1962-06

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Faulkner, Frank D.

INFORMS

Frank D. Faulkner, (1962) Determining Optimum Ship Routes. Operations Research 10(6):799-807.
http://hdl.handle.net/10945/70153

Published in Operations Research on December 01, 1962 as DOI: 10.1287/opre.10.6.799.
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DETERMINING OPTIMUM SHIP ROUTES

Frank D. Faulkner

United States Naval Postgraduate School, Monterey, Calif

(Received June, 1962)

A method is given for determining optimum ship routes on a digital computer A ship is assumed to be in a fallout field whose intensity is a known function $f(x,y,t)$ of position and time A typical problem is that of choosing a route to a point where $f \leqq f_m$, the maximum intensity that can be tolerated indefinitely The route is to be such as to minimize the dose $z=\int_0^T f\,dt$, the termininal time and point are not specified The problem of sending the ship to a given point with z minimized is also discussed These are equivalent to corresponding problems of choosing a route to minimize the probability of detection while going through a region where the probability of detection is a known function of position and time

THE METHOD of solution is based on differential formulas which G A Bliss introduced in Ballistics, based on the adjoint system of differential equations These can be combined with the Euler equations and inverted to determine constants associated with the solution, in a Newton-Raphson iteration The method is quite general and can be applied to, say, related problems of rendezvous, if the rendezvous point either follows a known course or cooperates The correction routine for the constants varies greatly with the end conditions, the method can best be demonstrated by some examples and a statement of the general conditions

To simplify the programming it was assumed that distances were small enough that negligible error was introduced by assuming the earth to be flat and that the speed of the ship was constant In most 'practical' problems it can be shown that if f is independent of the speed as above, maximum speed is optimal

BASIC FORMULAS FOR VARIATIONS

In this section formulas are derived for variations of the end values of the variables The method seems somewhat devious at first, but numerical methods for calculating differentials and variational theory generally rest on the equations of the type derived here, employing Lagrange multipliers and the adjoint system of differential equations When approached through these, introductory calculus of variations is quite straightforward

The governing equations for the course and the dose are

$$\begin{aligned} x &= v\cos p, \\ y &= v\sin p, \\ z &= f, \end{aligned} \qquad (1)$$

where x,y are coordinates, z is the dose accumulated, and p, a variable, is the course heading, which is to be found, a dot ( ) over a variable indicates its time derivative We shall be concerned with neighboring courses and it will be assumed that the variations $\delta x, \delta y$ etc , satisfy the equations

$$\begin{aligned} \delta x + v\sin p\,\delta p &= 0, \\ \delta y - v\cos p\,\delta p &= 0, \\ \delta z - f_x \delta x - f_y\,\delta y &= 0 \end{aligned} \tag{2}$$

We shall be interested in formulas for the change $\delta x(T)$, etc , in x at some future time T as a functional of the variation δp of the control variable p

Following Bliss's method, let us multiply each of the three equations (2) through by new unspecified variables λ,μ,ν, respectively, Lagrange multipliers, add, and integrate to get

$$\int_0^T [\lambda(\delta x + v\sin p\,\delta p) + \mu(\delta y - v\cos p\,\delta p) + \nu(\delta z - f_x\delta x - f_y\delta y)]\,dt = 0 \tag{3}$$

Now integrate this by parts to eliminate $\delta x,\delta y,\delta z$ from the integrand,

$$\begin{aligned} (\lambda\delta x + \mu\delta y + \nu\delta z)_0^T = \int_0^T [\delta x(\lambda + \nu f_x) + \delta y(\mu + \nu f_y) \\ + \delta z\nu + v\delta p(-\lambda\sin p + \mu\cos p)]\,dt \end{aligned} \tag{4}$$

To simplify this, let us *choose* λ,μ,ν as solutions to the differential equations

$$\begin{aligned} \lambda + \nu f_x &= 0, \\ \mu + \nu f_y &= 0, \\ \nu &= 0 \end{aligned} \tag{5}$$

These operations of integration by parts in (3) and setting to zero the coefficient of the variations $\delta x,\delta y,\delta z$ of the dependent variables in (4) defines the system (5) of equations *adjoint* to the system (2)

If also the terminal value T changes, the total differential changes in the terminal values of x,y,z are

$$\begin{aligned} \Delta x &= \delta x(T) + x(T)\,\delta T, \\ \Delta y &= \delta y(T) + y(T)\,\delta T, \\ \Delta z &= \delta z(T) + z(T)\,\delta T \end{aligned} \tag{6}$$

In the problems of interest here the initial values are assumed given so that $\delta x(0) = 0$, etc Equation (4) then reduces to

$$[\lambda\Delta x+\mu\Delta y+\nu\Delta z-(x\lambda+y\mu+\nu z)\delta T]_{t=T}$$
$$=\int_0^T(-\lambda\sin p+\mu\cos p)\delta p\,dt \tag{7}$$
$$=\int_0^T \vec{\Lambda}\ \vec{V}_p\,\delta p,$$

where $\vec{\Lambda}=\lambda\vec{\imath}+\mu\vec{\jmath}+\nu\vec{k}$, $\vec{V}=v(\vec{\imath}\cos p+\vec{\jmath}\sin p)$, and the subscript p indicates the partial derivative In the following, no distinction will be made between a vector $\vec{\Lambda}$ and a column matrix with the same elements $\begin{Vmatrix}\lambda\\ \mu\\ \nu\end{Vmatrix}$ Three solutions to (5) are of particular interest,

$$\Lambda_1=\begin{Vmatrix}1\\0\\0\end{Vmatrix},\Lambda_2=\begin{Vmatrix}0\\1\\0\end{Vmatrix},\Lambda_3=\begin{Vmatrix}\lambda_3\\ \mu_3\\ 1\end{Vmatrix}, \tag{8a}$$

where

$$\begin{aligned}\lambda_3+f_x&=0, &\lambda_3(T)&=0,\\ \mu_3+f_y&=0, &\mu_3(T)&=0,\end{aligned} \tag{8b}$$

these constitute a fundamental set For them we get

$$\begin{aligned}\Delta x&=x(T)\ \delta T+\int_0^T\vec{\Lambda}_1\ \vec{V}_p\,\delta p\,dt,\\ \Delta y&=y(T)\ \delta T+\int_0^T\vec{\Lambda}_2\ \vec{V}_p\,\delta p\,dt,\\ \Delta z&=z(T)\ \delta T+\int_0^T\vec{\Lambda}_3\ V_p\,\delta p\,dt\end{aligned} \tag{9}$$

Finally, with every optimum it can be shown that there is a solution Λ to the adjoint $c_1\ \Lambda_1+c_2\ \Lambda_2+c_3\ \Lambda_3$ such that

$$\vec{\Lambda}\ \vec{V}_p=0, \tag{10}$$

or

$$-\lambda\sin p+\mu\cos p=0, \tag{10$'$}$$

or

$$\vec{\Lambda}\ \vec{V}=\text{extremum}, \tag{10$''$}$$

as a function of p for each value of t, and a fourth equivalent form

$$\int_0^T\vec{\Lambda}\ \vec{V}dt=\text{extremum}, \tag{10$'''$}$$

the last extremum being achieved by choosing p properly The second of these, with the adjoint system, constitute the classic Euler, or Euler-Lagrange, equations in calculus of variations The last form (10$'''$) has attracted considerable attention recently due to Pontriagin as a maximum principle As a philosophy of approach, it is somewhat more general, but for our purposes all are equivalent Since equation (7) is homogeneous in the set λ,μ,ν, we may choose one relation among the c's it will be convenient to choose the coefficient c_3 of Δz in (7) to be one (1) This is taken to be the case henceforth

No philosophy eliminates the basic problem of determining the constants c_1,c_2,T

Curves on which (10) is satisfied for some choice of $\wedge$, a solution to the adjoint system, will be called *extremals* Curves which satisfy the constraints on the end values will be called *admissible*

Minimum Dose in Attaining a Specified Curve

Let us consider the following problem A ship is in a fallout area We wish to route it to the curve whereon $f=f_m$ = maximum allowed continual intensity, so that the resulting dose is a minimum

$$z(T)=\int_0^T f\,dt=\min \tag{11}$$

Suppose we have two neighboring admissible paths, the differential of the end values must satisfy the relation

$$\Delta f=f_x\,\Delta x+f_y\,\Delta y+f_t\,\delta T=0, \tag{12}$$

since both end on the hypersurface $f=f_m$ The transversal condition (*see* Bliss,[1] p 196) that the end values must satisfy may be expressed

$$\text{rank}\left\|\begin{matrix} f_x & f_y & 0 & f_t \\ 0 & 0 & 1 & 0 \\ \lambda & \mu & 1 & -(\lambda x+\mu y+z)\end{matrix}\right\|=2, \tag{13}$$

and the 2×4 submatrices must all have rank 2 The terms in the first row are from the constraint (12) The terms in the second row are from the quantity to be maximized (11) The terms in the last row are the coefficients of the variations on the left side of (7) From the transversal condition, we see that if we regard f_x, f_y, f_t and $\lambda,\mu,-(\lambda x+\mu y+f)$ as components of two vectors, those two must be parallel Hence at $t=T$

$$c_1/f_x=c_2/f_y=-(c_1x+c_1y+f)/f_t, \tag{14}$$

whence

$$\begin{aligned} c_1&=-f_xf_m/(df/dt),\\ c_2&=-f_yf_m/(df/dt) \end{aligned} \tag{15}$$

Since we want to minimize $z(T)$, the choice of p must be to minimize $\vec{\lambda}\ \vec{V}$

Computational Routine

Guess starting values for $\lambda(0),\mu(0)$ Integrate systems (1) and (5), using (10′). Continue until $f \leqq f_m$ Adjust the end time step

If n denotes an iteration index, note that

$$\begin{aligned}\lambda_{n+1}(T_n)-\lambda_{n+1}(0)&=\lambda_{3\,n+1}(T_n)-\lambda_{3\,n+1}(0)\\&=-\lambda_{3,n+1}(0)\end{aligned}$$

Now assume $\lambda_{3\,n+1}(0)=\lambda_{3\,n}(0)=\lambda_n(T_n)-\lambda_n(0)$

A similar relation holds for μ and we get

$$\begin{aligned}\lambda_{n+1}(0)&=c_{1n}+\lambda_n(0)-\lambda_n(T_n),\\ \mu_{n+1}(0)&=c_{2n}+\mu_n(0)-\mu_n(T_n)\end{aligned} \tag{16}$$

Since every term on the left is known, we are ready to start the $n+1$st round

Convergence Criterion

The above routine is continued until some convergence criterion is satisfied In this case, let

$$\epsilon=1-\frac{[f_x\lambda+f_y\mu-(v\lambda\cos p+v\mu\sin p+f)f_t]^2}{(f_x^2+f_y^2+f_t^2)(\lambda^2+\mu^2+[v\lambda\cos p+v\mu\sin p+f]^2)}$$

evaluated at $t=T$ The routine is assumed to have converged when ϵ is below some specified number The transversal condition specified that two vectors be parallel, ϵ is the square of the sign of the angle between them

Ship Sent to Prescribed Point

Let us now consider the problem of sending the ship to a specified point x_f,y_f with a minimum final value of z The transversal condition becomes

$$\operatorname{rank}\begin{Vmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \lambda & \mu & 1 & -(\lambda x+\mu y+z)\end{Vmatrix}=3, \tag{18}$$

the first two rows are from the constraints Hence

$$(\lambda x+\mu y+z)_T=0 \tag{19}$$

To get a corrective routine for the c's, note that on any extremal

$$\vec{\lambda}\ \vec{V}_p\equiv 0,$$

hence if we change c_1,c_2,p with the other variables fixed,

$$\vec{\lambda}_1\ \vec{V}_p\,\delta c_1+\vec{\lambda}_2\ \vec{V}_p\,\delta c_2+\vec{\lambda}\ \ \vec{V}_{pp}\,\delta p=0 \tag{20}$$

Since $\vec{\lambda}\ \ \vec{V}_{pp}=v|\vec{\lambda}|$, (20) reduces to

$$\delta p=(\sin p\ \delta c_1-\cos p\ \delta c_2)/|\vec{\lambda}|, \tag{21}$$

and

$$\begin{aligned}\Delta x&=x(T)\ \delta T-\int_0^T(v\sin^2 p/|\vec{\lambda}|)\ dt\ \delta c_1+\int_0^T(v\sin p\cos p/|\vec{\lambda}|)\ dt\ \delta c_2,\\ \Delta y&=y(T)\ \delta T+\int_0^T(v\sin p\cos p/|\vec{\lambda}|)\ dt\ \delta c_1-\int_0^T(v\cos^2 p/|\vec{\lambda}|)\ dt\ \delta c_2\end{aligned} \tag{22}$$

Numerical Routine

Let us guess $\lambda(0),\mu(0)$ Integrate to get $x,y,\lambda x+\mu y+f,\lambda,\mu$ at T and the integrals in (22) In general x,y will not assume the desired terminal values, nor will $\psi=\lambda x+\mu y+f$ be zero To correct ψ use

$$\Delta\psi=f_t\ \delta T+f_x\ \Delta x+f_y\ \Delta y+x\delta c_1+y\delta c_2, \tag{23}$$

everything evaluated at $t=T$ Now set

$$\begin{aligned}\Delta\psi&=-\psi,\\ \Delta x&=x_f-x(T),\\ \Delta y&=y_f-y(T),\end{aligned} \tag{24}$$

substituting from (22), (23) into (24), these equations determine $\delta c_1,\delta c_2,\delta T$ and hence c_1,c_2,T Equations (16) then determine $\lambda(0),\mu(0)$ to start the next iteration

The routine is continued until some convergence criterion is satisfied, one might take

$$(x_f-x)^2+(y_f-y)^2+\psi^2<\epsilon,$$

where ϵ is a preassigned number

COMMENTS

THE PROBLEM has been programmed and run No particular computational problems were encountered In some simple cases involving about 100 time steps, the route was determined in from four to twenty seconds on the CDC-1604, depending on the initial conditions assumed

The fallout problems above are equivalent to corresponding problems of finding routes that minimize the probability of detection in going from one point to another Assume that the probability of being detected at a point is given by

$$\Delta p=p(t+\Delta t)-p(t)=f(x,y,t)\ \Delta t,$$

if undetected prior to time t On the path the probability of being first detected at time t satisfies the relation

$$\Delta p = (1-p) f \, \Delta t,$$

set $z = \ln(1-p)$

In general we should keep maximum speed For if we find a path that furnishes a minimum with v a maximum for each value of t and then consider variations, $\delta v \leqq 0$, we have, from (7),

$$[\lambda \Delta x + \mu \Delta y + \Delta z - (\lambda x + \mu y + f)\, \delta T]_T = \int_0^T \frac{\vec{V} \cdot \vec{\lambda}}{v} \, \delta v \, dt$$

$$= -\int_0^T |\vec{\lambda}| \, \delta v \, dt,$$

all other terms in the integral vanish by virtue of (5), (10) The last minus sign occurs because we seek a minimum z and the integrand must be minimized If the speed v is maximal on the original path, then $\delta v < 0$ leads to a larger value of the integral and hence to a larger value of z This is intuitively obvious, the proof is constructive, and is equivalent to the proof of the fundamental lemma of the calculus of variations (*see* COURANT,[2] p 200)

The problem is more involved if f involves v and is an increasing function of v This may be the case for a submarine, where the additional noise might increase the likelihood of detection, or where the speed and probability of detection both change with depth The adjunction of another variable such as v leads to another Euler equation that must be adjoined to those above If more involved formulas are taken for the speed, the solutions to the adjoint system and the resulting correction formulas for the constants are more involved In the differential formulas above is the implicit assumption that the path has no corners in x,y,t space, if it has, a term must be added to each differential correction, formula (22), for each corner

In a sense this method is a simple application of methods that Bliss introduced for calculating differentials in artillery during World War I, summarized in his book (BLISS,[3] Chap V) The advent of large-scale digital computers has made it possible to use these formulas and invert them to determine the corresponding courses

If equations (1) govern the system, we may have from none to three side conditions of the form $G_m(x,y,z,t)_{t=T} = 0$, $m = 1, \quad , M < 4$, when all initial conditions are given, and a function $F(x,y,z,t)_{t=T} =$ extremum The transversal condition may then be expressed

$$\text{rank} \begin{Vmatrix} G_{m,x} & G_{m,y} & G_{m,z} & G_{m,t} \\ F_x & F_y & F_z & F_t \\ \lambda & \mu & \nu & -(\lambda x + \mu y + \nu z) \end{Vmatrix}_{t=T} \begin{matrix} \}(M \text{ rows}) \\ = M+1 \\ \end{matrix}$$

and the $(M+1) \times 4$ matrices obtained by eliminating either of the last two rows must also both be of rank $M+1$ The evolution of a routine to correct the c's and T, from the transversal conditions, seems to be the place where ingenuity is required in the routine, the initial values must also be guessed in a proper range or the Newton iteration may not converge

It is not possible to discuss in a short paper other conditions associated with the solution which ensure that the solution yields a minimum, the above routines yield only stationary values The conditions are discussed completely in Bliss[1] (Chaps VII, VIII) The treatment there is neat and precise but the problem is expressed in a symmetric form that may obscure for the control engineer the essential difference between the so-called state variables, x,y,z, which are properly dependent, and the control variable p, which must be determined to effect the extremum It should be stressed that the multipliers λ,μ,ν are introduced by the mathematician and he later chooses them to satisfy (5) to simplify Green's formula (4) to the form (7)

One problem that we may expect is that convergence will be poor when the initial point is near an envelope or a focal point of the extremals which satisfy the transversal conditions A trivial example of such a point is the center of a region which has circular symmetry with respect to a fixed point This may not be a serious problem since neighboring trajectories then generally lead to terminal values of z that differ 'little' However, problems such as these generally require an element of skill and judgment on the part of the operator

It seems to the author that there are two practical methods of determining optimum trajectories numerically One is the differential method presented here, always using extremals and approximating the transversal conditions The other is the method of 'steepest descent' of BRYSON AND DENHAM[4] and KELLEY (*see* KELLEY, KOPP, AND MOYER[5] for summary and comments) Discussions suggest we are all having typical small problems associated with computations

ACKNOWLEDGMENT

I WOULD like to thank LCDR K F COOK and MARY HAYNES for assistance in programming and running the problem A discussion of the problem of determining the function f is given in Cook's thesis [6] This work was supported by ONR A more complete discussion of some simpler related problems is available in a report[7] on minimum-time routes

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