-s)q\\ &= (p-s)\min(\theta,\delta)-(c-s)q.\end{aligned}$$

The firm must prefer a lower interest rate. Recall that $\delta = \frac{c+rc-s}{p-s}q$. A lower interest rate r leads to a smaller δ, while the q remains unchanged. It is then optimal for the firm to set $\delta \leq \frac{1}{2}\mu\left(1+\rho^2\right)$. Using Equation (A-2), we find

$$U_{wst}\left(r,q\right) = -\left(c-s\right)q + \frac{p-s}{1+\rho^2}\left(\frac{c+rc-s}{p-s}\right)q = \frac{q\left[rc-\rho^2\left(c-s\right)\right]}{1+\rho^2}. \tag{B-6}$$

The investor's worst case distribution is the distribution shown in Equation (A-3). Hence, the investor's IR constraint is simplified into

$$\frac{q\left[rc-\rho^2\left(c-s\right)\right]}{1+\rho^2} \geq r_0 cq.$$

When the investor's IR constraint is binding, we obtain that the firm's robust optimal interest rate is $r_f = \rho^2\left(1-\frac{s}{c}\right) + r_0\left(1+\rho^2\right)$. Once we determine the firm's optimal interest rate, we apply Proposition 3 to determine the firm's optimal production quantity. Q.E.D.

Proof of Proposition 5:

Letting $K\left(\theta,q\right)+I_0 = \left(1+r\right)\left(cq-I_1\right)^+$, we have

$$\delta\left(I_0,I_1\right) = \left(\frac{\left(1+r\right)\left(cq-I_1\right)^+ - I_0 - sq}{p-s}\right)^+,$$

which is the counterpart of Equation (4.2). Similar to Equation (B-6), we find

$$U_{wst}\left(r,q\right) = -\left(c-s\right)q + I_0 + \frac{p-s}{1+\rho^2}\frac{\left(\left(1+r\right)\left(cq-I_1\right)^+ - I_0 - sq\right)^+}{p-s} \geq r_0 cq. \tag{B-7}$$

The firm must prefer to reduce the interest rate. If $\left(1+r_0\right)\left(cq-I_1\right)^+ \leq I_0 + sq$, then $r = r_0$ is feasible to the investor's IR constraint (B-7). We conclude that the firm's optimal interest rate is $r_f = r_0$ and $\delta\left(r_0|I_0,I_1\right) = 0$ such that the loan does not involve any risk of defaulting. If $\left(1+r_0\right)\left(cq-I_1\right) > I_0 + sq$, then $r = r_0$ is infeasible to the investor's IR constraint (B-7). The loan involves a non-zero risk of defaulting. The investor's IR constraint (B-7) becomes:

$$U_{wst}\left(r,q\right) = -\left(c-s\right)q + I_0 + \frac{\left(1+r\right)\left(cq-I_1\right) - I_0 - sq}{1+\rho^2} \geq r_0 cq,$$

which yields

$$r \geq r_0 + \rho^2\left[r_0 - \frac{\left(I_0 - cq + sq\right)}{cq}\right] = r_0 + \rho^2\left(1 + r_0 - \frac{sq+I_0}{cq-I_1}\right).$$

Summarizing the two cases, we obtain that the firm's optimal interest rate satisfies

$$r_f(I_0) = r_0 + \rho^2 \left(1 + r_0 - \frac{sq + I_0}{(cq - I_1)^+}\right)^+,$$

which proves Equation (5.2).

Next, we explain how to solve the firm's optimal production quantity for any given r. Let $\bar{q}(r_0)$ be the central planner's optimal production quantity when the interest rate is r_0. We have two cases to consider.

Case i) $I_0 \geq (1 + r_0) c (\bar{q}(r_0) - I_1)^+ - s\bar{q}(r_0)$. The collateral is sufficiently valuable to make the loan risk-free. The firm's optimal interest rate is r_0, and the optimal production quantity is $q_r(I_0) = \bar{q}(r_0)$.

Case ii) $I_0 < (1 + r_0) c (\bar{q}(r_0) - I_1)^+ - s\bar{q}(r_0)$. The loan involves a risk of defaulting. To make Scarf's rule optimal, the critical condition is that $\theta_l = q - \sqrt{(\mu - q)^2 + \sigma^2} > \delta(I_0, I_1)$. We obtain that

$$h(q|I_0, I_1) = q^2 \left(1 - \left(\frac{p - c - rc}{p - s}\right)^2\right) - 2q\left[\mu + \frac{A_2 (p - c - rc)}{(p - s)^2}\right] + (\mu^2 + \sigma^2) - \left(\frac{A_2}{p - s}\right)^2,$$

the determinant of which changes to the following:

$$\Delta(r|I_0, I_1) = 4\left[\mu + \frac{A_2(p - c - rc)}{(p - s)^2}\right]^2 - 4\left[1 - \frac{(p - c - rc)^2}{(p - s)^2}\right]\left[(\mu^2 + \sigma^2) - \left(\frac{A_2}{p - c}\right)^2\right].$$

Sub-case 2a) $\Delta(r|I_0, I_1) \leq 0$. The firm's objective function equals

$$P_f(q|I_0, I_1) = \begin{cases} \frac{(\mu - \delta(I_0, I_1))^2 (p - c - rc) q}{\sigma^2 + (\mu - \delta(I_0, I_1))^2}, & 0 \leq q \leq \frac{\mu(p - s)}{p - c - rc} \\ 0, & q \geq \frac{\mu(p - s)}{c + rc - s} \end{cases},$$

which is similar to Equation (4.12) except that the threshold of defaulting changes to $\delta(I_0, I_1)$. Optimizing the above equation, we find that.

$$q = \frac{Y^2 + (A_1 + A_2) Y - A_3}{(c + rc - s) Y}$$

which proves Equation (5.1).

Sub-case 2b) $\Delta(r|I_0,I_1)>0$. Let $\hat{q}_1(I_0,I_1)$ and $\hat{q}_2(I_0,I_1)$ be the two roots of Equation $h(q|I_0,I_1)=0$. The firm's worst case expected profit equals

$$P_f(q|I_0,I_1)=\begin{cases}\frac{(\mu-\delta(I_0,I_1))^2(p-c-rc)q}{\sigma^2+(\mu-\delta(I_0,I_1))^2}, & 0\le q\le \hat{q}_1(I_0,I_1),\\ \frac{(p-s)}{2}\left(\mu+q-\sqrt{(\mu-q)^2+\sigma^2}\right)-(c+rc-s)q, & \hat{q}_1(I_0,I_1)\le q\le \hat{q}_2(I_0,I_1),\\ \frac{(\mu-\delta(I_0,I_1))^2(p-c-rc)q}{\sigma^2+(\mu-\delta(I_0,I_1))^2}, & \hat{q}_2(I_0,I_1)\le q\le \frac{\mu(p-s)}{c+rc-s},\\ 0, & q\ge \frac{\mu(p-s)}{c+rc-s},\end{cases}.$$

To solve the robust contracting equilibrium, we jointly determine $r_f(I_0,I_1)=r_f^*$ and $q_f(I_0,I_1)=q^*$ using Equations (5.2) and the firm's first order condition. Q.E.D.

Proof of Proposition 6:

The ironing procedure starts with the investor's robust optimization model as follows:

$$U(q|b)=\inf_{F\in\Omega}\int_0^\infty [bK(\theta,q)]\,dF(\theta)-cq.$$

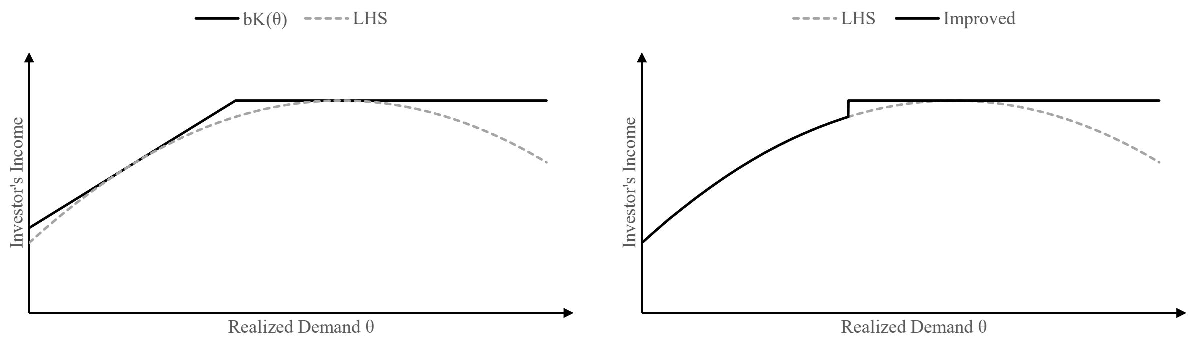
Because $-cq$ is a constant, we formulate the corresponding SIP model as follows:

$$\begin{aligned} &P_I(q|b)=\max_{y_i}\left\{y_0+y_1\mu+y_2\left(\mu^2+\sigma^2\right)\right\}\\ \text{s.t.}\quad &y_0+y_1\theta+y_2\theta^2\le b(p\theta+sq-s\theta),\ \forall 0\le\theta\le q,\\ &y_0+y_1\theta+y_2\theta^2\le bpq,\ \forall\theta\ge q.\end{aligned}$$

The above SIP model $P_I(q|b)$ is an affine transformation of Scarf's standard model. Similar to the analysis related to Equation (B-1), we obtain the investor's shadow prices as follows: $y_0^*=bpq+y_2^*\theta_h^2$, $y_1^*=-2y_2^*\theta_h$, and $y_2^*=-\frac{b(p-s)}{2(\theta_h-\theta_l)}$, where $\theta_l=q-\sqrt{(\mu-q)^2+\sigma^2}$ and $\theta_h=q+\sqrt{(\mu-q)^2+\sigma^2}$. The Lagrangian multipliers are $\lambda_l^*=\frac{1}{2}+\frac{q-\mu}{2\sqrt{(\mu-q)^2+\sigma^2}}$ and $\lambda_h^*=\frac{1}{2}-\frac{q-\mu}{2\sqrt{(\mu-q)^2+\sigma^2}}$.

Next, we point wise adjust the payment that the investor receives. To obtain an improved payment function $w_d(\theta)$, we use the quadratic function $y_0^*+y_1^*\theta+y_2^*\theta^2$ as the curved ironing board. We also observe that the ironing board is positive and increasing in θ for any $\theta\in[0,q]$. The ironing procedure involves two cases. i) If $\theta<q$, let $w_d(\theta)=y_0^*+y_1^*\theta+y_2^*\theta^2$. ii) If $\theta\ge q$, let $w_d(\theta)=bpq$. The improved payment function has a jump at point $\theta=q$ due to lost sales. When the realized demand $\theta>q$, the investor can verify only the sales quantity but not the realized demand. Hence, the improved payment function has a jump at point $\theta=q$. We visualize $w_d(\theta)$ in Figure 3. Specifically, in Figure 3a), we depict the investor's income $bK(\theta,q)$ as the dark solid curve and the LHS of the SIP constraints of model $P_I\left(q|b\right)$ as the light dashed curve. After ironing, the dark solid curve in Figure 3b) depicts the improved agreement $w_d\left(\theta\right)$ while the light dashed curve depicts the LHS of the SIP constraints for contrasting purposes.

Figure 3 Illustration of Ironing the Equity Agreement



(a) SIP Model for Equity Agreement

(b) Improved Payment Function

From the firm's perspective, the improved payment function is less expensive than the original equity agreement. The improved payment function has a quadratic component and a jump at the point $\theta = q$. If the firm keeps his production quantity unchanged, the improved payment function satisfies the investor's IR constraint and thus, dominates the equity agreement in Proposition 3.

Under the improved agreement $w_d\left(\theta\right)$, the firm's ex post cash position equals

$$\begin{aligned} Z\left(\theta|q,w_d\right) &= K\left(\theta,q\right) - w_d\left(\theta\right) \\ &= \begin{cases} \left(p\theta + sq - s\theta\right) - y_0^* - y_1^*\theta - y_2^*\theta^2, & 0 \le \theta \le q \\ \left(1-b\right)pq, & \theta \ge q \end{cases}. \end{aligned}$$

We formulate the firm's corresponding SIP model as follows:

$$\begin{aligned} & P_N\left(q|b\right) = \max_{x_i} \left\{x_0 + x_1\mu + x_2\left(\mu^2 + \sigma^2\right)\right\} \\ \text{s.t.}\quad & x_0 + x_1\theta + x_2\theta^2 \le \left(p\theta + sq - s\theta\right) - y_0^* - y_1^*\theta - y_2^*\theta^2,\ \forall 0 \le \theta \le q, \\ & x_0 + x_1\theta + x_2\theta^2 \le \left(1-b\right)pq,\ \forall \theta \ge q. \end{aligned}$$

Because we use y_i as the investor's shadow prices, we use x_i as the firm's shadow prices. Figure 2 visualizes the above SIP model. Let Γ be the Lagrangian with two binding constraints at $\theta = \theta_l$ and $\theta = \theta_h$. Similar to the analysis related to Equation (B-1), we obtain a system with 7 unknowns and

7 equations: $\frac{\partial\Gamma}{\partial y_0} = 1 - \lambda_l - \lambda_h = 0$, $\frac{\partial\Gamma}{\partial y_1} = \mu - \lambda_l\theta_l - \lambda_h\theta_h = 0$, $\frac{\partial\Gamma}{\partial y_2} = \mu^2(1+\rho^2) - \lambda_l\theta_l^2 - \lambda_h\theta_h^2 = 0$, $x_0 + x_1\theta_l + x_2\theta_l^2 = (p\theta_l + sq - s\theta_l) - y_0^* - y_1^*\theta_l - y_2^*\theta_l^2$, $x_0 + x_1\theta_h + x_2\theta_h^2 = (1-b)\,pq$, $x_1 + 2x_2\theta_l = (p-s) - y_1^* - 2y_2^*\theta_l$, and $x_1 + 2x_2\theta_h = 0$, where the last two are the tangent conditions. We obtain the firm's shadow prices as follows: $x_0^* = (1-b)\,pq + x_2^*\theta_h^2$, $x_1^* = -2x_2^*\theta_h$, and $x_2^* = -\frac{(1-b)(p-s)}{2(\theta_h-\theta_l)}$, where $\theta_l = q - \sqrt{(\mu-q)^2+\sigma^2}$ and $\theta_h = q + \sqrt{(\mu-q)^2+\sigma^2}$. The Lagrangian multipliers are $\lambda_l^* = \frac{1}{2} + \frac{q-\mu}{2\sqrt{(\mu-q)^2+\sigma^2}}$ and $\lambda_h^* = \frac{1}{2} - \frac{q-\mu}{2\sqrt{(\mu-q)^2+\sigma^2}}$. In particular,

$$x_1^* + 2x_2^*\theta_l + y_1^* + 2y_2^*\theta_l = (1-b)(p-s) + b(p-s) = p - s$$

such that the tangent condition at point $\theta = \theta_l$ holds. The binding SIP constraints and Lagrangian multipliers indicate that the firm's worst distribution remains unaffected by the improved contract w_d. Thus, the quadratic agreement dominates the equity agreement but does not strictly improve the firm's utility. Q.E.D.