1967

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Pollock, Stephen M.

INFORMS

Stephen M. Pollock, (1967) Minimum-Cost Checking Using Imperfect Information. Management Science 13(7):454-465.

http://hdl.handle.net/10945/70158

MANAGEMENT SCIENCE
Vol. 13, No. 7, March, 1967
Printed in U.S.A.

MINIMUM-COST CHECKING USING IMPERFECT INFORMATION*†

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An event takes place at time t, a discrete random variable with known probability function. At unit intervals of time, a measurement x is observed which yields information about the event; x is a random variable, with a known probability density function being dependent upon whether or not the event has yet occurred.

After each observation, a decision is made that the event has or has not yet occurred. The latter decision implies waiting for the next measurement. The former decision, if correct, ends the procedure. If incorrect, this fact is incorporated, and the procedure continues.

A decision cost structure is assumed that assigns:

(1) a fixed (false alarm) cost to deciding the event has occurred when, in fact, it has not;

(2) a (time late) cost proportional to the time between the occurrence of the event and the decision that it has occurred.

The minimum-expected-cost decision strategy and the minimum cost thus obtained are derived by means of dynamic programming.

1. Introduction

This paper discusses a common sequential decision process that occurs in equipment checking, target search, and other related problems. An event E occurs at some time, observations are made relating to whether or not it has yet occurred, and an appropriate cost structure is assumed. For example, the event E could be the failure of a piece of production equipment; the observations could be measurements of a critical parameter of the items produced, and the cost structure could be related to the penalties for shutting down the machinery unnecessarily, and for delaying the shutdown when necessary. A strategy is desired that will enable a decision maker to decide, on the basis of the observations made up to any point in time, whether to take action appropriate to assuming the event E has occurred (shutting down the machinery for repair) or to simply wait and continue observations.

The results bear a resemblance to Wald's classical sequential analysis [7] and also to the more recent minimum-cost sequential-analysis approach [3] and [4]. The difference is that in the present problem there are no longer two simple hypotheses H_0 and H_1 (E' and E) to be decided between, but an H_0 that at some time turns into an H_1. This becomes apparent in the solution presented, where it is shown that observations taken in the remote past have less effect on the decision process than more recent ones.

* Received March 1966.

† This work was sponsored in part by contract NONR-3963(06), Office of Naval Research.

‡ This research was performed while the author was a member of the staff of Arthur D. Little, Inc., Cambridge, Massachusetts.

The following is the general statement of the problem considered.

1. An event E occurs at time $t\,(t = 1, 2, \cdots)$ with known probability $p(t)$.
2. An observation is made at time $\tau\,(\tau = 1, 2, \cdots)$, the result being a random variable $x(\tau)$ which has p.d.f. $p_0(x)$ if $t > \tau$ and $p_1(x)$ if $t \leqq \tau$.
3. Immediately after each of these observations, one of the following decisions is made:

$$D = \text{“Decide the event has occurred”};$$

$$W = \text{“Wait for another observation”}.$$

4. The decision D, at time τ, may or may not be a terminal decision. If D is picked and $t > \tau$, then a false-decision cost F is incurred, the knowledge that $t > \tau$ is gained, and the process continues. If D is picked and $t \leqq \tau$, then the process is terminated with a cost of $c(t, \tau)$.
5. The objective is to minimize the total expected cost of the process. The strategy (i.e., the rule for deciding D or W at each point in time) for achieving this minimum cost is called the "optimal" strategy. For convenience in notation, we shall restrict $p_0(x)$ and $p_1(x)$ such that the likelihood function $L(x) = p_1(x)/p_0(x)$ is monotonic nondecreasing in x.

2. Linear Terminal Cost and Geometrical Arrival Time

Two reasonable assumptions will now be made in order to obtain a solution.

First, we shall assume that the terminal cost is proportional to the time "late", i.e., $c(t, \tau) = (\tau - t)w$. This form of the function is not necessary for a solution, but offers a minimum of algebraic difficulties that might otherwise cloud the development.

Second, and more restrictive, we shall assume that the occurrence time t of the event E has the distribution:

$$p(t) = a(1 - a)^{t-1} \qquad t = 1, 2, \cdots . \tag{1}$$

This geometric distribution has the advantage in describing the occurrence of E as being conceptually "random" by the fact that it provides a constant probability of occurrence per unit time (a), given that it has not yet occurred. This property also provides a simple representation for a state variable, a summing up of information concerning the state of nature (i.e., whether or not the event has occurred). This is outlined as follows.

If we let $Q(\tau)$ be the probability that the event has occurred at or before time τ, then by equation (1)

$$Q(\tau) = \textstyle\sum_{t=1}^{\tau} p(t) = 1 - (1 - a)^{\tau} = \text{prob.}\,\{t \leqq \tau\}. \tag{2}$$

Furthermore,

$$Q(\tau + 1) = (1 - a)\, Q(\tau) + a$$

so that (without any other information) to describe $Q(\tau + 1)$, all that is needed is $Q(\tau)$.

In addition, suppose that a value of $x(\tau)$ is observed at time τ. Then, by defining $Q\{\tau + 1 \mid x(\tau)\} \equiv \text{prob.}\ \{t \leqq \tau + 1 \mid x(\tau)\}$, we have

$$Q\{\tau + 1 \mid x(\tau)\} = \text{prob.}\ \{x(\tau), t \leqq \tau + 1\}/\text{prob.}\ \{x(\tau)\}.$$

To calculate prob. $\{x(\tau), t \leqq \tau + 1\}$, we note that this could happen in two ways: the event could have occurred at time τ or before, in which case $p_1(x)$ is the p.d.f. of $x(\tau)$; or, the event could have occurred at time $\tau + 1$, so that $p_0(x)$ is the p.d.f. of $x(\tau)$. Thus,

$$Q\{\tau + 1 \mid x(\tau)\} = \frac{Q(\tau)p_1(x) + [1 - Q(\tau)]ap_0(x)}{Q(\tau)p_1(x) + [1 - Q(\tau)]p_0(x)}; \tag{3}$$

and we see that given some observation x, the posterior probability of the event occurring at or before time $\tau + 1$ is still dependent only upon $Q(\tau)$, and not τ explicitly.

At the start of the process, $p(1) = Q(1) = a$. What is more important, we note that according to statement (4) of the problem definition, when the false decision $\{D \mid t > \tau\}$ is made, knowledge that t is greater than τ is automatically gained. This fact and equation (1) lead to $Q(\tau + 1 \mid t > \tau) = a$.

3. Functional Equation for the Minimum Expected Cost

We are now prepared to write a functional equation for the minimum expected cost. Let us define $V(P)$ as the minimum expected cost obtained using the optimal strategy at time τ, where $P = Q(\tau)$ is the present value of the probability that the event has occurred previous to, or at, time τ. There are two decision choices. One is D: decide the event has occurred, with resultant probability of being wrong of $1 - P$, and subsequent cost of F, *plus* what the process will cost from then on. The other is W: wait for more information contained in an observation of x, in which case the "late" cost w is incurred only if the event has occurred (probability P), and the process continues with the proper posteriori probability given by equation (3). Defining this probability to be $P'(x)$:

$$P'(x) \equiv (Pp_1(x) + (1 - P)p_0(x)a)/(Pp_1(x) + (1 - P)p_0(x)),$$

the minimum cost is then given by the dynamic programming equation:

$$V(P) = \min\left\{(1 - P)(F + V(a)),\ Pw + \int_{-\infty}^{\infty} [Pp_1(x) + (1 - P)p_0(x)]V[P'(x)]\, dx\right\}. \tag{4}$$

Action D follows if the first quantity is smaller, and action W follows otherwise. This equation holds for all τ, since only $P = Q(\tau)$ and not τ itself is needed to express the right-hand side.

One result is immediately apparent. By letting $P = 0$, we have

$$V(0) = \min[F + V(a), V(a)] = V(a) \quad \text{for} \quad F > 0.$$

This tells us that the cost of checking, if we know the event has not yet occurred, is the same as if we waited one time unit and started again. This is because there can be no "late" cost w if the event has not yet occurred.

In order to develop a feeling for the solution to equation (4), and to obtain an upper bound upon $V(P)$, we first consider the degenerate case involving getting no information from the observations.

4. Optimal Checking with No Information

Suppose that $p_0(x) = p_1(x)$. Then, as can be shown by equation (3), an observation of x does not affect the posteriori evaluation of P. In this case, the observation x is irrelevant, and the checker gains no information. This "non-informative" problem has been considered previously in the literature. For example, Barlow, et al. [1] assume a general form of the occurrence distribution, with t a continuous variable, and then assume that the optimal checking procedure will be to wait some time t_1, then check; if no failure is seen, wait some time t_2, check, etc. They go on to show that for the exponential failure-time density function (the equivalent of our geometric mass function) these t_i are equal. Their assumption of the fixed-checking-time form of solution (proven below) enables them to immediately write a cost expression in terms of the checking time t, which can then be minimized by differentiation.

Although the results of this section are not new, the dynamic programming approach adopted below has the unique advantage of providing the *form* of the optimal strategy, as well as its parameters, a result that is quite common in dynamic programming [5]. More important, as will be seen in Section 5, this section lays the groundwork so that the structure can be extended to allow for observations of a random variable relating to the state of nature.

When no information is gained between decision times, the checker's optimal strategy now consists simply of either waiting one time unit, or deciding the event has occurred. Equation (4) becomes

$$(5) \qquad V(P) = \min\{(1 - P)(F + V(a)), wP + V[P + (1 - P)a]\}.$$

Action D follows if the first quantity is smaller, and action W follows otherwise. As noted before, $V(0) = V(a)$, and we can also easily see that $V(1) = 0$.

From the form of equation (5), it is postulated that the structure of the strategy will be

$$\text{if } P \geqq q: D$$
$$P \leqq q: W$$

where q is a decision point to be determined as part of the solution.

That this is, indeed, the form of the strategy, and that it is not degenerate (that is, $0 < q < 1$), may be shown by the following proof by contradiction. (The discussion that follows can also be shown to be valid for the more general equation (4). Since it is the form of the proof that is of interest, it is carried out in this less complicated case.)

Let us define

$$D(P) = (1 - P)(F + V(a)),$$
$$G(P) = wP + V[P + (1 - P)a].$$

$D(P)$ is a straight line, with $D(0) = F + V(a) = F + V(0)$. $D(1) = 0$. $G(P)$ has an unknown functional dependence on P through $V[P + (1 - P)a]$, but it is continuous by the continuity of $V(P)$. The boundary values are known and are $G(0) = V(a) = V(0)$, $G(1) = w + V(1) = w$. Since $F > 0$ and $w > 0$, then $G(1) > D(1)$ and $G(0) < D(0)$ so that $G(P)$ and $D(P)$ must intersect at an odd number of points.

Suppose $G(P)$ was such that $G(P)$ and $D(P)$ intersect at more than one point, for example, three points q', q'', and q'''. Let us select a point P' such that

$$(6) \qquad \begin{aligned} q' &< P' < q'' \\ q'' &< P' + (1 - P')a < q''' \end{aligned}$$

where $[q', q'']$ is a D region and $[q'', q''']$ is a W region. Then, by equation (5),

$$(7) \quad V(P') = (1 - P')(F + V(a)) < wP' + V[P' + (1 - P')a]$$

and

$$(8) \quad \begin{aligned} V[P' + (1 - P')a] &= w(P' + (1 - P')a) + V[P' + (1 - P')a \\ &\quad + (1 - P')(1 - a)a] < (1 - P')(1 - a)[F + V(a)]. \end{aligned}$$

Combining equations (7) and (8), we find

$$(9) \qquad P' > a(F + V(a))/w.$$

If we select another point P'' so that

$$(10) \qquad \begin{aligned} q'' &< P'' < q''' \\ q''' &< P'' + (1 - P'')a < 1, \end{aligned}$$

we can show in a similar manner that

$$P'' < a(F + V(a))/w$$

which, with equation (9), implies that $P'' < P'$. But, since equations (6) and (10) require $P' < q'' < P''$, the contradiction is proven.

We have just shown then that $G(P)$ and $D(P)$ intersect at only one point; and defining this point as $P = q$, we rewrite equation (5)

$$(11) \qquad \begin{aligned} V(P) &= (1 - P)(F + V(a)) && \text{if } P \geqq q \\ &= wP + V[P + (1 - P)a] && \text{if } P \leqq q. \end{aligned}$$

As a first step in the solution of this equation, let $P = q$. Then, since $q + (1 - q)a > q$, we have

$$V(q) = (1 - q)(F + V(a)) = wq + V(q + (1 - q)a) = wq + (1 - q)(1 - a)(F + V(a))$$

from which we get

$$(12) \qquad F + V(a) = qw/(1 - q)a$$

so that equation (11) becomes

$$\begin{aligned} V(P) &= (1 - P)qw/(1 - q)a \qquad \text{if} \quad P \geqq q \\ &= wP + V[P + a(1 - P)] \quad \text{if} \quad P \leqq q. \end{aligned} \tag{13}$$

The next step is to find q in terms of w, F, and a. Once this is obtained, the optimal strategy is defined. (Determination of the functional form of $V(P)$ for $P \leqq q$ will then rely upon iterations of (13) in a manner to be described later.)

To determine q, let us assume that q *has* been obtained and is such that

$$\begin{aligned} 1 - (1 - a)^{k-1} &< q \quad (k = 2, 3, 4, \cdots, n - 1) \\ 1 - (1 - a)^{n} &> q \end{aligned} \tag{14}$$

where n is the smallest integer such that equation (14) holds. By $n - 1$ successive applications of equation (13), we get

$$\begin{aligned} V(a) &= wa + V[1 - (1 - a)^2] \\ &= wa + w(1 - (1 - a)^2) + V[1 - (1 - a)^3] \\ &\ \ \vdots \qquad\qquad \vdots \\ &= w[(n - 1) - ((1 - a)/a)\,(1 - (1 - a)^{n-1})] + V[1 - (1 - a)^n], \end{aligned}$$

all $(n - 1)$ steps being the result of W decisions. Finally, since the n^{th} must be a D decision,

$$\begin{aligned} V(a) = w[(n - 1) - ((1 - a)/a)\,(1 - (1 - a)^{n-1})] \\ + (1 - a)^n\,(q/(1 - q))\,(w/a). \end{aligned}$$

using the value of $V(a)$ from equation (12), we may solve for q in the terms of n

$$q = 1 - (1 - (1 - a)^n)/(a(F/w + n)). \tag{15}$$

By use of equation (14), we find that n is the smallest integer such that

$$(1 - a)^n \leqq [1 + a(F/w + n)]^{-1}. \tag{16}$$

Once n is found, q is then obtained from equation (15).

We have just proven that the form of the strategy consists of waiting for a fixed amount of time (number of time units) $n - 1$, then choosing a D decision. In the event that the checking process has just started, (or that a D decision has just been made but the event has not yet occurred, so that the process must be resumed with $P = 0$) this fixed amount of time is given by equation (16), from which q can be determined.

If the process starts out so that $P \neq a$, then by successive applications of equation (13) we can show in a calculation similar to that above that the time until a D decision, $n(P)$, is given by the smallest $n(P)$ that satisfies

$$(1 - P)\,(1 - a)^{n(P)-1} \leqq 1 - q. \tag{17}$$

In order to compute $V(P)$, we again simply apply equation (13), $(n(P) - 1)$

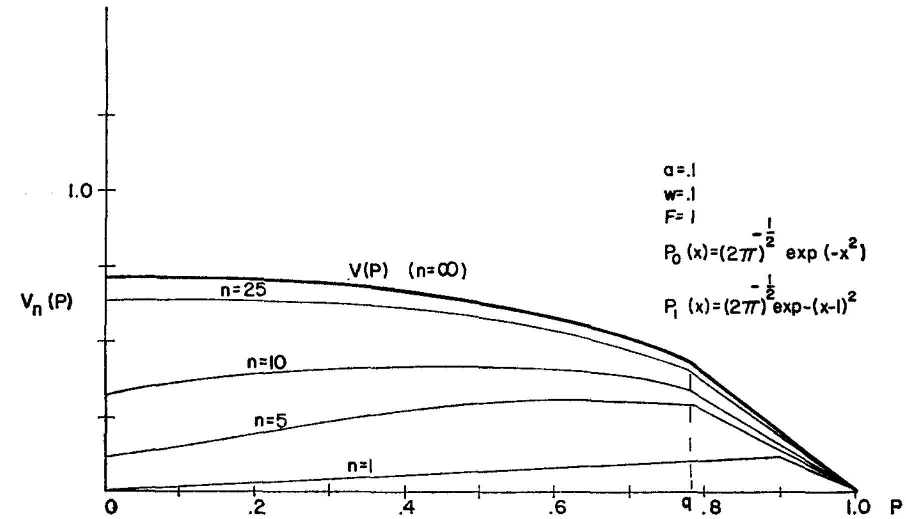

Fig. 1. V(P) with no information

times with decision W, and the $n(P)^{\text{th}}$ time with a D decision. This gives the following form of the minimum cost, where n now is the $n(P)$ defined in (17):

$$V(P) = w[(n-1) - ((1-P)/a)\,(1-(1-a)^{n-1})] + (1-P)\,(1-a)^{n-1}\,qw/(1-q)a. \tag{18}$$

When the expected occurrence time of the event gets very large (so that $a \ll 1$), an interesting approximation holds. Since n gets large, we may consider equation (16) to be an equality; and as $a \rightarrow 0$, we find that

$$n \cong (2F/wa)^{1/2}.$$

With this approximation, equation (15) becomes

$$q = (Fa/w + (2Fa/w)^{1/2})/(1 + Fa/w + (2Fa/w)^{1/2})$$

and so $V(0) = V(a) = (2Fw/a)^{1/2}$ from equation (12). Thus, if the process always starts with $P = 0$ or $P = a$, this expression gives the minimum expected cost.

For example, let $a = .1$, $w = .1$, $F = 1$. Using equation (16)

$$(.9)^n \leqq (2 + .1n)^{-1},$$

we find that the smallest n that satisfies this expression is $n = 11$. Thus, the optimal strategy, given that the event has not occurred at the start of the process, is to wait $(n-1) = 10$ time units before the first D decision and repeat this procedure until the event occurs.

The value of q, from equation (15), is .673. The expected cost at the beginning of the process is

$$V(0) = V(a) = qw/(1-q)a - F = .673/.327 - 1 = 1.06.$$

The resulting $V(P)$ is illustrated in Figure 1. Note that $V(P)$ is the lower envelope of 11 straight lines, each representing the expected cost if the strategy was to wait 1, 2, 3, $\cdots$ 11 units of time between D decisions.

5. Optimal Checking with Information

We are now ready to attack the problem involving observations of a random variable x with every W decision. First, we note that the proof in Section 4 concerning the form of the non-informative process may be carried over conceptually to the more general case represented by equation (4); so we state without formal proof that equation (4) may be written

$$V(P) = \min \{D(P), D(P)\} = D(P) \text{ if } P \geqq q \\ = H(P) \text{ if } P \leqq q, \tag{19}$$

where

$$D(P) = (1 - P)(F + V(a)),$$

$$H(P) = Pw + \int_{-\infty}^{\infty} [Pp_1(x) + (1 - P)p_0(x)]V[P'(x)]\,dx.$$

We are now faced with a functional equation for $V(\cdot)$ that does not apparently lend itself to an analytical solution. However, a standard technique for solving such a functional equation, and many similar transcendental equations, is a simple method of successive approximations. If certain convergence properties can be shown, then this method is valid, despite the non-physical character of intermediate solutions.

For this reason, we shall rewrite equation (19) with V as a function of an iteration index n. This index n can be considered to be a truncation index—one that prescribes the number of observation (and decision) points remaining before the procedure is arbitrarily stopped. Thus, $V_n(P)$ can be interpreted as the minimum expected cost of checking when P is the probability that the event has already happened and there are n remaining decisions allowed. Doing this yields

$$V_n(P) = \min \{D_n(P), H_n(P)\} = D_n(P) \text{ if } P \geqq q_n \\ = H_n(P) \text{ if } P \leqq q_n, \tag{20}$$

where

$$D_n(P) = (1 - P)(F + V_{n-1}(a)),$$

$$H_n(P) = Pw + \int_{-\infty}^{\infty} [Pp_1(x) + (1 - P)p_0(x)]V_{n-1}[P'(x)]\,dx.$$

All that is needed now is the selection of the boundary condition $V_0(P)$, and assurance that successive iterations will converge the process to $V(P)$ as $n \to \infty$. If we let $V_0(P) = 0$ for all P, then

$$V_1(P) = \min [(1 - P)F, Pw] \geqq 0 = V_0(P).$$

With the fact that we have found some $V_n(P) \geqq V_{n-1}(P)$, a proof very similar to that derived by Goode [4] for the minimum-cost sequential hypothesis test allows us to show that, in fact, all $V_n(P) \geqq V_{n-1}(P)$, so that the process will approach $V(P)$ from below. The proof is uninformative, and is deleted here. To complete the convergence, we need to show that $V(P)$ is bounded from above. This can be shown by noting that

$$V(a) \leqq (1 - a)\ (F + V(a)),$$

so that $V(a) \leqq F(1 - a)/a$; and, therefore,

$$V(P) \leqq (1 - P)\ (F + V(a)) \leqq (1 - P)F/a \leqq F/a. \tag{21}$$

It also may be proved that q_n, the solution of

$$D_n(q_n) = H_n(q_n),$$

converges to the limit q as $n \rightarrow \infty$. This proof is again deleted, and the interested reader is referred to [4] and [6].

An example of this iteration process is shown in Figure 2. The values of the parameters are comparable to the numerical example of Section 4, so that $a = .1$, $w = .1$, $F = 1$. The $p_0(x)$ and $p_1(x)$ are normal density functions, with unit variance and mean of 0 and 1, respectively. These calculations were obtained with an IBM 7090, being essentially additions, with an appropriate approximation for the integral.

From Figure 2, we note that $q \cong .78$. This is higher than the value of .673 obtained previously, and indicates that the availability of information will allow the checker to be less quick to respond. We also note that $V(0) = V(a) = .58$, which is a savings of close to 50 percent compared to the non-informative process. The strategy that gives these results follows.

Suppose $P = a = .1$ to start the process. Since $.1 \leqq q = .78$, an observation is required at the first time interval. Suppose a value x_1 is the result of this observation.

The posterior probability that the event has occurred is now given by equation (3) as

$$\frac{(.1)p_1(x_1) + (.1)(.9)p_0(x_1)}{(.1)p_1(x_1) + (.9)p_0(x_1)} = \frac{(.1)\exp\,(x_1 - \frac{1}{2}) + .09}{.1\exp\,(x_1 - \frac{1}{2}) + .9}.$$

Comparing this with $q = .78$, we see that if

$$x_1 \geqq .5 + \ln(27.8) = 5.9,$$

then the D decision should be made. If not, another observation x_2 should be made, the *a posteriori* probability based on x_1 and x_2 should be calculated and compared to q, etc.

6. A Comment on the Solution

As is shown by the example in Figure 2, although convergence of $V_n(P)$ to $V(P)$ is guaranteed, the rate of convergence is rather slow. In fact, as a gets very small, the convergence is even slower. This unfortunate practical difficulty is at present unresolved. One possible approach is suggested here.

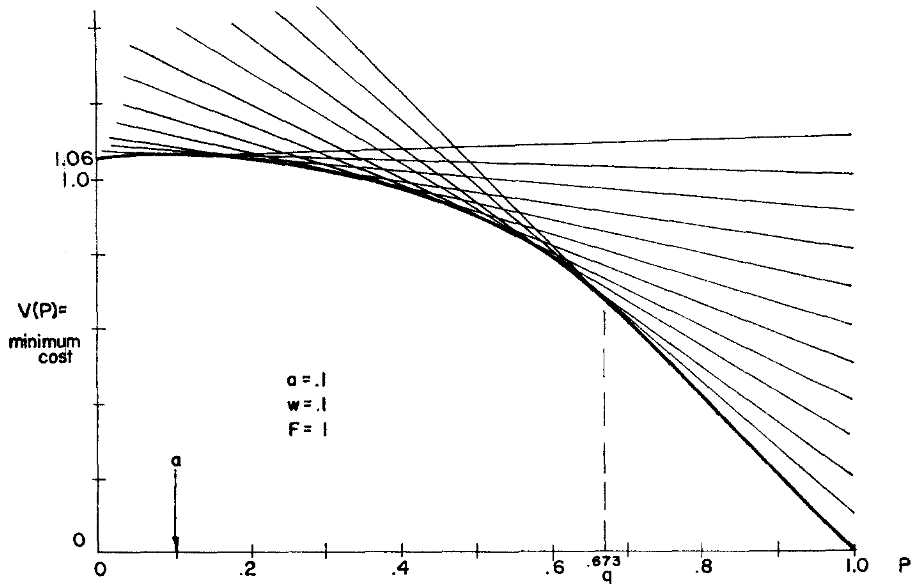

FIG. 2. Convergence of $V_n(P)$ to $V(P)$ with information

We decided in the previous section to start the iteration with $V_0(P) = 0$, which consequently assures convergence from below. It is equally possible to start the iteration at some appropriate large value, which will again assure convergence but then from above. One such value would be the right-hand side of the condition given in equation (21), i.e., $V_0(P) = F/a$. However, a lower starting point is available by noting that the minimum cost of the informative process is less than or equal to the minimum cost of the non-informative process for all values of P. This lower starting value of $V_0(P)$ could considerably decrease the number of iterations needed to provide a given degree of accuracy.

Other techniques for establishing a reasonable first guess of $V(P)$, and letting this equal $V_0(P)$, would be a valuable aid in the computation. In general, however, convergence proofs might be difficult for arbitrary starting functions $V_0(P)$.

7. Implementation of the Strategy and Comments on the Geometric Arrival Assumption

Previous work has shown that the use of a minimum-expected-cost sequential strategy results in essentially a Wald sequential-probability ratio test, where the decision boundaries are determined by cost considerations rather than by error probabilities [4]. A similar analysis of the implementation of the strategy developed in Section 5 is of interest in that it points out a basic limitation to the treatment of the problem.

From the form of the decision structure presented in equation (19), we see that P, the probability that the event has occurred up to some time, is constantly compared to some decision threshold q. When a D decision occurs, P automatically returns to 0 if the event has, in fact, not yet occurred. With a series of W

decisions, however, a series of observations $x_1, x_2, \cdots$ have been made; and P is the appropriately derived posterior probability that the event has occurred.

Specifically, let us consider the process to start with a W decision at $\tau = 1$, and that n successive observations of $x_1, x_2, \cdots, x_n$ are made. We shall also consider a completely general distribution for t, the time when the event occurs, $f(t)$, $(t = 1, 2, \cdots)$. Using the definition of conditional probability, where $X = (x_1, x_2, \cdots, x_n)$, we define

$$P_n \equiv \text{prob. } \{t \leqq n \mid X\} = \text{prob. } \{X, t \leqq n\}/\text{prob. } \{X\}.$$

The unconditional probability of receiving some vector X may be shown to be

$$\text{prob. } \{X\} = \sum_{j=1}^{n} f(j) \left[\prod_{i=1}^{j-1} p_0(x_i)\right] \left[\prod_{k=j}^{n} p_1(x_k)\right] + \sum_{j=n+1}^{\infty} f(j) \left[\prod_{i=1}^{n} p_0(x_i)\right]. \tag{22}$$

Similarly, we may calculate

$$\text{prob. } \{X, t \leqq n\} = \sum_{j=1}^{n} f(j) \left[\prod_{i=1}^{j-1} p_0(x_i)\right] \left[\prod_{k=j}^{n} p_1(x_k)\right]. \tag{23}$$

The condition for a D decision, given that X has been observed, is that $P_n \geqq q$. Using the above expressions and manipulating terms, this condition becomes

$$\sum_{j=1}^{n} f(j) \left[\prod_{i=j}^{n} L(x_i)\right] \geqq q/(1 - q) \sum_{j=n+1}^{\infty} f(j), \tag{24}$$

where

$$L(x_i) = p_1(x_i)/p_0(x_i)$$

is the likelihood ratio for each observed x_i.

This equation represents a very complicated process. Not only is the $\prod_{i=j}^{n} L(x_i)$ term weighted by the $f(j)$, but this weighting is successively compared to a term which gets smaller as n increases. Since the simple random walk with constant absorbing barriers has not been fully solved, there is no reason to believe that this non-Markovian (because P_n is more than just a function of P_{n-1}) process with nonconstant barriers would be any easier. Thus, a strictly Wald-type approach, depending as it does upon the statistics of such a process, would not seem too profitable.

If we let $f(t) = a(1 - a)^{t-1}$, $(t = 1, 2, \cdots)$, however, an interesting result is shown. By defining $A(x_i) = L(x_i)/(1 - a)$, equation (24) becomes

$$Z_n \equiv \sum_{j=1}^{n} \left[\prod_{i=j}^{n} A(x_i)\right] \geqq q/(1 - q)a, \tag{25}$$

which has the advantage of being a test that compares a variable Z_n, defined above, to the *constant* decision threshold, $q/(1 - q)a$.

In addition, the sequence Z_n describes a Markov process in that Z_{n+1} only depends upon Z_n (as well as $A(x_{n+1})$, of course). To show this, we note that

$$Z_{n+1} = Z_n A(x_{n+1}) + A(x_{n+1}), \tag{26}$$

which can be verified by direct substitution into the definition of Z_n.

The geometric distribution of arrival times thus imparts a Markov character to the decision process. And, indeed, it is just this character that has allowed us to approach the problem from the dynamic programming point of view. By allowing the argument of V to be P, we have been assuming that P is completely descriptive of the checker's state of knowledge about the system, and that the history of events that led to P is unimportant.

Conversely, since for a general arrival-time distribution it *cannot* be shown that equation (24) represents a Markov process, we cannot write a general equation similar to equation (4), since P alone is not sufficient to represent a "state" of the process. Thus, although equation (24) is descriptive of the optimal strategy, the evaluation of the resultant expected cost by dynamic programming can be obtained by at least a consideration of a minimum cost that is a function of both P and some other state variable. (A referee has suggested that the additional variable be the number of observations made since the process started.)

Equation (25), derived above for the special case of random occurrence, (and equation (24), for the general case) resembles the Wald sequential-probability ratio test (SPRT). In the SPRT, the variable $\prod_{i=1}^{n} L(x_i)$ is compared to fixed-decision thresholds. This variable can be seen to weight all x_i equally in the decision process. However, equation (25) indicates that the more recent observations are more important. It is interesting to note that equation (26) bears great similarity to equations found in exponential smoothing.

8. Conclusion

Our analysis of the problem of checking for the random occurrence of an event with a minimum-expected-cost strategy yielded two important results. First, the form of the strategy may be shown to be the obvious "wait until Probability {the event has occurred up till now} is greater than some critical value q, then check". Second, the minimum cost obtained (and the value of q) may be solved by successive iterations of specific functional equations.

The general approach to this sort of problem and, in fact, to all non-deterministic sequential-decision problems, has been outlined in [2]. References [3] and [4] use a similar dynamic programming solution for analysis of the minimum-cost sequential hypothesis test. Further development of the work discussed here may be found in [6], with an emphasis placed on applications to search theory.

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Published in Management Science on March 01, 1967 as DOI: 10.1287/mnsc.13.7.454.
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