STABILIZATION OF LINEAR *(L,M)* SHIFT INVARIANT PLANT

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Abstract: In this paper, a lifting technique is employed to realize a single input single output linear *(L,M)* shift invariant plant as a filter bank system. Based on the filter bank structure, a controller is designed so that the aliasing components in the control loop are cancelled and the loop gain becomes a time invariant transfer function. Pole placement technique is applied to stabilize the overall system and ensure the causality of the filters in the controller. An example on the control of a linear *(L,M)* shift invariant plant with simulation result is illustrated. The result shows that our proposed algorithm is simple and effective.

Keywords
Lifting technique, linear *(L,M)* shift invariant plant, filter bank system, aliasing components, pole placement

I. INTRODUCTION

A linear *(L,M)* shift invariant plant is a linear discrete-time single input single output (SISO) system in which the output will shift by L samples when the input shifts by M samples, where L and M form a pair of minimum values of positive integers [1, 2, 3]. A particular example is a linear time periodic varying (LTPV) system, which plays an important role in real world applications [4, 5].

There are some control strategies to deal with those LTPV plants, such as employing the linear quadratic regulation (LQR) method [6]. However, it requires to solve the Riccati equation, which is quite complicated. Also, the dimension of the controller is, in general, higher than that of the plant.

Besides, output stabilization via pole placement is proposed [7]. Although the poles of the system can be arbitrarily assigned except at the origin, there are a lot of constraints on the plant, and some LTPV systems cannot be controlled using this approach.

An H_∞ sampled-data control method is also suggested [8]. However, it requires the Riccati equation corresponding to the Hamilton matrix to have a stabilizing solution, which is also complicated, and some plants do not satisfy this condition.

There is not much research work on the control of a plant with different input and output data rates and this problem is studied in this paper. A lifting technique is reviewed in section II. A time invariance condition in the control loop, a stability condition for the overall system and a causality condition for the filters in the controller are stated in section III, section IV and section V, respectively. The formulation of the controller and the design procedures are discussed in section VI and section VII, respectively. An example on the control of a linear *(L,M)* shift invariant plant with simulation result is illustrated in section VIII. Finally, a concluding remark is discussed in section IX.

II. REVIEW ON THE LIFTING TECHNIQUE

The input-output relationship of a linear *(L,M)* shift invariant system [1, 3] can be characterized by:

$$y[n]=\sum_{k\to-\infty}^{+\infty} g[n,k]\cdot x[k] \qquad \textbf{(1),}$$

where $g[n,k]$ is a two-dimensional kernel function of the system satisfying:

$$g[n,k]=g[n-L,k-M] \qquad (2).$$

Although this kernel function is an infinite dimension matrix, it has L independent rows and M independent columns. So by defining L different linear time invariant (LTI) filters, $h_j[n]$, where $h_j[n]=g[j,-n]$, for $j=0,1,\ldots,L-1$, all linear *(L,M)* shift invariant systems can be realized via a filter bank structure as shown in figure 1.

III. TIME INVARIANCE CONDITION IN THE CONTROL LOOP

Since the pole placement techniques are essentially for LTI systems, the overall control loop should be effectively time invariant. In the filter bank language, it should be free from aliasing. Since the plant is lifted as an analysis bank, the controller may be designed as a synthesis bank as shown in figure 2.

By constructing a closed loop feedback system as shown in figure 3, we have:

$$P(z)=\frac{\mathbf{1}}{M}\cdot\left[\mathbf{E}(z)\quad \mathbf{E}(z\cdot W)\quad \cdots\quad \mathbf{E}\left(z\cdot W^{M-1}\right)\right]\cdot \begin{bmatrix} \mathbf{H}_0(z) & \mathbf{H}_1(z) & \cdots & \mathbf{H}_{L-1}(z) \\ \mathbf{H}_0(z\cdot W) & \mathbf{H}_1(z\cdot W) & \cdots & \mathbf{H}_{L-1}(z\cdot W) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_0\left(z\cdot W^{M-1}\right) & \mathbf{H}_1\left(z\cdot W^{M-1}\right) & \cdots & \mathbf{H}_{L-1}\left(z\cdot W^{M-1}\right) \end{bmatrix}\cdot\begin{bmatrix}\mathbf{F}_0(z)\\ \mathbf{F}_1(z)\\ \vdots \\ \mathbf{F}_{L-1}(z)\end{bmatrix} \textbf{(3),}$$

where

$$W=e^{-\frac{j\cdot 2\cdot\pi}{M}} \qquad \textbf{(4).}$$

If the controller is designed in such a way that:

$$\begin{bmatrix} \mathbf{H_0}(z) & \mathbf{H_1}(z) & \cdots & \mathbf{H_{L\text{-}1}}(z) \\ \mathbf{H_0}(z\cdot W) & \mathbf{H_1}(z\cdot W) & \cdots & \mathbf{H_{L\text{-}1}}(z\cdot W) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H_0}(z\cdot W^{M-1}) & \mathbf{H_1}(z\cdot W^{M-1}) & \cdots & \mathbf{H_{L\text{-}1}}(z\cdot W^{M-1}) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F_0}(z) \\ \mathbf{F_1}(z) \\ \vdots \\ \mathbf{F_{L\text{-}1}}(z) \end{bmatrix} = \begin{bmatrix} \mathbf{M}\cdot\mathbf{T}(z) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \quad \textbf{(5)},$$

then the aliasing components, $E(z\cdot W^l)$, for $l=1,2,\ldots,M-1$, are cancelled and the control loop becomes time invariant with transfer function $T(z)$.

IV. STABLITY CONDITION FOR THE OVERALL SYSTEM

Since the control loop is time invariant now, we have $T(z)=P(z)/E(z)$. As $E(z)=X(z)-P(z)$, we have:

$$E(z)=\frac{X(z)}{\mathbf{1}+T(z)} \quad \textbf{(6)}.$$

The overall system becomes a new linear (L,M) shift invariant system with the corresponding filters in the filter bank structure having transfer functions $H_j(z)/[1+T(z)]$, for $j=0,1,\ldots, L-1$. The overall system is stable if and only if $H_j(z)/[1+T(z)]$, for $j=0,1,\ldots, L-1$, are all stable.

Suppose $H_j(z)$, for $j=0,1,\ldots,L-1$, and $T(z)$ are rational, that is:

$$H_j(z)=H_{j,\mathbf{0}}\cdot\frac{\prod_{i=\mathbf{0}}^{B_j-\mathbf{1}}\left(\mathbf{1}-b_{i,j}z^{-\mathbf{1}}\right)}{\prod_{i=\mathbf{0}}^{A_j-\mathbf{1}}\left(\mathbf{1}-a_{i,j}z^{-\mathbf{1}}\right)} \textbf{ and}$$

$$T(z)=T_{\mathbf{0}}\cdot\frac{\prod_{i=\mathbf{0}}^{D-\mathbf{1}}\left(\mathbf{1}-d_i z^{-\mathbf{1}}\right)}{\prod_{i=\mathbf{0}}^{C-\mathbf{1}}\left(\mathbf{1}-c_i z^{-\mathbf{1}}\right)} \quad \textbf{(7)},$$

then

$$\frac{H_j(z)}{\mathbf{1}+T(z)}=H_{j,\mathbf{0}}\cdot\frac{\prod_{i=\mathbf{0}}^{B_j-\mathbf{1}}\left(\mathbf{1}-b_{i,j}z^{-\mathbf{1}}\right)}{\prod_{i=\mathbf{0}}^{A_j-\mathbf{1}}\left(\mathbf{1}-a_{i,j}z^{-\mathbf{1}}\right)}\cdot\frac{\prod_{i=\mathbf{0}}^{D-\mathbf{1}}\left(\mathbf{1}-c_i z^{-\mathbf{1}}\right)}{T_{\mathbf{0}}\cdot\prod_{i=\mathbf{0}}^{D-\mathbf{1}}\left(\mathbf{1}-d_i z^{-\mathbf{1}}\right)+\prod_{i=\mathbf{0}}^{C-\mathbf{1}}\left(\mathbf{1}-c_i z^{-\mathbf{1}}\right)} \quad \textbf{(8)}.$$

Hence, the overall system is stable if the unstable poles in $H_j(z)$ are cancelled by that in $T(z)$ and the sum of the numerator and the denominator of $T(z)$ is stable.

V. CAUSALITY CONDITION FOR THE FILTERS IN THE CONTROLLER

By applying a polyphase decomposition on $H_j(z)$ and $F_j(z)$, for $j=0,1,\ldots,L-1$, we have:

$$H_j(z)=H_{j,\mathbf{0}}\cdot\frac{\prod_{i=\mathbf{0}}^{B_j-\mathbf{1}}\left(\mathbf{1}-b_{i,j}z^{-\mathbf{1}}\right)\cdot\prod_{i=\mathbf{0}}^{A_j-\mathbf{1}}\left(\mathbf{1}+a_{i,j}z^{-\mathbf{1}}+\cdots+a_{i,j}{}^{M-\mathbf{1}}z^{-(M-\mathbf{1})}\right)}{\prod_{i=\mathbf{0}}^{A_j-\mathbf{1}}\left(\mathbf{1}-a_{i,j}{}^{M}z^{-M}\right)}$$

$$=\sum_{l=\mathbf{0}}^{M-\mathbf{1}}z^{-l}\cdot Q_{j,l}\left(z^M\right) \textbf{and}$$

$$F_j(z)=\sum_{l=\mathbf{0}}^{M-\mathbf{1}}z^{-(M-\mathbf{1}-l)}\cdot R_{l,j}\left(z^M\right) \quad \textbf{(9)}.$$

The control loop can be simplified as in figure 4. If $H_j(z)$ and $F_j(z)$, for $j=0,1,\ldots,L-1$, are all causal, then the polyphase matrices $Q(z)$ and $R(z)$ are also causal. Because the delay chain in the filter bank structure causes a delay of $M-1$, the minimum delay of $T(z)$ is $M-1$. Hence, we should design $T(z)$ in the form of $T(z)=z^{-(M-1)}\cdot T_0(z)$, for some causal $T_0(z)$.

VI. FORMULATION OF THE FILTERS IN THE CONTROLLER

As $T(z)$ is designed, $F_j(z)$, for $j=0,1,\ldots,L-1$, can be solved by equation (5), which consists of a system of M linear equations and L unknowns. If $M=L$ and the aliasing matrix does not drop rank, then there is a unique solution for $F_j(z)$, for $j=0,1,\ldots,L-1$, as follows:

$$\begin{bmatrix} \mathbf{F_0}(z) \\ \mathbf{F_1}(z) \\ \vdots \\ \mathbf{F_{M\text{-}1}}(z) \end{bmatrix} = \begin{bmatrix} \mathbf{H_0}(z) & \mathbf{H_1}(z) & \cdots & \mathbf{H_{M\text{-}1}}(z) \\ \mathbf{H_0}(z\cdot W) & \mathbf{H_1}(z\cdot W) & \cdots & \mathbf{H_{M\text{-}1}}(z\cdot W) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H_0}(z\cdot W^{M-1}) & \mathbf{H_1}(z\cdot W^{M-1}) & \cdots & \mathbf{H_{M\text{-}1}}(z\cdot W^{M-1}) \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{M}\cdot\mathbf{T}(z) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \quad \textbf{(10)}.$$

If $M<L$, there are infinitely many solutions. By selecting $F_M(z),F_{M+1}(z),\ldots,F_{L-1}(z)$ properly, we can solve $F_0(z),F_1(z),\ldots,F_{M-1}(z)$ as follows:

$$\begin{bmatrix} \mathbf{F_0}(z) \\ \mathbf{F_1}(z) \\ \vdots \\ \mathbf{F_{M\text{-}1}}(z) \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{H_0}(z) & \mathbf{H_1}(z) & \cdots & \mathbf{H_{M\text{-}1}}(z) \\ \mathbf{H_0}(z\cdot W) & \mathbf{H_1}(z\cdot W) & \cdots & \mathbf{H_{M\text{-}1}}(z\cdot W) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H_0}(z\cdot W^{M-1}) & \mathbf{H_1}(z\cdot W^{M-1}) & \cdots & \mathbf{H_{M\text{-}1}}(z\cdot W^{M-1}) \end{bmatrix}^{-1} \cdot$$

$$\left(\begin{bmatrix} \mathbf{M}\cdot\mathbf{T}(z) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{H_M}(z) & \mathbf{H_{M+1}}(z) & \cdots & \mathbf{H_{L\text{-}1}}(z) \\ \mathbf{H_M}(z\cdot W) & \mathbf{H_{M+1}}(z\cdot W) & \cdots & \mathbf{H_{L\text{-}1}}(z\cdot W) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H_M}(z\cdot W^{M-1}) & \mathbf{H_{M+1}}(z\cdot W^{M-1}) & \cdots & \mathbf{H_{L\text{-}1}}(z\cdot W^{M-1}) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F_M}(z) \\ \mathbf{F_{M+1}}(z) \\ \vdots \\ \mathbf{F_{L\text{-}1}}(z) \end{bmatrix} \right) \quad \textbf{(11)}.$$

VII. DESIGN PROCEDURE OF THE CONTROLLER

Our proposed design procedure of the controller is as follows:

Step 1: Employ a lifting technique to realize a linear (L,M) shift invariant plant as a filter bank structure as figure 1.

Step 2: From the filter bank structure, highlight all the unstable poles of $H_j(z)$, for $j=0,1,\ldots,L-1$. Then put those unstable poles into the denominator of $T(z)$.

Step 3: Design the roots of the sum of the numerator and the denominator of $T(z)$. Then employ pole placement techniques to solve the numerator of $T(z)$ such that the roots of the sum of the numerator and the denominator of $T(z)$ are the desirable ones and the minimum delay of $T(z)$ is $M-1$.

Step 4: Solve $F_j(z)$ for $j=0,1,\ldots,L-1$ by equation (10) or (11).

Step 5: Construct the controller in a filter bank structure as figure 2 and connect the controller and the plant together as shown in figure 3.

VIII. EXAMPLE

Consider a linear *(L,M)* shift invariant plant with $L=3$, $M=2$, and with kernel function as follows:

$$g[\mathbf{0},k]=\mathbf{2}^{-k}\cdot u[-k],$$
$$g[\mathbf{1},k]=\mathbf{2}^{k}\cdot u[-k],$$
$$g[\mathbf{2},k]=\mathbf{4}^{k}\cdot u[-k]\ \textbf{and}$$
$$g[n,k]=g[n-\mathbf{3},k-\mathbf{2}],\forall n,k\in Z \qquad \textbf{(12).}$$

Step 1

Applying the lifting technique to the plant, we have:

$$h_0[n]=\mathbf{2}^{n}\cdot u[n]\Rightarrow H_0(z)=\frac{\mathbf{1}}{\mathbf{1}-\mathbf{2}\cdot z^{-1}}$$
$$h_1[n]=\mathbf{2}^{-n}\cdot u[n]\Rightarrow H_1(z)=\frac{\mathbf{1}}{\mathbf{1}-\mathbf{0.5}\cdot z^{-1}}\ \textbf{and}$$
$$h_2[n]=\mathbf{4}^{-n}\cdot u[n]\Rightarrow H_2(z)=\frac{\mathbf{1}}{\mathbf{1}-\mathbf{0.25}\cdot z^{-1}} \qquad \textbf{(13).}$$

Step 2

Since $H_0(z)$ is unstable with unstable pole at $z=2$ and $H_1(z)$ and $H_2(z)$ are stable, the denominator of $T(z)$ is $1\text{-}2{\cdot}z^{-1}$.

Step 3

As $M=2$, the minimum delay of $T(z)$ is 1. We can let the numerator of $T(z)$ be $T_1{\cdot}z^{-1}{\cdot}(1\text{-}d{\cdot}z^{-1})$. Since the order of the numerator is 2 and that of the denominator is 1, we can place two poles arbitrarily. By selecting two stable poles at $z=0.6$ and $z=0.8$, then we have:

$T_1{\cdot}z^{-1}{\cdot}(1\text{-}d{\cdot}z^{-1})+1\text{-}2{\cdot}z^{-1}=(1\text{-}0.6{\cdot}z^{-1}){\cdot}(1\text{-}0.8{\cdot}z^{-1})$,

$\Rightarrow T_1=0.6$ and $d=\text{-}0.8$ (14).

That is:

$$T(z)=\frac{\mathbf{0.6}\cdot z^{-1}\cdot\left(\mathbf{1}+\mathbf{0.8}\cdot z^{-1}\right)}{\mathbf{1}-\mathbf{2}\cdot z^{-1}} \qquad \textbf{(15).}$$

We can check that:

$$\frac{H_0(z)}{\mathbf{1}+T(z)}=\frac{\mathbf{1}}{\left(\mathbf{1}-\mathbf{0.6}\cdot z^{-1}\right)\cdot\left(\mathbf{1}-\mathbf{0.8}\cdot z^{-1}\right)}$$
$$\frac{H_1(z)}{\mathbf{1}+T(z)}=\frac{\mathbf{1}-\mathbf{2}\cdot z^{-1}}{\left(\mathbf{1}-\mathbf{0.6}\cdot z^{-1}\right)\cdot\left(\mathbf{1}-\mathbf{0.8}\cdot z^{-1}\right)\cdot\left(\mathbf{1}-\mathbf{0.5}\cdot z^{-1}\right)}\ \textbf{and}$$
$$\frac{H_2(z)}{\mathbf{1}+T(z)}=\frac{\mathbf{1}-\mathbf{2}\cdot z^{-1}}{\left(\mathbf{1}-\mathbf{0.6}\cdot z^{-1}\right)\cdot\left(\mathbf{1}-\mathbf{0.8}\cdot z^{-1}\right)\cdot\left(\mathbf{1}-\mathbf{0.25}\cdot z^{-1}\right)} \qquad \textbf{(16),}$$

are all stable.

Step 4

Since

$$\begin{bmatrix} H_0(z) & H_1(z) & H_2(z) \\ H_0(-z) & H_1(-z) & H_2(-z) \end{bmatrix}\cdot\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix}=\begin{bmatrix} \mathbf{2}\cdot T(z) \\ \mathbf{0} \end{bmatrix} \qquad \textbf{(17),}$$

where $W=\text{-}1$. We have:

$$\begin{bmatrix} \mathbf{F}_0(z) \\ \mathbf{F}_1(z) \end{bmatrix}=\begin{bmatrix} \mathbf{H}_0(z) & \mathbf{H}_1(z) \\ \mathbf{H}_0(-z) & \mathbf{H}_1(-z) \end{bmatrix}^{-1}\cdot\left(\begin{bmatrix} \mathbf{2}\cdot T(z) \\ \mathbf{0} \end{bmatrix}-\begin{bmatrix} \mathbf{H}_2(z) \\ \mathbf{H}_2(-z) \end{bmatrix}\cdot\mathbf{F}_2(z)\right) \textbf{(18),}$$

by selecting $F_2(z)=0$, then we have:

$$\begin{bmatrix} \mathbf{F}_0(z) \\ \mathbf{F}_1(z) \end{bmatrix}=\begin{bmatrix} \mathbf{0.4}\cdot\left(\mathbf{1}+\mathbf{0.8}\cdot z^{-1}\right)\cdot\left(\mathbf{1}+\mathbf{2}\cdot z^{-1}\right)\cdot\left(\mathbf{1}-\mathbf{0.5}\cdot z^{-1}\right) \\ -\mathbf{0.4}\cdot\left(\mathbf{1}+\mathbf{0.8}\cdot z^{-1}\right)\cdot\left(\mathbf{1}+\mathbf{0.5}\cdot z^{-1}\right)\cdot\left(\mathbf{1}-\mathbf{0.5}\cdot z^{-1}\right) \end{bmatrix} \qquad \textbf{(19).}$$

It can been seen that the filters in the controller are causal, FIR and stable. The simulation result is based on a unit step input. Figure 5 shows the output without any controller. Figure 6 shows the output with a controller designed by our proposed algorithm.

IX. CONCLUDING REMARKS

In this paper, a control method on a linear *(L,M)* shift invariant plant is proposed. The algorithm is based on the lifting technique. The time invariance condition in the control loop, the stability condition for the overall system and the causality condition for the filters in the controller are discussed. Based on those conditions, a detail design procedure of the controller is proposed and an example is illustrated. The proposed algorithm is simple and effective, and can be applied not only to LTPV plants, but also linear *(L,M)* shift invariant plants.

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REFERENCES

[1] Tongwen Chen, Li Qiu and Er-Wei Bai, "General Multirate Building Structures with Application to Nonuniform Filter Banks," *IEEE Transactions on Circuits and Systems-II: Analog and Digital Signal Processing*, vol. 45, NO. 8, pp. 948-958, August, 1998.

[2] William M. Campbell and Thomas W. Parks, "Design of a Class of Multirate Systems Using a Maximum Relative l^2-Error Criterion," *IEEE Transactions on Signal Processing*, vol. 45, NO. 3, pp. 561-571, March, 1997.

[3] David G. Meyer, "A New Class of Shift-Varying Operators, Their Shift Invariant Equivalents, and Multirate Digital Systems," *IEEE Transactions on Automatic Control*, vol. 35, NO. 4, pp. 429-433, April, 1990.

[4] K. Onogi and M. Matsubara, "Structure Analysis of Periodically Controlled Chemical Processes," *Chemical Engineering Science*, vol. 34, pp. 1009-1019, 1980.

[5] K. Schadlich, U. Hoffmann and H. Hoffmann, "Periodical Operation of Chemical Processes and Evaluation of Conversion Improvements," *Chemical Engineering Science*, vol. 38, pp. 1375-1384, 1983.

[6] Hamed M. Al-Rahmani and Gene F. Franklin, "A New Optimal Multirate Control of Linear Periodic and Time-Invariant Systems," *IEEE Transactions*

on Automatic Control, vol. 35, NO. 4, pp. 429-433, April, 1990.

[7] Dirk Aeyels and Jacques L. Willems, "Pole Assignment for Linear Periodic Systems by Memoryless Output Feedback," *IEEE Transactions on Automatic Control*, vol. 40, NO. 4, pp. 735-739, April, 1995.

[8] Andrey M. Ghulchak, "Parameterization of All Suboptimal Controllers in H_∞ Sampled-Data Control Problem", *Proceedings on the 35th Conference on Decision and Control*, vol. 1, pp. 434-439, December, 1996.

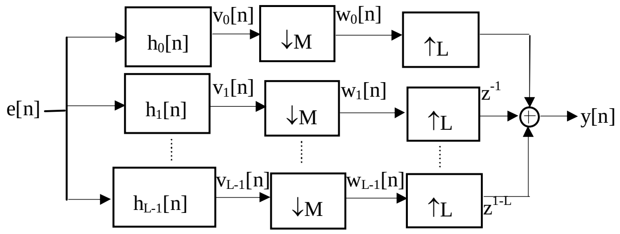

Fig. 1. Filter bank realization of a linear *(L,M)* shift invariant plant

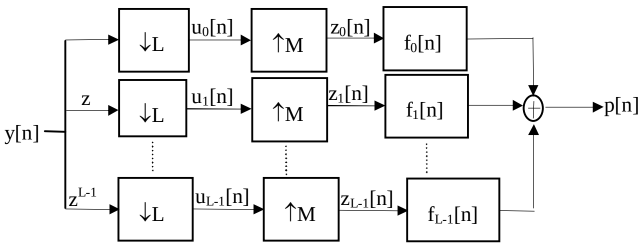

Fig. 2. Filter bank realization of a controller

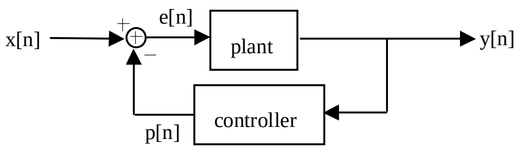

Fig. 3. Control loop in the feedback system

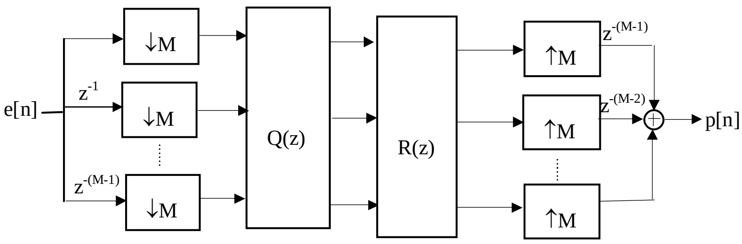

Fig. 4. Polyphase structure of the control loop

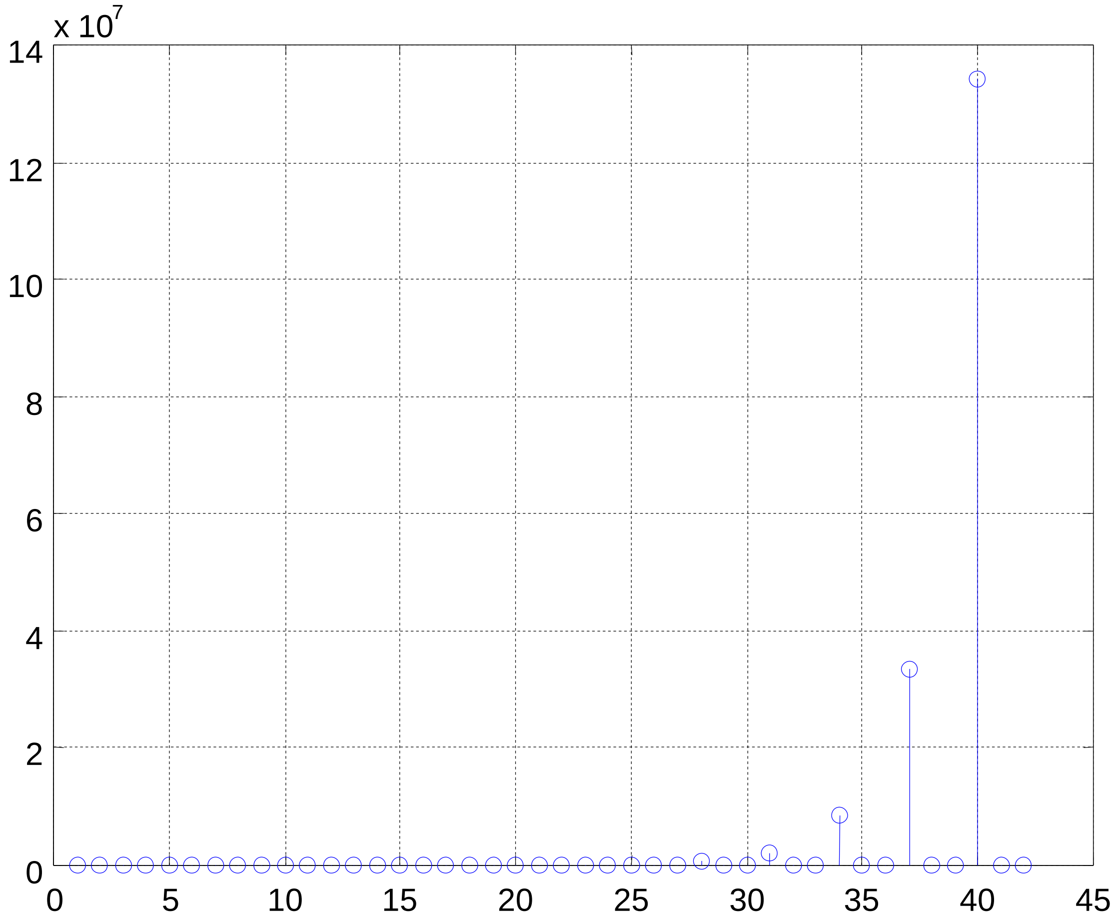

Fig. 5. Step response of the plant without controller

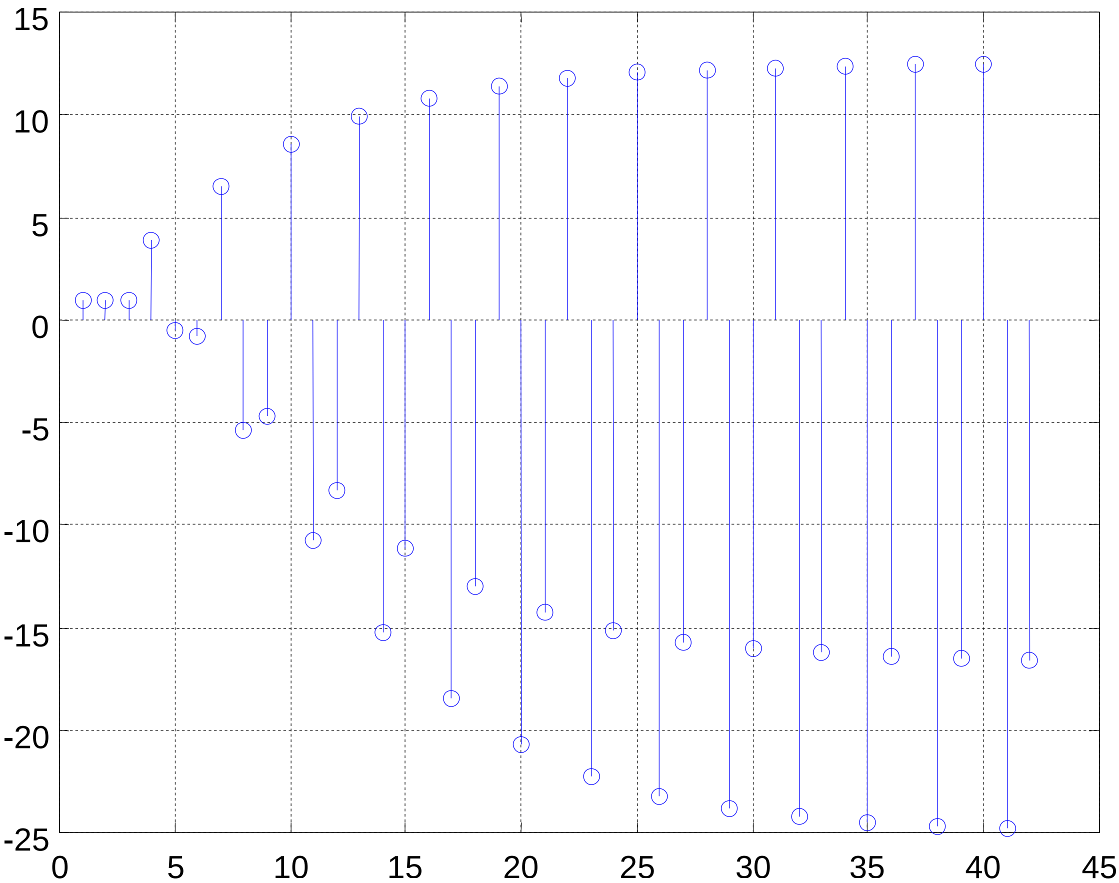

Fig. 6. Step response of the plant with our proposed controller