Linear Functional Analysis

Lectures Fall Term 1988

Hörmander, Lars

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Lars Hörmander

Preface

These lectures might have been called applied linear functional analysis, for the purpose is not to present functional analysis for its own sake but rather as a tool for the working analyst. The general results are therefore illustrated by a large number of examples. They are on the whole taken from natural contexts. However, functional analysis alone rarely solves an analytical problem; its role is to clarify what is essential in it. It has therefore been necessary to simplify and modify many of the examples so that they can be handled by elementary arguments. In spite of that the reader may not always have the necessary background to follow the analytical details, but he should then rest assured that the examples are not strictly necessary for the understanding of the main theme.

It is not possible to draw a sharp dividing line between analysis and functional analysis. On the contrary, the vital parts of functional analysis have always developed from proofs of theorems in analysis. When some arguments are felt to occur frequently they are isolated and put in an abstract form, involving rather little structure so that they can be applied in apparently different circumstances. This development is still going on, and what is perceived as central in functional analysis depends to some extent on what parts of analysis that one is interested in. These lecture notes are undoubtedly biased towards applications to differential equations and harmonic analysis, but nevertheless I hope that the material chosen is of a wide interest. They are a slightly edited version of lectures given at the University of Lund in 1969 and in 1988.

I am grateful to Anders Melin who read the whole manuscript and suggested a number of improvements. I would also like to thank the students in the course just finished for their attention to details which has helped improve the exposition. Some of the exercises added at the end have been taken over from earlier courses on the same subject given in Lund or in Copenhagen.

Lund in February 1989

Lars Hörmander

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Chapter I

Linear Algebra

1.1 Vector spaces and linear transformations. Let K be a field. (Later on K will always be either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers.) We recall the following definition:

Definition 1.1.1. A vector space V over K is an abelian group, with elements called vectors and group operation denoted by $+$, such that to any $x \in V$ and $a \in K$ there is assigned an element $ax \in V$ such that

$$\begin{aligned} a(x+y) &= ax + ay && \text{for all } x, y \in V \text{ and } a \in K, \\ (a+b)x &= ax + bx && \text{for all } x \in V \text{ and } a, b \in K, \\ (ab)x &= a(bx) && \text{for all } x \in V \text{ and } a, b \in K, \\ 1 \cdot x &= x && \text{for all } x \in V. \end{aligned}$$

Here 1 is the unit element in K.

These conditions are of course not independent of each other. Note that they imply that $x = (1+0)x = 1 \cdot x + 0 \cdot x = x + 0 \cdot x$, thus $0 \cdot x = 0$ for every $x \in V$. Here 0 denotes both an element in K and one in V, the origin.

Example. Let M be an arbitrary set and denote by V the set of all functions on M with values in K, sometimes denoted K^M. With the operations defined by

$$(af + bg)(m) = af(m) + bg(m); \quad a, b \in K, \ f, g \in V, \ m \in M,$$

it is clear that V is a vector space over K. If $M = \{1, \ldots, n\}$, the space is usually denoted by K^n and is the set of all n-tuples of elements of K. Similarly the set V^M of functions from M to a vector space V over K is a vector space over K.

More general examples are obtained as follows:

Proposition 1.1.2. *Let W be a subset of a vector space V over K such that for all $x, y \in W$ and $a, b \in K$ we have $ax + by \in W$. Then W is a vector space with the operations inherited from V.*

The proof is obvious. One calls W a *linear subspace* of V.

Example. The set V of all $f \in K^M$ such that $\sum_M |f(m)| < \infty$ is clearly a linear subspace of K^M. (Here $K = \mathbf{R}$ or $K = \mathbf{C}$.)

Given a vector space V and a linear subspace V_1 we can construct another vector space V/V_1, called the *quotient space* of V by V_1, as follows: If $x, y \in V$ and $x - y \in V_1$, we write $x \equiv y$ and say that x is congruent to $y \mod V_1$. This is an equivalence relation. Since $x \equiv x_1$ and $y \equiv y_1$ implies that $ax + by \equiv ax_1 + by_1$, the addition and multiplication by scalars in V induce such operations in V/V_1 which will clearly inherit the properties required in Definition 1.1.1.

Proposition 1.1.3. *If $V_2 \subset V_1 \subset V$ where V is a vector space and V_1, V_2 linear subspaces, then the map $V/V_2 \to V/V_1$ gives an isomorphism*

$$(V/V_2)/(V_1/V_2) \to V/V_1.$$

The simple proof is left to the reader. Instead we pass to introducing the appropriate maps between vector spaces.

Definition 1.1.4. Let V_1 and V_2 be two vector spaces over K. A map T from V_1 to V_2 is called a linear map (or linear transformation) if it commutes with the vector operations, that is,

$$T(ax+by) = aTx + bTy; \quad x, y \in V; \ a, b \in K.$$

The linear maps from V_1 to V_2 form a vector space $L(V_1, V_2)$, which is a linear subspace of $V_2^{V_1}$. Thus

$$(a_1T_1 + a_2T_2)x = a_1T_1x + a_2T_2x \quad \text{when } a_1, a_2 \in K \text{ and } x \in V_1.$$

It is also obvious that the composition of two linear maps is a linear map.

Recall that a (linear) map T from V_1 to V_2 is called:

(1) *injective* if $Tx = Ty$ implies $x = y$,
(2) *surjective* if for every $y \in V_2$ we have $Tx = y$ for some $x \in V_1$,
(3) *bijective* if it is both injective and surjective, and thus an isomorphism.

Since $Tx = Ty$ is equivalent to $T(x-y) = 0$, injectivity means precisely that $Tx = 0$ implies $x = 0$.

Example. If V_1 is a linear subspace of V, then the inclusion map $V_1 \to V$ is injective, and the quotient map $V \to V/V_1$ is surjective.

In general a linear map T from V_1 to V_2 is of course neither injective nor surjective, so one has to consider the kernel

$$\operatorname{Ker} T = \{x; x \in V_1, Tx = 0\}$$

and the range

$$\operatorname{Im} T = \{Tx; x \in V_1\}.$$

It is obvious that the kernel is a linear subspace of V_1 and that the range is a linear subspace of V_2. The map T induces a bijection T' from $V_1/\operatorname{Ker} T$ to $\operatorname{Im} T$ since T maps two elements x_1, x_2 in V_1 to the same element in V_2 if and only if x_1 and x_2 are congruent modulo $\operatorname{Ker} T$. Often we shall somewhat incorrectly denote T' by T, although T is really a composition

$$V_1 \to V_1/\operatorname{Ker} T \xrightarrow{T'} \operatorname{Im} T \to V_2$$

where the first map is the quotient map and the last the inclusion.

Definition 1.1.5. If V_1 and V_2 are linear subspaces of a vector space V, then V is said to be the *direct sum* of V_1 and V_2 if every $x \in V$ in one and only one way can be written $x = x_1 + x_2$ with $x_j \in V_j$. One also calls V_2 a supplement of V_1. We write $V = V_1 \oplus V_2$.

Let P_i be the map $x \mapsto x_i$. This is then a linear map, and we have $\operatorname{Im} P_i = V_i$, $\operatorname{Ker} P_1 = V_2$, $\operatorname{Ker} P_2 = V_1$. Moreover, the restriction of P_j to V_j is the identity map of V_j. It is obvious that, with $I =$ identity map,

$$I = P_1 + P_2; \quad P_1P_2 = P_2P_1 = 0; \quad P_j^2 = P_j. \tag{1.1.1}$$

Conversely, given a linear map $P : V \to V$ with $P^2 = P$, if we set $P_1 = P$ and $P_2 = I - P$, then the relations (1.1.1) are fulfilled. If $V_1 = \operatorname{Ker} P_2$, $V_2 = \operatorname{Ker} P_1$, then $x = P_1x + P_2x = P_1x$ when $x \in V_1$, and $x = P_2x$ when $x \in V_2$. Thus P_j leaves the elements of V_j fixed. The equation $x = x_1 + x_2$ with $x_j \in V_j$ implies that $P_jx = P_jx_j = x_j$. Conversely, $x = x_1 + x_2$ if we set $x_j = P_jx \in V_j$. Thus V is the direct sum of V_1 and V_2, and P_1, P_2 are precisely the maps corresponding to this decomposition. We have thus found that there is a one-to-one correspondence between direct sum decompositions of V and projections:

Definition 1.1.6. A linear map $P : V \to V$ is called a projection if

$$P^2 = P.$$

That V is the direct sum of V_1 and V_2 means precisely that the restriction to V_2 of the map $V \to V/V_1$ is a bijection, in other words, that each equivalence class modulo V_1 contains a unique element of V_2.

Note that given two vector spaces V_1 and V_2 we can construct a vector space

$$V = \{(x_1, x_2); x_j \in V_j\}$$

with vector operations obtained from those in V_j for each component. We can regard V_1 (resp. V_2) as the subspace of V for which $x_2 = 0$ (resp. $x_1 = 0$). Then we have the situation in Definition 1.1.5.

1.2. Dimension and rank. We shall now assign to every vector space V over K its dimension $\dim_K V$ or $\dim V$ for short. This shall be a non-negative integer or $+\infty$ with the following properties:

a) $\dim K^n = n$,
b) If $T : V_1 \to V_2$ is a surjective (injective) linear map, then

$$\dim V_1 \geq \dim V_2 \quad (\text{resp. } \dim V_1 \leq \dim V_2).$$

We shall see that this is possible in one and only one way.

Indeed, let V be a vector space over K and consider linear maps

$$T : K^n \to V.$$

Such a map is completely determined by the vectors $Te_i = x_i \in V$, where e_i is the element in K^n with ith coordinate 1 and the other coordinates 0. For using the linearity of T we obtain

$$T(a_1, \ldots, a_n) = \sum_1^n a_jx_j, \quad a_j \in K; \tag{1.2.1}$$

conversely, given $x_1, \dots, x_n \in V$ the equation (1.2.1) defines a linear map $K^n \to V$ with $Te_i = x_i$. We recall the following terminology:

Definition 1.2.1. The vectors $x_1, \dots, x_n \in V$ are called *linearly independent* if the map T defined by (1.2.1) is injective; they are said to *generate* V if T is surjective; and they are called a *basis* if T is bijective.

To use conditions a) and b) above we shall consider both surjective and injective maps from spaces K^n to V. To relate them we need the following lemma — the only non-trivial point in the discussion.

Lemma 1.2.2. *If $T_1 : K^n \to V$ is a surjective and $T_2 : K^m \to V$ is an injective linear map, then $m \le n$.*

Proof. Since T_1 is surjective, we can choose a linear map $T : K^m \to K^n$ such that $T_2 = T_1 T$. In fact, it suffices to choose Te_j with $T_1(Te_j) = T_2 e_j$ when e_j is a basis vector in K^m. Then T is injective since T_2 is injective, so it suffices to prove that if

$$T : K^m \to K^n$$

is an injective linear map, then $m \le n$ (or equivalently that a homogeneous system of equations with fewer equations than unknowns always has a non-trivial solution). This is obvious if $n = 1$ since two arbitrary elements in K^n are proportional then. We prove the statement in general assuming that it is already known for smaller values of n. Define $Q : K^n \to K^{n-1}$ by

$$Q\,(a_1, \dots, a_n) = (a_1, \dots, a_{n-1})$$

and $R_k : K^{m-1} \to K^m$ for $1 \le k \le m$ by

$$R_k(a_1, \dots, a_{m-1}) = (a_1, \dots, a_{k-1}, 0, a_k, \dots, a_{m-1}).$$

If $QT : K^m \to K^{n-1}$ is injective it follows by the inductive hypothesis that $m \le n - 1$. Otherwise, if a and b are two non-zero elements in $\operatorname{Ker} QT$, then Ta and Tb are proportional and not zero, so a is proportional to b since T is injective. Choose k so that the kth coordinate $a_k \ne 0$, hence $b_k \ne 0$. Then the map $QTR_k : K^{m-1} \to K^{n-1}$ is injective, so $m - 1 \le n - 1$ by the inductive hypothesis. The proof is complete.

Theorem 1.2.3. *Let V be a vector space over K such that there exists a surjective map $T : K^n \to V$ for some n. Every system of linearly independent vectors $x_1, \dots, x_k$ in V can then be extended to a basis $x_1, \dots, x_k, x_{k+1}, \dots, x_d$. Every system of generators contains a basis. All bases in V have the same number of elements d. The number d is also the smallest such that there exists a surjective linear map $K^d \to V$ as well as the largest such that there exists an injective linear map $K^d \to V$; such maps are automatically bijective.*

Proof. Let D be the smallest integer such that there is a surjective map $K^D \to V$, and let d be the largest integer such that there is an injective map $K^d \to V$. The existence of D is guaranteed by the hypothesis, and it follows from Lemma 1.2.2 that d is defined and $\le D$. Let now $x_1, \dots, x_k$ be a system of linearly independent elements in V. Then we have $k \le d$. If they do not form a basis we can choose

x_{k+1} which is not a linear combination of $x_1, \dots, x_k$. Then $x_1, \dots, x_{k+1}$ are linearly independent, for if

$$a_1 x_1 + \cdots + a_{k+1} x_{k+1} = 0$$

we cannot have $a_{k+1} \neq 0$ since division by a_{k+1} would then show that x_{k+1} is a linear combination of $x_1, \dots, x_k$. Thus $a_{k+1} = 0$ and since $x_1, \dots, x_k$ are linearly independent it follows that $a_1 = \cdots = a_k = 0$ also. We can thus extend the system $x_1, \dots, x_k$ until we get a basis $x_1, \dots, x_N$. Then we have $N \leq d \leq D \leq N$, so $N = d = D$.

Now assume that $x_1, \dots, x_k$ is a system of generators. If they are not linearly independent then one of them, say x_k is a linear combination of the others, so $x_1, \dots, x_{k-1}$ is also a system of generators. We can continue dropping elements until we have a linearly independent system of generators, that is, a basis. The theorem is proved.

The proof also shows that if there is no surjective map $K^n \to V$ for any n, then one can find an injective map $K^n \to V$ for any n. The following definition of the dimension is therefore the only one which can have the properties a) and b) stated at the beginning of the section.

Definition 1.2.4. A vector space V over K is said to have finite dimension (over K) if there exists a surjective linear map $K^n \to V$ for some n. The smallest such integer n is equal to the largest integer n such that there is an injective linear map $K^n \to V$. It is called the *dimension* of V. If V does not have finite dimension we say that V is infinite dimensional and write $\dim V = \infty$.

Remark. In case there may be some doubt which scalar field K is being used we shall make this clear by writing $\dim_K V$ for the dimension of V as a vector space over K.

It is clear that the dimension of K^n is equal to n as desired, and we also have the property b):

Theorem 1.2.5. *If the linear map $T : V_1 \to V_2$ is surjective resp. injective or bijective, then*

$$\dim V_1 \geq \dim V_2 \quad \textit{resp.} \ \dim V_1 \leq \dim V_2 \quad \textit{or} \ \dim V_1 = \dim V_2.$$

Proof. Let T be surjective. There is nothing to prove unless $\dim V_1 < \infty$. From any surjective map $K^n \to V_1$ we then obtain by composition with T a surjective map $K^n \to V_2$, which proves that $\dim V_1 \geq \dim V_2$. If T is injective, we conclude that $\dim V_1 \leq \dim V_2$ if we consider injective maps $K^n \to V_1$ instead. This proves the theorem.

In particular, the dimension of a subspace or a quotient space of V is thus smaller than or equal to the dimension of V. If we recall that a linear map $T : V_1 \to V_2$ gives rise to a bijection $T' : V_1/\operatorname{Ker} T \to \operatorname{Im} T$, hence that

$$\dim(V_1/\operatorname{Ker} T) = \dim(\operatorname{Im} T), \tag{1.2.2}$$

we recover Theorem 1.2.5 from these special cases. The number occurring in (1.2.2) is so important that it has a special name:

Definition 1.2.6. If T is a linear map $V_1 \to V_2$, then the *rank* of T is defined as $\dim(\operatorname{Im} T)$ or equivalently as $\dim(V_1/\operatorname{Ker} T)$.

In a moment we shall write (1.2.2) in various other ways, but first it is useful to introduce another concept:

Definition 1.2.7. If V is a vector space and W a linear subspace, then the *codimension* of W in V, denoted $\operatorname{codim}_V W$ or simply $\operatorname{codim} W$, is defined to be the dimension of the quotient space V/W.

In finite dimensional spaces this notion is not indispensable, for we have

Theorem 1.2.8. *If V is a vector space and W a linear subspace, then*

$$\dim W + \operatorname{codim} W = \dim V.$$

Proof. Since $\dim W \le \dim V$ and $\operatorname{codim} W \le \dim V$, we may assume in the proof that the left hand side is finite. Let $T_1 : K^n \to W$ and $T_2 : K^m \to V/W$ be bijections, thus $n = \dim W$ and $m = \operatorname{codim} W$. We lift T_2 to a map $T_2' : K^m \to V$ such that the composition with the quotient map $V \to V/W$ is equal to T_2. Then

$$T = T_1 \oplus T_2' : K^{n+m} = K^n \oplus K^m \ni (a,b) \mapsto T_1 a + T_2' b \in V$$

is bijective. In fact, if $T(a,b) = 0$ then composition with the quotient map $V \to V/W$ shows that $T_2 b = 0$, hence $b = 0$. It follows that $T_1 a = 0$ so $a = 0$ also. In a similar way one concludes that T is surjective, which proves the theorem.

We can write (1.2.2) in the form

$$\operatorname{codim} \operatorname{Ker} T = \dim \operatorname{Im} T. \tag{1.2.2$'$}$$

Using Theorem 1.2.8 we therefore obtain

Theorem 1.2.9. *Let $T : V_1 \to V_2$ be a linear map. Then*

$$\dim \operatorname{Im} T + \dim \operatorname{Ker} T = \dim V_1, \quad \operatorname{codim} \operatorname{Im} T + \operatorname{codim} \operatorname{Ker} T = \dim V_2.$$

If V_1 and V_2 have finite dimension, then

$$\dim \operatorname{Ker} T - \dim \operatorname{Coker} T = \dim V_1 - \dim V_2. \tag{1.2.3}$$

Here we have used the notation $\operatorname{Coker} T = V_2/\operatorname{Im} T$. In the special case when $\dim V_1 = \dim V_2$ the result (1.2.3) is classically phrased as follows:

The number of linear conditions for solvability of the equation $Tx = y$ is equal to the number of linearly independent solutions of the equation $Tx = 0$.

One of our main goals is to discuss infinite dimensional extensions of this rule or the more general formula (1.2.3). In the algebraic case this discussion will begin in the following section. We shall then need some identities concerning dimensions of vector spaces which will now be derived.

Let

$$0 \to V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \dots \xrightarrow{T_{N-1}} V_N \to 0 \tag{1.2.4}$$

be a sequence of vector spaces and linear maps, starting with the 0 dimensional vector space consisting of the origin only and ending in the same way. One calls (1.2.4) a *complex* if $T_{j+1}T_j = 0$ for every j. This means that

$$\operatorname{Im} T_j \subset \operatorname{Ker} T_{j+1}.$$

The complex is called *exact* if $\operatorname{Im} T_j = \operatorname{Ker} T_{j+1}$ for every j, which in particular shall mean that T_0 is injective and that T_{N-1} is surjective. If $N = 1$ exactness therefore means that $T_0 : V_0 \to V_1$ is bijective, and then we know that $\dim V_0 = \dim V_1$. This fact is generalised as follows:

Theorem 1.2.10. *If (1.2.4) is an exact complex of vector spaces and linear maps, then*

$$\sum_j \dim V_{2j} = \sum_j \dim V_{2j+1} \tag{1.2.5}$$

or, if all dimensions are finite,

$$\sum_j (-1)^j \dim V_j = 0. \tag{1.2.6}$$

Proof. Let $R_j = \operatorname{Ker} T_j = \operatorname{Im} T_{j-1}$. By Theorem 1.2.8 and (1.2.2) we have

$$\dim V_j = \dim R_j + \operatorname{codim} R_j = \dim R_j + \dim R_{j+1}$$

which shows that both sides of (1.2.5) are equal to $\sum \dim R_j$.

As a first example we note that given a linear map $T : V_1 \to V_2$ we have an exact sequence

$$0 \to \operatorname{Ker} T \to V_1 \to \operatorname{Im} T \to 0$$

where the first map is a restriction of T and the second is a quotient map. Hence we obtain

$$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim V_1.$$

Similarly, we have an exact sequence

$$0 \to V_1/\operatorname{Ker} T \to V_2 \to V_2/\operatorname{Im} T \to 0$$

where the first map is induced by T and the second is a quotient map. This gives

$$\operatorname{codim} \operatorname{Ker} T + \operatorname{codim} \operatorname{Im} T = \dim V_2$$

as we already knew from Theorem 1.2.9. Further important examples are given in the following

Theorem 1.2.11. *Let V_1 and V_2 be linear subspaces of the vector space V, and let $V_1 + V_2$ be the subspace $\{x_1 + x_2;\ x_1 \in V_1, x_2 \in V_2\}$. Then we have*

$$\dim(V_1 \cap V_2) + \dim(V_1 + V_2) = \dim V_1 + \dim V_2, \tag{1.2.7}$$

$$\operatorname{codim}(V_1 \cap V_2) + \operatorname{codim}(V_1 + V_2) = \operatorname{codim} V_1 + \operatorname{codim} V_2, \tag{1.2.8}$$

$$\dim(V_1 \cap V_2) + \operatorname{codim} V_2 = \operatorname{codim}(V_1 + V_2) + \dim V_1. \tag{1.2.9}$$

Note that these equalities are equivalent if V has finite dimension but not otherwise.

Proof. We have an exact sequence

$$0 \to V_1 \cap V_2 \to V_1 \oplus V_2 \to V_1 + V_2 \to 0$$

where the first map is $V_1 \cap V_2 \ni x \mapsto (x, x) \in V_1 \oplus V_2$, and the second map is $V_1 \oplus V_2 \ni (x_1, x_2) \mapsto x_1 - x_2$. The trivial verification of exactness is omitted. This gives (1.2.7). Similarly the exact sequence

$$0 \to V/(V_1 \cap V_2) \to (V/V_1) \oplus (V/V_2) \to V/(V_1 + V_2) \to 0$$

leads to (1.2.8). Finally (1.2.9) is obtained from the exact sequence

$$0 \to V_1 \cap V_2 \to V_1 \to V/V_2 \to V/(V_1 + V_2) \to 0.$$

The definition of the various maps and the proof of exactness are left as an exercise for the reader.

Finally we shall make a remark on *duality*:

Definition 1.2.12. Let V_1 and V_2 be two vector spaces over K, and let $V_1 \times V_2 \ni (x, y) \mapsto \langle x, y\rangle \in K$ be a bilinear form, that is, a function which is linear in x (resp. y) when y (resp. x) is fixed. The form is said to define a duality between V_1 and V_2 if

$$\langle x, y\rangle = 0 \quad \forall y \in V_2 \implies x = 0, \tag{1.2.10}$$

$$\langle x, y\rangle = 0 \quad \forall x \in V_1 \implies y = 0. \tag{1.2.11}$$

An example is $V_1 = V_2 = K^n$ and

$$\langle x, y\rangle = \sum_1^n x_j y_j; \qquad x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n). \tag{1.2.12}$$

This is more than an example:

Theorem 1.2.13. *If V_1 and V_2 are two vector spaces which are dual with respect to a bilinear form $\langle \cdot, \cdot\rangle$, then* $\dim V_1 = \dim V_2$. *If this dimension is finite, then (1.2.12) holds for the coordinates with respect to suitable bases in V_1 and V_2.*

Proof. There is nothing to prove unless one of the spaces, say V_1, is finite dimensional. Then we choose a basis $e_1, \dots, e_n$ for V_1 and consider the linear map

$$\varphi : V_2 \ni y \mapsto (\langle e_1, y\rangle, \dots, \langle e_n, y\rangle) \in K^n.$$

φ is injective since $\langle e_j, y\rangle = 0$, $j = 1, \dots, n$, implies $\langle x, y\rangle = 0$, $x \in V_1$, hence $y = 0$. This proves that $\dim V_2 \le n = \dim V_1$, and interchanging the roles of V_1 and V_2 we conclude that $\dim V_1 = \dim V_2$. Thus φ is bijective. The inverse can be written

$$K^n \ni (y_1, \dots, y_n) \mapsto \sum_1^n y_i f_i,$$

where $f_1, \dots, f_n$ is a basis for V_2. Hence

$$\langle \sum x_j e_j, \sum y_i f_i\rangle = \sum x_j (\varphi(\sum y_i f_i))_j = \sum x_j y_j$$

as claimed.

1.3. The index of linear transformations. We shall now resume the discussion of the quantity which occurs in the left-hand side of (1.2.3), without assuming that $\dim V_1$ and $\dim V_2$ are finite. If $T : V_1 \to V_2$ is a linear map such that either $\dim \operatorname{Ker} T$ or $\dim \operatorname{Coker} T$ is finite, we thus define the index of T by

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T. \tag{1.3.1}$$

The index has a stability property generalizing (1.2.3).

Theorem 1.3.1. *If T and S are linear maps from V_1 to V_2 such that $\operatorname{ind} T$ is defined and S has finite rank, then $\operatorname{ind}(T+S)$ is defined and*

$$\operatorname{ind} T = \operatorname{ind}(T+S). \tag{1.3.2}$$

Before the proof of Theorem 1.3.1 we give another property of the index, sometimes referred to as the logarithmic law, which is convenient in computations involving the index.

Theorem 1.3.2. *Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be linear maps. For the linear map $T_2T_1 : V_1 \to V_3$ we have*

$$\operatorname{ind}(T_2T_1) = \operatorname{ind} T_2 + \operatorname{ind} T_1 \tag{1.3.3}$$

provided that the right-hand side is defined, that is, either $\dim \operatorname{Ker} T_j < \infty$ for $j = 1, 2$ or $\dim \operatorname{Coker} T_j < \infty$ for $j = 1, 2$.

Proof. It suffices to verify the exactness of the complex

$$\begin{aligned} 0 \to \operatorname{Ker} T_1 \xrightarrow{i} \operatorname{Ker} T_2T_1 \xrightarrow{T_1'} \operatorname{Ker} T_2 \xrightarrow{q} V_2/\operatorname{Im} T_1 \\ \xrightarrow{T_2'} V_3/\operatorname{Im} T_2T_1 \xrightarrow{q} V_3/\operatorname{Im} T_2 \to 0, \end{aligned} \tag{1.3.4}$$

where i and q denote inclusion and quotient maps and T_1' and T_2' are derived from T_1 and T_2 in an obvious way. For then it follows that the left-hand side of (1.3.3) is defined if the right-hand side is, and using Theorem 1.2.10 we obtain (1.3.3). The easy verification that (1.3.4) is an exact complex is left for the reader except for the exactness at $V_2/\operatorname{Im} T_1$ which may be somewhat less trivial than the other statements. So assume that $x_2 \in V_2$ and that the class of x_2 in $V_2/\operatorname{Im} T_1$ is mapped to 0 by T_2'. This means that $T_2x_2 = T_2T_1x_1$ for some $x_1 \in V_1$. Thus $x_2 - T_1x_1 \in \operatorname{Ker} T_2$, and since $x_2 - T_1x_1$ is equal to x_2 modulo $\operatorname{Im} T_1$, this proves the statement.

The following special case of Theorem 1.3.2 is often useful:

Corollary 1.3.3. *Let $T : V_1 \to V_2$ be a linear map such that $\operatorname{ind} T$ is defined, let W_1 be a subspace of V_1 of finite codimension and W_2 a subspace of V_2 of finite dimension. Let $i : W_1 \to V_1$ and $q : V_2 \to V_2/W_2$ be the inclusion and quotient maps. Then the index of $qTi : W_1 \to V_2/W_2$ is defined and equals*

$$\operatorname{ind} T + \dim W_2 - \operatorname{codim} W_1.$$

Proof of Theorem 1.3.1. By hypothesis $W_1 = \operatorname{Ker} S$ has finite codimension. With the notation of Corollary 1.3.3 we have $\operatorname{ind}(Ti) = \operatorname{ind} T + \operatorname{ind} i$, if i is the inclusion $W_1 \to V_1$. Since $Si = 0$, it follows that

$$\operatorname{ind}((T+S)i) = \operatorname{ind}(Ti) = \operatorname{ind} T + \operatorname{ind} i,$$

which shows that either $\operatorname{Ker}(T+S)$ or $\operatorname{Coker}(T+S)$ has finite dimension, for by (1.2.9) we have $\dim \operatorname{Ker}(T+S) \le \dim \operatorname{Ker}((T+S)i) + \operatorname{codim} W_1$, and $\dim \operatorname{Coker}(T+S) \le \dim \operatorname{Coker}((T+S)i)$. Thus $\operatorname{ind}(T+S)$ is defined, and another application of Theorem 1.3.2 shows that

$$\operatorname{ind}((T+S)i) = \operatorname{ind}(T+S) + \operatorname{ind} i,$$

hence that $\operatorname{ind}(T+S) = \operatorname{ind} T$.

When T is a linear map $V \to V$ and $\dim V < \infty$ we know that $\operatorname{ind} T = 0$. This is not always true in the infinite dimensional case, however, which is one of the reasons for the interest of the index.

Example 1.3.4. Let $V = K^{\{1,2,\dots\}}$, that is, the set of all sequences $x = (x_1, x_2, \dots)$ with elements in K, the vector operations being defined coordinatewise. Let n be a fixed integer and set

$$Tx = (x_{n+1}, x_{n+2}, \dots)$$

where coordinates with index ≤ 0 should be read as 0. Then we have

$$\dim \operatorname{Ker} T = \max(n, 0), \qquad \dim \operatorname{Coker} T = \max(-n, 0),$$

and it follows that $\operatorname{ind} T = n$.

1.4. Hyperplanes and linear forms. A linear subspace W of a vector space V is called a hyperplane if $\operatorname{codim} W = 1$. Equivalently, this means that W is a proper subspace of V contained in no strictly larger proper subspace. A hyperplane can always be defined by one linear equation. For if V/W has dimension 1, there is a bijection $V/W \to K$. By composition we get a *linear form*

$$L : V \to V/W \to K$$

such that W is the inverse image of 0 in K. Thus we have

$$L(x+y) = L(x) + L(y), \quad x, y \in V; \qquad L(ax) = aL(x), \quad a \in K, x \in V; \tag{1.4.1}$$

and $L(x) = 0$ if and only if $x \in W$. Conversely, assume that we have a linear form L, that is, a map $V \to K$ satisfying (1.4.1), and that L does not vanish identically. Then $W = \{x; L(x) = 0\}$ is a hyperplane, for the map $V/W \to K$ induced by L is a bijection.

Now consider a linear subspace W of finite codimension n. Composing the quotient map $V \to V/W$ with a bijection $V/W \to K^n$ we then get a linear map $L : V \to K^n$ such that L is surjective and $Lx = 0$ is equivalent to $x \in W$. Writing $L = (L_1, \dots, L_n)$, where L_j are linear forms, we conclude that W can be defined by n linear equations, that is, W is the intersection of n hyperplanes. On the other hand, the intersection of k hyperplanes has codimension $\le k$ in view of (1.2.8). Thus we have

Theorem 1.4.1. *A linear subspace W of V of codimension n can always be obtained as the intersection of n but not fewer than n hyperplanes.*

We shall now discuss an analogue of Theorem 1.4.1 for spaces of arbitrary codimension.

Theorem 1.4.2. *Every linear subspace of V is the intersection of hyperplanes.*

An equivalent statement is the following:

Theorem 1.4.2′. *If V_1 is a subspace of V and x an element in $V \setminus V_1$, then one can find a hyperplane V_2 in V such that $V_1 \subset V_2$ but $x \notin V_2$.*

Proof. a) We prove first that if $\operatorname{codim} V_1 > 1$ then one can choose V_2 containing V_1 strictly but not containing x. To do so we note that V/V_1 has dimension > 1 and that the residue class $\dot{x}$ of x there is not 0. We can therefore find another element, say the residue class $\dot{y}$ of some $y \in V$, which is linearly independent of $\dot{x}$. Now set

$$V_2 = \{z + ty; z \in V_1, t \in K\}.$$

This is a linear subspace of V which contains V_1 strictly since $y \in V_2 \setminus V_1$. On the other hand, $x \notin V_2$ for the image of V_2 in V/V_1 is generated by $\dot{y}$ so it does not contain $\dot{x}$.

b) Now consider the set $\mathcal{F}$ of all linear subspaces V_2 of V containing V_1 but not x. The union of a completely ordered subset of $\mathcal{F}$ is obviously an element of $\mathcal{F}$. According to Zorn's lemma it follows that there exists at least one maximal element $V_2 \in \mathcal{F}$. Since $x \notin V_2$ we have $\operatorname{codim} V_2 \geq 1$, and by a) we know that V_2 would not be maximal if $\operatorname{codim} V_2 > 1$. Hence $\operatorname{codim} V_2 = 1$, which proves the theorem.

With $V_1 = \{0\}$ we conclude in particular that hyperplanes exist and that for every $x \in V \setminus 0$ there is some linear form L on V with $L(x) \neq 0$. Another useful consequence is given by

Corollary 1.4.3. *If $W_1 \subset V$ is a linear subspace of finite dimension, then there exists a linear subspace $W_2 \subset V$ with* $\operatorname{codim} W_2 = \dim W_1$ *such that $W_1 \cap W_2 = \{0\}$. Thus V is the direct sum $W_1 \oplus W_2$.*

Proof. The assertion is obvious if $W_1 = \{0\}$. If $0 \neq x \in W_1$ we can find a hyperplane H_1 with $x \notin H_1$, thus $\dim(W_1 \cap H_1) = \dim W_1 - 1$ by (1.2.9) since $H_1 + W_1 = V$. Repeating the argument we obtain hyperplanes H_k, $k = 1, \dots, d$, $d = \dim W_1$, such that

$$\dim (W_1 \cap H_1 \cap \cdots \cap H_k) = d - k, \qquad k = 1, \dots, d.$$

Thus $W_1 \cap W_2 = \{0\}$ if $W_2 = H_1 \cap \cdots \cap H_d$, and $d \leq \operatorname{codim} W_2 \leq d$ by (1.2.9) and (1.2.8).

Remark. Using Zorn's lemma it is also easy to prove directly that Corollary 1.4.3 holds for an arbitrary linear subspace W_1. We leave this as an exercise for the reader.

As an application we prove a supplement to Theorem 1.3.1.

Theorem 1.4.4. *Let $T : V_1 \to V_2$ be a linear map of index* 0*. Then one can find a linear map $S : V_1 \to V_2$ of finite rank such that $T + S$ is a bijection.*

Proof. Using Theorem 1.4.3 we choose a linear subspace W_1 of V_1 such that $W_1 \cap \operatorname{Ker} T = \{0\}$ and $\operatorname{codim} W_1 = \dim \operatorname{Ker} T$. The range of T is then equal to the range of the restriction of T to W_1, which is injective. Now choose a linear subspace $W_2 \subset V_2$ such that $W_2 \cap \operatorname{Im} T = \{0\}$ and $\dim W_2 = \dim \operatorname{Coker} T$. We just have to take W_2 spanned by vectors whose images in $\operatorname{Coker} T$ form a basis there. By hypothesis

$$\dim \operatorname{Ker} T = \dim \operatorname{Coker} T = \dim W_2;$$

hence we can define S so that S vanishes on W_1 and the restriction to $\operatorname{Ker} T$ is a bijection with range W_2. Then the image of $T + S$ contains $\operatorname{Im} T$ and W_2 so it is equal to V_2. The construction also shows that $T + S$ is injective. This proves the theorem.

The origin in V belongs to every linear subspace. Sometimes it is convenient to remove this special role of the origin by introducing the following concept:

Definition 1.4.5. A subset W of the vector space V is called an *affine* subspace (hyperplane) if W can be transformed to a linear subspace (hyperplane) by a translation.

The definition means that $\{x - y; x \in W\}$ is a linear subspace (hyperplane) for every fixed $y \in W$, and that this set is independent of y. Conversely, if $\{x - y; x \in W\}$ is a linear subspace (hyperplane) for some fixed y , then W is an affine subspace (hyperplane) through y. With this terminology Theorem 1.4.2′ extends immediately to affine spaces.

Every linear form L on K^n is obviously of the form

$$K^n \ni (a_1, \dots, a_n) \mapsto \sum_1^n c_j a_j$$

for some $c_j \in K$. However, in an infinite dimensional space there may be so many linear forms that they are hard to describe. This is one reason for the study of vector spaces with topology where one only has to consider *continuous* linear forms. As a preparation for that we shall now give an extension of Theorem 1.4.2′ with $K = \mathbf{R}$ and the point x replaced by a larger set.

Definition 1.4.6. A subset A of the vector space V over $\mathbf{R}$ is called *convex* if for arbitrary $x, y \in V$ the set $\{t; t \in \mathbf{R}, x + ty \in A\}$ is an interval (open, closed or half open; finite or infinite). We say that A is convex and *linearly open* if the interval is always open.

Theorem 1.4.7. *(Geometric form of the Hahn-Banach theorem.) Let A be a convex, linearly open set in the vector space V over $\mathbf{R}$, and let V_1 be a linear subspace which does not intersect A. Then there exists a hyperplane V_2 such that $V_1 \subset V_2$ and $V_2 \cap A = \emptyset$.*

Proof. This is analogous to the proof of Theorem 1.4.2′ although the 2-dimensional case is not equally trivial now. We discuss it first.

a) $\dim V = 2$. Then $V_1 = \{0\}$ or else there is nothing to prove. From the convexity of A it follows that if a half ray $\{tx; t \geq 0\}$ through 0 intersects A, then the opposite half ray $\{tx; t \leq 0\}$ does not. Identifying half rays with points on the unit circle by means of a basis in V, we denote by O_+ and O_- the set of half rays intersecting A and their opposites. These are open sets since A is linearly open, and we have just seen that they are disjoint. Since the unit circle is connected, it follows that it cannot be the union of O_+ and O_-, so we can find a half ray which is neither in O_+ nor in O_-. The corresponding line has no point in common with A then.

b) Assuming that $\operatorname{codim} V_1 > 1$ we now prove that there exists a strictly larger subspace V_2 which does not intersect A. Thus form $V' = V/V_1$ and let A' be the image of A in V'. Then A' is convex and linearly open. For if $\xi_j \in A'$ for $j = 1, 2$, we can find $x_j \in A$ with residue class ξ_j for $j = 1, 2$. Thus we have $tx_1 + (1-t)x_2 \in A$ for all t in an open interval I containing $[0, 1]$, hence $t\xi_1 + (1-t)\xi_2 \in A'$, $t \in I$, which proves the assertion. In view of a) we can find a straight line W' in V' which does not intersect A'. In fact, we can choose W' in any two dimensional subspace of V' (which by assumption has dimension > 1). But then the inverse image of W' in V by the quotient map $V \to V'$ has the required properties.

c) We can now apply Zorn's lemma precisely as in part b) of the proof of Theorem 1.4.2$'$. The repetition is left as an exercise for the reader.

Chapter II
Topological vector spaces

2.1. Topological and metric spaces. Let us first recall that a set E is called a topological space if there is given a family $\mathcal{O}$ of subsets, which are said to be *open*, such that

a) $E \in \mathcal{O}$,
b) the union of the sets in any subfamily of $\mathcal{O}$ is always a set in $\mathcal{O}$,
c) the intersection of finitely many sets in $\mathcal{O}$ is also in $\mathcal{O}$.

The space E is called a Hausdorff space if in addition

d) for two arbitrary different points $x, y \in E$ one can always find two disjoint sets in $\mathcal{O}$ containing x and y respectively.

The complements in E of the open sets are said to be *closed*. According to b) and c) *arbitrary intersections and finite unions of closed sets are closed.* For every set $M \subset E$ the intersection $\overline{M}$ of all closed sets containing M is therefore closed, hence the smallest closed set containing M.

A subset N of E is called a neighborhood of $x \in E$ (and x is said to be an interior point of N) if $x \in O \subset N$ for some $O \in \mathcal{O}$. The neighborhoods of x have the following properties:

i) E is a neighborhood of x, and x belongs to every neighborhood of x,
ii) the intersection of a finite number of neighborhoods of x is a neighborhood of x,
iii) if N is a neighborhood of x and $N' \supset N$, then N' is a neighborhood of x,
iv) if N is a neighborhood of x then there is another neighborhood N' of x such that N is a neighborhood of every $y \in N'$.

From the neighborhoods one can reconstruct the open sets: If O is a subset of E then $O \in \mathcal{O}$ if (and only if) O is a neighborhood of all of its points. In fact, for every $x \in O$ we can then find $O_x \in \mathcal{O}$ with $x \in O_x \subset O$. Hence $\cup_{x \in O} O_x = O$ is in $\mathcal{O}$ by condition b).

Suppose now instead that for every $x \in E$ we are given a family $\mathcal{V}_x$ of subsets of E, called neighborhoods of x, so that i) and ii) are fulfilled. We can then define a topology $\mathcal{T}$ in E by declaring that a set $O \subset E$ is open if for every $x \in O$ we have $N \subset O$ for some $N \in \mathcal{V}_x$. Then a) follows from i), b) is trivial, and c) follows from ii) or even the weaker version

ii)′ the intersection of a finite number of neighborhoods of x contains a neighborhood of x.

If $\mathcal{N}$ is a neighborhood of x in the topology $\mathcal{T}$ thus defined, then we can find an open set O with $x \in O \subset \mathcal{N}$, hence some $N' \in \mathcal{V}_x$ with $x \in N' \subset O \subset \mathcal{N}$, so $\mathcal{N}$ contains one of our original "neighborhoods". Now suppose that also

iv)′ If $N \in \mathcal{V}_x$ then there is a set $N' \ni x$ such that N contains a neighborhood of y for every $y \in N'$.

Let $\widehat{\mathcal{V}}_x$ be the family of subsets of E containing some set in $\mathcal{V}_x$. Then (i) and (iii) are fulfilled, (ii)′, (iv)′ reduce to (ii) and (iv), and the topology $\mathcal{T}$ is also defined by the families $\widehat{V}_x$.

Let $N \in \mathcal{V}_x$, and denote by O the set of all $y \in N$ such that $N \in \widehat{\mathcal{V}}_y$. By iv) we can then find $N'_y \in \widehat{\mathcal{V}}_y$ such that $N \in \widehat{\mathcal{V}}_z$ for every $z \in N'_y$, thus $N'_y \subset O$. This means that O is open in the topology $\mathcal{T}$, and since $x \in O \subset N$ we see that N is also a neighborhood of x in the topology $\mathcal{T}$. The neighborhoods of x in the topology $\mathcal{T}$ are therefore precisely the sets containing some set belonging to $\mathcal{V}_x$. If iii) is fulfilled the neighborhoods are precisely those we were given from the beginning.

To define a topology we can therefore either give a family of open sets satisfying a), b), c) or give the families $\mathcal{V}_x$ of neighborhoods satisfying at least i), ii)$'$, iv)$'$. If condition iii) is not required one calls the sets $\mathcal{V}_x$ a *neighborhood basis* or a *fundamental system of neighborhoods*. The neighborhoods of x in the topology defined are precisely the sets $\widehat{\mathcal{V}}_x$ which contain some set belonging to $\mathcal{V}_x$.

Any subset E_1 of E is itself a topological space with the restriction of the topology in E: The open sets in E_1 are by definition the sets $O \cap E_1$ with $O \in \mathcal{O}$.

A map $f : E_1 \to E_2$ between topological spaces is called *continuous* if for every open set $O \subset E_2$ the *inverse image*

$$f^{-1}(O) \overset{\text{def}}{=} \{x \in E_1; f(x) \in O\}$$

is an open set. Equivalently this means that for every neighborhood N_2 of $f(x)$ one can find a neighborhood N_1 of x with $f(N_1) \subset N_2$.

A sequence $x_j \in E$ is said to converge to $x \in E$ if for any neighborhood N of x all but a finite number of the points x_j belong to N. If the Hausdorff separation axiom d) above is satisfied, then this cannot happen for more than one value of x. A closed set $F \subset E$ contains all limits of sequences contained in F, for $E \setminus F$ is a neighborhood of all of its points and contains no element of such a sequence.

A topology can sometimes be defined by means of a *metric*, that is, a function d on $E \times E$ with values in the non-negative reals such that

a) $d(x, z) \le d(x, y) + d(y, z)$, (the triangle inequality),
b) $d(x, y) = d(y, x)$,
c) $d(x, y) = 0 \iff x = y$,

for all $x, y, z \in E$. The metric gives rise to a system of neighborhoods of x,

$$N_{x,\varepsilon} = \{y \in E; d(x, y) < \varepsilon\}, \qquad 0 < \varepsilon \le \infty,\ x \in E,$$

satisfying i), ii), iv)$'$ above, so the metric defines a topology. It follows from a) that $d(x, y)$ is a continuous function of x (or y) in this topology. It is clear that the topology satisfies the Hausdorff separation condition d) above. (Note that all topologies cannot be defined by a metric, and that different metrics may define the same topology.)

A metric space is called *complete* if for every *Cauchy sequence*, that is, every sequence $x_n \in E$ with $d(x_n, x_m) \to 0$ as $n, m \to \infty$, there is an element $x \in E$ such that $x_n \to x$, that is, $d(x_n, x) \to 0$. The real numbers $\mathbf{R}$ and the complex numbers $\mathbf{C}$ with the usual distance $d(x, y) = |x - y|$ for $x, y \in \mathbf{C}$ are examples of complete metric spaces.

Some of the most important theorems in functional analysis follow from a fundamental classical theorem of Baire concerning complete metric spaces:

Theorem 2.1.1. *(Baire) Let E be a complete metric space, and let F_n, $n = 1, 2, \dots,$ be closed subsets containing no interior points. Then the union $\cup_1^\infty F_n$ has no interior point either.*

Proof. For any $x \in E$ and $\varepsilon > 0$ we want to show that

$$N_{x,\varepsilon} = \{y \in E; d(x, y) \le \varepsilon\}$$

is not contained in $\cup F_n$. To do so we want to choose sequences $x_j \in E$ and $\varepsilon_j > 0$ so that

$$N_{x,\varepsilon} \supset N_{x_1,\varepsilon_1} \supset N_{x_2,\varepsilon_2} \supset \cdots, \qquad F_j \cap N_{x_j,\varepsilon_j} = \emptyset \quad \text{for every } j.$$

When $j < k$ this implies that $x_k \in N_{x_j,\varepsilon_j}$, hence $d(x_j, x_k) < \varepsilon_j$, so x_k is a Cauchy sequence if $\varepsilon_j \to 0$. The limit y belongs to N_{x_j,ε_j} for every j since N_{x_j,ε_j} is closed. Hence $y \notin \cup F_j$ so the theorem will follow if we show that the sequences x_j, ε_j can be constructed. Suppose that $x_1, \varepsilon_1, \ldots, x_{j-1}, \varepsilon_{j-1}$ have already been chosen. By hypothesis the ball $N_{x_{j-1},\varepsilon_{j-1}/2}$ contains some point x_j which is not in F_j. Since F_j is closed we can choose ε_j so that $F_j \cap N_{x_j,\varepsilon_j} = \emptyset$. If $\varepsilon_j < \varepsilon_{j-1}/2$, we have $N_{x_j,\varepsilon_j} \subset N_{x_{j-1},\varepsilon_{j-1}}$ by the triangle inequality, and $\varepsilon_j < 2^{1-j}\varepsilon_1 \to 0$. This proves the theorem.

Example. We give an example with $E = \mathbf{R}$. Let f be a function from $\mathbf{R}$ to $\mathbf{R}$ which is differentiable at every point. The example $f(x) = x^2 \sin(x^{-2})$, $x \neq 0$; $f(0) = 0$; shows that $\overline{\lim}_{y\to x} |f'(y)|$ need not be finite for every x. However, we shall prove that

$$E = \{x \in \mathbf{R}; \varlimsup_{y\to x} |f'(y)| < \infty\}$$

is an open dense set (that is, the complement is closed and has no interior point). That E is open is obvious. Let I be a closed interval in $\mathbf{R}$, not reduced to a point, and set

$$F_n = \{x \in I; |f(x) - f(y)| \le n|x-y|, y \in I\} \subset \{x \in I; |f'(x)| \le n\}.$$

Since f is continuous it is clear that F_n is closed, and $\cup F_n = I$ since $f'(x)$ exists for every x. Hence F_n has an interior point for some n, so $I \cap E$ is not empty, which means that E is dense in $\mathbf{R}$.

Definition 2.1.2. A subset A of a complete metric space is said to be of the *first category* if there exist countably many closed sets $F_1, F_2, \ldots$ in E without interior points such that $A \subset \cup_1^\infty F_j$. All other sets are said to be of the *second category.*

The definition implies that any countable union of sets of the first category is of the first category, and so is any subset of a set of the first category. By Theorem 2.1.1 a set of the first category has no interior point, so the complement is dense. One is therefore justified in thinking of sets of the first category as quite small although they may of course be dense (such as the set of rational numbers $\subset \mathbf{R}$).

We shall finally recall the main facts concerning compact spaces. A topological space E is called *compact* if it is a Hausdorff space and the Borel-Lebesgue lemma is valid, that is, if for every family of open subsets $O_\alpha, \alpha \in A$, with $\cup_A O_\alpha = E$ it is possible to find a finite subfamily $O_{\alpha_1}, \ldots, O_{\alpha_n}$ with union equal to E. Here A may be any set of indices. An equivalent property is that for closed subsets F_α of E with $\cap_A F_\alpha = \emptyset$ a finite number of the sets already have an empty intersection. This follows by considering the complements. Negation gives a third equivalent and useful statement:

If F_α are closed subsets of the compact set E and if any finite number of the sets F_α have a non-empty intersection, then all have a non-empty intersection.

It is obvious that *a closed subset of a compact space is compact.* The converse is contained in the following:

Theorem 2.1.3. *(i) Every compact subset of a Hausdorff space is closed. (ii) If $f : E \to F$ is continuous, E is compact and F is Hausdorff, then $f(E)$ is compact. (iii) If in addition f is injective, then f is a homeomorphism $E \to f(E)$, that is, the inverse is also continuous. (iv) If E is compact, then every $x \in E$ has a fundamental system of compact neighborhoods.*

Proof. (i) Let K be a compact subset of the Hausdorff space E and let $x \in E \setminus K$. For every $y \in K$ we can choose disjoint open sets $O'_y \ni x$ and $O''_y \ni y$. Since K is compact, we can find $y_1, \dots, y_k$ so that $K \subset O''_{y_1} \cup \dots \cup O''_{y_k}$. Then $O'_{y_1} \cap \dots \cap O'_{y_k}$ is an open set containing x which does not intersect K. Hence $E \setminus K$ is an open set. (ii) If O_α are open subsets of F with $f(E) \subset \cup O_\alpha$, then $E \subset \cup f^{-1} O_\alpha$, so $E \subset \cup_1^k f^{-1} O_{\alpha_j}$ for suitable $\alpha_1, \dots, \alpha_k$. This implies that $f(E) \subset \cup O_{\alpha_j}$ so $f(E)$ is compact since it is clearly Hausdorff. (iii) From (ii) it follows that f maps closed subsets of E to closed subsets of $f(E)$, hence open subsets to open subsets, which means that f^{-1} is continuous. (iv) If O is an open neighborhood of x then the proof of (i) applied to $K = E \setminus O$ shows that there are disjoint open sets $O' \ni x$ and $O'' \supset K$; then $O' \subset N = E \setminus O'' \subset O$, and since N is closed, hence compact, the statement is proved.

Let A be an arbitrary set and let E_α be a compact set for every $\alpha \in A$. The infinite product

$$E = \prod_{\alpha \in A} E_\alpha$$

is defined as the set of all systems $\{x_\alpha\}_{\alpha \in A}$ with $x_\alpha \in E_\alpha$. Let p_α be the projection $E \to E_\alpha$ on the αth component. A topology in E is defined by taking as a basis for open sets the finite intersections of sets of the form $p_\alpha^{-1} O_\alpha$ where O_α is open in E_α; note that this makes the projections p_α continuous.

Theorem 2.1.4. *(Tychonov) The infinite product $E = \prod_{\alpha \in A} E_\alpha$ is compact if each E_α is compact.*

Proof. Let $\mathcal{F}$ be a family of closed subsets of E such that finitely many members of $\mathcal{F}$ never have an empty intersection. Using Zorn's lemma we can extend $\mathcal{F}$ to a maximal family $\mathcal{F}'$ having the same property. In particular, finite intersections of sets in $\mathcal{F}'$ are themselves in $\mathcal{F}'$. Since E_α is compact we can find $x_\alpha \in E_\alpha$ so that $x_\alpha \in \overline{p_\alpha F}$ for every $F \in \mathcal{F}'$, where the bar denotes closure. If U_α is a compact neighborhood of x_α, then it follows that $p_\alpha^{-1} U_\alpha \in \mathcal{F}'$ since $\mathcal{F}'$ is maximal. Hence x_α is uniquely determined. Choosing all U_α except finitely many equal to E_α, we obtain

$$\prod U_\alpha = \bigcap p_\alpha^{-1} U_\alpha \in \mathcal{F}', \quad \text{hence } \Big(\prod U_\alpha\Big) \cap F \neq \emptyset \quad \text{if } F \in \mathcal{F}'.$$

Thus an arbitrary neighborhood of $x = \{x_\alpha\}_{\alpha \in A}$ intersects F, for every $F \in \mathcal{F}'$, and since F is closed this implies that $x \in F$. The proof is complete.

2.2. Vector space topologies. A vector space V which is also a topological space is called a *topological vector space* if the vector operations

$$V \times V \ni (x, y) \mapsto x + y \in V; \quad K \times V \ni (a, x) \mapsto ax \in V \tag{2.2.1}$$

are continuous. Here and in what follows K denotes the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers with the usual metric topologies. In particular, (2.2.1) implies that for fixed y and fixed $a \in K$, $a \neq 0$, the maps

$$x \mapsto x + y \quad \text{(translation)} \qquad \text{and } x \mapsto ax \quad \text{(dilation)}$$

are homeomorphisms. Thus the open sets are invariant under translations, so the topology is completely determined by the neighborhoods of 0. Since

$$x + x_0 + y + y_0 = x + y + x_0 + y_0, \qquad (a + a_0)(x + x_0) = ax + ax_0 + a_0x + a_0x_0,$$

the continuity of the operations (2.2.1) reduces to the following conditions on the neighborhoods of 0:

$$(2.2.1)' \qquad \begin{aligned} (x, y) &\mapsto x + y && \text{is continuous at } (0, 0); \\ (a, x) &\mapsto ax && \text{is continuous at } (0,0); \\ a &\mapsto ax_0 && \text{is continuous at 0 for every } x_0; \\ x &\mapsto a_0x && \text{is continuous at 0 for every } a_0. \end{aligned}$$

Explicitly the first three conditions mean that for every neighborhood N of 0 in V there shall exist a neighborhood N_1 of 0 and $\varepsilon > 0$ such that

$$(2.2.2) \qquad \begin{gathered} N_1 + N_1 = \{x + y;\ x, y \in N_1\} \subset N; \quad ax \in N \text{ if } |a| \leq \varepsilon \text{ and } x \in N_1; \\ \text{for every } x \in V\ \exists \varepsilon_x > 0 \text{ such that } ax \in N \quad \text{if } |a| < \varepsilon_x. \end{gathered}$$

The last condition in $(2.2.1)'$ is a consequence of (2.2.2), for the second part of (2.2.2) gives $a_0x \in N$ if $2^kx \in N_1$ for some integer $k > 0$ with $2^{-k}|a_0| < \varepsilon$, and repeated use of the first part of (2.2.2) shows that there is a neighborhood N_2 of 0 such that $2^kN_2 \subset N_1$, thus $a_0x \in N$ if $x \in N_2$.

To use the second part of (2.2.2) we form

$$N_2 = \bigcup_{|a| \leq \varepsilon} aN_1 \subset N.$$

This is also a neighborhood of 0, and we have $aN_2 \subset N_2$ if $|a| \leq 1$.

Definition 2.2.1. A set M in a vector space V over K is called *balanced* if $ax \in M$ for all $x \in M$ and $a \in K$ with $|a| \leq 1$. It is called *absorbing* if for every $x \in V$ we have $ax \in M$ when $|a|$ is sufficiently small.

Every neighborhood of 0 is absorbing, and we have found that there is a fundamental system of balanced neighborhoods of 0. Conversely, for any system of balanced absorbing sets satisfying the first condition in (2.2.2) the finite intersections can be taken as a fundamental system of neighborhoods of 0 for a vector space topology in V.

So far we have not required the Hausdorff separation axiom to be valid. This axiom is equivalent to

(2.2.3) the intersection of all neighborhoods of 0 is equal to $\{0\}$.

In fact, (2.2.3) is obviously a consequence of the Hausdorff axiom. On the other hand, if (2.2.3) is valid we can for every $x \neq 0$ find a balanced neighborhood N of 0 such that $x \notin N + N$. The neighborhoods N and $\{x\} + N$ of 0 and x are then disjoint, for if $x + y = z$; $y, z \in N$; then $x = z + (-y) \in N + N$.

It follows from (2.2.2) that in any topological vector space V the intersection V_0 of all neighborhoods of 0 is a linear subspace. If O is an open set in V and if $x \in O$, $y \in V_0$, it follows that $x + y \in O$. The open sets in V are therefore unions of residue classes modulo V_0. We obtain a vector space topology in V/V_0 if we take as open sets there the maps of the open sets in V into V/V_0; conversely, the open sets in V are the inverse images of the open sets in V/V_0. The definition of V_0 and (2.2.3) imply that V/V_0 satisfies the Hausdorff axiom.

If W is any linear subspace of a topological vector space V, we can also topologize V/W by taking as open sets the images of the open sets in V. It is clear that V/W is a Hausdorff space if and only if 0 is closed there, that is, W is a *closed* linear subspace of V.

Theorem 2.2.2. *If V is a finite dimensional Hausdorff topological vector space and if $T : K^n \to V$ is a linear bijection, then T is a homeomorphism if K^n is given the product topology.*

Thus there is only one Hausdorff vector space topology possible in the finite dimensional case.

Proof. Since for some $x_1, \dots, x_n \in V$

$$Ta = \sum_1^n a_j x_j, \quad a = (a_1, \dots, a_n) \in K^n,$$

it follows from (2.2.1) that T is continuous. To prove that T^{-1} is continuous we have to show that T maps open sets to open sets. This follows if we prove that TI is a neighborhood of 0 when

$$I = \{a \in K^n; |a_j| < 1, j = 1, \dots, n\}.$$

If ∂I is the boundary of I then

$$I = \{a \in K^n; wa \notin \partial I \text{ when } w \in K \text{ and } |w| \le 1\}.$$

Thus

$$T(I) = \{x \in V; wx \notin T(\partial I) \text{ when } w \in K \text{ and } |w| \le 1\}.$$

Since $T(\partial I)$ is a compact set which does not contain 0 (Theorem 2.1.3), it follows, again by Theorem 2.1.3, that there is a neighborhood N of 0 with $N \cap T(\partial I) = \emptyset$. If we take N balanced it follows that $N \subset T(I)$.

Corollary 2.2.3. *Let W be a linear subspace of finite codimension in a topological vector space V. Then all linear forms vanishing in W are continuous if and only if W is closed.*

Proof. It follows from Theorem 1.4.2 that W is the intersection of the zero sets of linear forms vanishing on W, so W is closed if these are all continuous. On the other hand, if W is closed then V/W is a finite dimensional Hausdorff space. A linear form f on V vanishing on W is a composition

$$V \to V/W \xrightarrow{\tilde{f}} K$$

where $\tilde{f}$ is continuous by Theorem 2.2.2. Hence f is also continuous.

Let N be an open and balanced neighborhood of 0 in the topological vector space V. Then there is a unique function $p : V \to \mathbf{R}_+$ (the non-negative reals) such that

$$(2.2.4) \qquad p(ax) = |a|p(x), \quad a \in K, x \in V; \qquad N = \{x \in V; p(x) < 1\}.$$

Moreover, the function p is continuous at 0. In fact, if N is defined by p in this way and if $t > 0$, then

$$p(x) = tp(x/t) \geq t \text{ if } x/t \notin N; \qquad p(x) < t \text{ if } x/t \in N,$$

hence

$$(2.2.5) \qquad p(x) = \inf\{t; x/t \in N\}.$$

Conversely, the function defined by (2.2.5) obviously satisfies (2.2.4) if N is balanced and absorbing. If $p(x) < 1$ we have $x/t \in N$ for some $t < 1$, hence $x \in N$, and if $p(x) \geq 1$ then $x/t \notin N$ if $t < 1$. Since N is open we may conclude that $x \notin N$. Finally $p(x) < \varepsilon$ if $x \in \varepsilon N$, so p is continuous at 0.

If N_1 is another neighborhood of 0 of the same kind, and if $N_1 + N_1 \subset N$ as in (2.2.2), we have $p(x + y) < 1$ if $p_1(x) < 1$ and $p_1(y) < 1$, where p_1 is defined by (2.2.5) with N replaced by N_1. In view of the homogeneity it follows that

$$p(x + y) \leq \max(p_1(x), p_1(y)).$$

Conversely, if we give a family $\mathcal{P}$ of functions from V to $\mathbf{R}_+$ satisfying (2.2.4) such that for every $p \in \mathcal{P}$

$$p(x + y) \leq C \max(p_1(x), p_1(y)), \quad x, y \in V,$$

for some $p_1 \in \mathcal{P}$ and some constant C, then the finite intersections of the sets

$$N_{p,\varepsilon} = \{x; p(x) < \varepsilon\}, \qquad \varepsilon > 0, \quad p \in \mathcal{P},$$

are a basis of neighborhoods of 0 for a vector space topology in V. The verification is left for the reader.

Example. Let V be the space of K valued continuous functions on $[0, 1] \subset \mathbf{R}$, and set for some $r > 0$

$$\|f\|_r = \left(\int |f|^r \, dx \right)^{1/r}.$$

Then $\|f\|_r$ satisfies (2.2.4) and

$$\|f + g\|_r \leq 2^{(r+1)/r} \max(\|f\|_r, \|g\|_r).$$

Thus the sets $\{f; \|f\|_r < \varepsilon\}$ form a basis for neighborhoods of 0 in a vector space topology.

The general spaces discussed so far are of little use in analysis so we shall narrow down our discussion to more and more special classes of spaces which occur frequently and have useful properties. The first condition we impose is the existence of a fundamental system of convex neighborhoods. This is required for the application of the Hahn-Banach theorem proved in Section 1.4.

Definition 2.2.4. A topological vector space is called *locally convex* if there is a fundamental system of neighborhoods of 0 which are open, convex, balanced and absorbing.

Let N be such a neighborhood and define again a function p by means of (2.2.5). Then we have

$$p(ax) = |a|p(x), \quad a \in K, x \in V; \qquad p(x+y) \le p(x) + p(y), \quad x, y \in V. \tag{2.2.6}$$

For if $s, t > 0$ and $x/s \in N, y/t \in N$, then the convexity of N gives

$$(x+y)/(s+t) = (s/(s+t))(x/s) + (t/(s+t))(y/t) \in N,$$

thus $p(x+y) \le s+t$, which implies the second part of (2.2.6). The first part is just (2.2.4). It follows from (2.2.6) that p is continuous, for

$$|p(x+y) - p(y)| \le p(x)$$

and p is continuous at 0. Conversely, (2.2.6) implies that $\{x; p(x) < 1\}$ is convex, balanced and absorbing.

Definition 2.2.5. If V is a vector space over $K = \mathbf{R}$ or $\mathbf{C}$, then a function $p : V \to \mathbf{R}_+$ satisfying (2.2.6) is called a *semi-norm.*

In a locally convex topological vector space V the sets

$$N_p = \{x \in V; p(x) < 1\},$$

where p is a continuous semi-norm, will thus form a fundamental system of neighborhoods of 0. Conversely, assume that in the vector space V over K we are given a family of semi-norms p_i, $i \in I$. Finite intersections of sets of the form

$$N_{p_i,\varepsilon} = \{x \in V; p_i(x) < \varepsilon\}, \quad i \in I, \varepsilon > 0,$$

are then a fundamental system of neighborhoods of 0 in a locally convex topology in V. A semi-norm p in V is continuous if and only if

$$p(x) \le C \sum_1^J p_{i_j}(x), \quad x \in V, \tag{2.2.7}$$

for some constant C and a finite subset $i_1, \dots, i_J$ of I. The sufficiency of (2.2.7) is obvious. To prove the necessity we note that if p is continuous then one can find i_j and ε such that

$$p_{i_j}(x) < \varepsilon, \quad j = 1, \dots, J \implies p(x) < 1.$$

Replacing x by tx, $t>0$, we find that $tp(x)<1$ if $tp_{i_j}(x)<\varepsilon$, $j=1,\dots,J$, hence

$$p(x) \le \max p_{i_j}(x)/\varepsilon \le \varepsilon^{-1}\sum p_{i_j}(x).$$

Another system of semi-norms $\{q_k\}_{k\in K}$ may of course define the same topology; it defines a weaker topology if and only if for any k an estimate (2.2.7) is valid with $p=q_k$. The topologies are the same if this remains true when the roles of the p's and the q's are reversed.

Example. The continuous functions on $\mathbf{R}$ form a locally convex vector space V with the topology defined by the semi-norms

$$p_n(f)=\sup_{|x|\le n}|f(x)|,\quad n=1,2,\dots.$$

(This is designed so that $f_j\to f$ means that $f_j\to f$ uniformly on any bounded interval.) The topology cannot be defined by means of a single semi-norm $p(f)$. For then there would exist a constant C_n such that

$$p_n(f)\le C_np(f),\quad f\in V.$$

In particular, $|f(n)|\le C_np(f)$. Since we can choose a continuous function f with $f(n)=nC_n$ for every n, this is a contradiction when n is large.

Remark. In the example preceding Definition 2.2.4 there is no continuous semi-norm p except 0 if $r<1$. In fact, if $p(f)\le C\|f\|_r$ and we write $f=f_1+\dots+f_N$ it follows that

$$p(f)\le\sum_1^N p(f_j)\le C\sum_1^N\|f_j\|_r.$$

If we allowed f_j to have a jump, we could subdivide the interval [0,1] into N intervals and take each f_j equal to f in one of the intervals and 0 in the others, so that $\int_0^1|f_j|^r\,dx$ is equal to $\int_0^1|f|^r\,dx/N$; smoothing out the discontinuities we can certainly achieve that

$$\int_0^1|f_j|^r\,dx\le 2\|f\|_r^r/N,\quad j=1,\dots,N.$$

Hence $p(f)\le CN(2/N)^{1/r}\to 0$ as $N\to\infty$, if $r<1$, which proves the assertion.

On the other hand, $\|f\|_r$ is a semi-norm if $r\ge 1$ (Minkowski's inequality). In fact, let $\|f\|_r\le 1$, $\|g\|_r\le 1$, and let $a,b\ge 0$, $a+b=1$. Then

$$|af+bg|^r\le(a|f|+b|g|)^r\le a|f|^r+b|g|^r$$

since the function $t\mapsto t^r$ is convex when $t\ge 0$. Hence $\|af+bg\|_r\le 1$, so $\{f;\|f\|_r\le 1\}$ is convex which means that $f\mapsto\|f\|_r$ is a semi-norm.

We shall next discuss the conditions required when we want to apply Baire's theorem. Let V be a locally convex topological vector space over K such that the topology can be defined by means of a metric. Then the topology has to be Hausdorff, and there must exist a countable fundamental system of neighborhoods of 0. Thus there must exist at most countably many semi-norms $p_1, p_2, \dots$ which define the topology of V. That the topology is Hausdorff means that $p_j(x) = 0$ for every j implies that $x = 0$. Conversely, suppose that V has a topology defined by countably many semi-norms with no common zero $\neq 0$. Then the function

$$d(x) = \sum_1^\infty 2^{-n} p_n(x)/(1 + p_n(x)), \quad x \in V, \tag{2.2.8}$$

has the properties

$$\begin{aligned} d(x+y) &\le d(x) + d(y); \quad && x, y \in V; \\ d(ax) &\le d(x); && |a| \le 1, x \in V; \\ d(x) &> 0; && 0 \neq x \in V \end{aligned} \tag{2.2.9}$$

In particular, $d(x, y) = d(x - y)$ is a metric. To prove (2.2.9) we need only verify that the function $f(t) = t/(1 + t)$, $t \in \mathbf{R}_+$ has the properties

$$f(t) \le f(t + s) \le f(t) + f(s) \qquad \text{if } s, t \in \mathbf{R}_+.$$

The first inequality is clear since $f(t) = 1 - (1 + t)^{-1}$. To prove the second we note that $f(t)/t = 1/(1 + t)$ is decreasing. Hence

$$f(s)/s \ge f(s + t)/(s + t), \quad f(t)/t \ge f(s + t)/(s + t),$$

which gives $f(s) + f(t) \ge f(s + t)(s + t)/(s + t) = f(s + t)$.

The topology defined by the metric $d(x - y)$ is identical to the topology defined by the semi-norms $p_1, p_2, \dots$. In fact, we have on one hand

$$d(x) < \varepsilon 2^{-N} \implies p_n(x) < \varepsilon/(1 - \varepsilon) \quad \text{if } n \le N,$$

and on the other hand

$$p_n(x) \le \varepsilon/2 \text{ for } n \le N \implies d(x) < \varepsilon, \qquad \text{if } 2^{-N} < \varepsilon/2.$$

We sum up our conclusions as a theorem:

Theorem 2.2.6. *A locally convex topological vector space is metrizable if and only if the topology is Hausdorff and can be defined by a countable number of semi-norms. The metric can then be chosen translation invariant, that is, of the form $d(x, y) = d(x - y)$, where d satisfies (2.2.9). For given seminorms $p_1, p_2, \dots$ the metric may be defined by (2.2.8).*

Most of the spaces encountered in analysis have all the properties discussed so far:

Definition 2.2.7. A locally convex metrizable and *complete* topological vector space is called a *Fréchet space.*

Note that a sequence $x_j \in V$ is a Cauchy sequence if and only if for every n

$$p_n(x_j - x_k) \to 0 \text{ as } j, k \to \infty.$$

When using the completeness one can therefore avoid working with the metric; the important point is just to know that it exists.

If the topology can be defined by finitely many semi-norms, it can of course also be defined by a single semi-norm, for instance their sum, but an example above shows that a topology defined by countably many semi-norms may not be possible to define by a single semi-norm.

Definition 2.2.8. If the topology in a vector space V can be defined by a single semi-norm p we shall say that V is a *semi-normed space*. If V is in addition a Hausdorff space, that is, if

$$p(x) = 0 \implies x = 0 \tag{2.2.10}$$

then p is called a *norm* and V a *normed space*. Usually we write $\|x\|$ instead of $p(x)$ then. A complete normed space is called a *Banach space*.

If q is another semi-norm defining the same topology as p then there is a constant C such that

$$C^{-1}p \le q \le Cp;$$

conversely, this implies that q defines the same topology as p. Such (semi-)norms are called *equivalent*.

If V is a semi-normed space and $V_0 = \{x \in V; p(x) = 0\}$, then V_0 is a linear subspace and V/V_0 is a normed space with $\|\xi\| = p(x)$ if x is in the residue class $\xi \in V/V_0$. A familiar example of this is when V is the set of all measurable functions on $\mathbf{R}$ with $\int |f|\,dx < \infty$. Then V_0 consists of the functions which vanish almost everywhere, and V/V_0 is the space of equivalence classes of integrable functions which are equal almost everywhere. This is the space which is usually denoted by $L^1(\mathbf{R})$; it will be discussed further below.

If V is a semi-normed vector space and $W \subset V$ is a linear subspace, then V/W is semi-normed with

$$\tilde{p}(\xi) = \inf_{x\in\xi} p(x), \quad \xi \in V/W.$$

$\tilde{p}$ may not be a norm even if p is a norm; clearly $\tilde{p}$ is a norm if and only if W is closed in V.

A linear transformation $T : V_1 \to V_2$ where V_1, V_2 are locally convex topological vector spaces is continuous if and only if $\{x \in V_1; q(Tx) < 1\}$ is a neighborhood of 0 in V_1 for every continuous semi-norm q in V_2. Thus $x \mapsto q(Tx)$ shall be a continuous semi-norm in V_1, that is, (2.2.7) shall be valid with $p(x)$ replaced by $q(Tx)$ and suitable continuous semi-norms in V_1 on the right. The continuous linear transformations form a linear subspace $\mathcal{L}(V_1, V_2)$ of the vector space $L(V_1, V_2)$ of all linear transformations from V_1 to V_2.

If V_1 and V_2 are semi-normed spaces, then a linear transformation $T : V_1 \to V_2$ is continuous if and only if for some C

$$\|Tx\|_2 \le C\|x\|_1, \quad x \in V_1,$$

where $\|\cdot\|_j$ is the semi-norm in V_j. The smallest possible constant C is called the semi-norm of T,

$$\|T\| = \sup\{\|Tx\|_2; \|x\|_1 \le 1\}.$$

It is a norm if V_2 is a normed space. When both V_1 and V_2 are normed spaces one usually calls the elements in $\mathcal{L}(V_1, V_2)$ *bounded* operators (from V_1 to V_2). We leave as an easy exercise for the reader to verify that $\mathcal{L}(V_1, V_2)$ is a Banach space if V_1 is a normed space and V_2 is a Banach space.

We shall finally consider a still more special class of spaces

Definition 2.2.9. A vector space H over K is called a pre-Hilbert space over K if there is given a function $(\cdot, \cdot)$ from $H \times H$ to K with the properties

$$(x_1 + x_2, y) = (x_1, y) + (x_2, y); \quad x_1, x_2, y \in H, \tag{2.2.11}$$

$$(ax, y) = a(x, y); \quad x, y \in H, \; a \in K, \tag{2.2.12}$$

$$(x, y) = \overline{(y, x)}; \quad x, y \in H, \tag{2.2.13}$$

$$(x, x) \geq 0; \quad x \in H. \tag{2.2.14}$$

It follows from these conditions that

$$(a_1 x_1 + a_2 x_2, y) = a_1(x_1, y) + a_2(x_2, y), \quad (x, b_1 y_1 + b_2 y_2) = \bar{b}_1(x, y_1) + \bar{b}_2(x, y_2).$$

Moreover, since

$$0 \leq (ax + by, ax + by) = a\bar{a}(x, x) + a\bar{b}(x, y) + b\bar{a}(y, x) + b\bar{b}(y, y),$$

we have the Cauchy-Schwarz inequality

$$|(x, y)|^2 \leq (x, x)(y, y). \tag{2.2.15}$$

This follows if we take $a = w\sqrt{(y, y)}$, $b = \sqrt{(x, x)}$ where $w \in K$, $|w| = 1$, and w is chosen so that $w(x, y) \leq 0$. (If (x, x) or (y, y) is 0 we obtain $(x, y) = 0$ by choosing either b or a small.) By (2.2.15) we have $|(x, y)|^2 \leq (x, x)$ if $(y, y) \leq 1$, and if $(x, x) \neq 0$ then equality is attained when $y = x/\sqrt{(x, x)}$. Hence

$$\sqrt{(x, x)} = \sup\{|(x, y)|; \; (y, y) \leq 1\}. \tag{2.2.16}$$

Since $x \mapsto |(x, y)|$ is a semi-norm, it follows that $x \mapsto \sqrt{(x, x)} = \|x\|$ is a semi-norm. A pre-Hilbert space can therefore always be considered as a semi-normed space with this semi-norm. If

$$(x, x) = 0 \implies x = 0, \tag{2.2.17}$$

the semi-norm is a norm.

Definition 2.2.10. A pre-Hilbert space where (2.2.17) is valid and which is complete with respect to the topology defined by the norm $\|x\| = \sqrt{(x, x)}$ is called a *Hilbert space.*

We end this section with a diagram which recalls the different levels of generality of the spaces we have introduced, and with some examples.

$$
\begin{array}{ccccc}
\multicolumn{5}{c}{\text{Topological vector spaces}} \\
\multicolumn{5}{c}{\Downarrow} \\
\multicolumn{5}{c}{\text{Locally convex topological vector spaces}} \\
\Downarrow & & \multicolumn{3}{c}{\Downarrow} \\
 & & \multicolumn{3}{c}{\text{Semi-normed space}} \\
\Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{Normed spaces} & & \text{Pre-Hilbert spaces} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{Fréchet spaces} & \Longrightarrow & \text{Banach spaces} & \Longrightarrow & \text{Hilbert spaces}
\end{array}
$$

Here the spaces in the last line are complete.

A linear subspace W of a topological vector space is itself a topological vector space. W is locally convex, semi-normed, normed or pre-Hilbert if V is. If W is closed it also inherits completeness from V, so W is in any one of the classes in the diagram where V is. The quotient space V/W is also a topological vector space. As already observed, it is Hausdorff if and only if W is closed in V. The reader should verify as an exercise that completeness properties of V are then inherited by V/W.

We shall now give some examples.

1) If M is an arbitrary set, then the bounded functions $f : M \to K$ form a linear subspace $l^\infty(M)$ of K^M with the norm

$$\|f\|_\infty = \sup_{x\in M} |f(x)|.$$

This is a Banach space. For if $f_j \in l^\infty(M)$, $\|f_j - f_k\|_\infty \to 0$ as $j, k \to \infty$, then

$$|f_j(x) - f_k(x)| \le \|f_j - f_k\|_\infty \to 0$$

for every x so $\lim_{j\to\infty} f_j(x) = f(x)$ exists. For every $\varepsilon > 0$ we have $|f_j(x)-f_k(x)| < \varepsilon$, $x \in M$, if $j, k > N(\varepsilon)$. Letting $k \to \infty$ we obtain $|f_j(x) - f(x)| \le \varepsilon$, $x \in M$, when $j > N(\varepsilon)$. Hence $f \in l^\infty(M)$ and $\|f_j - f\|_\infty \le \varepsilon$ for $j > N(\varepsilon)$, which proves that $\|f_j - f\|_\infty \to 0$.

The set $c(M)$ of all $f \in l^\infty(M)$ such that for every $\delta > 0$ there are only finitely many $x \in M$ with $|f(x)| > \delta$ is a closed subspace and therefore also a Banach space. — When $M = \{1, 2, \dots\}$ one usually drops M from the notation.

2) When M is a topological space then the continuous functions in $l^\infty(M)$ form a closed subspace and therefore a Banach space. In fact, a uniform limit of a sequence of continuous functions is continuous.

3) If Ω is an open subset of $\mathbf{R}^n$ then the set $C(\Omega)$ of all continuous functions in Ω is a Fréchet space with the topology defined by the semi-norms

$$u \mapsto \sup_{x\in K} |u(x)|$$

where K is a compact subset of Ω. In fact, the topology is metrizable since it is enough to consider a sequence $K_1, K_2, \dots$ of compact sets in Ω such that every compact subset of Ω is contained in the union of finitely many K_j. We may for example take all closed balls $\subset \Omega$ with rational center and radius. If u_j is a Cauchy sequence it follows from 2) that u_j converges uniformly to a continuous function on every compact set in Ω. Thus there is a function $u \in C(\Omega)$ such that $u_j \to u$ in $C(\Omega)$, so $C(\Omega)$ is complete.

4) If j is a positive integer or $+\infty$ then the space $C^j(\Omega)$ of all j times continuously differentiable functions in Ω is a Fréchet space with the topology defined by the semi-norms

$$u \mapsto \sup_{x \in K} |\partial^\alpha u(x)|,$$

where K is a compact subset of Ω and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers with $|\alpha| \le j$. We have written

$$|\alpha| = \sum \alpha_k, \qquad \partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}.$$

The metrizability follows as in 3). If u_i is a Cauchy sequence it follows from 3) that for every α with $|\alpha| \le j$ there is a continuous function u^α such that $\partial^\alpha u_j \to u^\alpha$ uniformly on every compact set. But then it is well known that $u^0 \in C^j$ and that $\partial^\alpha u^0 = u^\alpha$, $|\alpha| \le j$, which proves that $u_j \to u^0$ in $C^j(\Omega)$.

5) Let M be an arbitrary set and let $1 \le r < \infty$. Then the space $l^r(M)$ of all functions $M \to K$ with

$$\|f\|_r = \Big(\sum_{x \in M} |f(x)|^r \Big)^{\frac{1}{r}} < \infty$$

is a Banach space. The proof that $\|f\|_r$ is a norm is essentially the same as in the integral case discussed in a remark after Definition 2.2.5, so it is left as an exercise for the reader. If $f_j \in l^r(M)$ is a Cauchy sequence then $|f_j(x) - f_k(x)| \le \|f_j - f_k\|_r \to 0$ as $j, k \to \infty$, so $f(x) = \lim_{j \to \infty} f_j(x)$ exists for every $x \in M$. For every $\varepsilon > 0$ we have for $j, k > N(\varepsilon)$ and an arbitrary finite subset F of M

$$\Big(\sum_{x \in F} |f_j(x) - f_k(x)|^r \Big)^{\frac{1}{r}} \le \varepsilon.$$

Letting $k \to \infty$ we conclude since F is arbitrary that

$$\Big(\sum_{x \in M} |f_j(x) - f(x)|^r \Big)^{\frac{1}{r}} \le \varepsilon, \qquad j > N(\varepsilon),$$

so $f \in l^r(M)$ and $\|f_j - f\|_r \to 0$ as $j \to \infty$. This proves the completeness.

When $p = 2$ we have a Hilbert space with the scalar product

$$(f, g) = \sum_{x \in M} f(x)\overline{g(x)}.$$

6) Let $d\mu$ be a positive measure in an open set $\Omega \subset \mathbf{R}^n$, and let $L^p(d\mu)$ be the space of functions measurable with respect to $d\mu$ such that

$$\|f\|_r = \Big(\int_\Omega |f|^r \, d\mu \Big)^{\frac{1}{r}} < \infty,$$

or more correctly the quotient by the subspace of functions with $\|f\|_r = 0$, that is, vanishing almost everywhere. Here $1 \le r < \infty$. When $r = \infty$ we introduce instead $\|f\|_\infty = \operatorname{ess\,sup} |f|$, that is, the smallest upper bound for $|f|$ valid almost everywhere with respect to $d\mu$. For the proof of completeness we must refer to a textbook on Lebesgue integration; the completeness is the main reason for introducing the Lebesgue integral; it is not true for the Riemann integral. In the examples below we shall use that $L^r(d\mu)$ is a Banach space and follow the tradition of neglecting the distinction between a measurable function and its equivalence class.

2.3. The Hahn-Banach theorem. Already Theorem 1.4.7 gives a geometrical statement of the Hahn-Banach theorem. Here we shall specialize it to locally convex vector spaces and derive several variants which are important in the applications.

Theorem 2.3.1. *Let A be a convex non-empty open set in the topological vector space V over $K = \mathbf{R}$ or $\mathbf{C}$, and let F be an affine subspace with $F \cap A = \emptyset$. Then there exists a* closed *affine hyperplane H which contains F and does not meet A either.*

Proof. We may assume that $0 \in F$. First assume that $K = \mathbf{R}$. According to Theorem 1.4.7 there exists a hyperplane $H \supset F$ with $H \cap A = \emptyset$, for an open convex set is clearly linearly open and convex. The closure $\overline{H}$ is also a linear subspace. It contains H so the codimension must be 0 or 1. Since A is open we have $\overline{H} \cap A = \emptyset$ which proves that the codimension cannot be 0. Hence $\overline{H} = H$, which proves the theorem if $K = \mathbf{R}$.

If $K = \mathbf{C}$ we can also regard V as a vector space over $\mathbf{R}$, so there is a closed real hyperplane $H \supset F$ with $H \cap A = \emptyset$. The intersection $H_1 = H \cap (iH)$ is then a vector space over $\mathbf{C}$ since it is invariant under multiplication by arbitrary complex numbers. The quotient space V/H_1 is a vector space over $\mathbf{C}$ with real dimension ≤ 2, hence complex dimension ≤ 1. It follows that H_1 is a closed complex hyperplane, and the theorem is now completely proved.

Theorem 2.3.1 is obviously of little interest if there are no non-trivial convex open sets in the space V. In the following theorem we therefore restrict attention to locally convex spaces.

Theorem 2.3.2. *Let A be a closed convex set in the locally convex topological vector space V, and let $x \notin A$. Then there exists a continuous linear form f on V such that*

$$\inf_{y \in A} |f(y) - f(x)| > 0.$$

In particular, the affine hyperplane $\{y; f(y) = f(x)\}$ does not intersect A.

Proof. Choose an open balanced convex neighborhood N of 0 such that $x + N$ does not intersect A. The set

$$N + A = \{y + z;\ y \in N, z \in A\}$$

is then open (for it is the union of the open sets $N + z$ when $z \in A$), and it is obviously convex since both N and A are convex. Furthermore, $x \notin N + A$ in view of the choice of N. Hence there exists a closed hyperplane through x which does not intersect $N + A$. By Corollary 2.2.3 there is a continuous linear form f on V such that the hyperplane is $\{y; f(y - x) = 0\}$. Thus

$$f(x) \neq f(y) + f(z) \qquad \forall y \in A, z \in N,$$

and since $f(N) \neq \{0\}$ is a balanced subset of K, the theorem follows.

The following simple but important consequence should be compared with Theorem 1.4.2.

Theorem 2.3.3. *Let W be a linear subspace of the locally convex topological vector space V. Then the closure $\overline{W}$ of W consists of all $x \in V$ such that $f(x) = 0$ for all continuous linear forms on V vanishing on W.*

Another formulation is that a closed linear subspace is the intersection of all closed hyperplanes containing it.

Proof. Every continuous linear form f vanishing on W must also vanish on the closure $\overline{W}$. On the other hand, if $x \notin \overline{W}$ it follows from Theorem 2.3.2 since $\overline{W}$ is closed and linear, hence convex, that one can find a continuous linear form f with $f(y) \neq f(x)$ for all $y \in \overline{W}$. Since $y \in \overline{W}$ implies that $ay \in \overline{W}$ for all $a \in K$, we obtain $af(y) \neq f(x)$ for every $a \in K$, hence $f(y) = 0 \neq f(x)$ for all $y \in \overline{W}$. This proves the theorem.

The last but equally important formulation is the following one.

Theorem 2.3.4. *Let V be a vector space, W a linear subspace and f a linear form defined in W such that*

$$|f(x)| \leq p(x), \qquad x \in W, \tag{2.3.1}$$

where p is a semi-norm in V. Then there exists a linear form f_1 on V which coincides with f on W and has the same bound on V,

$$|f_1(x)| \leq p(x), \qquad x \in V. \tag{2.3.2}$$

Proof. Let W_1 be the affine subspace of W which is defined by the equation $f(x) = 1$. (We may assume that f is not identically 0 since the statement is trivial then.) By (2.3.1) W_1 does not meet the convex set

$$A = \{x; p(x) < 1\}$$

which is open for the topology defined by p. Hence there exists a hyperplane

$$V_1 = \{x \in V; f_1(x) = 1\}$$

which contains W_1 and does not meet A. Thus $af_1(x) = f_1(ax) \neq 1$ if $p(ax) = |a|p(x) < 1$, which proves that $|f_1(x)| \leq p(x)$. Now W is the smallest linear space containing W_1. Since $f_1 = f$ in W_1 it follows that $f_1 = f$ in W, and the theorem is proved.

Before passing to examples of applications we shall discuss the special case of a Hilbert space where simple alternative proofs are available which also give additional information.

Theorem 2.3.5. *Let H be a Hilbert space, A a closed convex subset of H, and x a point $\notin A$. Then there exists one and only one point $y \in A$ such that*

$$\|x - y\| \leq \|x - z\|, \qquad \forall z \in A. \tag{2.3.3}$$

The continuous linear form $L(z) = (z, x - y)$, $z \in H$, does not vanish identically, and we have

$$\operatorname{Re} L(z - y) \leq 0, \qquad z \in A, \tag{2.3.4}$$

which implies that

$$|L(z) - L(x)| \geq |L(y) - L(x)| = \|x - y\|^2 > 0, \qquad z \in A.$$

Proof. Let $d = \inf\{\|x - y\|;\ y \in A\}$, and choose a sequence $y_n \in A$ so that $\|x - y_n\| \to d$. We claim that the sequence y_n converges. To prove this we note that if $z_n = x - y_n$ then

$$\|z_n - z_m\|^2 + \|z_n + z_m\|^2 = 2(\|z_n\|^2 + \|z_m\|^2),$$

which is the classical "diagonal theorem". We have $\|(z_n + z_m)/2\| \geq d$ since $(y_n + y_m)/2 \in A$. Hence

$$\|z_n - z_m\|^2 \leq 2(\|z_n\|^2 + \|z_m\|^2 - 2d^2) \to 0 \qquad \text{when } n, m \to \infty.$$

It follows that the limit $y = \lim_{n\to\infty} y_n$ exists, that $y \in A$ (since A is closed) and that $\|x - y\| = \lim_{n\to\infty} \|x - y_n\| = d$. That y is unique follows from the fact that if y' is another point in A with $\|x - y'\| = d$, we have proved that the sequence $y, y', y, y', \dots$ converges.

If $z \in A$ we have $(1 - t)y + tz \in A$ when $0 \leq t \leq 1$. Hence

$$\begin{aligned}\|x - y\|^2 &\leq \|x - (1 - t)y - tz\|^2 = \|x - y - t(z - y)\|^2 \\ &= \|x - y\|^2 - 2t \operatorname{Re}(x - y, z - y) + t^2\|z - y\|^2, \qquad 0 < t < 1.\end{aligned}$$

If we divide by t and let $t \to +0$, it follows that $\operatorname{Re}(x - y, z - y) \leq 0$, that is, $\operatorname{Re} L(z - y) \leq 0$. Since

$$L(z) - L(x) = L(z - y) + L(y - x) = L(z - y) - \|y - x\|^2,$$

the last assertion follows immediately.

Corollary 2.3.6. *Let H be a Hilbert space, G a closed subspace, and set*

$$G^\perp = \{x \in H;\ (x, y) = 0 \text{ for all } y \in G\}.$$

Then $G^\perp$ is a closed subspace and H is the direct sum of G and $G^\perp$. When $x \in H$ is written in the form $x = y + z$ with $y \in G$ and $z \in G^\perp$, then

$$\|x\|^2 = \|y\|^2 + \|z\|^2.$$

One calls $G^\perp$ the orthogonal complement of G in H.

Proof. If $x = y + z$ and $(y, z) = 0$, then

$$\|x\|^2 = (y + z, y + z) = (y, y) + (z, z) = \|y\|^2 + \|z\|^2.$$

In particular, $y = z = 0$ if $x = 0$. Since $G^\perp$ is obviously closed, it is therefore sufficient to show that every $x \in H$ can be written $x = y + z$ with $y \in G$ and $z \in G^\perp$. To do so we apply Theorem 2.3.5 with $A = G$ assuming as we may that $x \notin G$. For the closest point $y \in G$ we have

$$\operatorname{Re}(w - y, x - y) \leq 0, \qquad w \in G.$$

Since $w - y$ is an arbitrary element in G, it follows that $z = x - y \in G^\perp$, and since $x = y + z$, this gives the required decomposition.

Corollary 2.3.7. *Every continuous linear form on a Hilbert space H can in one and only one way be written in the form*

$$x \mapsto (x, y) \tag{2.3.5}$$

where $y \in H$. The norm of the linear form is $\|y\|$.

Proof. (2.2.16) shows that the norm of the form (2.3.5) is $\|y\|$ so it only remains to show that every continuous linear form L can be written in this way. This is obvious if $L \equiv 0$. If not, then

$$G = \{x \in H; L(x) = 0\}$$

is a closed linear subspace of codimension 1. Hence $G^\perp$ is one dimensional. Take $y \in G^\perp$, $y \neq 0$. Then the linear form

$$x \mapsto L(x) - (x, y)L(y)/(y, y)$$

vanishes in G and for $x = y$ so it is identically 0. Replacing y by $y\overline{L(y)}/(y, y)$ we have proved the statement.

We have now given alternative proofs of Theorems 2.3.2 and 2.3.3 in the case of a Hilbert space. There is also a simple proof of Theorem 2.3.4 when V is a Hilbert space. In fact, if $|f(x)| \le C\|x\|$, $x \in W$, we can first extend f by continuity to the closure $\overline{W}$. In view of Corollary 2.3.7 there exists then a unique element $y \in \overline{W}$ with $\|y\| \le C$ such that $f(x) = (x, y)$, $x \in \overline{W}$. The form $x \mapsto (x, y)$ is defined on H and has the required properties. We remark that it is the only extension of f which vanishes on the orthogonal complement of W. When dealing with extension of forms depending on parameters it is often essential to use this unique extension in order to control the dependence on the parameters.

We shall now give some examples of applications of the Hahn-Banach theorem. This requires knowledge of the continuous linear forms on some spaces, so we first recall some such results. Proofs may be found in textbooks on integration theory. For the notation see the end of Section 2.2.

1) For any set M and $1 \le r < \infty$ every continuous linear form L on $l^r(M)$ can be written in the form

$$L(f) = \sum_{x \in M} f(x)g(x)$$

where $g \in l^{r'}(M)$ and $1/r + 1/r' = 1$. The norm of the linear form is $\|g\|_{r'}$. Every continuous linear form L on $c(M)$ can be written in the same way with $g \in l^1(M)$, and the norm of the form is $\|g\|_1$.

2) Let $d\mu$ be a positive measure in an open set $\Omega \in \mathbf{R}^n$. Every continuous linear form L on $L^r(d\mu)$, $1 \le r < \infty$, can be written in the form

$$L(f) = \int f(x)g(x)\, d\mu(x)$$

where $g \in L^{r'}(d\mu)$. The norm of the form is $\|g\|_{r'}$.

3) Let V be the space $C_0(\Omega)$ where Ω is an open set in $\mathbf{R}^n$. This is defined as the set of continuous functions f in Ω tending to 0 at ∞ in the sense that to every

$\varepsilon > 0$ there is a compact set $K \subset \Omega$ such that $|f(x)| < \varepsilon$ in $\Omega \setminus K$. V is a Banach space with the norm $\|f\| = \sup |f|$. A continuous linear form L on V can be written

$$L(f) = \int f \, d\mu$$

where $d\mu$ is a measure with finite total mass $\int |d\mu|$, equal to the norm of the linear form. An analogous statement is valid for the space $C(K)$ of continuous functions on any compact space, with maximum norm.

We now give some examples of *approximation theorems* which can be proved using Theorem 2.3.3 in combination with various *uniqueness theorems.*

Example 2.3.8. Let $\zeta_1, \zeta_2, \ldots$ be a sequence of different complex numbers, and let $N(t)$ be the number of ζ_j with $|\zeta_j| < t$. If

$$\varlimsup_{t\to\infty} N(t)/2t > 1$$

it follows that the linear combinations of the exponentials $x \mapsto e^{ix\zeta_j}$ are dense in $C([-\pi, \pi])$, that is, the closed linear hull of these exponentials is equal to $C([-\pi, \pi])$.

Proof. We have to show that every continuous linear form vanishing for the exponentials must vanish identically. Such a linear form can be written

$$C([-\pi, \pi]) \ni f \mapsto \int f \, d\mu$$

where $d\mu$ is a measure on $[-\pi, \pi]$, so what we must prove is that if

$$\int e^{ix\zeta_j} \, d\mu(x) = 0, \qquad j = 1, 2, \ldots$$

then $d\mu = 0$. Introduce the Fourier-Laplace transform

$$F(\zeta) = \int e^{ix\zeta} \, d\mu(x), \qquad \zeta \in \mathbf{C},$$

which is an entire analytic function of exponential type:

$$|F(\zeta)| \le Ce^{\pi|\operatorname{Im}\zeta|}, \qquad \zeta \in \mathbf{C}.$$

If F is not identically 0 and $N_1(t)$ is the number of zeros of F in the disc $\{\zeta;\ |\zeta| < t\}$, counted with multiplicites, then a classical theorem of Titchmarsh states that $N_1(t)/2t$ converges when $t \to \infty$ to the distance between the extreme points in the support of $d\mu$, divided by 2π. Hence $\lim N_1(t)/2t \le 1$. Now our hypothesis that $F(\zeta_j) = 0$ implies that $N(t) \le N_1(t)$. This is a contradiction which proves that $F = 0$, hence $\mu = 0$.

Example 2.3.9. (Müntz-Szász) Let ζ_n be a sequence of different complex numbers with $\operatorname{Re}\zeta_n > -1/2$. Then the closed linear hull in $L^2(0,1)$ of the functions $x \mapsto x^{\zeta_n}$ is equal to $L^2(0,1)$ if (and only if)

$$\sum_1^\infty \operatorname{Re}(\zeta_n + \tfrac{1}{2})/(1 + |\zeta_n|^2) = \infty.$$

Proof. If we set $x = e^{-t}$, the interval $(0,1)$ is transformed to $(0,\infty)$ and

$$\int_0^1 |f(x)|^2\,dx = \int_0^\infty |f(e^{-t})|^2 e^{-t}\,dt = \int_0^\infty |F(t)|^2\,dt,$$

where $F(t) = f(e^{-t})e^{-t/2}$. If $\eta_n = \zeta_n + \frac{1}{2}$ we have $\operatorname{Re}\eta_n > 0$ and the problem is reduced to showing that the exponentials $\exp(-t\eta_n)$ are dense in $L^2(0,\infty)$. By Theorem 2.3.3 this means that we must show that a function $F \in L^2(0,\infty)$ must vanish almost everywhere if

$$\int_0^\infty F(t)e^{-t\eta_n}\,dt = 0, \qquad n = 1, 2, \dots .$$

Introduce the Fourier-Laplace transform

$$\widehat{F}(\zeta) = \int_0^\infty F(t)e^{it\zeta}\,dt,$$

which is analytic for $\operatorname{Im}\zeta > 0$ and satisfies

$$\int_{-\infty}^\infty |\widehat{F}(\xi + i\eta)|^2\,d\xi = 2\pi \int_0^\infty |F(t)|^2 e^{-2t\operatorname{Re}\eta}\,dt \le 2\pi \int_0^\infty |F(t)|^2\,dt, \qquad \operatorname{Re}\eta > 0.$$

If $\widehat{F} \not\equiv 0$ it is well known that the zeros $\lambda_1, \lambda_2, \dots$ satisfy the condition for convergence of a Blaschke product in the upper half plane, that is,

$$\sum \operatorname{Im}\lambda_j/(1 + |\lambda_j|^2) < \infty.$$

Now $i\eta_1, i\eta_2, \dots$ are zeros of $\widehat{F}$, so if $\widehat{F}$ is not identically 0 we must have

$$\sum \operatorname{Re}\eta_j/(1 + |\eta_j|^2) < \infty.$$

If this sum diverges, which is the condition in the statement, we can therefore conclude that $\widehat{F} = 0$ and that the exponentials are dense. On the other hand, if the sum converges, then the Blaschke product

$$B(\zeta) = \prod_1^\infty \frac{\zeta - i\eta_j}{\zeta + i\bar{\eta}_j} e^{i\theta_j}, \qquad \theta_j = \arg(1 + \bar{\eta}_j)/(1 - \eta_j),$$

is analytic and $|B(\zeta)| < 1$ in the upper half plane. Hence it follows by the Paley-Wiener theorem that $\zeta \mapsto (\zeta + i)^{-2}B(\zeta)$ is the Fourier-Laplace transform of a function $F \in L^2(0,\infty)$. Since $\widehat{F}(i\eta_j) = 0$ for all j it follows that the exponentials are not dense then.

From the preceding result it is easy to make a conclusion on density in $C([0,1])$ with the maximum norm when $\operatorname{Re}\zeta_n > 0$. For density in that space implies density in L^2, hence that

$$\sum (\operatorname{Re}\zeta_n + \frac{1}{2})/(1 + |\zeta_n|^2) = \infty.$$

Conversely, if this condition is satisfied and $\underline{\lim}_{n\to\infty} \operatorname{Re}\zeta_n > \frac{1}{2}$, the numbers $\zeta_n - 1$ with $\operatorname{Re}\zeta_n > \frac{1}{2}$ will satisfy the same condition. For every $f \in C^1([0,1])$ and every $\varepsilon > 0$ we can then find an approximation to f' with

$$\int_0^1 |f'(x) - \sum a_j x^{\zeta_j - 1}|^2\, dx < \varepsilon^2$$

where the sum is finite and $\operatorname{Re}\zeta_j > \frac{1}{2}$ for every term. Since

$$f(x) = f(0) + \int_0^x f'(t)\, dt,$$

we conclude that

$$\sup_{(0,1)} |f(x) - f(0) - \sum a_j x^{\zeta_j}/\zeta_j| < \varepsilon.$$

When $\underline{\lim} \operatorname{Re}\zeta_j > \frac{1}{2}$ the condition

$$\sum \operatorname{Re}\zeta_j/(1 + |\zeta_j|^2) = \infty$$

is therefore necessary and sufficient for $C([0,1])$ to be the closed linear hull of the functions 1 and $x \mapsto x^{\zeta_j}$, $j = 1, 2, \dots$.

Example 2.3.10. Let K be a compact set in $\mathbf{C}$ with connected complement, and let f be a function which is analytic in a neighborhood of K. Then one can for every $\varepsilon > 0$ find a polynomial g in the complex variable z such that $|f - g| < \varepsilon$ on K. (By a famous theorem of Mergelyan it suffices to assume that f is analytic in the interior of K, but that requires much more delicate arguments than the classical Runge theorem stated here.)

Proof. By Theorem 2.3.3 we have to show that if $d\mu$ is a measure on K with

$$\int z^n\, d\mu(z) = 0, \qquad n = 0, 1, 2, \dots,$$

then $\int f(z)\, d\mu(z) = 0$. Let $\varphi \in C^1$ be a function which is equal to 1 in a neighborhood of K and vanishes outside a compact subset of another neighborhood where f is analytic. By the Cauchy integral formula we have, $d\lambda$ denoting the Lebesgue measure in $\mathbf{C} = \mathbf{R}^2$,

$$f(z)\varphi(z) = -1/\pi \int f(\zeta)\partial\varphi(\zeta)/\partial\bar{\zeta}(z - \zeta)^{-1}\, d\lambda(\zeta).$$

Since $\varphi = 1$ on K this gives after a change of order of integration

$$\int f(z)\, d\mu(z) = -1/\pi \int f(\zeta)\partial\varphi(\zeta)/\partial\bar{\zeta}\, d\lambda(\zeta) \int (z - \zeta)^{-1} d\mu(z).$$

The assertion would therefore be proved if we show that

$$M(z) = \int (z - \zeta)^{-1}\, d\mu(\zeta)$$

vanishes in the complement of K. Now $M(z)$ is clearly analytic in the complement of K, and at infinity the Laurent series expansion is

$$M(z) = \sum_0^\infty z^{-j-1} \int \zeta^j \, d\mu(\zeta).$$

All the coefficients here are 0 by hypothesis, so $M(z) = 0$ for large $|z|$ and therefore in the entire complement of K, for it is connected by hypothesis. This proves the Runge theorem

We now pass to examples of *existence theorems* which illustrate the use of Theorem 2.3.4.

Example 2.3.11. Let $\alpha > 1$ and let $a_0, a_1, \dots$ be a sequence of complex numbers with

$$|a_n| \le C_0^{n+1} n^{\alpha n}, \qquad n = 0, 1, \dots,$$

for some constant C_0. Then there exists a function $f \in C^\infty(\mathbf{R})$ such that $f^{(n)}(0) = a_n$ for every n and for another constant C_1

$$|f^{(n)}(x)| \le C_1^{n+1} n^{\alpha n}, \qquad n = 0, 1, \dots, \quad x \in \mathbf{R}.$$

Proof. We want to represent f as a Fourier-Stieltjes transform

$$f(x) = \int e^{ix\xi} \, d\mu(\xi).$$

Since $|f^{(n)}(x)| \le \int |\xi|^n \, |d\mu(\xi)|$, we conclude that if

$$\int e^{c|\xi|^{1/\alpha}} \, |d\mu(\xi)| < \infty, \tag{2.3.6}$$

then $|f^{(n)}(x)| \le C \sup |\xi|^n e^{-c|\xi|^{1/\alpha}}$. The supremum is attained when we have $n/|\xi| = (c/\alpha)|\xi|^{(1/\alpha)-1}$, that is, $|\xi|^{1/\alpha} = \alpha n/c$. Thus

$$|f^{(n)}(x)| \le C e^{-\alpha n} (\alpha n/c)^{\alpha n}$$

which is a bound of the desired form. That $f^{(n)}(0) = a_n$ means that

$$\int \xi^n \, d\mu(\xi) = i^{-n} a_n, \qquad n = 0, 1, \dots,$$

so if $p(\xi) = \sum p_j \xi^j$ is a polynomial, we must have

$$\int p(\xi) \, d\mu(\xi) = \sum i^{-j} a_j p_j = L(p).$$

Here the last equality is a definition. Now let V be the Banach space of all continuous functions u on $\mathbf{R}$ such that $u(\xi) e^{-c|\xi|^{1/\alpha}} \to 0$, $\xi \to \infty$, and set

$$\|u\| = \sup |u(\xi)| e^{-c|\xi|^{1/\alpha}}.$$

The continuous linear forms on this space can be written

$$u \mapsto \int u(x)\,d\mu(x)$$

where $d\mu$ is a measure satisfying (2.3.6). The proof is therefore reduced to the construction of a continuous linear form L' on V which coincides with L on the subspace of polynomials. By Theorem 2.3.4 there exists such a linear form if (and only if) the estimate

$$|L(p)| \le C\|p\|$$

is valid for all polynomials p.

Now the inequality $|p(\xi)|e^{-c|\xi|^{1/\alpha}} \le \|p\|$, valid for real ξ, implies by the maximum principle applied to $\zeta \mapsto p(\zeta)e^{-c'(\zeta/\pm i)^{1/\alpha}}$ in the upper and lower half planes respectively that

$$|p(\zeta)|e^{-c' \operatorname{Re}(\zeta/\pm i)^{1/\alpha}} \le \|p\|, \qquad \zeta \in \mathbf{C}.$$

Here $c' = c/\cos(\pi/2\alpha)$. Hence

$$|p(\zeta)| \le e^{c'|\zeta|^{1/\alpha}}\|p\|, \qquad \zeta \in \mathbf{C},$$

which by Cauchy's inequalities implies that

$$|p_j| \le \|p\| \inf_{r>0} e^{c' r^{1/\alpha}} r^{-j} = \|p\| e^{j\alpha}(\alpha j/c')^{-\alpha j}.$$

It follows that

$$|L(p)| \le \|p\| \sum C_0^{j+1} j^{\alpha j} e^{j\alpha}(\alpha j/c')^{-\alpha j} = \|p\| C_0 \sum (C_0 e^\alpha c'^\alpha \alpha^{-\alpha})^j.$$

If we choose c so small that the geometric series converges, we have proved the desired estimate and so the existence of the function f.

Example 2.3.12. Let $P(\xi) = P(\xi_1, \dots, \xi_n)$ be a polynomial with complex coefficients not all zero, and let $P(D)$ be the differential operator obtained if every ξ_j is replaced by $D_j = -i\partial/\partial x_j$. For every $f \in L^2(\mathbf{R}^n)$ with respect to the measure $e^{-|x|^2}\,dx$ one can then find a solution u of the equation $P(D)u = f$ (in the sense of distribution theory) such that u is also in $L^2(\mathbf{R}^n)$ with respect to this measure.

Proof. That $P(D)u = f$ in the sense of distribution theory means, if $Q(D) = P(-D)$, that

$$\int uQ(D)v\,dx = \int fv\,dx, \qquad v \in C_0^\infty(\mathbf{R}^n).$$

In other words, the unknown linear form $L(v) = \int uv\,dx$ shall have the property

$$L(Q(D)v) = \int fv\,dx, \qquad v \in C_0^\infty(\mathbf{R}^n).$$

Moreover, L shall be a continuous linear form on L^2 with respect to the measure $e^{|x|^2}\,dx$. By the Hahn-Banach theorem (Theorem 2.3.4) the existence of such a form L is equivalent to the estimate

$$|\int fv\,dx|^2 \le C\int |Q(D)v|^2 e^{|x|^2}\,dx, \qquad v \in C_0^\infty(\mathbf{R}^n).$$

By the Cauchy-Schwarz inequality this follows if we prove that

$$(2.3.7)\qquad \int |v|^2 e^{|x|^2}\,dx \le C \int |Q(D)v|^2 e^{|x|^2}\,dx, \qquad v \in C_0^\infty(\mathbf{R}^n).$$

This will be done by means of an identity based on commutation relations.

First we assume that $n=1$. We shall integrate by parts noting that

$$\int (Dv)\overline{w}e^{x^2}\,dx = \int v\overline{\delta w}e^{x^2}\,dx; \qquad v,w \in C_0^\infty; \quad \delta w = Dw - 2ixw.$$

If $\overline{Q}$ is the polynomial obtained by taking complex conjugates of the coefficients of Q, we conclude that

$$\int |Q(D)v|^2 e^{x^2}\,dx = \int (\overline{Q}(\delta)Q(D)v)\overline{v}e^{x^2}\,dx, \qquad v \in C_0^\infty.$$

Here we want to change the order of the two operators. To do so we note that $\delta D = D\delta + 2$. Since

$$\begin{aligned}\delta D^k = (\delta D - D\delta)D^{k-1} + D(\delta D - D\delta)D^{k-2} + \cdots + D^{k-1}(\delta D - D\delta) + D^k\delta \\ = 2kD^{k-1} + D^k\delta,\end{aligned}$$

it follows that for every polynomial p we have $\delta p(D) = p(D)\delta + 2p'(D)$. More generally, for every polynomial q we have

$$q(\delta)p(D) = \sum p^{(j)}(D)q^{(j)}(\delta)2^j/j!.$$

In fact, if this is true for q and if $q_1(\delta) = \delta q(\delta)$, then

$$\begin{aligned} q_1(\delta)p(D) &= q(\delta)(p(D)\delta + 2p'(D)) \\ &= \sum p^{(j)}(D)q^{(j)}(\delta)\delta 2^j/j! + \sum p^{(j+1)}2^{j+1}q^{(j)}(\delta)/j! \\ &= \sum p^{(j)}(D)(\delta q^{(j)}(\delta) + jq^{(j-1)}(\delta))2^j/j! = \sum p^{(j)}(D)q_1^{(j)}(\delta)2^j/j!, \end{aligned}$$

so the assertion follows in general by induction. Hence we obtain

$$\int |Q(D)v|^2 e^{x^2}\,dx = \sum 2^j/j! \int |\overline{Q}^{(j)}(\delta)v|^2 e^{x^2}\,dx.$$

A similar identity is valid in the case of several variabales. Then we just introduce the operators $\delta_j w = D_j w - 2ix_j w$, which commute with each other and with all D_k for $k \ne j$; we have

$$\delta_j D_j - D_j \delta_j = 2.$$

Writing $p^{(\alpha)}(\xi) = \partial^\alpha p(\xi)$ and $\alpha! = \alpha_1! \dots \alpha_n!$ when $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, we have

$$q(\delta)p(D) = \sum p^{(\alpha)}(D)q^{(\alpha)}(\delta)2^{|\alpha|}/\alpha!,$$

for this follows from the one-dimensional case when p and q are monomials. This leads to the identity

$$\int |Q(D)v|^2 e^{|x|^2}\,dx = \sum 2^{|\alpha|}/\alpha! \int |\overline{Q}^{(\alpha)}(\delta)v|^2 e^{|x|^2}\,dx, \qquad v \in C_0^\infty(\mathbf{R}^n).$$

Since $\overline{Q}^{(\alpha)}$ is a constant $\neq 0$ for some α with $|\alpha|$ equal to the order of Q, the inequality (2.3.7) follows.

Example 2.3.13. Let V be a semi-normed space and T a linear transformation from V to V with $\|T\| \le 1$, that is, $\|Tx\| \le \|x\|$ for all $x \in V$. Assume that T has a fixed point, that is, $Tx_0 = x_0$ for some x_0, $\|x_0\| \neq 0$. Then there exists a continuous linear form f on V with $f(x_0) = \|x_0\|$, $\|f\| = 1$, and $f(Tx) = f(x)$, $x \in V$.

Before the proof we make an application. Let V be the space of all bounded sequences $x = (x_1, x_2, \dots)$ with $x_j \in \mathbf{C}$, and define $\|x\| = \sup |x_j|$. Denote by T the *shift operator* $(Tx)_j = x_{j+1}$. The sequence $(1, 1, \dots)$ is then a fixed point of T, so we obtain a continuous linear form f on V with $f(x) = c$ if $x = (c, c, \dots)$. Moreover, $\|f\| \le 1$ and $f(Tx) = f(x)$, $x \in V$. This implies that for every $x \in V$

$$|f(x)| = |f(T^n x)| \le \sup_{j>n} |x_j|, \qquad \text{hence } |f(x)| \le \varlimsup_{n\to\infty} |x_n|.$$

It follows that for every $c \in \mathbf{C}$

$$|f(x) - c| \le \varlimsup_{n\to\infty} |x_n - c|,$$

that is, $f(x)$ lies in every closed disc containing all limit points of the sequence x. In particular, $f(x) = \lim_{j\to\infty} x_j$ if the limit exists, and

$$\varliminf_{n\to\infty} x_n \le f(x) \le \varlimsup_{n\to\infty} x_n$$

if the sequence is real. One calls $f(x)$ a generalized Banach limit of the sequence $x = (x_1, x_2, \dots)$. We sum up its properties:

1) $f(x)$ is defined for every bounded sequence x and is equal to $\lim_{n\to\infty} x_n$ if the limit exists.
2) $f(x)$ depends linearly on x.
3) $f(x) = f(y)$ if x and y only differ by a power of the shift operator.

Proof of the claim in Example 2.3.13. Introduce a new semi-norm in V by

$$|||x||| = \inf \|\sum_0^\infty s_n T^n x\|$$

where the infimum is taken over all non-negative $s_0, s_1, \dots$ with sum 1 such that only finitely many are $\neq 0$. This is a semi-norm. For if

$$\|\sum s_n T^n x\| < |||x||| + \varepsilon, \qquad \|\sum t_n T^n y\| < |||y||| + \varepsilon,$$

it follows that

$$\|(\sum s_n T^n)(\sum t_m T^m)(x+y)\| < |||x||| + |||y||| + 2\varepsilon,$$

since $\sum s_n T^n$ and $\sum t_m T^m$ have norms ≤ 1. This gives the triangle inequality, and the homogeneity of $|||\cdot|||$ is evident. We have $|||Tx - x||| = 0$ for all x, for

$$\|n^{-1}(I + T + \cdots + T^{n-1})(Tx - x)\| = \|T^n x - x\|/n \leq 2\|x\|/n \to 0, \quad n \to \infty.$$

Since $Tx_0 = x_0$ we have $|||x_0||| = \|x_0\|$. By Theorem 2.3.4 it follows that there exists a linear form f on V with $f(x_0) = \|x_0\|$ and

$$|f(x)| \leq |||x|||, \qquad x \in V.$$

This implies that $|f(x)| \leq \|x\|$ and that $f(Tx) - f(x) = f(Tx - x) = 0$ for all $x \in V$.

As an application of Corollary 2.3.6 we shall now discuss the spectral theory of the operator on $L^2(0, 2\pi)$ consisting of multiplication by $e^{i\theta}$.

Example 2.3.14. In $H = L^2(0, 2\pi)$ let T be the operator defined by

$$(Tf)(\theta) = e^{i\theta} f(\theta), \qquad f \in H.$$

Let G be a closed subspace of H with $TG \subset G$. Then either

(1) there exists a measurable set $E \subset (0, 2\pi)$ such that G consists of all $f \in L^2$ vanishing almost everywhere in E, or
(2) there exists a function $f_0 \in G$ with $|f_0| = 1$ almost everywhere such that every $f \in G$ is of the form $f = f_0 \varphi$ where $\varphi \in H_+$, that is, $\varphi \in H$ and the Fourier coefficients

$$c_n = (2\pi)^{-1} \int_0^{2\pi} \varphi(\theta) e^{-in\theta}\, d\theta = 0$$

vanish when $n < 0$. This means that φ is the boundary value of the analytic function

$$z \mapsto \sum_0^\infty c_n z^n, \qquad |z| < 1.$$

Proof. Since $\|Tf\| = \|f\|$, it is clear that TG is a closed subspace of G.

1° If $TG = G$ we have $T^n G = G$ for every integer n. Hence $f \in G$ implies $uf \in G$ if

$$u(\theta) = \sum a_k e^{ik\theta}$$

is an arbitrary trigonometrical polynomial. Since these are dense in the space of continuous periodic functions, we have $uf \in G$ for all continuous periodic u if $f \in G$. If $h \in G^\perp$, then the fact that

$$\int_0^{2\pi} uf\bar{h}\, d\theta = 0,$$

for every continuous periodic function u, implies that $f\bar{h} = 0$ almost everywhere. Let E_f be the set where $f = 0$. Since h vanishes almost everywhere in the complement of E_f, it follows that every $F \in H$ which is 0 almost everywhere in E_f must be orthogonal to $G^\perp$ and so belong to G. More generally, let $f_1, f_2, \dots$ be a dense set in G. Then $h = 0$ almost everywhere in the complement of $E = \cap E_{f_j}$, so we conclude that G contains the set H_E of all $f \in H$ vanishing almost everywhere in E. H_E is a closed set, and since $f_j \in H_E$ for every j, we have $G \subset H_E$. This proves that $G = H_E$.

2° Assume that $TG \neq G$. Then we can choose some $f_0 \in G$ with $\|f_0\| = \sqrt{2\pi}$ and $f_0 \perp TG$. This implies that f_0 is orthogonal to $T^n f_0$ if $n > 0$, thus

$$(2\pi)^{-1} \int_0^{2\pi} |f_0(\theta)|^2 e^{-in\theta}\, d\theta = 0, \qquad n > 0.$$

By complex conjugation it follows that the Fourier coefficient also vanishes for $n < 0$, and for $n = 0$ it is equal to 1. Hence $|f_0(\theta)| = 1$ almost everywhere. If there exists another function g_0 with the same properties as f_0 and orthogonal to f_0, then

$$|f_0(\theta)| = |g_0(\theta)| = |af_0(\theta) + bg_0(\theta)| = 1 \text{ for almost all } \theta \text{ if } |a|^2 + |b|^2 = 1.$$

Thus $\operatorname{Re} a\bar{b} f_0(\theta)\overline{g_0(\theta)} = 0$ which implies that $f_0(\theta)\overline{g_0(\theta)} = 0$ almost everywhere. This is a contradiction which proves that

$$G = \mathbf{C}f_0 \oplus TG \qquad \text{(orthogonal sum)}$$

where $\mathbf{C}f_0$ is the space spanned by f_0. Since $(f,g) = 0$ implies $(Tf, Tg) = 0$, we conclude after n applications of our result that

$$G = \mathbf{C}f_0 \oplus \mathbf{C}Tf_0 \oplus \cdots \oplus \mathbf{C}T^n f_0 \oplus T^{n+1}G.$$

The intersection $G_0 = \cap_{n>0} T^n G$ is a closed linear space with $TG_0 = G_0$. According to 1° it must therefore consist of all $f \in H$ vanishing on a certain measurable set E. Since all such f must be orthogonal to f_0 and $|f_0| = 1$ almost everywhere, we conclude that the complement of E is of measure 0 and that $G_0 = \{0\}$.

For an arbitrary $f \in G$ we can now write in a unique way

$$f = af_0 + a_1 Tf_0 + \cdots + a_n T^n f_0 + R_n$$

where $R_n \in T^{n+1}G$ and

$$\|f\|^2 = 2\pi(|a_0|^2 + \cdots + |a_n|^2) + \|R_n\|^2.$$

It follows that the partial sums of $\sum a_j T^j f_0$ form a Cauchy sequence, hence that R_n has a limit. This limit belongs to G_0 so it must be 0. Thus

$$f = \sum_0^\infty a_j T^j f_0,$$

which proves the statement.

We shall finally use the Hahn-Banach theorem to study direct sum decompositions.

Definition 2.3.15. A topological vector space V is said to be the direct topological sum of two subspaces V_1 and V_2 if $V = V_1 \oplus V_2$ and the projection P_1 on V_1 along V_2 (and therefore the projection P_2 on V_2 along V_1) is continuous. One calls V_2 a topological supplement of V_1.

We recall that $P_1 + P_2 = I$, that V_1 is the kernel of P_2 and that V_2 is the kernel of P_1. If V is a Hausdorff space which is the direct topological sum of V_1 and V_2, it follows therefore that V_1 and V_2 are closed. On the other hand, repeating the proof of Corollary 1.4.3 with Theorem 2.3.3 substituted for Theorem 1.4.2′ we obtain

Theorem 2.3.16. *Let V_1 be a finite dimensional subspace of the locally convex Hausdorff topological vector space V. Then there exists a continuous projection $V \to V_1$, thus a topological direct sum decomposition $V = V_1 \oplus V_2$. In particular, V_1 is closed.*

The corresponding theorem where V_1 has finite codimension is completely elementary and does not require local convexity:

Theorem 2.3.17. *Let V_1 be a closed subspace of finite codimension of the topological vector space V. Then there exists a topological supplement V_2 of V_1; in fact, any algebraic supplement is a topological one.*

Proof. Let V_2 be a space spanned by elements $x_1, \dots, x_n$ whose equivalence classes mod V_1 form a basis for V/V_1. Then V_2 is an algebraic supplement. The projection $P_2 : V \mapsto V_2$ vanishes on V_1 so it can be factored

$$V \to V/V_1 \to V_2.$$

Here V/V_1 is a finite dimensional Hausdorff topological vector space, since V_1 is closed, so the second map and therefore the composed map P_2 is continuous. This proves the statement.

2.4. Applications of Baire's theorem. We shall now prove some results which often make it possible to derive quantitative information from a qualitative one. For example, if it is known that a certain equation has a solution for arbitrary data one may conclude that the solution necessarily depends continuously on the data.

Theorem 2.4.1. *(Banach) Let T be a continuous injective linear map from a Fréchet space F_1 to another Fréchet space F_2. Then either* $\operatorname{Im} T$ *is of the first category or else* $\operatorname{Im} T = F_2$ *and T is a homeomorphism. (By Baire's theorem F_2 is not of the first category.)*

Proof. If U is a convex symmetric neighborhood of 0 in F_1 and $nU = \{nx;\ x \in U\}$, then

$$\operatorname{Im} T = \bigcup_1^\infty T(nU) \subset \bigcup_1^\infty \overline{T(nU)}.$$

If $\operatorname{Im} T$ is not of the first category it follows that $\overline{T(nU)}$ has an interior point for some n, and since these sets are homothetic it follows that $\overline{T(U)} = n^{-1}\overline{T(nU)}$ also has an interior point. We can therefore choose a point $y \in F_2$ and a convex symmetric neighborhood V of 0 so that $V + \{y\} \subset \overline{T(U)}$. Since $\overline{T(U)}$ is symmetric with respect to 0, it follows that $V + \{-y\} \subset \overline{T(U)}$, and since $\overline{T(U)}$ is convex we have

$$V = \tfrac{1}{2}(V + \{y\}) + \tfrac{1}{2}(V + \{-y\}) \subset \overline{T(U)}.$$

Now choose a fundamental system $U_1 \supset U_2 \supset \dots$ of convex symmetric neighborhoods of 0 in F_1 such that $2U_k \subset U_{k-1}$. For every k we can find a convex symmetric neighborhood V_k of 0 in F_2 such that $V_k \subset \overline{T(U_k)}$, and we can make sure that they form a fundamental system of neighborhoods of 0 in F_2. To prove that $\operatorname{Im} T = F_2$ it suffices to show that every $y \in V_1$ belongs to $\operatorname{Im} T$. Now we can choose $x_1 \in U_1$ so that

$$y_1 = y - Tx_1 \in V_2$$

and then $x_2 \in U_2$ so that

$$y_2 = y_1 - Tx_2 \in V_3.$$

Continuing in this way we get a sequence $x_n \in U_n$ such that

$$y_n = y_{n-1} - Tx_n \in V_{n+1}.$$

The sequence $X_n = x_1 + \dots + x_n$ is convergent in F_1, for

$$x_{n+1} + \dots + x_{n+m} \in U_{n+1} + \dots + U_{n+m} \subset 2^{-1}U_n + \dots + 2^{-m}U_n \subset U_n,$$

since $2^{-1} + \dots + 2^{-m} < 1$ and U_n is convex and symmetric. Let X be the limit of the Cauchy sequence X_n. Since addition of the preceding equations gives

$$y_n = y - TX_n$$

and $y_n \to 0$ when $n \to \infty$, we conclude using the continuity of T that $y = TX$. Thus $\operatorname{Im} T = F_2$. Moreover, we have found that if $y \in V_1$ then the unique solution x of the equation $Tx = y$ belongs to $2\overline{U_1}$. Since this may be taken as an arbitrarily small neighborhood of 0, we have also proved the continuity of the inverse of T.

Corollary 2.4.2. *If T is a continuous linear map $F_1 \to F_2$, where F_1, F_2 are Fréchet spaces, then* $\operatorname{Im} T$ *is either of the first category or else equal to F_2.*

Proof. The quotient space $F_1/\operatorname{Ker} T$ is also a Fréchet space, and T induces a continuous map $T' : F_1/\operatorname{Ker} T \to F_2$ with the same range as T. Since T' is injective, the statement now follows from Theorem 2.4.1.

Example 2.4.3. Let $P(D)$, $D = -i\partial/\partial x$, be a partial differential operator with constant coefficients and order m. Assume that for some open non empty set $\Omega \subset \mathbf{R}^n$ every solution $u \in C^m(\Omega)$ of the equation $P(D)u = 0$ is in fact in $C^{m+1}(\Omega)$. Then we have $\operatorname{Im} \zeta \to \infty$ if $\zeta \to \infty$ on the surface in $\mathbf{C}^n$ defined by the equation $P(\zeta) = 0$. (Conversely, if P has this property then every distribution solution of the equation $P(D)u = 0$ is in C^∞, but we shall not prove that here.)

Proof. Let F_1, F_2 be the set of all $u \in C^{m+1}(\Omega)$ resp. $C^m(\Omega)$ satisfying the equation $P(D)u = 0$. As closed subspaces of $C^{m+1}(\Omega)$ and $C^m(\Omega)$ these are Fréchet spaces. Our hypothesis is that the inclusion map $F_1 \to F_2$ is surjective. By Theorem 2.4.1 it follows that the inverse map is continuous. For every compact set $K \subset \Omega$ one can therefore find another compact set $K' \subset \Omega$ and a constant C such that

$$\sum_{|\alpha| \le m+1} \sup_K |D^\alpha u| \le C \sum_{|\alpha| \le m} \sup_{K'} |D^\alpha u|, \qquad u \in F_1 = F_2.$$

In particular, we can choose $u(x) = e^{i\langle x,\zeta\rangle}$ if $P(\zeta) = 0$, $\zeta \in \mathbf{C}^n$, for $P(D)u = P(\zeta)u = 0$ then. If A is the maximum distance between points in K' and points in K it follows that

$$\sum_{|\alpha|\le m+1} |\zeta^\alpha| \le Ce^{A|\operatorname{Im}\zeta|} \sum_{|\alpha|\le m} |\zeta^\alpha|.$$

Since the quotient of the sum on the left by that on the right converges to ∞ as $(1+|\zeta|)$ when $\zeta \to \infty$, we conclude that $\operatorname{Im}\zeta \to \infty$ then.

As an example we obtain that the Schrödinger equation $(D_1^2 - D_2)u = 0$ has solutions $u \in C^2 \setminus C^3$.

We shall now give a variant of Banach's theorem which is often more convenient in applications.

Definition 2.4.4. If T is a map from a subset $\mathcal{D}_T$ of a Fréchet space F_1 to a Fréchet space F_2, then T is said to be closed if $x_n \in \mathcal{D}_T$, $x_n \to x$, $Tx_n \to y$ implies that $x \in \mathcal{D}_T$ and $Tx = y$. Equivalently, this means that the *graph* of T

$$G = \{(x, Tx),\ x \in \mathcal{D}_T\}$$

is a closed subset of the direct sum $F_1 \oplus F_2$.

If T is linear, the graph is thus a Fréchet space too. Note that the two projections

$$G \ni (x, Tx) \mapsto x, \qquad G \ni (x, Tx) \mapsto Tx$$

are continuous and that the first is injective. If we apply Theorem 2.4.1 to the first map and Corollary 2.4.2 to the second map, we obtain:

Theorem 2.4.5. *(Closed graph theorem) If T is a closed linear map from F_1 to F_2 where F_1 and F_2 are Fréchet spaces, then either the domain $\mathcal{D}_T$ is of the first category or else it is equal to F_1 and T is continuous. The range* $\operatorname{Im} T$ *is either of the first category or equal to F_2.*

Example 2.4.6. Let $P(D) = P_m(D) + P_{m-1}(D) + \dots$ be a partial differential operator where P_k is a homogeneous polynomial in $D = -i\partial/\partial x$ of degree k. Assume that for some real $\xi \neq 0$ we have

$$P_m(\xi) = 0, \qquad 0 \neq P_m'(\xi) = \operatorname{grad} P_m(\xi) \in \mathbf{R}^n.$$

Let $L = \{tP_m'(\xi),\ t \in \mathbf{R}\}$ (a bicharacteristic line). Then one can find $u \in C^m(\mathbf{R}^n)$ such that $P(D)u \in C^\infty(\mathbf{R}^n)$ and $u \in C^\infty(\complement L)$ whereas u is not in C^{m+1} in any open set which intersects L. Thus L is the set of singularities of the function u.

Since the equation $P(D)v = f$ has a solution $v \in C^\infty$ for every $f \in C^\infty$ (see Example 2.6.38 below), it is easy to modify the function u in the example so that $P(D)u = 0$. If for example $P(D)$ is the wave operator, the example then shows that there is a solution of the wave equation with singularities precisely on any given light ray.

Proof. Let F be the set of all $u \in C^m(\mathbf{R}^n)$ such that $u \in C^\infty(\complement L)$ and $P(D)u \in C^\infty(\mathbf{R}^n)$. This is a Fréchet space with the topology defined by the semi-norms

$$u \mapsto \sup_K |D^\alpha u|, \text{ where } \begin{cases} |\alpha| \le m,\ K \text{ is compact in } \mathbf{R}^n, \text{ or} \\ \alpha \text{ is arbitrary and } K \text{ is compact} \subset \complement L, \end{cases}$$

in addition to the semi-norms

$$u \mapsto \sup_K |D^\alpha P(D)u|, \quad K \text{ compact in } \mathbf{R}^n \text{ and } \alpha \text{ arbitrary.}$$

The verification of completeness is left as an exercise for the reader. Let V be an open set in $\mathbf{R}^n$ containing some point $x_0 \in L$. We shall prove then that

$$M_V = F \cap C^{m+1}(V)$$

is of the first category in F. Taking the union of the sets M_V when V varies over all balls with radius $r > 0$ and center $tP'_m(\xi)$ where r and t are rational, we conclude that the set of all $u \in F$ belonging to $C^{m+1}(V)$ for some V intersecting L is also of the first category, so the assertion follows.

Assuming that M_V is not of the first category we apply the closed graph theorem to the restriction map

$$T : F \to C^{m+1}(V)$$

with domain equal to $F \cap C^{m+1}(V)$. That T is closed is obvious. It follows that T must be continuous. Hence we can find a constant C, an integer N, and compact sets $K \subset \mathbf{R}^n$, $K' \subset \complement L$, such that

$$\begin{aligned}\sum_{|\alpha| \le m+1} |D^\alpha u(x_0)| \le C\Big(\sum_{|\alpha|\le m} \sup_K |D^\alpha u| + \sum_{|\alpha|\le N} \sup_K |D^\alpha P(D)u| \\ + \sum_{|\alpha|\le N} \sup_{K'} |D^\alpha u|\Big), \qquad u \in F.\end{aligned} \tag{2.4.1}$$

To show that this is impossible we must construct good approximate solutions of the equation $P(D)u = 0$.

To do so we set

$$u_t(x) = e^{it\langle x,\xi\rangle} v_t(x)$$

where t shall $\to \infty$. Then

$$P(D)u_t(x) = e^{it\langle x,\xi\rangle} P(D + t\xi) v_t(x).$$

Since $P_m(\xi) = 0$ we obtain using Taylor's formula

$$P(D + t\xi) = t^{m-1}(\langle P'_m(\xi), D\rangle + a + R_t(D))$$

where a is a constant and R_t is a polynomial in $1/t$ without constant term. Thus we should choose v_t so that

$$(\langle P'_m(\xi), D\rangle + a + R_t(D))v_t$$

is very small. This can be done by successive approximation. We take v_0 as a solution of the first order differential equation

$$(\langle P'_m(\xi), D\rangle + a)v_0 = 0, \qquad v_0(x_0) = 1.$$

Since this is a differential equation which only involves differentiations in the direction of L, we can choose $v_0 \in C^\infty$ so that the support, which is a cylinder Γ with the generator in the direction L, does not meet K'. Then we choose successively $v_{1,t}$, $v_{2,t}$, ... so that

$$(\langle P'_m(\xi), D\rangle + a)v_{k,t} = -R_t(D)v_{k-1,t}.$$

Then $v_{k,t}$ will be a finite sum of terms each of which contains a factor t^{-j} with $j \geq k$. We can choose all $v_{k,t}$ with support in Γ. Now set

$$V_{k,t} = v_0 + v_{1,t} + \cdots + v_{k,t}.$$

Then

$$(\langle P'_m(\xi), D\rangle + a + R_t(D))V_{k,t} = R_t(D)v_{k,t},$$

and all semi-norms in C^∞ of the right-hand side are $O(t^{-k-1})$. With $u(x) = e^{it\langle x,\xi\rangle}V_{k,t}$ the left-hand side of (2.4.1) is $\geq ct^{m+1}$ for some $c > 0$. The first sum on the right is $O(t^m)$, the second is $O(t^{N+m-1-k-1})$ and the third is 0. If $m+1 > N+m-k-2$, that is, $k > N-3$, we conclude that (2.4.1) is not valid and have proved the statement.

Before passing to the next consequence of Baire's theorem we must introduce some notation.

Definition 2.4.7. A subset M of a locally convex topological vector space V is said to be *bounded* if for every neighborhood U of 0 in V there exists some $\varepsilon > 0$ such that $\varepsilon M \subset U$.

Since U is absorbing it is clear that every finite set is bounded. We can set $U = \{x;\ p(x) < 1\}$ where p is a continuous semi-norm. That $\varepsilon M \subset U$ means that $p(\varepsilon x) < 1$ for all $x \in M$, that is, $p(x) < 1/\varepsilon$, $x \in M$. In a locally convex topological vector space an equivalent definition is therefore: M is bounded if every continuous semi-norm is bounded on M. If V is a normed space, a set $M \subset V$ is therefore bounded if and only if the norms of its elements are bounded.

Now let V_1, V_2 be two locally convex topological vector spaces and let Φ be a subset of the space $\mathcal{L}(V_1, V_2)$ of continuous linear maps from V_1 to V_2. We shall say that Φ is *equi-continuous* if for every neighborhood U_2 of 0 in V_2 there is a neighborhood U_1 of 0 in V_1 such that $TU_1 \subset U_2$ for every $T \in \Phi$. If U_1, U_2 are defined by the semi-norms p_1, p_2, this means that $p_2(Tx) \leq p_1(x)$ for $x \in V_1$ and $T \in \Phi$. If V_1, V_2 are normed spaces, the definition of equi-continuity thus means that

$$\|Tx\|_2 \leq C\|x\|_1, \qquad x \in V_1,\ T \in \Phi,$$

for some constant C, that is, $\|T\| \leq C$ for every $T \in \Phi$.

Theorem 2.4.8. *(Banach-Steinhaus; the principle of uniform boundedness) Let F be a Fréchet space and V a locally convex topological vector space. If Φ is a subset of $\mathcal{L}(F, V)$ such that $\{Tx;\ T \in \Phi\}$ is a bounded subset of V for every fixed $x \in F$, then Φ is equi-continuous. On the other hand, if Φ is* not *equi-continuous, then the set of all $x \in F$ such that $\{Tx;\ T \in \Phi\}$ is bounded forms a set of the first category.*

Proof. Let U be a convex, closed, symmetric neighborhood of 0 in V and set

$$A = \{x \in F;\ Tx \in U \text{ for every } T \in \Phi\}.$$

Then A is convex and symmetric, and A is closed since every $T \in \Phi$ is continuous. We now distinguish two different cases.

1° A has an interior point for every choice of U. As in the proof of Theorem 2.4.1 we conclude that 0 is an interior point of A. Since $TA \subset U$ for all $T \in \Phi$, the family Φ is thus equi-continuous.

2° If for some U the set A has no interior point, then $\cup_1^\infty(nA)$ is of the first category and contains all x such that $\{Tx;\ T \in \Phi\}$ is bounded. Indeed, for such x we have $\varepsilon x \in A$ for some $\varepsilon > 0$, hence $x \in nA$ if $n\varepsilon > 1$. This completes the proof.

Corollary 2.4.9. *Let F be a Fréchet space and V a locally convex Hausdorff topological vector space. If $T_1, T_2, \dots \in \mathcal{L}(F, V)$ and $T_j x \to Tx$ for every $x \in F$, it follows that $T \in \mathcal{L}(F, V)$.*

Proof. Passing to the limit in the equation $T_j(ax + by) = aT_jx + bT_jy$; $x, y \in F$, $a, b \in K$; we obtain $T(ax + by) = aTx + bTy$ since V is Hausdorff. By Theorem 2.4.8 the sequence T_j is equi-continuous which implies that T is continuous.

The following statement is closely related to Theorem 2.4.8 but it is quite elementary.

Theorem 2.4.10. *Let $T_j \in \mathcal{L}(F, V)$, $j = 1, 2, \dots,$ be equi-continuous, F and V locally convex topological vector spaces. If $T_0 \in \mathcal{L}(F, V)$ then*

$$F_1 = \{x \in F;\ T_j x \to T_0 x, \text{ as } j \to \infty\}$$

is a closed *linear subspace of F.*

Proof. The linearity is obvious. Assume that x is in the closure of F_1. Let U be any convex symmetric neighborhood of 0 in V and choose a neighborhood U' of 0 in F such that $T_jU' \subset U$ when $j \geq 0$. For some $y \in F_1$ we have $y - x \in U'$. Then

$$T_jx - T_0x = T_jy - T_0y + T_j(x - y) - T_0(x - y) \in 3U$$

if j is sufficiently large. Hence $x \in F_1$.

Example 2.4.11. There exists a continuous function with period 2π such that the partial sums of its Fourier series are not bounded at 0.

Proof. The space C of continuous functions with period 2π is a Banach space with the maximum norm $\|f\| = \max |f(x)|$. The Fourier coefficients of f are

$$c_k = (2\pi)^{-1} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$$

and the partial sums are

$$s_n(f, x) = \sum_{-n}^{n} c_k e^{ikx}.$$

In particular, we have

$$s_n(f, 0) = \sum_{-n}^{n} c_k = \int_{-\pi}^{\pi} f(x) D_n(x)\, dx$$

where the *Dirichlet kernel* D_n is given by

$$D_n(x) = (2\pi)^{-1}\sum_{-n}^{n} e^{ikx} = (2\pi)^{-1}\sin((n+\tfrac{1}{2})x)/\sin(\tfrac{1}{2}x).$$

The norm of the linear form $f \mapsto s_n(f,0)$ on C is equal to $\int_{-\pi}^{\pi}|D_n(x)|\,dx$. By splitting the interval of integration where $(n+\frac{1}{2})x = k\pi$ with k an integer, we obtain since $\sin(\frac{1}{2}x) \le \frac{1}{2}x$

$$\int_{-\pi}^{\pi}|D_n(x)|\,dx \ge 4\pi^{-2}\sum_{1}^{n} 1/k = (4\log n)/\pi^2 + O(1).$$

Thus the norm of the linear form $f \mapsto s_n(f,0)$ tends to ∞ with n, so Theorem 2.4.8 shows that the set of all $f \in C$ such that $s_n(f,0)$ is bounded must be of the first category.

Example 2.4.12. Let M_n be a sequence of positive numbers such that $M_n/\log n \to 0$ as $n \to \infty$. Then there exists a continuous function f with period 2π such that $s_n(f,x)/M_n$ is unbounded for every rational number x.

Proof. The proof in the preceding example shows that the set of all $f \in C$ such that $s_n(f,0)/M_n$ is bounded is of the first category. The same conclusion is valid for $s_n(f,x)/M_n$ for every fixed x in view of the translation invariance. If we let x vary over any countable set A, the union of the sets of $f \in C$ for which $s_n(f,x)/M_n$ is bounded for some $x \in A$ and all n will be of the first category, hence $\neq C$. This proves the assertion.

Let us note that $s_n(f,x) = o(\log n)$ uniformly in x for $f \in C$ since s_n is bounded for f in the dense subset of C^1 functions and the lower bound for $\int|D_n|\,dx$ in Example 2.4.11 is also valid apart from a constant factor as an upper bound. Furthermore, Carleson has proved that $s_n(f,x) \to f(x)$ for almost every x if $f \in C$, so we could not extend the conclusion in Example 2.4.11 to any set of positive measure or any M_n of faster increase.

Example 2.4.13. Let $a_{jk};\ j,k = 1,2,\dots;$ be a matrix of complex numbers which is row finite, that is, $a_{jk} = 0$ for large k when j is fixed. For every sequence $s_1, s_2, \dots$ of complex numbers we set

$$(As)_j = \sum a_{jk}s_k, \qquad j = 1,2,\dots$$

and say that the sequence s_k is summable with the method (A) if $(As)_j$ has a limit when $j \to \infty$. Every convergent sequence is then summable to the usual limit with the method (A) if and only if for some constant M

$$\sum_k |a_{jk}| \le M, \qquad j = 1,2,\dots \tag{2.4.2}$$

$$\lim_{j\to\infty} a_{jk} = 0 \text{ for fixed } k; \qquad \lim_{j\to\infty}\sum_k a_{jk} = 1. \tag{2.4.3}$$

Proof. Assume that every convergent sequence is summable to the usual limit. Then (2.4.3) follows if we consider the sequence $s_j = 1,\ j = k;\ s_j = 0,\ j \neq k$, or the sequence with $s_j = 1$ for every j. To prove the necessity of (2.4.2) we consider the Banach space B of all sequences with $s_n \to 0$ and $\|s\| = \sup |s_n|$. The map

$$B \ni s \mapsto (As)_j \in \mathbf{C}$$

is then for fixed j a linear form with the norm $\sum_k |a_{jk}|$, and since $(As)_j \to 0$ when $j \to \infty$ for every fixed $s \in B$, we conclude in view of Theorem 2.4.8 that the norms must be uniformly bounded. This means that (2.4.2) must be valid. — To prove the sufficiency of (2.4.2), (2.4.3) we note that (2.4.2) gives

$$\overline{\lim}\, |(As)_j| \le \sup |(As)_j| \le M\|s\|, \quad s \in B.$$

For every N we can write $s = s' + s''$ where $s'_j = s_j$ for $j \le N$ and $s''_j = s_j$ for $j > N$. From (2.4.3) it follows then that $(As')_j \to 0$, as $j \to \infty$, hence

$$\overline{\lim}\, |(As)_j| = \overline{\lim}\, |(As'')_j| \le M\|s''\| = M \sup_{j>N} |s_j|.$$

Thus $(As)_j \to 0$ if $s \in B$. The second part of (2.4.3) allows us to conclude that $(As)_j \to c$ if $s_j \to c$, for $s_j - c \to 0$ then. This proves the statement.

As an example we can take $a_{jk} = 1/j$, $k \le j$, $a_{jk} = 0$, $k > j$. Then $(As)_j$ is the arithmetic mean of the first j elements in the sequence. Note that $(As)_j$ may then converge even if s_j does not converge. An example is $s_j = (-1)^j$ for which $(As)_j \to 0$.

We shall now discuss a result on the continuity of bilinear maps which is closely related to Theorem 2.4.8. Let E, F, G be topological vector spaces and $B: E \times F \to G$ a *bilinear map*. This means that $B(x, y)$, $x \in E$, $y \in F$, is linear in x for fixed y and linear in y for fixed x. If B is continuous at the origin, then we can for every neighborhood U_G of 0 in G find neighborhoods U_E, U_F of 0 in E and in F such that $B(x, y) \in U_G$ when $x \in U_E$ and $y \in U_F$. Then it follows that $B(x, y)$ is continuous everywhere in $E \times F$. In fact,

$$B(x + x_0, y + y_0) = B(x, y) + B(x, y_0) + B(x_0, y) + B(x_0, y_0).$$

We can choose $\varepsilon > 0$ so that $\varepsilon x_0 \in U_E$ and $\varepsilon y_0 \in U_F$. Then we have

$$B(x, y_0) \in U_G \text{ if } x/\varepsilon \in U_E, \qquad B(x_0, y) \in U_G \text{ if } y/\varepsilon \in U_F.$$

Hence

$$B(x + x_0, y + y_0) - B(x_0, y_0) \in U_G + U_G + U_G \text{ if } x \in U_E \cap \varepsilon U_E,\ y \in U_F \cap \varepsilon U_F.$$

If E, F, G are locally convex, then the continuity means that for every continuous semi-norm p_G in G one can find continuous semi-norms p_E and p_F in E and in F so that

$$p_G(B(x, y)) \le p_E(x) p_F(y); \qquad x \in E,\ y \in F.$$

In fact, if the neighborhoods U_G, U_E, U_F are the sets where p_G, p_E, p_F are < 1, the preceding inequality follows in view of the homogeneity.

A bilinear form is called separately continuous if the linear forms obtained by giving one of the arguments a fixed value are continuous.

Theorem 2.4.14. *Let E be a locally convex metrizable vector space, F a Fréchet space and G a locally convex topological vector space. Every* separately *continuous bilinear map from $E \times F$ to G is then continuous.*

Proof. Let U be a closed, convex, symmetric neighborhood of 0 in G. Choose a fundamental system $V_1 \supset V_2 \supset \dots$ of neighborhoods of 0 in E, and set

$$A_j = \{y \in F;\ B(x,y) \in U \text{ for every } x \in V_j\}.$$

It is then clear that A_j is convex and symmetric. Since U is closed and B is continuous with respect to y for fixed x, the sets A_j are closed. Since B is continuous with respect to x for fixed y, we have $\cup_1^\infty A_j = F$. Hence some A_j has an interior point, and as in the proof of Theorem 2.4.1 we conclude that 0 is an interior point. Now $B(x,y) \in U$ if $x \in V_j$ and $y \in A_j$, so B is continuous.

The proof did not require G to be Hausdorff, and it also gives

Theorem 2.4.15. *Let E be a locally convex metrizable vector space, F a Fréchet space and G a locally convex topological vector space. Let B be a bilinear map from $E \times F$ to G such that $B(x,y)$ is a continuous function of y for fixed x. If B is not continuous, then the set of all $y \in F$ such that $B(x,y)$ is continuous with respect to x is of the first category.*

We shall use this improvement in the following example.

Example 2.4.16. For some $f \in C^\infty(\mathbf{R}^3)$ the differential equation

$$Pu = (D_1 + iD_2 + 2i(x_1 + ix_2)D_3)u = f$$

does not have a distribution solution u in any open set in $\mathbf{R}^3$.

Proof. If Ω is open in $\mathbf{R}^3$ and the equation has a solution in Ω, the definitions of differentiation and multiplication in the space of distributions mean that

$$\langle u, -Pv\rangle = \langle f, v\rangle, \qquad v \in C_0^\infty(\Omega).$$

If K is a compact subset of Ω the definition of distributions shows that

$$|\langle f, v\rangle| \le C \sum_{|\alpha|\le m} \sup |D^\alpha Pv|, \qquad v \in C_0^\infty(K), \tag{2.4.4}$$

where C and m depend on f. Let E be the space $C_0^\infty(K)$ with the topology defined by the semi-norms occurring in the right-hand side of (2.4.4). Only countably many occur. Let F be the Fréchet space $C^\infty(\mathbf{R}^3)$. The bilinear form

$$B:\ E \times F \ni (v, f) \mapsto \int fv\, dx$$

is continuous with respect to v in view of (2.4.4) if the equation $Pu = f$ has a solution $u \in \mathcal{D}'(\Omega)$. On the other hand, the map is automatically continuous with respect to f for fixed v. If we prove that B is not continuous it follows therefore from Theorem 2.4.15 that the set of all $f \in F$ such that the equation $Pu = f$ has a distribution solution in Ω is of the first category.

Assume that B is continuous. Then we have for some C and m

(2.4.5)
$$|\int fv\,dx| \le C \sum_{|\alpha|\le m} \sup|D^\alpha Pv| \sum_{|\alpha|\le m} \sup|D^\alpha f|, \quad v\in C_0^\infty(K),\ f\in C^\infty(\mathbf{R}^3).$$

Our aim is to prove that this estimate is not valid if K has interior points. First we assume that 0 is an interior point of K. We want to choose v so that Pv is nearly equal to 0 while v is not. To do so we note first that the equation $Pv = 0$ is satisfied by $v = x_1 + ix_2$ and by $v = x_1^2 + x_2^2 + ix_3$, hence also by all polynomials in these two solutions. In particular,

$$w(x) = -x_1^2 - x_2^2 - ix_3 + (x_1^2 + x_2^2 + ix_3)^2$$

satisfies the equation $Pw = 0$, and $w(x) = -ix_3 - |x|^2 + O(|x|^3)$, as $x \to 0$. In a neighborhood of 0 it follows that $\operatorname{Re} w(x) \le -|x|^2/2$.

Now choose $\chi \in C_0^\infty(K)$ so that $\chi = 1$ in a neighborhood of 0 and $\operatorname{Re} w(x) \le -|x|^2/2$ in $\operatorname{supp}\chi$. Writing

$$v_t(x) = \chi(x)e^{tw(x)}$$

we have $Pv_t = (P\chi)e^{tw}$, hence $|D^\alpha Pv_t| = O(e^{-ct}t^{|\alpha|})$ when $t\to\infty$, where c is a positive constant. Set

$$f_t(x) = e^{itx_3}t^3h(tx)$$

with $h \in C_0^\infty$ and $\int h\,dx = 1$. With $v = v_t$ and $f = f_t$ the limit of the left-hand side in (2.4.5) when $t\to\infty$ is equal to 1, for

$$\int f_t v_t\,dx = \int h(x)\chi(x/t)e^{tw(x/t)+ix_3}\,dx \to \int h(x)\,dx = 1.$$

On the other hand, the right-hand side of (2.4.5) can be estimated by $Ct^{3+2m}e^{-ct} \to 0$ as $t\to\infty$, which contradicts (2.4.5).

Since the origin has a countable fundamental system of neighborhoods we have now proved that the set of all $f \in C^\infty(\mathbf{R}^3)$ such that the equation $Pu = f$ can be solved in some neighborhood of 0 is of the first category. We shall now prove that this is also true for any other point. This will show that the set of all $f \in C^\infty(\mathbf{R}^3)$ such that $Pu = f$ can be solved in some open set (and therefore in some neighborhood of a rational point) is also of the first category.

To study the equation $Pu = f$ near $y\in\mathbf{R}^3$ we replace x by $x+y$ and obtain the equation

$$(D_1 + iD_2 + (2i(y_1+iy_2) + 2i(x_1+ix_2))D_3)u = f.$$

This we write in the form

$$((D_1 - 2y_2D_3) + i(D_2 + 2y_1D_3) + 2i(x_1+ix_2)D_3)u = f.$$

If we introduce as new coordinates $x_1' = x_1$, $x_2' = x_2$, $x_3' = x_3 + 2y_2x_1 - 2y_1x_2$, the equation assumes its original form and our assertion is proved.

2.5. Fredholm theory. In this section we shall study the index of linear operators between topological vector spaces. For the sake of simplicity we study only Banach spaces at first and make some comments on more general spaces afterwards.

If T is a continuous linear map from a Banach space B_1 to another B_2, it is clear that the kernel

$$\operatorname{Ker} T = \{x \in B_1; Tx = 0\}$$

is a *closed* linear subspace of B_1. On the other hand, the range $\operatorname{Im} T$ need not be closed. For example, it is not closed if $B_1 = B_2 =$ the space of continuous functions of $t \in [0,1]$ and T is integration from 0 to t or multiplication by t. However, we can easily prove

Theorem 2.5.1. *If* $T \in \mathcal{L}(B_1, B_2)$ *and* $\operatorname{codim} \operatorname{Im} T < \infty$*, then* $\operatorname{Im} T$ *is closed.*

Proof. We may assume that T is injective, for otherwise we can consider instead the map from $B_1/\operatorname{Ker} T$ to B_2 induced by T. If n is the codimension of $\operatorname{Im} T$, we can choose a linear map

$$S : K^n \to B_2$$

such that $\operatorname{Im} S$ is an algebraic supplement of $\operatorname{Im} T$. Then the map

$$T_1 : B_1 \oplus K^n \ni (x, y) \mapsto Tx + Sy \in B_2$$

is a continuous bijection so by Banach's theorem it is a homeomorphism. It follows that $\operatorname{Im} T = T_1(B_1 \oplus \{0\})$ is closed.

Passage to the graph of T shows that the preceding result is also true if T is just closed and not necessarily continuous.

Definition 2.5.2. An operator $T \in \mathcal{L}(B_1, B_2)$ is called a Fredholm operator if $\dim \operatorname{Ker} T$ and $\dim \operatorname{Coker} T$ are both finite, thus

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T$$

is a finite integer.

In Section 1.3 we proved the stability of the index under perturbations of finite rank. For continuous linear operators we shall now prove that the index is stable under perturbations which are small in various other respects.

Theorem 2.5.3. *Let* $T \in \mathcal{L}(B_1, B_2)$ *be a Fredholm operator. If* $S \in \mathcal{L}(B_1, B_2)$ *and* $\|S\|$ *is sufficiently small, it follows that* $T + S$ *is a Fredholm operator with*

$$\operatorname{ind}(T+S) = \operatorname{ind} T, \qquad \dim \operatorname{Ker}(T+S) \le \dim \operatorname{Ker} T. \tag{2.5.1}$$

More generally, if $T \in \mathcal{L}(B_1, B_2)$, $\dim \operatorname{Ker} T < \infty$ *and* $\operatorname{Im} T$ *is closed, then* $T + S$ *has the same properties and (2.5.1) holds if* $\|S\|$ *is sufficiently small.*

The extension stated here will be convenient in the proofs; the obvious lack of symmetry in kernel and cokernel will be removed in Section 2.6. We first prove a simple special case:

Lemma 2.5.4. *Let I be the identity operator in the Banach space B and let S be an operator in $\mathcal{L}(B,B)$ with $\|S\| < 1$. Then it follows that $I - S$ has an inverse in $\mathcal{L}(B,B)$, hence that* $\operatorname{ind}(I-S) = 0$.

Proof. The Neumann series

$$R = \sum_0^\infty S^k$$

converges in $\mathcal{L}(B,B)$ since

$$\sum_0^\infty \|S^k\| \le \sum_0^\infty \|S\|^k \le 1/(1 - \|S\|).$$

Since $R(I-S) = (I-S)R = I$, the lemma is proved.

Proof of Theorem 2.5.3. a) First assume that T is bijective. Then T^{-1} is continuous by Banach's theorem and

$$T + S = T(I + T^{-1}S).$$

When $\|T^{-1}\|\|S\| < 1$ this is a product of two invertible operators, so $T + S$ is invertible and the index is 0.

b) If T is a general Fredholm operator then $B_1 = V_1 \oplus \operatorname{Ker} T$ and $B_2 = V_2 \oplus \operatorname{Im} T$, by Theorems 2.3.16 and 2.3.17, where V_2 is finite dimensional, V_1 is closed and the sums are direct topological. If T' and S' denote the maps $V_1 \to B_2/V_2$ induced by T and S, then T' is bijective and $\|S'\| \le \|S\|$. From a) it follows then that $T' + S'$ is bijective when $\|S\| < \varepsilon$. Hence

$$\dim \operatorname{Ker}(T+S) \le \dim \operatorname{Ker} T, \quad \dim \operatorname{Coker}(T+S) \le \dim V_2 = \dim \operatorname{Coker} T.$$

Since $T' + S'$ is the composition of the injection $V_1 \to B_1$, $T + S$, and the quotient map $B_2 \to B_2/V_2$, Theorem 1.3.2 gives

$$0 = \operatorname{ind}(T' + S') = \dim V_2 + \operatorname{ind}(T+S) - \dim \operatorname{Ker} T = \operatorname{ind}(T+S) - \operatorname{ind} T,$$

which proves (2.5.1).

c) Assume now that $\dim \operatorname{Ker} T < \infty$ and that $\operatorname{Im} T$ is closed, $\dim \operatorname{Coker} T = \infty$. Choose V_1 as in case b) above. Then T is a bijection of V_1 on $\operatorname{Im} T$, so it follows from Banach's theorem that

$$\|x\| \le C\|Tx\|, \quad x \in V_1.$$

Hence

$$\|x\| \le C\|(T+S)x\| + C\|S\|\|x\|, \quad x \in V_1,$$

and if $S \in \Omega = \{S \in \mathcal{L}(B_1, B_2); \|S\| < 1/2C\}$ it follows that

$$\|x\| \le 2C\|(T+S)x\|, \quad x \in V_1.$$

Hence $T+S$ is then injective with closed range on V_1, which implies that $(T+S)B_1$ is closed and that $\dim \operatorname{Ker}(T+S) \le \dim \operatorname{Ker} T$. It remains to prove that

$$\Sigma = \{S \in \Omega; \dim \operatorname{Coker}(T+S) < \infty\}$$

is empty. Σ is the union of the sets $\Sigma_n = \{S \in \Omega; \operatorname{ind}(T+S) = n\}$, which are open by part b) of the proof. If Σ_n is not empty, then $\Omega \setminus \Sigma_n$ is not open, for $0 \in \Omega \setminus \Sigma_n$ and Ω is connected. Hence Σ_n must have a boundary point $S_0 \in \Omega$. Since Σ_k are open and disjoint, it is clear that $S_0 \notin \cup \Sigma_k$. Thus $\dim \operatorname{Coker}(T + S_0) = \infty$, but in every neighborhood of S_0 one can find S such that $\dim \operatorname{Coker}(T+S) = \dim \operatorname{Ker}(T+S) - n \leq \dim \operatorname{Ker} T - n$. Choose a finite dimensional subspace W of B_2 with $\dim W > \dim \operatorname{Ker} T - n$ which intersects $\operatorname{Im}(T + S_0)$ only at the origin. Then the composed map

$$V_1 \to B_1 \xrightarrow{T+S} B_2 \to B_2/W$$

is injective with closed range when $S = S_0$, hence also when S is close to S_0. Thus the index is ≤ 0, so we obtain by Theorem 1.3.2

$$0 \geq \operatorname{ind}(T+S) - \dim \operatorname{Ker} T + \dim W = n - \dim \operatorname{Ker} T + \dim W > 0$$

which is a contradiction completing the proof.

We shall now extend Theorem 2.5.3 to certain perturbations which, like operators of finite rank, may have a large norm:

Definition 2.5.5. A linear operator T from a Banach space B_1 to a Banach space B_2 is called *compact* (or *completely continuous*) if the closure of the image of the unit ball of B_1 is compact in B_2. Equivalently: If $x_n \in B_1$ and $\|x_n\| \leq 1$, $n = 1, 2, \dots$, then the sequence Tx_n has a convergent subsequence.

Compactness obviously implies continuity, which motivates the term complete continuity.

Example 2.5.6. Let B_1 and B_2 be the Banach spaces $C^k(I)$ and $C^j(I)$ where I is a compact interval on $\mathbf{R}$. Then we have a continuous inclusion map $B_1 \to B_2$ if $j \leq k$, and it is compact by Ascoli's theorem if $j < k$.

Proposition 2.5.7. *The compact linear operators from a Banach space B_1 to another B_2 form a closed subspace $\mathcal{L}_c(B_1, B_2)$ of the Banach space $\mathcal{L}(B_1, B_2)$. If $T_1 \in \mathcal{L}(B_1, B_2)$ and $T_2 \in \mathcal{L}(B_2, B_3)$ where B_3 is another Banach space, and if either T_1 or T_2 is compact, then $T_2 T_1 \in \mathcal{L}_c(B_1, B_3)$. Every $T \in \mathcal{L}(B_1, B_2)$ of finite rank is compact. In particular, $\mathcal{L}_c(B_1, B_2) = \mathcal{L}(B_1, B_2)$ if B_1 or B_2 is finite dimensional.*

Proof. Let $T_n \in \mathcal{L}_c(B_1, B_2)$ and $T \in \mathcal{L}(B_1, B_2)$, $\|T - T_n\| \to 0$. To prove that T is compact we consider a sequence $x_k \in B_1$ with $\|x_k\|_1 \leq 1$. Then there exists an increasing sequence $k(1, j), j = 1, 2, \dots$ of indices such that $T_1 x_{k(1,j)}$ converges in B_2 when $j \to \infty$. This sequence in turn has a subsequence $k(2, j)$ such that $T_2 x_{k(2,j)}$ has a limit when $j \to \infty$, and so on. For the diagonal sequence $x'_j = x_{k(j,j)}$, which apart from a finite number of elements is a subsequence of all the sequences constructed, we know then that $T_n x'_j$ converges in B_2 for every n as $j \to \infty$. Now we have for every n

$$\overline{\lim_{i,j \to \infty}} \|Tx'_i - Tx'_j\| \leq \overline{\lim_{i,j \to \infty}} \|T_n x'_i - T_n x'_j\| + 2\|T - T_n\| \leq 2\|T - T_n\|,$$

so Tx'_j is a Cauchy sequence. Hence T is compact. The next statement in the proposition is trivial and it implies the last if we observe that the identity operator in a finite dimensional space is compact since the unit ball is compact. Now a continuous operator T of finite rank can be factored as a product of continuous operators

$$B_1 \to B_1/\operatorname{Ker} T \to B_2$$

where the space in the middle is finite dimensional. Hence T is compact.

The following classical theorem of F. Riesz shows that the last statements in Proposition 2.5.7 cannot be improved, and it will also be of crucial importance later on.

Theorem 2.5.8. *If the identity map in the Banach space B is compact, then B has finite dimension.*

The theorem is an immediate consequence of the following

Lemma 2.5.9. *If $B_1 \subset B_2 \subset \dots$ is a strictly increasing sequence of finite dimensional subspaces of the Banach space B, then one can find $x_j \in B_j$ so that*

$$\|x_j - x\| \geq \|x_j\| = 1, \qquad \text{when } x \in B_{j-1}.$$

In particular, $\|x_k - x_j\| \geq 1$ when $k < j$, so the sequence x_j has no convergent subsequence.

Proof. Choose $y_j \in B_j \setminus B_{j-1}$. Since $\|y - y_j\|$ is a positive continuous function of $y \in B_{j-1}$ which $\to \infty$ at ∞, it has a minimum point $z_j \in B_{j-1}$. Thus

$$0 < \|z_j - y_j\| \leq \|x + z_j - y_j\|, \qquad \text{if } x \in B_{j-1}.$$

It follows that $x_j = (y_j - z_j)/\|y_j - z_j\|$ has the desired properties.

Remark. If B is a Hilbert space H there is an even simpler proof of Theorem 2.5.8. For any sequence $x_1, x_2, \dots$ of linearly independent vectors gives rise to an *orthonormal* sequence $e_1, e_2, \dots$ by the Gram-Schmidt orthogonalisation procedure:

$$e_1 = x_1/\|x_1\|, \dots, e_k = y_k/\|y_k\|; \; y_k = x_k - \sum_{j=1}^{k-1} e_j(x_k, e_j).$$

It is an easy exercise to verify inductively that $(e_j, e_k) = 0$ when $j \neq k$ and that $(e_j, e_j) = 1$. Since $\|e_j - e_k\|^2 = 2$ when $j \neq k$ it is not possible to find a convergent subsequence, so the identity map is not compact unless H is finite dimensional.

If B_1 and B_2 are Hilbert spaces it is known that $\mathcal{L}_c(B_1, B_2)$ is equal to the closure of the space of continuous operators of finite rank, but a theorem of Per Enflo states that this is false in general. However, we shall nevertheless find that the index is stable for all compact perturbations.

Theorem 2.5.10. *Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator and $S \in \mathcal{L}_c(B_1, B_2)$. Then the operator $T + S$ is also a Fredholm operator, and*

$$\operatorname{ind}(T + S) = \operatorname{ind} T. \tag{2.5.2}$$

More generally, if $T \in \mathcal{L}(B_1, B_2)$, $\dim \operatorname{Ker} T < \infty$ and $\operatorname{Im} T$ is closed, then $T + S$ also has these properties and (2.5.2) holds.

This important theorem is due to I. Fredholm and F. Riesz when $B_1 = B_2$ and $T = I$. The general version has been given by many authors, notably by F. V. Atkinson. We shall prepare the proof with two lemmas.

Lemma 2.5.11. *Let $T \in \mathcal{L}(B_1, B_2)$ and assume that $\dim \operatorname{Ker} T < \infty$ and that $\operatorname{Im} T$ is closed. If $x_j \in B_1$ is bounded and Tx_j is convergent, then the sequence x_j has a convergent subsequence.*

Proof. Write B_1 as a topological direct sum $V_1 \oplus \operatorname{Ker} T$, and let P be the corresponding projection on V_1. Then $Tx_j = TPx_j$. Since the restriction of T to V_1 is a homeomorphism on $\operatorname{Im} T$, by Banach's theorem, it follows that Px_j converges. But $(I - P)x_j$ is a bounded sequence in the finite dimensional vector space $\operatorname{Ker} T$, so it has a convergent subsequence. This proves the lemma.

Lemma 2.5.12. *Let $S \in \mathcal{L}_c(B_1, B_2)$, $T \in \mathcal{L}(B_1, B_2)$, $\dim \operatorname{Ker} T < \infty$, and assume that $\operatorname{Im} T$ is closed. If $x_j \in B_1$ is a bounded sequence and $(T + S)x_j \to y$ in B_2, then the sequence x_j has a limit point $x \in B_1$ with $(T + S)x = y$.*

Proof. For a subsequence such that Sx_j converges, it is clear that Tx_j converges, so x_j converges by Lemma 2.5.11 for some still sparser sequence.

Proof of Theorem 2.5.10. Every sequence $x_j \in \operatorname{Ker}(T + S)$ with $\|x_j\| \le 1$ has a convergent subsequence by Lemma 2.5.12. Hence $\operatorname{Ker}(T + S)$ is finite dimensional by Theorem 2.5.8. If $B_1 = V_1 \oplus \operatorname{Ker}(T + S)$ then $(T + S)B_1 = (T + S)V_1$, and there is a constant C such that

$$\|x\| \le C\|(T + S)x\|, \quad x \in V_1,$$

For otherwise we could find $x_j \in V_1$ with $\|x_j\| = 1$ and $\|(T+S)x_j\| \to 0$; by Lemma 2.5.12 a subsequence has a limit $x \in V_1$ with $\|x\| = 1$ and $(T + S)x = 0$, which contradicts the choice of V_1. Hence $\operatorname{Im}(T+S)$ is closed. If we apply Theorem 2.5.3 to the operators $T + \lambda S$ with $0 \le \lambda \le 1$ it follows that the index is independent of λ, which completes the proof.

Example 2.5.13. Let a and b be continuous functions in $[0, 1]$, and consider the boundary problem

$$u'' + au' + bu = f, \qquad u(0) = u(1) = 0,$$

where $u \in C^2([0, 1])$ and $f \in C([0, 1])$. Then the number of linearly independent conditions for solvability is equal to the number of linearly independent solutions of the problem when $f = 0$, thus at most equal to 1.

Proof. Let B_1 be the set of all $u \in C^2([0, 1])$ with $u(0) = u(1) = 0$ and the norm inherited from $C^2([0, 1])$, and let $B_2 = C([0, 1])$ with maximum norm. Then B_1 and B_2 are Banach spaces. The operator

$$T : \ B_1 \ni u \mapsto u'' \in B_2$$

is clearly bijective so it has index 0. By Example 2.5.6 the operator

$$K : B_1 \ni u \mapsto au' + bu \in B_2$$

is compact. Hence $T + K$ is a Fredholm operator with index 0, which proves the assertion. If a is real valued and $b < 0$, then a solution of the homogeneous problem must be 0. In fact, since $u' = 0$ and $u'' \le 0$ (resp $u'' \ge 0$) at a maximum (minimum) point in (0,1), the equation $u'' + au' + bu = 0$ shows that $u \le 0$ (resp.

$u \geq 0$) there. Since $u(0) = u(1) = 0$ it follows that $u = 0$. (A slight elaboration of this maximum principle argument shows that it suffices to assume $b \leq 0$.) Hence the inhomogeneous problem is always solvable then.

Example 2.5.14. Let H be the subspace of $L^2(0, 2\pi)$ consisting of functions u with

$$\int_0^{2\pi} u(\theta) e^{in\theta}\, d\theta = 0, \qquad n > 0.$$

This means that H consists of boundary values of analytic functions in the unit disc. Let P be the orthogonal projection on H. If f is a continuous periodic function we define $T_f : H \to H$ by

$$T_f u = P(fu), \qquad u \in H.$$

Thus the equation $T_f u = v$ means that $fu = v + w$ where u and v are boundary values of analytic functions in the unit disc and w is the boundary value of an analytic function outside the unit disc vanishing at ∞. We claim that if f has no zeros, then T_f is a Fredholm operator and the index of T_f is the *winding number* of $1/f$, that is, the argument variation of $f(\theta)$ when θ varies from 0 to 2π, divided by -2π.

Proof. A standard proof (Wiener-Hopf) is based on factorization of f with one factor analytic inside and another analytic outside the unit disc. This also gives the values of the dimensions of kernel and cokernel, but we shall give a different proof which is also applicable if H consists of functions with values in $\mathbf{C}^N$ and the values of f are invertible $N \times N$ matrices.

First we prove that for all continuous f and g

$$T_{fg} - T_f T_g \quad \text{is a compact operator.}$$

To do so we first assume that f and g are trigonometric polynomials, that is, finite sums

$$f = \sum a_n e^{in\theta}, \qquad g = \sum b_n e^{in\theta}.$$

Then $T_g u = gu$, $T_f T_g u = fgu$ and $T_{fg} u = fgu$ if sufficiently many Fourier coefficients of u are equal to 0. Hence $T_{fg} - T_f T_g$ is of finite rank and therefore compact. Now $\|T_f\| \leq \sup |f|$, so if we approximate f and g uniformly by trigonometric polynomials it follows in view of Proposition 2.5.7 that $T_{fg} - T_f T_g$ is always compact. If f is never 0 we can take $g = 1/f$ and conclude that $T_f T_g - I$ and $T_g T_f - I$ are compact. Hence the cokernel and kernel of T_f are finite dimensional. We have also

$$\operatorname{ind} T_f + \operatorname{ind} T_g = \operatorname{ind} T_{fg}$$

for arbitrary continuous f and g without zeros. Moreover, when $f(\theta) = e^{in\theta}$ the index is $-n$. (This is essentially Example 1.3.4.) Therefore it only remains to show that the index of T_f is 0 when the argument variation of f is 0. Then we can write $f = \exp F$ where F is also a continuous periodic function. If we set $f_t = \exp(tF)$, Theorem 2.5.3 shows that $\operatorname{ind} T_{f_t}$ is independent of t so $\operatorname{ind} T_f = \operatorname{ind} T_{f_1} = \operatorname{ind} T_{f_0} = \operatorname{ind} I = 0$.

When the two Banach spaces B_1 and B_2 coincide one can give more specific information:

Theorem 2.5.15. *(F. Riesz) Let S be a compact operator $B \to B$. Then*

$$N = \{x \in B; (I+S)^k x = 0 \text{ for some } k > 0\} \tag{2.5.3}$$

is a finite dimensional subspace of B, thus $N = \operatorname{Ker}(I+S)^j$ when $j \geq J$. Moreover,

$$F = \bigcap_1^\infty (I+S)^k B = (I+S)^j B, \quad j \geq J, \tag{2.5.4}$$

is a closed subspace of B, and B is the direct topological sum of N and F. The restriction of $I+S$ to N is a nilpotent *operator $N \to N$, and the restriction of $I+S$ to F is a bijection $F \to F$.*

Proof. The spaces $N_k = \operatorname{Ker}(I+S)^k$ are finite dimensional, $N_1 \subset N_2 \subset \dots$, and $(I+S)N_k \subset N_{k-1}$. If they were all different we could by Lemma 2.5.9 choose $x_k \in N_k$ with $\|x_k\| = 1$ so that $\|x_k - x\| \geq 1$, $x \in N_{k-1}$. If $j < k$ we can take $x = (I+S)x_k - Sx_j$ and obtain $\|Sx_k - Sx_j\| \geq 1$, so the sequence Sx_k has no convergent subsequence contrary to the hypotheses. Hence $N_k = N_{k+1}$ for some k, which inductively implies $N_k = N_j$ if $j > k$, thus $N = N_k$. Since $\operatorname{ind}(I+S)^j = 0$ for every j, it follows that $(I+S)^j B = (I+S)^k B$, $j > k$, for the codimensions are the same and one space contains the other. Thus $F = (I+S)^k B$ so it follows from Theorem 2.5.10 that F is closed and $\operatorname{codim} F = \dim N$. Since S maps F into F and $I+S$ is a surjective map $F \to F$, it follows from Theorem 2.5.10 that $I+S$ is a bijection $F \to F$. Hence $N \cap F = \{0\}$, which completes the proof.

The operator $I + \lambda S$ is a bijection of F for λ close to 1 in view of Theorem 2.5.3, and $I + \lambda S$ is a bijection of N for every $\lambda \neq 1$ since $(I+\lambda S)x = 0$, $x \in N$, gives

$$0 = (\lambda I + \lambda S)^k x = (\lambda - 1)^k x.$$

It follows that $I + \lambda S$ is a bijection of B for $\lambda \neq 1$ sufficiently close to 1. Replacing S by a constant times S, we conclude more generally that $I + \lambda S$ is a bijection of B except for a closed set of isolated values of λ, which leads to the following:

Theorem 2.5.16. *Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator and $S \in \mathcal{L}_c(B_1, B_2)$. Set*

$$k = \min_{\lambda \in K} \dim \operatorname{Ker}(T + \lambda S), \qquad w = \min_{\lambda \in K} \dim \operatorname{Coker}(T + \lambda S).$$

Then we have $\dim \operatorname{Ker}(T + \lambda S) = k$ *and* $\dim \operatorname{Coker}(T + \lambda S) = w$ *except when λ belongs to a closed set with only isolated points.*

Proof. We may assume that $\dim \operatorname{Ker} T = k$, for we can otherwise replace T by $T + \lambda S$ for a suitable λ to make this come true. Since the index of $T + \lambda S$ is independent of λ it suffices to prove that for all λ outside a closed set with only isolated points we have $\dim \operatorname{Ker}(T + \lambda S) \leq k$. We can follow the proof of Theorem 2.5.3.

a) If T is a bijection then $T + \lambda S = T(I + \lambda T^{-1}S)$ is bijective except for a closed set of isolated values of λ.

b) In the general case $B_1 = V_1 \oplus \operatorname{Ker} T$ and $B_2 = V_2 \oplus \operatorname{Im} T$ where V_2 is finite dimensional, V_1 is closed and the sums are direct topological. If T' and S' denote the maps $V_1 \to B_2/V_2$ induced by T and S, then T' is bijective and S' is compact, so $T' + \lambda S'$ is bijective except when λ is in a closed set with only isolated points. Then we have $\dim \operatorname{Ker}(T + \lambda S) \leq \dim \operatorname{Ker} T = k$, which completes the proof.

Theorem 2.5.15 also gives precise information concerning the *resolvent* $(S - zI)^{-1}$ when $z \neq 0$:

Theorem 2.5.17. *Let S be a compact operator $B \to B$. For arbitrary $\lambda \neq 0$ such that* $\operatorname{Ker}(S - \lambda I) = \{0\}$ *we have, with $A_j \in \mathcal{L}(B, B)$ and convergence in operator norm for small z, a power series expansion*

$$(S - (\lambda + z)I)^{-1} = \sum_0^\infty A_j z^j. \tag{2.5.5}$$

At any other $\lambda \neq 0$ we have for small $z \neq 0$ a Laurent series expansion

$$(S - (\lambda + z)I)^{-1} = \sum_{-n}^\infty A_j z^j \tag{2.5.6}$$

where $-A_{-1}$ is the projection on $N_\lambda = \bigcup_1^\infty \operatorname{Ker}(S - \lambda I)^k$ along $F_\lambda = \bigcap_1^\infty (S - \lambda I)^k B$.

Proof. In the first case the resolvent is given by the Neumann series (see Lemma 2.5.4)

$$(S - (\lambda + z)I)^{-1} = (S - \lambda I)^{-1}(I - z(S - \lambda I)^{-1})^{-1} = \sum_0^\infty z^j (S - \lambda I)^{-j-1},$$

provided that $|z|\|(S - \lambda I)^{-1}\| < 1$. In the second case we can by Theorem 2.5.15 applied to $-S/\lambda$ use this expansion on the space F_λ. On the space N_λ the resolvent is

$$(S - (\lambda + z)I)^{-1} = -z^{-1}(I - (S - \lambda I)/z)^{-1} = -z^{-1} \sum_0^\infty z^{-j}(S - \lambda I)^j,$$

where the sum is finite since high powers of $(S - \lambda I)$ vanish on N_λ. The first term is $-z^{-1}$ times the identity operator in N_λ. This proves the theorem.

Remark. We derived Theorem 2.5.17 from Theorem 2.5.15 but one can also argue in the opposite direction. In fact, multiplication of (2.5.6) to the left or right by $S - (\lambda + z)I$ gives after identification of the coefficients

$$\begin{aligned} (S - \lambda I)A_{-n} &= A_{-n}(S - \lambda I) = 0, \\ (S - \lambda I)A_j - A_{j-1} &= A_j(S - \lambda I) - A_{j-1} = \begin{cases} 0, & \text{if } j > -n, j \neq 0 \\ I, & \text{if } j = 0. \end{cases} \end{aligned}$$

It follows that all A_j commute with S and that $A_{-1}(S - \lambda I)^n = 0$. Hence

$$A_0(S - \lambda I)^{n+1} = (S - \lambda I)^n,$$

which shows that $(S - \lambda I)^{n+1}x = 0$ implies $(S - \lambda I)^n x = 0$, so

$$(S - \lambda I)^n x = 0 \quad \text{if } (S - \lambda I)^j x = 0 \text{ for some } j.$$

Let N be the kernel of $(S - \lambda I)^n$. Then we know that the range of A_{-1} is contained in N, and on the other hand

$$x + A_{-1}x = (S - \lambda I)A_0 x = (S - \lambda I)^2 A_1 x = \cdots = (S - \lambda I)^n A_n x = 0, \quad x \in N.$$

Hence $-A_{-1}$ is a projection on N. If $A_{-1}x = 0$ then $A_{-1}(S-\lambda I)x = (S-\lambda I)A_{-1}x = 0$, so $S-\lambda I$ maps the kernel of A_{-1} into itself. Since $\operatorname{Ker}(S-\lambda I) \subset N$ it follows that $S-\lambda I$ is a bijection on the kernel F of the projection $-A_{-1}$. Thus we have recovered Theorem 2.5.15 from (2.5.6).

Theorem 2.5.17 is closely related to the classical Jordan canonical form for matrices, which is obtained by further decomposition of the space N. The details can be found in the appendix. Instead we shall now discuss Fredholm operators depending on parameters. The results have important applications in topology but they are outside the scope of these notes. Let T be a Hausdorff topological space and let P_t, $t \in T$ be a Fredholm operator in $\mathcal{L}(B_1, B_2)$ which depends continuously on t in the uniform topology, that is, $\|P_t - P_s\| < \varepsilon$ for an arbitrary $\varepsilon > 0$ if t is in a neighborhood of s depending on ε. When ε is sufficiently small we know from Theorem 2.5.3 that the index is then independent of t. If $\operatorname{Coker} P_t = \{0\}$, then the kernel varies continuously:

Lemma 2.5.18. *Let P_t, $t \in T$, be a uniformly continuous family of Fredholm operators in $\mathcal{L}(B_1, B_2)$ which is surjective for every t. For every $s \in T$ one can then find a neighborhood N_s and continuous functions $x_j : N_s \to B_1$ such that $x_1(t), \dots, x_\nu(t)$ are a basis for* $\operatorname{Ker} P_t$ *for every $t \in N_s$.*

Proof. We can write $B_1 = V_1 \oplus \operatorname{Ker} P_s$ where V_1 is closed and the sum is direct topological. Then P_s is a continuous bijection of V_1 onto B_2, and by Theorem 2.5.3 this remains true for P_t when t is sufficiently close to s. Let R_t be the inverse, and let $x_1, \dots, x_\nu$ be a basis for $\operatorname{Ker} P_s$. Then

$$x_j(t) = x_j - R_t P_t x_j \in \operatorname{Ker} P_t;$$

and $x_1(t), \dots, x_\nu(t)$ are linearly independent since $x_j(t) \equiv x_j \mod V_1$. Since the dimension of $\operatorname{Ker} P_t$ cannot exceed the codimension ν of V_1, we have obtained the desired basis.

In Lemma 2.5.18 we have in fact encountered the notion of vector bundle which we shall now define and study:

Definition 2.5.19. Let T be a Hausdorff topological space. A complex *vector bundle* over T is a Hausdorff topological space V together with

i) a continuous map $p : V \to T$, called the *projection*,
ii) a finite dimensional vector space structure on each $V_t = p^{-1}(t)$, $t \in T$, compatible with the topology induced on the *fiber* V_t from V,
iii) such that for every $s \in T$ there is a neighborhood N_s and a homeomorphism $p^{-1}(N_s) \to N_s \times \mathbf{C}^n$ for some n, which restricts to a linear isomorphism $V_t \to \{t\} \times \mathbf{C}^n$ for every $t \in N_s$.

Two vector bundles V and W over T are called isomorphic if there exists a homeomorphism between them which restricts to a linear isomorphism in each fiber.

Under the hypotheses in Lemma 2.5.18 the kernel is thus a vector bundle on the parameter space, and it is natural to define the index of the family as this bundle. To study general families it is necessary to introduce a construction which allows us to form differences between vector bundles. This is quite analogous to the extension from positive integers to arbitrary integers.

First note that the set Vect T of isomorphism classes of vector bundles on a compact space T is an abelian semi-group if we define the sum of two vector bundles V_1 and V_2 as the set of all $(v_1, v_2) \in V_1 \times V_2$ with $p_1 v_1 = p_2 v_2$, where p_j is the projection $V_j \to T$, and the addition in the fiber over $t \in T$ is defined as usual in a direct sum. If A is any abelian semigroup, we can associate to A an abelian group $K(A)$, determined up to isomorphism, with the properties

i) there is a semigroup homomorphism $A \to K(A)$,
ii) if G is any abelian group and $A \to G$ is a semigroup homomorphism, then there is a unique homomorphism $K(A) \to G$ such that the map $A \to G$ can be factored as

$$\begin{array}{l} A \longrightarrow K(A) \\ \downarrow \swarrow \\ G \end{array}$$

The uniqueness of $K(A)$ is obvious: If G is another group with the same property as $K(A)$, we obtain in this way homomorphisms $G \to K(A)$ and $K(A) \to G$ which must be inverses of each other by the uniqueness required in (ii). To construct $K(A)$ we let A^2 be the set of all pairs (a_1, a_2), $a_j \in A$, with the obvious addition, and let A^2_D be the subset of diagonal pairs (a, a), $a \in A$. The quotient $K(A) = A^2/A^2_D$ defined by

$$(2.5.7)\quad (a_1, a_2) \sim (b_1, b_2) \ \text{ if } \ (a_1 + a, a_2 + a) = (b_1 + b, b_2 + b) \ \text{ for some } \ a, b \in A,$$

is then an abelian group where the 0 element is the class of A^2_D and the inverse of the class defined by (a_1, a_2) is the class defined by (a_2, a_1). We have a homomorphism $A \to K(A)$ mapping $a \in A$ to the class $[a]$ of $(a + b, b)$, $b \in A$, which does not depend on b in view of (2.5.7). Note that if $a, b \in A$ then $[a] = [b]$ if and only if $a+c = b+c$ for some $c \in A$. The map $A \to K(A)$ is therefore injective if and only if the cancellation law ($a + c = b + c$ implies $a = b$) is valid in A. It is also easily seen to be bijective if and only if A is a group. If B is another abelian semigroup and $A \to B$ a homomorphism, the construction gives a unique group homomorphism $K(A) \to K(B)$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & K(A) \\ \downarrow & & \downarrow \\ B & \longrightarrow & K(B) \end{array}$$

commutes. In particular, this shows that property ii) is fulfilled.

In particular, we can therefore introduce $K(\text{Vect } T)$, which we denote simply by $K(T)$. If T and S are compact spaces and $f : T \to S$ is a continuous map, then we have a homomorphism

$$f^* : \text{Vect } S \to \text{Vect } T$$

which for a given vector bundle V on S gives one on T with the fiber at t equal to the fiber of V at $f(t)$. More precisely, the *pullback* f^*V is defined by

$$f^*V = \{(t, v) \in T \times V; pv = f(t)\}.$$

The verification that this is a vector bundle on T is an easy exercise. Now we obtain from ii) a homomorphism $f^* : K(S) \to K(T)$. In particular, if T is a point, then $K(T)$ is isomorphic to the integers and f^*a is an integer valued function of $a \in K(S)$, which can be regarded as the "virtual dimension" of the element a at $f(t)$.

We shall prove next that there is homotopy invariance in K-theory:

Proposition 2.5.20. *Let X and Y be compact spaces and let $f : X \times I \to Y$ be a continuous map, $I = [0,1]$; set $f_t(x) = f(x,t)$ if $x \in X$, $t \in I$. Then $f_t^* : K(Y) \to K(X)$ is independent of $t \in I$.*

Proof. It is sufficient to show that for every vector bundle V on Y the isomorphism class of $f_t^* V$ is independent of t. Let $f^*V = W$ be the corresponding vector bundle over $X \times I$. It is then sufficient to show that for every vector bundle W over $X \times I$ the isomorphism class of the pullback W_t of W to X by the map $x \mapsto (x,t)$ is independent of t. To do so we just have to show that W_t and W_s are isomorphic if s is sufficiently close to t. Let $\widetilde{W}(t)$, $t \in I$, be the bundle on $X \times I$ obtained by pulling back W by the map $X \times I \ni (x,s) \to (x,t)$. Then there is a bundle homomorphism $\varphi : \widetilde{W}(t) \to W$, that is, a continuous map which commutes with the projections and is linear in the fibers, such that φ is the identity over $X \times \{t\}$. In fact, the existence of such a homomorphism is trivial in a neighborhood of any point $(x,s) \in X \times I$, since bundles are locally trivial. Piecing such local constructions together using a partition of unity we obtain the map φ. For reasons of continuity the homomorphism is an isomorphism at (x,s) if s is sufficiently close to t, which proves that W_s and W_t are isomorphic then. The proof is complete.

If X is a compact space, we shall say that we have a *Fredholm family* of operators from a Banach space B_1 to another B_2, parametrized by X, if for every $x \in X$ we have a Fredholm operator $P_x \in \mathcal{L}(B_1, B_2)$ which is uniformly continuous as a function of x.

Theorem 2.5.21. *There is a unique way of assigning to each Fredholm family P parametrized by a compact space X an* index $\operatorname{ind} P \in K(X)$ *such that*

i) $\operatorname{ind} P = [\operatorname{Ker} P]$ *if P_x is surjective for every $x \in X$,*
ii) $\operatorname{ind}(P \oplus Q) = \operatorname{ind} P + \operatorname{ind} Q$ *if P and Q are two Fredholm families which are parametrized by X.*
iii) *the index is functorial: if Y is another compact space and $f : Y \to X$ is a continuous map, and $(f^*P)_y$ is defined as $P_{f(y)}$, $y \in Y$, then* $\operatorname{ind} f^*P = f^* \operatorname{ind} P$.
iv) *if $P_{x,t}$, $x \in X$, $t \in I = [0,1]$, is a Fredholm family parametrized by $X \times I$, then the Fredholm families on X defined by $P_{x,1}$ and $P_{x,0}$ have the same index.*
v) *if P, Q are Fredholm families of operators from B_1 to B_2 and from B_2 to B_3 respectively, parametrized by X, then* $\operatorname{ind}(QP) = \operatorname{ind} Q + \operatorname{ind} P$.

Proof. Concerning i) we first recall that $\operatorname{Ker} P$ is a vector bundle by Lemma 2.5.18 if P_x is surjective for every $x \in X$. Let us also recall that if P_x is surjective then P_y is surjective for all y in a neighborhood of x. In fact, $\dim \operatorname{Ker} P_y \le \dim \operatorname{Ker} P_x$ when y is sufficiently close to x, by Theorem 2.5.3, and since $\operatorname{ind} P_y = \operatorname{ind} P_x$ it follows that

$$\dim \operatorname{Coker} P_y = \dim \operatorname{Coker} P_x - \dim \operatorname{Ker} P_x + \dim \operatorname{Ker} P_y \le 0.$$

For every $x \in X$ we can find a finite dimensional space $W \subset B_2$ such that $B_2 = W \oplus \operatorname{Im} P_x$. Thus the map

$$Q_x : B_1 \oplus W \ni (b, w) \mapsto P_x b + w \in B_2 \tag{2.5.8}$$

is surjective, so Q_y remains surjective for all y near x. In view of the compactness we can therefore find a finite number of finite dimensional subspaces $W_1, \dots, W_k$ such that for every $x \in X$ the map (2.5.8) is surjective when $W = W_j$ for some j. If W is the linear hull of $W_1, \dots, W_k$ then Q_x is surjective for every $x \in X$. Thus we must have $\operatorname{ind} Q = [\operatorname{Ker} Q]$ in view of i). Now the operators

$$\widetilde{Q}_{x,t} : (b, w) \mapsto P_x b + tw \in B_2, \qquad (x, t) \in X \times I,$$

define a Fredholm family parametrized by $X \times I$ which for $t = 0$ reduces to the direct sum of P and the map $W \to \{0\}$ which has the index $[W]$ and for $t = 1$ is the family Q. If ii) and iv) hold, it follows that

$$\operatorname{ind} P + [W] = \operatorname{ind} \widetilde{Q}_0 = \operatorname{ind} \widetilde{Q}_1 = [\operatorname{Ker} Q],$$

hence

$$\operatorname{ind} P = [\operatorname{Ker} Q] - [W]. \tag{2.5.9}$$

Thus the index is *unique* if it exists.

We shall define the index by (2.5.9) after proving that the right-hand side is independent of the choice of W. Let W_0 and W_1 be two such spaces W and let Q_0, Q_1 be the corresponding Fredholm families defined by (2.5.8). Consider the Fredholm family

$$B_1 \oplus W_0 \oplus W_1 \ni (b, w_0, w_1) \mapsto P_x b + (1-t) w_0 + t w_1 \in B_2, \qquad (x, t) \in X \times I.$$

It is surjective so the kernel is a vector bundle V over $X \times I$. Thus the bundles V_0 and V_1 obtained by restricting to $t = 0$ and to $t = 1$ are isomorphic by Proposition 2.5.20. However,

$$V_0 = \operatorname{Ker} Q_0 \oplus W_1, \quad V_1 = \operatorname{Ker} Q_1 \oplus W_0, \qquad \text{hence}$$
$$[\operatorname{Ker} Q_0] + [W_1] = [\operatorname{Ker} Q_1] + [W_0], \quad \text{that is,} \quad [\operatorname{Ker} Q_0] - [W_0] = [\operatorname{Ker} Q_1] - [W_1],$$

so it is legitimate to define the index by (2.5.9). ii) is then an obvious consequence of the definition, and so is iii), which implies iv). v) follows from ii) and iii) if we use the following homotopy of families of operators from $B_1 \oplus B_2$ to $B_2 \oplus B_3$:

$$\begin{pmatrix} I_2 & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \cos(\pi t/2) I_2 & \sin(\pi t/2) I_2 \\ -\sin(\pi t/2) I_2 & \cos(\pi t/2) I_2 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & I_2 \end{pmatrix}$$

where I_2 is the identity operator in B_2. For $t = 0$ this is the family $P \oplus Q$ and for $t = 1$ the family $\begin{pmatrix} 0 & I_2 \\ -QP & 0 \end{pmatrix}$, which is the direct sum of the identity in B_2 and $-QP$. It follows that $\operatorname{ind} QP = \operatorname{ind} P + \operatorname{ind} Q$. The proof is complete.

Remark. The preceding homotopy argument could also have been used to prove the "logarithmic law" for the index of a single operator.

For further developments of the connections between Fredholm theory and K-theory, and applications to geometry, we must refer to the literature. Instead we shall discuss Fredholm theory in general locally convex topological vector spaces. First we consider a semi-normed space.

Lemma 2.5.22. *Let E be a vector space with semi-norm q, and let $S : E \to E$ be a linear map such that $q(x) = 0$ implies $Sx = 0$ and for every sequence $x_n \in E$ with $q(x_n) \le 1$ there is a subsequence x_{n_j} and an element $y \in E$ such that $q(Sx_{n_j} - y) \to 0$, $j \to \infty$. Then* $\operatorname{Ker}(I+S)$ *is finite dimensional,* $\operatorname{Im}(I+S)$ *is closed, in the topology defined by q, and* $\dim \operatorname{Ker}(I+S) = \operatorname{codim} \operatorname{Im}(I+S)$.

Proof. The new features here are that q may not be a norm and that E may not be complete. We shall remove these obstacles successively.

Set $N = \{x \in E; q(x) = 0\}$ and $E_1 = E/N$, which is a normed space with $\|\xi\| = q(x)$ when x is in the residue class $\xi \in E_1$. Since $Sx = 0$ on N, the operator S induces an operator $S_1 : E_1 \to E_1$, and $S_1\xi_j$ contains a subsequence converging in E_1 for every bounded sequence $\xi_j \in E_1$. If I_1 is the identity in E_1 then $\xi \in \operatorname{Ker}(I_1 + S_1)$ means that $(I+S)x = y$ for some $y \in N$, if x is in the class of ξ. Thus $x - y$ is the only element in ξ belonging to $\operatorname{Ker}(I+S)$, so the map $E \to E/N$ restricts to a bijection $\operatorname{Ker}(I+S) \to \operatorname{Ker}(I_1+S_1)$. Similarly, $\operatorname{Im}(I+S)$ is the inverse image of $\operatorname{Im}(I_1 + S_1)$. It is therefore sufficient to prove the lemma for S_1, and we are then in a normed space.

We can complete the space E_1 to a Banach space E_2 where E_1 is a dense subset. This familiar procedure consists in forming the space C of all Cauchy sequences $X = (\xi_1, \xi_2, \dots)$ in E_1, that is, sequences with $\|\xi_j - \xi_k\| \to 0$ as $j, k \to \infty$. We define $\|X\| = \lim_{j\to\infty} \|\xi_j\|$. This is a semi-normed space where the subspace $C_0 = \{X \in C; \|X\| = 0\}$ consists of sequences converging to 0. The quotient $E_2 = C/C_0$ is a Banach space; mapping $\xi \in E_1$ to the class of $(\xi, \xi, \dots)$ in E_2 gives an isometric embedding of E_1 as a dense subset. The proof of the completeness of E_2 and the other statements are the same as for the completion of rational numbers to real numbers, and they are left for the reader.

The hypotheses imply that $\|S_1\xi\| \le M\|\xi\|$, $\xi \in E_1$, so S_1 can be extended by continuity to a linear map $S_2 : E_2 \to E_2$. Since every element of E_2 is the limit of a sequence $\xi_j \in E_1$ and $S_1\xi_j$ has a subsequence converging in E_1, it follows that $S_2E_2 \subset E_1$. If $(I_2 + S_2)\xi = \eta$, where $\xi \in E_2$ and $\eta \in E_1$, we can therefore conclude that $\xi \in E_1$. Hence

$$\operatorname{Ker}(I_1 + S_1) = \operatorname{Ker}(I_2 + S_2), \qquad \operatorname{Im}(I_1 + S_1) = E_1 \cap \operatorname{Im}(I_2 + S_2).$$

The second equality shows that the obvious map

$$E_1/\operatorname{Im}(I_1 + S_1) \to E_2/\operatorname{Im}(I_2 + S_2)$$

is injective, and it is surjective since E_1 is dense in E_2 and $E_2/\operatorname{Im}(I_2 + S_2)$ is finite dimensional. The assertions concerning S_1 are therefore consequences of the corresponding statements about S_2 contained in Theorem 2.5.10. The lemma is proved.

Definition 2.5.23. If E and F are locally convex topological vector spaces, then a linear operator $S : E \to F$ is called *compact* if there is a neighborhood U of 0 in E such that SU is contained in a compact subset of F.

If $U = \{x \in E; q(x) \le 1\}$, where q is a continuous semi-norm in E, then the hypothesis means that $\{Sx; x \in E; q(x) \le 1\}$ is contained in a compact subset of F.

Theorem 2.5.24. *Let E be a locally convex, Hausdorff topological vector space and S a compact linear operator $E \to E$, I the identity operator in E. Then* $\operatorname{Ker}(I+S)$ *is finite dimensional,* $\operatorname{Im}(I+S)$ *is closed, and*

$$\dim \operatorname{Ker}(I+S) = \dim \operatorname{Coker}(I+S), \quad \textit{that is,} \quad \operatorname{ind}(I+S) = 0.$$

Proof. Let q be a continuous semi-norm such that $\{Sx; x \in E, q(x) \le 1\}$ is contained in a compact set. Then $q(x) = 0$ implies that $tS(x)$ belongs to that set for all scalars t, hence $Sx = 0$. By Lemma 2.5.22 it follows that $\dim \operatorname{Ker}(I+S) < \infty$, that the range of $I+S$ is closed even in the topology defined by the semi-norm q alone, and that $\operatorname{ind}(I+S) = 0$.

Theorem 2.5.25. *Let E, F be Fréchet spaces and let $T : E \to F$ be a continuous linear operator with finite dimensional kernel and cokernel, that is, a Fredholm operator, Let S be a compact linear operator $E \to F$. Then $T+S$ is also a Fredholm operator, and* $\operatorname{ind} T = \operatorname{ind}(T+S)$.

Proof. The proof of Theorem 2.5.1 shows that $\operatorname{Im} T$ is closed. The rest of the proof follows the lines of the proof of Theorem 2.5.10 and will be left for the reader to carry out.

2.6. Duality. We shall first restrict ourselves to duality of Banach spaces but at the end of the section the most important results will be extended to locally convex topological vector spaces.

If B_1, B_2 are Banach spaces, then the space $\mathcal{L}(B_1, B_2)$ of continuous linear maps $T : B_1 \to B_2$ is a Banach space with the norm

$$\|T\| = \sup_{0 \ne x \in B_1} \|Tx\|/\|x\| = \sup_{\|x\| \le 1} \|Tx\|.$$

The completeness follows from the completeness of B_2; we could have allowed B_1 to be any normed space. In fact, if $\|T_j - T_k\| \to 0$, then $\|T_j x - T_k x\| \to 0$, so the completeness of B_2 shows that $Tx = \lim_{j\to\infty} T_j x$ exists for every $x \in B_1$. Obviously T is linear and

$$\|Tx - T_j x\| = \lim_{k\to\infty} \|T_k x - T_j x\| \le \varlimsup_{k\to\infty} \|T_k - T_j\| \|x\|,$$

which proves that $T - T_j$ and therefore T is continuous and that

$$\|T - T_j\| \le \varlimsup_{k\to\infty} \|T_k - T_j\| \to 0, \quad j \to \infty.$$

(See also Corollary 2.4.9.)

Definition 2.6.1. If B is a Banach space over $K = \mathbf{R}$ or $\mathbf{C}$, then the dual space B^* of B is the Banach space $\mathcal{L}(B, K)$ of continuous K-linear forms on B.

If $x \in B$ and $\xi \in B^*$ we shall use the notation $\langle x, \xi \rangle$ instead of $\xi(x)$ to denote the value of the linear form ξ at x. The reason for this is that we shall see that there is a farreaching (though not complete) symmetry between the roles of B and B^*. First note that the map $x \mapsto \langle x, \xi \rangle$ is linear for fixed ξ since ξ is a linear form.

Secondly the map $\xi \mapsto \langle x, \xi \rangle$ is linear for fixed x in view of the definition of addition and multiplication by scalars in B^*. Thus the form

$$B \times B^* \ni (x, \xi) \mapsto \langle x, \xi \rangle$$

is bilinear. There is a fundamental flaw in the symmetry though, for there may exist continuous linear forms on B^* which are *not* of the form $\xi \mapsto \langle x, \xi \rangle$ for any $x \in B$. An example is given by $B = L^1(\mathbf{R})$, $B^* = L^\infty(\mathbf{R})$. In fact, by the Hahn-Banach theorem there exists a continuous linear form L on $L^\infty(\mathbf{R})$ such that

$$L(u) = u(0), \quad \text{if } u \in C(\mathbf{R}) \cap L^\infty(\mathbf{R}).$$

If we assume that for some $f \in L^1(\mathbf{R})$

$$L(u) = \langle f, u \rangle, \quad u \in L^\infty(\mathbf{R}),$$

then we must have

$$u(0) = \int u(x) f(x)\, dx, \quad u \in C \cap L^\infty,$$

which implies $f(x) = 0$ for almost every x, which is a contradiction.

The definition of the norm of a linear form means that

$$\|\xi\| = \sup_{0 \neq x \in B} |\langle x, \xi \rangle| / \|x\|, \quad \xi \in B^*, \tag{2.6.1}$$

(Note that the same notation is used for the norm in B and for that in B^*. This should cause no confusion since we shall use the latin alphabet for elements in B and the greek alphabet for elements in B^*.) In (2.6.1) the roles of B and B^* may be reversed:

Theorem 2.6.2. *For every $x \in B$ we have*

$$\|x\| = \sup_{0 \neq \xi \in B^*} |\langle x, \xi \rangle| / \|\xi\|. \tag{2.6.2}$$

Proof. By (2.6.1) we have $|\langle x, \xi \rangle| \leq \|x\| \|\xi\|$ for all $x \in B$, $\xi \in B^*$. Hence

$$\sup_{0 \neq \xi \in B^*} |\langle x, \xi \rangle| / \|\xi\| \leq \|x\|.$$

On the other hand, according to the Hahn-Banach theorem there is a continuous linear form ξ with $\|\xi\| = 1$ such that the value of ξ at x is equal to $\|x\|$. This proves (2.6.2) and shows at the same time that the supremum in (2.6.2) is attained (which is not always true in (2.6.1)).

Theorem 2.6.2 means that the natural map $B \to (B^*)^* = B^{**}$ of B into the dual of B^*, obtained by assigning to $x \in B$ the linear form $B^* \ni \xi \mapsto \langle x, \xi \rangle$, is a linear *isometry*, that is, norm preserving. Thus the range is closed but as we have seen in an example above it may be strictly smaller than B^{**}.

Definition 2.6.3. If the isometry $B \to B^{**}$ is bijective, that is, if every continuous linear form on B^* can be written $\xi \mapsto \langle x, \xi \rangle$ for some $x \in B$, then B is called *reflexive.*

For reflexive spaces the symmetry between B and B^* is thus perfect. Later on we shall give a characterization of reflexive spaces.

Having discussed the duality of Banach spaces we pass to studying maps between them.

Theorem 2.6.4. *For every $T \in \mathcal{L}(B_1, B_2)$ there exists one and only one operator $T^* \in \mathcal{L}(B_2^*, B_1^*)$ such that*

$$\langle Tx, \eta\rangle_2 = \langle x, T^*\eta\rangle_1, \quad x \in B_1,\ \eta \in B_2^*, \tag{2.6.3}$$

where $\langle \cdot, \cdot\rangle_j$ denotes the bilinear form in $B_j \times B_j^*$. The map

$$\mathcal{L}(B_1, B_2) \ni T \mapsto T^* \in \mathcal{L}(B_2^*, B_1^*)$$

is a linear isometry. It is surjective if B_2 is reflexive.

Proof. For fixed $\eta \in B_2^*$ the map

$$B_1 \ni x \mapsto \langle Tx, \eta\rangle_2$$

is a linear form on B_1 with norm $\le \|T\|\|\eta\|_2$. Thus there exists a unique element $\xi \in B_1^*$ such that

$$\langle Tx, \eta\rangle_2 = \langle x, \xi\rangle_1, \quad x \in B_1,$$

and we have $\|\xi\|_1 \le \|T\|\|\eta\|_2$. The map $\eta \mapsto \xi$ is then linear and the norm is $\le \|T\|$. This shows that there exists a unique operator T^* satisfying (2.6.3) and that $\|T^*\| \le \|T\|$. Taking the supremum of the two sides of (2.6.3) when $\|x\|_1 \le 1$, $\|\eta\|_2 \le 1$, we conclude that there is in fact equality.

The argument can be reversed if B_2 is reflexive. Given $T^* \in \mathcal{L}(B_2^*, B_1^*)$ the equality (2.6.3) defines an element Tx which is a continuous linear form on B_2^*, hence can be identified with an element in B_2. The proof is complete.

Note that if we take $B_1 = K$, then $\mathcal{L}(B_1, B_2) = B_2$, $\mathcal{L}(B_2^*, B_1^*) = B_2^{**}$, and the map $T \mapsto T^*$ is the embedding of B_2 in B_2^{**}. Thus the last conclusion in Theorem 2.6.4 is false unless B_2 is reflexive.

Definition 2.6.5. The operator T^* defined by (2.6.3) is called the *adjoint* operator of T.

Theorem 2.6.6. *If $T_1 \in \mathcal{L}(B_1, B_2)$, $T_2 \in \mathcal{L}(B_2, B_3)$ then $(T_2T_1)^* = T_1^*T_2^*$.*

The proof is obvious. We shall now study the relations between T and T^* starting with an important special case.

Theorem 2.6.7. *Let W be a closed linear subspace of the Banach space B and let W° be its annihilator in B^*, that is,*

$$W^\circ = \{\xi \in B^*; \langle x, \xi\rangle = 0 \text{ for all } x \in W\},$$

which is a closed linear subspace of B^*. Let i, i° be the inclusion maps $W \to B$ and $W^\circ \to B^*$, and let q, q° be the quotient maps $B \to B/W$ and $B^* \to B^*/W^\circ$. Then $i^* : B^* \to W^*$ vanishes on W° and induces an isometric bijection $B^*/W^\circ \to W^*$. If we identify W^* with B^*/W°, it follows that $i^* = q^\circ$. Furthermore, $q^* : (B/W)^* \to B^*$ is an isometry with range W°. If we identify $(B/W)^*$ with W° it follows that $q^* = i^\circ$.

Thus inclusions and quotients are dual to each other.

Proof. If $\xi \in B^*$ then $i^*\xi$ is by definition the restriction of the linear form ξ to W, so it is 0 if and only if $\xi \in W^\circ$. Since every continuous linear form on W can be extended to a continuous linear form on B with the same but not with a smaller norm, it follows that $i^* : B^* \to W^*$ is surjective with kernel W° and that

$$\|i^*\xi\|_{W^*} = \inf_{\eta \in W^\circ} \|\xi - \eta\|.$$

This proves the first assertion. If $\eta \in (B/W)^*$ then $q^*\eta$ is the linear form $B \ni x \mapsto \langle qx, \eta\rangle$ which vanishes on W, hence belongs to W°. Conversely, every linear form $B \to K$ which vanishes on W can be factored through B/W to a linear form $B/W \to K$ which by definition of the norm in a quotient space has the same norm. This shows that q^* is an isometry with range W° and completes the proof.

We shall always use the natural identifications of W^* with B^*/W° and $(B/W)^*$ with W° given in Theorem 2.6.7 without mentioning this explicitly.

Theorem 2.6.7 implies that

$$W^{\circ\circ} = \{x \in B; \langle x, \xi\rangle = 0, \xi \in W^\circ\} = W.$$

In fact, $\operatorname{Im} q^* = W^\circ$ so $x \in W^{\circ\circ}$ if and only if

$$\langle qx, \theta\rangle = \langle x, q^*\theta\rangle = 0, \quad \theta \in (B/W)^*,$$

which means that $qx = 0$, that is, that $x \in W$. This gives

Corollary 2.6.8. *If W is an arbitrary linear subspace of B, closed or not, then $W^{\circ\circ}$, the annihilator in B of the annihilator W° of W in B^*, is equal to the closure in B of W.*

Proof. The annihilator of W is equal to the annihilator of the closure $\overline{W}$, so the assertion follows from the preceding remarks. The corollary is of course just another way of stating Theorem 2.3.3.

It should be emphasized that if we start with a closed subspace W of B^*, take its annihilator W° in B and then the annihilator $W^{\circ\circ}$ of W° in B^*, we may obtain a strictly larger space than W. This question will be discussed later. Obviously it only comes up for non-reflexive spaces.

Theorem 2.6.9. *Let $T \in \mathcal{L}(B_1, B_2)$ and assume that $\operatorname{Im} T$ is closed. Then it follows that $\operatorname{Im} T^*$ is closed. The annihilator of $\operatorname{Ker} T$ is $\operatorname{Im} T^*$ and the annihilator of $\operatorname{Im} T$ is $\operatorname{Ker} T^*$. Hence $\dim \operatorname{Ker} T = \dim \operatorname{Coker} T^*$ and $\dim \operatorname{Coker} T = \dim \operatorname{Ker} T^*$.*

The same conclusion is valid if $\operatorname{Im} T^*$ but not $\operatorname{Im} T$ is assumed to be closed. However, the proof is much more difficult and has to be postponed to Theorem 2.6.31 below.

Proof. Assume first that T is bijective. According to Banach's theorem we have a bounded inverse $S \in \mathcal{L}(B_2, B_1)$ then. Thus $ST = I_1$, $TS = I_2$, which implies

$$T^*S^* = I_1^*, \quad S^*T^* = I_2^*,$$

so T^* is a bijection. In the general case we can write $T = T_3T_2T_1$ where T_1 is the quotient map from B_1 to $B_1/\operatorname{Ker} T$, T_2 is the bijection $B_1/\operatorname{Ker} T \to \operatorname{Im} T$ induced by T, and T_3 is the inclusion $\operatorname{Im} T \mapsto B_2$. Then we have $T^* = T_1^*T_2^*T_3^*$ where by Theorem 2.6.7 T_1^* is the inclusion map $(\operatorname{Ker} T)^\circ \to B_1^*$ and T_3^* is the quotient map $B_2^* \to B_2^*/(\operatorname{Im} T)^\circ$. According to the first part of the proof T_2^* is a bijection, which proves that $\operatorname{Im} T^* = (\operatorname{Ker} T)^\circ$ and that $\operatorname{Ker} T^* = (\operatorname{Im} T)^\circ$. Since $\operatorname{Ker} T$ and $\operatorname{Coker} T^*$ (resp. $\operatorname{Ker} T^*$ and $\operatorname{Coker} T$) are dual, the last statement follows from Theorem 1.2.13.

The reader should also give a direct proof using the Hahn-Banach theorem, Banach's theorem, and the definition of the adjoint operator.

Corollary 2.6.10. *If T satisfies the hypotheses of Theorem 2.6.9 and* $\operatorname{ind} T$ *is defined, then* $\operatorname{ind} T^* = -\operatorname{ind} T$.

The advantage of passing to the adjoint operator, which will be more clear later on when we have perfected Theorem 2.6.9, is that T^* may often be easier to study than T. For example, if we expect that T is surjective but has a kernel of infinite dimension, then T^* should be injective. By Banach's theorem the question whether $\operatorname{Im} T^*$ is closed is then equivalent to the question whether there is an estimate of the form $\|\xi\| \le C\|T^*\xi\|$, $\xi \in B_2^*$. Thus the proof of existence theorems may be transferred to the proof of estimates, which is a much more concrete problem.

We shall now show the advantage of duality arguments by using them to give a partly different derivation of Fredholm theory.

Theorem 2.6.11. *If $T \in \mathcal{L}(B_1, B_2)$ has finite rank, then T^* has also finite rank. If $T \in \mathcal{L}_c(B_1, B_2)$ then $T^* \in \mathcal{L}_c(B_2^*, B_1^*)$.*

Proof. The first assertion is a trivial consequence of Theorem 2.6.9. To prove the second assertion we need a lemma.

Lemma 2.6.12. *If $T \in \mathcal{L}(B_1, B_2)$ is compact, then* $\operatorname{Im} T$ *has a dense countable subset.*

Proof. Let $A = \{Tx; x \in B_1, \|x\|_1 \le 1\}$. We have $\operatorname{Im} T = \cup(nA)$ so it suffices to show that A and therefore nA has a dense countable subset. To prove this we note that for every $\varepsilon > 0$ there exist finitely many $y_1, \dots, y_N \in A$ such that $\|y_j - y_k\|_2 \ge \varepsilon$ for $j \ne k$ but $\|y - y_j\| < \varepsilon$ for some $j = 1, \dots, N$ for any $y \in A$. This follows from the compactness of T since there would otherwise exist an infinite sequence in A which has no limit point. If we choose such elements $y_1, \dots, y_N$ for a sequence of values of ε converging to 0, the assertion follows.

Proof of Theorem 2.6.11 continued. Let $\eta_n \in B_2^*$, $n = 1, 2, \dots$ and $\|\eta_n\|_2 \le 1$. For every fixed $y \in B_2$ we can choose a subsequence η_{n_k} such that

$$\lim_{k\to\infty} \langle y, \eta_{n_k}\rangle_2 \tag{2.6.4}$$

exists. Using the Cantor diagonal procedure as in the proof of Proposition 2.5.7 we can select the subsequence so that the limit (2.6.4) exists for every y in the dense countable subset M of $\operatorname{Im} T$ constructed in Lemma 2.6.12. The sequence $T^*\eta_{n_k}$ is a Cauchy sequence. For otherwise there would exist subsequences $\eta_{n_k'}$ and $\eta_{n_k''}$ with $\|T^*\eta_{n_k'} - T^*\eta_{n_k''}\|_1 \ge c > 0$ for all k. We can choose $x_k \in B_1$ with $\|x_k\|_1 = 1$ and

$$|\langle x_k, T^*\eta_{n_k'} - T^*\eta_{n_k''}\rangle| > c/2$$

for every k. Since T is compact the sequence Tx_k contains a convergent subsequence. Changing notation if necessary we may assume that $Tx_k \to y$ where $y \in \overline{\operatorname{Im} T} = \overline{M}$. Then we have for every $Y \in M$

$$\langle x_k, T^*\eta_{n_k'} - T^*\eta_{n_k''}\rangle = \langle Tx_k - y, \eta_{n_k'} - \eta_{n_k''}\rangle + \langle y - Y, \eta_{n_k'} - \eta_{n_k''}\rangle + \langle Y, \eta_{n_k'} - \eta_{n_k''}\rangle.$$

When $k \to \infty$ we obtain $c/2 \le 2\|y - Y\|$. Since $y \in \overline{M}$ this is a contradiction proving the theorem.

Fredholm theory can now be developed in the following way. Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator and $S \in \mathcal{L}_c(B_1, B_2)$. Then the kernel of $T + S$ is finite dimensional, by Lemma 2.5.12 and Theorem 2.5.8, and so is the kernel of $T^* + S^*$ in view of Theorem 2.6.11. The range of $T + S$ is closed. For if F is a topological supplement of $\operatorname{Ker}(T + S)$ and $x_n \in F$, $(T + S)x_n \to y$, then the sequence x_n must be bounded. In fact, otherwise there exists a subsequence x_{n_k} such that $(T + S)x_{n_k}/\|x_{n_k}\| \to 0$. We can choose the subsequence so that also $Sx_{n_k}/\|x_{n_k}\|$ converges. Then $Tx_{n_k}/\|x_{n_k}\|$ converges, so Lemma 2.5.11 shows that a subsequence of $x_{n_k}/\|x_{n_k}\|$ converges to a limit $x \in F$. Since $\|x\| = 1$ and $(T+S)x = 0$ we obtain a contradiction. In the same way we then conclude that the sequence x_n has a convergent subsequence. If the limit is x then $(T+S)x = y$, so $\operatorname{Im}(T+S)$ is closed. In view of Theorem 2.6.9 it follows that $\dim \operatorname{Coker}(T + S) = \dim \operatorname{Ker}(T^* + S^*)$ which is finite. Hence $T + S$ is a Fredholm operator. By Theorem 2.5.3 the index of $T + tS$ is therefore independent of t, thus equal to $\operatorname{ind} T$.

The adjoint operator can also be defined for certain *unbounded* linear operators. By an unbounded operator from B_1 to B_2 we shall mean a linear operator with domain $\mathcal{D}_T$ contained in B_1 and range contained in B_2. We want to define T^* so that η is in the domain of T^* and $T^*\eta = \xi$ if and only if (2.6.3) is valid, that is,

$$\langle Tx, \eta\rangle_2 = \langle x, \xi\rangle_1, \quad x \in \mathcal{D}_T.$$

In other words, if

$$G_T = \{(x, Tx); x \in \mathcal{D}_T\} \subset B_1 \oplus B_2$$

is the graph of T, we want to have

$$\langle x, \xi\rangle_1 - \langle y, \eta\rangle_2 = 0, \quad \text{if } (x, y) \in G_T.$$

Now $B_1^* \oplus B_2^*$ is the dual space of $B_1 \oplus B_2$ with respect to the bilinear form

$$\langle (x, y), (\xi, \eta)\rangle = \langle x, \xi\rangle - \langle y, \eta\rangle; \quad x \in B_1,\ y \in B_2, \quad \xi \in B_1^*,\ \eta \in B_2^*.$$

In fact, if $B_1 \oplus B_2 \ni (x, y) \mapsto L(x, y)$ is a continuous linear form, then $L(x, 0) = \langle x, \xi\rangle$, $x \in B_1$, $-L(0, y) = \langle y, \eta\rangle$, $y \in B_2$, where $\xi \in B_1^*$, $\eta \in B_2^*$. This proves that $L(x, y) = \langle x, \xi\rangle - \langle y, \eta\rangle$. (The minus sign is introduced for the sake of convenience later on.) The desired definition of T^* then means that G_{T^*} shall be the annihilator of G_T. Denote the annihilator of G_T by W. In order that W shall be the graph of a function from B_2^* to B_1^* it is necessary and sufficient that $(\xi, 0) \in W$ implies $\xi = 0$. But $(\xi, 0) \in W$ means precisely that ξ is orthogonal to $\mathcal{D}_T$, so the adjoint operator T^* exists if and only if T is *densely defined*, that is, $\mathcal{D}_T$ is dense. If B_2 is reflexive then T^* is densely defined if and only if the only element $(0, y)$ orthogonal to W is $(0, 0)$. But we know that the annihilator of W in $B_1 \oplus B_2$ is the closure of G_T. Thus T^* is densely defined if and only if the closure of G_T is a graph, that is, if T has a closed extension. One calls T *preclosed* then. Summing up, we have proved:

Theorem 2.6.13. *If $T : B_1 \to B_2$ is a closed, densely defined linear operator, then (2.6.3) defines a closed operator $B_2^* \to B_1^*$; η is in the domain of T^* and $T^*\eta = \xi$ if and only if*

$$\langle Tx, \eta\rangle_2 = \langle x, \xi\rangle_1, \quad \text{for all } x \in \mathcal{D}_T.$$

Conversely, x is in the domain of T and $Tx = y$ if and only if

$$\langle y, \eta\rangle_2 = \langle x, T^*\eta\rangle_1 \quad \text{for all } \eta \in \mathcal{D}_{T^*}.$$

T^* is densely defined if B_2 is reflexive.

It is often convenient to note that T is closed if and only if $\mathcal{D}_T$ is a Banach space with the *graph norm*

$$x \mapsto \|x\|_1 + \|Tx\|_2.$$

In fact, this means that G_T is complete which is equivalent to G_T being closed as a subset of $B_1 \times B_2$.

We shall now discuss some important special features of the Hilbert space case. Let H be a Hilbert space. Then $H \ni x \mapsto (x, y)$ is for every fixed $y \in H$ a linear form with norm $\|y\|$, and by Corollary 2.3.7 every continuous linear form on H can be written in this way. Thus we have an isometric bijection $\theta : H \to H^*$ assigning to $y \in H$ the linear form above. θ is *antilinear*, that is,

$$\theta(ax + by) = \bar{a}\theta(x) + \bar{b}\theta(y), \quad x, y \in H,\ a, b \in K.$$

It follows that H is reflexive.

If $T : H_1 \to H_2$ is a closed densely defined linear map between Hilbert spaces, then the diagram

$$\begin{array}{ccc} H_2 & \xrightarrow{\widetilde{T}} & H_1 \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\theta} \\ H_2^* & \xrightarrow{T^*} & H_1^* \end{array}$$

defines a closed densely defined map $\widetilde{T} : H_2 \to H_1$. This is also called the adjoint of T; we have

$$(Tx, y)_2 = (x, \widetilde{T}y)_1, \quad x \in \mathcal{D}_T,\ y \in \mathcal{D}_{\widetilde{T}}.$$

T and $\widetilde{T}$ determine each other by this relation just as T and T^* above. In what follows we shall use the conventional notation T^* instead of $\widetilde{T}$ since it will always be clear from the context if we are defining the adjoint operator with respect to Hilbert space scalar products.

Definition 2.6.14. If H is a Hilbert space and $T : H \to H$ is a densely defined linear operator, then T is called *self-adjoint* if $T^* = T$; this implies that T is closed.

In Chapter III we shall give a detailed study of the structure of self-adjoint operators. Here we shall content ourselves with some additions to Theorem 2.6.13 which follow from inspection of the proof.

Let $G_T \subset H_1 \oplus H_2$ be the graph of a closed densely defined linear map from H_1 to H_2. Then (x_1, x_2) is orthogonal to G_T if and only if

$$(x_1, x) + (x_2, Tx) = 0, \quad x \in \mathcal{D}_T.$$

This means that x_2 is in the domain of T^* and that $T^*x_2 = -x_1$. By Corollary 2.3.6 we now conclude that every element $(\xi, \eta) \in H_1 \oplus H_2$ can be written in one and only one way in the form

$$(\xi, \eta) = (x, Tx) + (-T^*y, y), \quad x \in \mathcal{D}_T,\ y \in D_{T^*}.$$

Thus $\xi = x - T^*y$ and $\eta = Tx + y$. If we take $\xi = 0$ it follows that $T^*y = x$, so y is in the domain of TT^* and $(I_2 + TT^*)y = \eta$. Scalar product with y gives

$$(y, y) + (T^*y, T^*y) = (\eta, y) \le \|\eta\|\|y\|,$$

which implies that $\|y\| \le \|\eta\|$ and that $\|T^*y\| \le \|\eta\|$. Thus the inverse of $I_2 + TT^*$ is an everywhere defined operator of norm ≤ 1, so its graph and therefore the graph of $I_2 + TT^*$ is closed. $I_2 + TT^*$ is a self-adjoint operator. For let (x, y) belong to the graph of the adjoint. This means that

$$(x, (I_2 + TT^*)\xi) = (y, \xi), \quad \xi \in \mathcal{D}_{TT^*}.$$

We can write $y = (I_2 + TT^*)\eta$ and obtain

$$(x, (I_2 + TT^*)\xi) = (\eta, (I_2 + TT^*)\xi).$$

Since $I_2 + TT^*$ is surjective this proves that $x = \eta$, that is, that $(I_2 + TT^*)x = y$. On the other hand, if (x, y) and (x', y') are in the graph of $I_2 + TT^*$ then

$$(x, y') - (y, x') = (x, x' + TT^*x') - (x + TT^*x, x') = 0,$$

so the graph of $I_2 + TT^*$ is equal to the graph of its adjoint, that is, $I_2 + TT^*$ is self-adjoint. Now the sum of a self-adjoint operator and a *bounded* self-adjoint operator is obviously self-adjoint, so we have proved:

Theorem 2.6.15. *If T is a closed densely defined operator from a Hilbert space H_1 to another H_2, then T^*T and TT^* are self-adjoint operators (thus closed and densely defined). The operators $(I_1 + T^*T)^{-1}$, $(I_2 + TT^*)^{-1}$, $T(I_1 + T^*T)^{-1}$ and $T^*(I_2 + TT^*)^{-1}$ are everywhere defined and of norm ≤ 1.*

We shall now discuss some properties of complexes of unbounded operators in Hilbert space. As preparation we give a variant of Theorem 2.6.9 for unbounded operators between Hilbert spaces. Properly modified it is also valid for arbitrary Banach spaces.

Theorem 2.6.16. *Let T be a closed densely defined linear operator from one Hilbert space H_1 to another H_2. Then the following conditions are equivalent:*

a) $\operatorname{Im} T$ *is closed.*
b) *There is a constant C such that*

$$\|x\|_1 \le C\|Tx\|_2, \quad x \in \mathcal{D}_T \cap [\operatorname{Im} T^*].$$

Here $[W]$ denotes the closure of a linear space W.

c) $\operatorname{Im} T^*$ *is closed.*
d) *There exists a constant C such that*

$$\|y\|_2 \le C\|T^*y\|_1, \quad y \in \mathcal{D}_{T^*} \cap [\operatorname{Im} T].$$

Proof. Let us first note that by the definition of the adjoint operator the orthogonal space of $\operatorname{Im} T$ is equal to $\operatorname{Ker} T^*$. The orthogonal space of $\operatorname{Ker} T^*$ is therefore the closure of $\operatorname{Im} T$. Now assume that a) holds. Since the orthogonal space of $[\operatorname{Im} T^*]$ is the kernel of T, the restriction of T to $\mathcal{D}_T \cap [\operatorname{Im} T^*]$ is closed and injective. The range is the closed subspace $\operatorname{Im} T$ of H_2. Hence the inverse is continuous by the closed graph theorem, which proves b). Conversely, b) implies a), for $T(\mathcal{D}_T \cap [\operatorname{Im} T^*]) = \operatorname{Im} T$ since T vanishes on $[\operatorname{Im} T^*]^\perp$. In view of the symmetry between T and T^* it is now clear that c) and d) are also equivalent, and it suffices to prove that b) implies d). From b) we obtain, if $y \in \mathcal{D}_{T^*}$ and $x \in \mathcal{D}_T \cap [\operatorname{Im} T^*]$,

$$|(y, Tx)_2| = |(T^*y, x)_1| \le \|T^*y\|_1 \|x\|_1 \le C\|T^*y\|_1 \|Tx\|_2.$$

Hence $|(y, \eta)_2| \le C\|T^*y\|_1 \|\eta\|_2$ if $y \in \mathcal{D}_{T^*}$ and $\eta \in [\operatorname{Im} T]$, and this proves d).

Let us now assume that in addition to the operator T we have another closed densely defined linear operator S from H_2 to a third Hilbert space H_3 and that $ST = 0$, which means that $\operatorname{Im} T \subset \operatorname{Ker} S$. An example of this situation occurs if T is defined by the exterior differential operator from k forms to $k+1$ forms and S by the exterior differential operator from $k+1$ forms to $k+2$ forms. We are interested in studying the possible gap between $\operatorname{Im} T$ and $\operatorname{Ker} S$, the *homology* of the complex.

Theorem 2.6.17. *A necessary and sufficient condition for* $\operatorname{Im} T$ *and* $\operatorname{Im} S$ *both to be closed is that*

$$\text{(2.6.5)}\quad \|y\|_2^2 \le C^2(\|T^*y\|_1^2 + \|Sy\|_3^2), \quad y \in \mathcal{D}_{T^*} \cap \mathcal{D}_S \cap N^\perp,\ N = \operatorname{Ker} T^* \cap \operatorname{Ker} S.$$

$\operatorname{Im} T$ *and* $\operatorname{Im} S$ *are closed and* N *is finite dimensional if and only if from every sequence* $y_k \in \mathcal{D}_{T^*} \cap \mathcal{D}_S$ *with* $\|y_k\|_2$ *bounded and* $T^*y_k \to 0$ *in* $H_1, Sy_k \to 0$ *in* H_3, *one can select a convergent subsequence.*

Proof. First we note that

$$H_2 = [\operatorname{Im} T] \oplus N \oplus [\operatorname{Im} S^*].$$

In fact, since $\operatorname{Im} T \subset \operatorname{Ker} S$ and $\operatorname{Ker} S$ is orthogonal to $[\operatorname{Im} S^*]$, we know that $[\operatorname{Im} T]$ and $[\operatorname{Im} S^*]$ are orthogonal. The intersection of their orthogonal complements is N by definition. Now S vanishes on $[\operatorname{Im} T]$, and T^* vanishes on $[\operatorname{Im} S^*]$ since $T^*S^* = 0$. In view of Theorem 2.6.16 the estimate (2.6.5) is therefore valid for $y \in \mathcal{D}_{T^*} \cap [\operatorname{Im} T]$ if and only if $\operatorname{Im} T$ is closed, and it is valid for $y \in \mathcal{D}_S \cap [\operatorname{Im} S^*]$ if and only if $\operatorname{Im} S$ is closed. Since every y occurring in (2.6.5) can be split into two such orthogonal components, it follows that (2.6.5) holds if and only if $\operatorname{Im} T$ and $\operatorname{Im} S$ are both closed. Now assume that this is true and that N is finite dimensional. If $y_k \in \mathcal{D}_{T^*} \cap \mathcal{D}_S$ we write $y_k = z_k + n_k$ where $n_k \in N = \operatorname{Ker} S \cap \operatorname{Ker} T^*$ and z_k is orthogonal to N. Using (2.6.5) we conclude that $z_k \to 0$ if $T^*z_k = T^*y_k \to 0$ and $Sz_k = Sy_k \to 0$. Since N is finite dimensional we can extract a convergent subsequence from the sequence n_k, if y_k is bounded.

On the other hand, assume that the compactness condition in Theorem 2.6.17 is fulfilled. Then the unit ball in N is compact, so N is finite dimensional by Theorem 2.5.8. If (2.6.5) were not valid we could choose a sequence $y_k \in \mathcal{D}_{T^*} \cap \mathcal{D}_S$ orthogonal to N such that $\|y_k\|_2 = 1$ but $\|T^*y_k\|_1 \to 0$ and $\|Sy_k\|_3 \to 0$. Let y be a strong limit of this sequence, which exists by hypothesis. Then we have $\|y\|_2 = 1$, y is orthogonal to N, and $T^*y = Sy = 0$ since S and T^* are closed. Thus $y \in N$ which is a contradiction proving (2.6.5) and so the theorem.

Theorem 2.6.17 has important applications in the theory of functions of several complex variables. It would take us too far to develop them here, so the reader is referred to Chapter IV of L. Hörmander, An introduction to complex analysis in several variables.

We shall now discuss some deeper results concerning duality which are related to the use of another topology in the dual space. We introduce this notion in a quite general context:

Definition 2.6.18. Let F and G be two vector spaces over $K = \mathbf{R}$ or $\mathbf{C}$, and let $\langle x, y\rangle$ be a bilinear form on $F \times G$ defining a duality between F and G (see Definition 1.2.12). The locally convex topology in F defined by the semi-norms

$$F \ni x \mapsto |\langle x, y\rangle|$$

with $y \in G$ is called the *weak topology* in F and denoted by $\sigma(F, G)$. Correspondingly we have a weak topology $\sigma(G, F)$ in G.

Note that the definitions imply that $\sigma(F, G)$ and $\sigma(G, F)$ are Hausdorff topologies, and that the linear form $x \mapsto \langle x, y\rangle$ on F is continuous for every $y \in G$. The same is true if F and G are interchanged, but from now on we usually just mention one of two such symmetric statements. On the other hand, we have

Theorem 2.6.19. *Every linear form on F which is continuous for $\sigma(F, G)$ can for one and only one $y \in G$ be written in the form*

$$F \ni x \mapsto \langle x, y\rangle. \tag{2.6.6}$$

Proof. Let L be a linear form on F which is continuous for $\sigma(F, G)$. Then there exist $y_1, \dots, y_n \in G$ and a constant C such that

$$|L(x)| \le C \sum_1^n |\langle x, y_i\rangle|, \quad x \in F. \tag{2.6.7}$$

Thus $L(x) = 0$ if $x \in N = \{x \in F; \langle x, y_i\rangle = 0, i = 1, \dots, n\}$. The forms L and $x \mapsto \langle x, y_i\rangle$ induce linear forms L' and Y_i on the finite dimensional vector space F/N, and 0 is the only common zero of the forms Y_i. Hence L' is a linear combination of $Y_1, \dots, Y_n$, which proves the theorem.

We shall now examine the duality between a Banach space B and its dual space B^*.

Theorem 2.6.20. *If $\xi_1, \xi_2, \dots \in B^*$ and $\xi_n - \xi_m \to 0$ in $\sigma(B^*, B)$ when $n, m \to \infty$, it follows that there exists an element $\xi \in B^*$ such that $\xi_n \to \xi$ in $\sigma(B^*, B)$ when $n \to \infty$.*

Proof. The completeness of the scalars implies that $\langle x, \xi_n\rangle$ has a limit for every $x \in B$, so the theorem is an immediate consequence of Corollary 2.4.9.

Now we turn to the fundamental property of the weak topology $\sigma(B^*, B)$ (usually called the weak* topology).

Theorem 2.6.21 (Tychonov). *If B is a Banach space, then the unit ball*

$$U = \{\xi \in B^*; \|\xi\| \leq 1\}$$

is compact for $\sigma(B^*, B)$.

Proof. For every $\xi \in U$ and $x \in B$ we have $|\langle x, \xi \rangle| \leq \|x\|$. If for every $x \in B$ we let D_x denote the disc $\{z \in K; |z| \leq \|x\|\}$, we therefore obtain an injective map

$$\gamma : U \ni \xi \mapsto \{\langle x, \xi \rangle\}_{x \in B} \in \prod_{x \in B} D_x = D.$$

The range consists of all $f = \{f_x\}_{x \in B} \in \prod_{x \in B} D_x = D$ such that

$$f_{ax+by} = af_x + bf_y \qquad \forall\, x, y \in B,\ a, b \in K.$$

Each of these relations defines a closed subset of D. Thus γU is a closed subset of D, and since D is compact (Theorem 2.1.4) it follows that γU is compact. Now a basis for open sets in D is obtained by imposing on $f = \{f_x\}_{x \in B}$ a finite number of inequalities

$$|f_x - c_x| < \varepsilon_x, \quad \varepsilon_x > 0.$$

The inverse image in U is the set of all $\xi \in U$ such that

$$|\langle x, \xi \rangle - c_x| < \varepsilon_x$$

for finitely many values of x. It follows that γ is a homeomorphism, hence that U is compact.

Remark. Note that the proof is valid without change if B is just semi-normed.

In the cases which occur most frequently in the applications, the proof of Theorem 2.6.21 is completely elementary:

Definition 2.6.22. The Banach space B is called *separable* if it contains a countable dense subset.

Theorem 2.6.23. *Let B be separable and let $x_1, x_2, \dots$ be a dense subset. Then the semi-norms*

$$\xi \mapsto |\langle x_j, \xi \rangle|, \quad j = 1, 2, \dots, \tag{2.6.8}$$

define the same topology as $\sigma(B^*, B)$ on the unit ball U in B^*.

Proof. It suffices to show that if N is a neighborhood in U of a point $\xi_0 \in U$ with respect to the topology $\sigma(B^*, B)$, then N contains a neighborhood in the topology defined by means of the semi-norms (2.6.8). We may assume that

$$N = \{\xi \in U; |\langle x, \xi - \xi_0 \rangle| < \varepsilon\}; \qquad \varepsilon > 0, x \in B.$$

Choose k so that $\|x - x_k\| < \varepsilon/4$. Then $|\langle x - x_k, \xi - \xi_0 \rangle| < \varepsilon/2$ if $\xi, \xi_0 \in U$, so

$$N \supset \{\xi \in U; |\langle x_k, \xi - \xi_0 \rangle| < \varepsilon/2\},$$

which proves the theorem.

Combination of Theorems 2.6.21 and 2.6.23 gives

Theorem 2.6.24. *The unit ball in the dual space of a separable Banach space is a compact metrizable space in the weak topology $\sigma(B^*, B)$.*

This result can also easily be proved directly. It suffices to show that every sequence in U has a weakly convergent subsequence. This can be proved using the Cantor diagonal procedure and Theorem 2.6.23. The details are left as an exercise.

Example 2.6.25. From every bounded sequence $f_n \in L^p$, $1 < p \le \infty$, one can choose a subsequence f_{n_k} with a weak limit $f \in L^p$, that is,

$$\int f_{n_k} g\, dx \to \int fg\, dx, \quad g \in L^{p'},\ 1/p + 1/p' = 1.$$

An analogous "selection theorem" is valid for a sequence of measures with uniformly bounded total mass.

We shall now prove a certain converse of the preceding theorems. The first step is a general but elementary result concerning vector spaces in duality.

Let F, G be two vector spaces in duality with respect to the bilinear form $\langle x, y\rangle$. Let $A \subset F$ be *convex* and *balanced* (that is, $x \in A$ and $a \in K$, $|a| \le 1$ implies $ax \in A$). We set

$$A^\circ = \{y \in G; |\langle x, y\rangle| \le 1 \quad \forall x \in A\},$$

which is called the *polar* of A. It is clear that A° is convex, balanced and closed for $\sigma(G, F)$, so $A^{\circ\circ}$ has the corresponding properties and contains A.

Note that if A is conic, that is, $tx \in A$ if $t > 0$ and $x \in A$, then

$$A^\circ = \{y \in G; \langle x, y\rangle = 0 \quad \forall x \in A\}.$$

In particular the polar of a linear subspace of F is its annihilator in G, so there is no conflict with our earlier notation.

Theorem 2.6.26. *$A^{\circ\circ}$ is the closure of A in the topology $\sigma(F, G)$.*

Proof. Assume that x_0 is not in the closure of A. According to the Hahn-Banach theorem (Theorem 2.3.2) there exists a linear form f on F, which is continuous for $\sigma(F, G)$, such that

$$|f(x) - f(x_0)| \ge \varepsilon > 0, \quad x \in A.$$

Since $x \in A$ implies $ax \in A$ if $|a| \le 1$, it follows that $|f(x_0)| \ge |f(x)| + \varepsilon$ when $x \in A$. Let $g(x) = f(x)/a$ where $|f(x_0)| - \varepsilon \le a < |f(x_0)|$. Then

$$|g(x)| \le 1 < |g(x_0)|, \quad x \in A.$$

By Theorem 2.6.19 we have $g(x) = \langle x, y\rangle$ for some $y \in G$. Hence $y \in A^\circ$ and $x_0 \notin A^{\circ\circ}$, so the proof is complete.

We can now prove a converse of Theorem 2.6.21.

Theorem 2.6.27. *Let F and G be two Banach spaces which are in duality with respect to the bilinear form $B(x, y)$. Assume that*

$$\|x\| = \sup_{0 \ne y \in G} |B(x, y)|/\|y\|, \quad x \in F, \tag{2.6.9}$$

and that the unit ball in G is compact for $\sigma(G,F)$. Every linear form on F which is continuous with respect to the norm topology can then be written $x \mapsto B(x,y)$ for suitable $y \in G$, and the norm of the linear form is $\|y\|$. Thus the map $G \to F^*$ defined by B is a bijective isometry.

Proof. From (2.6.9) it follows that the linear form $z : x \mapsto B(x,y)$ is continuous with norm $\leq \|y\|$. The map $y \mapsto z = \varphi(y)$ is therefore an injection $G \to F^*$ of norm ≤ 1. Let U_G be the unit ball in G, let $U = \varphi(U_G)$ be the image in F^*, and denote the unit ball in F^* by U^*. The theorem will be proved if we show that $U = U^*$. First note that it follows from (2.6.9) that

$$U^\circ = \{x \in F; |\langle x, z\rangle| \leq 1 \quad \forall z \in U\} = \{x \in F; |B(x,y)| \leq 1 \quad \forall y \in G, \|y\| \leq 1\}$$

is the unit ball in F. Here $\langle \cdot, \cdot \rangle$ is the bilinear form on $F \times F^*$. If we apply Theorem 2.6.26 to the spaces F and F^* with this duality, it follows that U^* is the closure of U in $\sigma(F^*, F)$. By hypothesis U_G is compact in the topology $\sigma(G,F)$, and φ is continuous for the topologies $\sigma(G,F)$ and $\sigma(F^*,F)$ since

$$\langle x, \varphi(y)\rangle = B(x,y) \quad \text{if } x \in F, y \in G.$$

Hence U is compact and therefore closed in the topology $\sigma(F^*,F)$ (Theorem 2.1.3), so $U = U^*$.

Corollary 2.6.28. *A Banach space B is reflexive if and only if the unit ball in B is compact for the weak topology $\sigma(B, B^*)$.*

We shall now complete Theorem 2.6.9 to a more symmetric statement. The main tool in the proof is

Theorem 2.6.29 (Banach). *A linear subspace M of the dual B^* of a Banach space is closed for $\sigma(B^*, B)$ if and only if the intersection of M and the unit ball in B^* is compact in this topology.*

The necessity follows at once from Tychonov's theorem. The main step in the proof of the sufficiency is the following:

Lemma 2.6.30. *Let M be a linear subspace of the dual B^* of a Banach space such that the intersection with the unit ball in B^* is compact for the topology $\sigma(B^*,B)$, and let ξ_0 be a fixed element $\in B^* \setminus M$. Then there exists a sequence $x_n \in B$ with $x_n \to 0$ such that $\xi \notin M$ if $|\langle x_n, \xi - \xi_0\rangle| \leq 1$ for all n.*

Proof. Let $M_n = \{\xi \in M; \|\xi - \xi_0\| \leq n\}$. Assume that we have already proved that there exist finitely many elements $x_1, \dots, x_k \in B$ such that

$$|\langle x_j, \xi - \xi_0\rangle| \leq 1, \quad 1 \leq j \leq k \implies \xi \notin M_n.$$

We shall prove that there exist elements $x_{k+1}, \dots, x_{k+m} \in B$ with norms $\leq 1/n$ such that

$$|\langle x_j, \xi - \xi_0\rangle| \leq 1, \quad 1 \leq j \leq k+m \implies \xi \notin M_{n+1}. \tag{2.6.10}$$

Repeated use of this construction will of course yield a sequence with the required properties.

M_{n+1} is compact for $\sigma(B^*, B)$, for

$$M_{n+1} = \{\xi \in M; \|\xi\| \le n+1+\|\xi_0\|\} \cap \{\xi \in B^*; \|\xi - \xi_0\| \le n+1\},$$

where the first set is compact for $\sigma(B^*, B)$ by hypothesis, and the second is closed. By assumption

$$M_{n+1} \bigcap \bigcap_{j=1}^{k} \{\xi; |\langle x_j, \xi - \xi_0\rangle| \le 1\} \bigcap_{n\|x\|\le 1} \{\xi; |\langle x, \xi - \xi_0\rangle| \le 1\} = \emptyset, \tag{2.6.11}$$

for the last intersection is $\{\xi; \|\xi - \xi_0\| \le n\}$. Since M_{n+1} is compact for $\sigma(B^*, B)$ and the other sets are closed, it follows that already a finite number of the sets in (2.6.11) have an empty intersection. But this means that (2.6.10) holds for suitable $x_{k+1}, \dots, x_{k+m}$ with norm $\le 1/n$, so the proof is complete.

Proof of Theorem 2.6.29. Let $\xi_0 \notin M$ and choose a sequence x_n according to Lemma 2.6.30. Write

$$p(\xi) = \sup_n |\langle x_n, \xi\rangle|, \quad \xi \in B^*.$$

Then the distance from ξ_0 to M is at least 1 with respect to the semi-norm p, so by the Hahn-Banach theorem there exists a linear form L on B^* which is continuous for p, and vanishes on M although $L(\xi_0) = 1$. To prove the theorem we have to show that L is the scalar product with an element of B.

Let c be the Banach space of sequences $a = (a_1, a_2, \dots)$ converging to 0 in K, with the norm $\|a\| = \sup |a_j|$. The dual space is l^1. We define a linear map $\varphi : B^* \to c$ by

$$\varphi(\xi) = (\langle x_1, \xi\rangle, \langle x_2, \xi\rangle, \dots).$$

Since $p(\xi) = \|\varphi(\xi)\|$ we have $L(\xi) = L'(\varphi(\xi))$ where L' is a continuous linear form on the range of φ. Let $\widetilde{L}$ be a continuous linear extension of L' to all of c. Then we have $L(\xi) = \widetilde{L}(\varphi(\xi))$, and

$$\widetilde{L}(a) = \sum t_j a_j, \qquad \sum |t_j| < \infty.$$

Now

$$L(\xi) = \widetilde{L}(\varphi(\xi)) = \sum t_j \langle x_j, \xi\rangle = \langle x, \xi\rangle, \quad \xi \in B^*,$$

where $x = \sum t_j x_j$. The proof is complete.

It is now easy to complete Theorem 2.6.9.

Theorem 2.6.31. *Let $T \in \mathcal{L}(B_1, B_2)$ and assume that* $\operatorname{Im} T^*$ *is closed (in the norm topology of B_1^*). Then it follows that* $\operatorname{Im} T$ *is closed, the annihilator of* $\operatorname{Ker} T$ *is* $\operatorname{Im} T^*$*, and the annihilator of* $\operatorname{Im} T$ *is* $\operatorname{Ker} T^*$*.*

Proof. As in the proof of Theorem 2.6.9 we can reduce the proof to the case where $\operatorname{Ker} T = \{0\}$ and $\operatorname{Im} T$ is dense in B_2. Then we have $\operatorname{Ker} T^* = \{0\}$, and we shall first show that $\operatorname{Im} T^* = B_1^*$.

Since $\operatorname{Im} T^*$ is closed and $\operatorname{Ker} T^* = \{0\}$, it follows from Banach's theorem that there is a constant C such that

$$\|\eta\|_2 \le C\|T^*\eta\|_1, \quad \eta \in B_2^*. \tag{2.6.12}$$

Let U_1^* be the unit ball in B_1^*. Then we have

$$U_1^* \cap \operatorname{Im} T^* = U_1^* \cap \{T^*\eta; \eta \in B_2^*, \|\eta\|_2 \le C\}. \tag{2.6.13}$$

Now T^* is continuous for the weak topologies, for if $x \in B_1$ and $\varepsilon > 0$, then $|\langle x, T^*\eta\rangle_1| < \varepsilon$ for all $\eta \in B_2^*$ with $|\langle Tx, \eta\rangle_2| < \varepsilon$, which is a neighborhood of 0 in $\sigma(B_2^*, B_2)$. Hence the last set occurring in (2.6.13) is compact and therefore closed for $\sigma(B_1^*, B_1)$, for it is the continuous image of a compact set for $\sigma(B_2^*, B_2)$. Thus $U_1^* \cap \operatorname{Im} T^*$ is closed for $\sigma(B_1^*, B_1)$, so Theorem 2.6.29 shows that $\operatorname{Im} T^*$ is closed for $\sigma(B_1^*, B_1)$. But if $x \in B_1$ is orthogonal to $\operatorname{Im} T^*$, we have

$$\langle x, T^*\eta\rangle_1 = \langle Tx, \eta\rangle_2 = 0, \quad \eta \in B_2.$$

This implies that $Tx = 0$, hence that $x = 0$ since we have assumed that $\operatorname{Ker} T = \{0\}$. By the Hahn-Banach theorem applied to B_1^* with the topology $\sigma(B_1^*, B_1)$ it follows that $\operatorname{Im} T^* = B_1^*$.

Using (2.6.12) we now obtain

$$|\langle x, T^*\eta\rangle_1| = |\langle Tx, \eta\rangle_2| \le C\|Tx\|_2\|T^*\eta\|_1, \qquad x \in B_1,\ \eta \in B_2^*$$

and since $\operatorname{Im} T^* = B_1^*$, it follows that

$$\|x\|_1 \le C\|Tx\|_2.$$

Since $\operatorname{Im} T$ is dense in B_2 we conclude that $\operatorname{Im} T = B_2$, and the theorem is proved.

We can also extend Theorems 2.5.3 and 2.5.10 now so that symmetry between kernel and cokernel is restored:

Theorem 2.6.32. *Let $T \in \mathcal{L}(B_1, B_2)$ and assume that $\operatorname{Im} T$ has finite codimension. If $S \in \mathcal{L}(B_1, B_2)$ and $\|S\|$ is sufficiently small, it follows that the range of $T + S$ also has finite codimension and that*

$$\operatorname{ind}(T+S) = \operatorname{ind} T, \qquad \dim \operatorname{Coker}(T+S) \le \dim \operatorname{Coker} T.$$

For an arbitrary $S \in \mathcal{L}_c(B_1, B_2)$ the range of $T + S$ has finite codimension and

$$\operatorname{ind}(T+S) = \operatorname{ind} T.$$

Proof. By Theorem 2.5.1 we know that $\operatorname{Im} T$ is closed. Hence it follows from Theorem 2.6.9 that $\operatorname{Im} T^*$ is closed, and $\dim \operatorname{Ker} T^* = \operatorname{codim} \operatorname{Im} T = \dim \operatorname{Coker} T$. When $\|S^*\| = \|S\|$ is small, it follows from Theorem 2.5.3 that $(T+S)^* = T^* + S^*$ has finite dimensional kernel and closed range, and that

$$\operatorname{ind}(T+S)^* = \operatorname{ind} T^*, \qquad \dim \operatorname{Ker}(T+S)^* \le \dim \operatorname{Ker} T^* = \dim \operatorname{Coker} T.$$

For arbitrary $S \in \mathcal{L}_c(B_1, B_2)$ we have $S^* \in \mathcal{L}_c(B_2^*, B_1^*)$ by Theorem 2.6.11, so we can then conclude from Theorem 2.5.10 that $(T+S)^*$ has finite dimensional kernel and closed range, and that $\operatorname{ind}(T+S)^* = \operatorname{ind} T^*$. Now it follows from Theorem 2.6.31 that $T+S$ has closed range and that the statements in the theorem are valid.

Before giving an example of applications we shall discuss the extension of some of the preceding results to the case of Fréchet spaces. The dual E^* of a locally convex topological vector space E is always defined as the space of continuous linear forms on E. There are several interesting topologies on E^* but we shall only be concerned with the weak* topology $\sigma(E^*, E)$.

Theorem 2.6.33. *If E is a locally convex topological vector space and p is a continuous semi-norm on E, then*

$$U_p = \{\xi \in E^*; |\langle x, \xi\rangle| \le p(x),\ x \in E\}$$

is compact for $\sigma(E^*, E)$.

This was already observed in a remark following Theorem 2.6.21.

Theorem 2.6.34. *A linear subspace M of the dual E^* of a Fréchet space is closed for $\sigma(E^*, E)$ if and only if $U_p \cap M$ is closed for $\sigma(E^*, E)$ when p is any continuous semi-norm in E.*

Proof. The necessity follows from Theorem 2.6.33. To prove the sufficiency we first have to extend Lemma 2.6.30 and prove that if $\xi_0 \notin M$, then there exists a sequence $x_n \in E$ with $x_n \to 0$ such that $\xi \notin M$ if $|\langle x_n, \xi - \xi_0\rangle| \le 1$ for all n. To do so we take a sequence of continuous semi-norms $p_1, p_2, \dots$ on E such that

$$\{x \in E; p_j(x) \le 1\}$$

is a decreasing fundamental sequence of neighborhoods of 0 and $\xi_0 \in U_{p_1}$. Let

$$M_n = M \cap (U_{p_n} + \{\xi_0\}) \subset M \cap U_{2p_n}.$$

Assume that we have already found finitely many $x_1, \dots, x_k \in E$ such that

$$|\langle x_j, \xi - \xi_0\rangle| \le 1, \quad 1 \le j \le k \implies \xi \notin M_n.$$

We claim that there exist elements $x_{k+1}, \dots, x_{k+m} \in E$ such that $p_n(x_j) \le 1$, $k < j \le k+m$, and

$$|\langle x_j, \xi - \xi_0\rangle| \le 1, \quad 1 \le j \le k+m \implies \xi \notin M_{n+1}.$$

If this were not true we would as in the proof of Lemma 2.6.30 obtain an element $\xi \in M_{n+1}$ such that

$$\begin{aligned} |\langle x_j, \xi - \xi_0\rangle| &\le 1, \quad j \le k, \qquad \text{thus } \xi \notin M_n, \\ |\langle x, \xi - \xi_0\rangle| &\le 1 \qquad \text{for all } x \in E \text{ with } p_n(x) \le 1,\ \text{thus } \xi \in M_n. \end{aligned}$$

The existence of the sequence x_n is therefore established. The proof of Theorem 2.6.29 is now applicable with no further change, for $\sum t_j x_j$ converges in E if $\sum |t_j| < \infty$ since $\sum |t_j| p_k(x_j) < \infty$ for every semi-norm p_k.

Now let E and F be two Fréchet spaces and $T : E \to F$ a continuous linear map. As before we can define the adjoint $T^* : F^* \to E^*$ by

$$\langle Tx, \eta\rangle = \langle x, T^*\eta\rangle, \quad x \in E,\ \eta \in F^*,$$

for this means that $T^*\eta$ is the continuous linear form $x \mapsto \langle Tx, \eta\rangle$ on E.

Theorem 2.6.35. *The range of T is closed if and only if the range of T^* is weak* closed in E^*. The annihilator of* $\operatorname{Im} T$ *is then* $\operatorname{Ker} T^*$*, and the annihilator of* $\operatorname{Ker} T$ *is* $\operatorname{Im} T^*$.

In applications of this theorem one will of course use Theorem 2.6.34 in verifying that the range of T^* is weak* closed.

Proof. a) Assume that TE is closed. Then we shall prove that $\xi \in T^*F^*$ if and only if ξ belongs to the annihilator of $\operatorname{Ker} T$. The necessity is easy, for if $\xi = T^*\eta$, then

$$\langle \operatorname{Ker} T, \xi\rangle = \langle \operatorname{Ker} T, T^*\eta\rangle = \langle T \operatorname{Ker} T, \eta\rangle = 0.$$

On the other hand, assume that ξ is orthogonal to $\operatorname{Ker} T$, and set

$$L(Tx) = \langle x, \xi\rangle, \quad x \in E.$$

L is uniquely defined since $\langle x, \xi\rangle = 0$ if $x \in \operatorname{Ker} T$. L is also continuous since the map $T_1 : E/\operatorname{Ker} T \to \operatorname{Im} T$ is a homeomorphism by Banach's theorem. Hence there exists a continuous extension of L to all of F, that is, there is an element $\eta \in F^*$ with

$$\langle x, \xi\rangle = \langle Tx, \eta\rangle, \quad x \in E.$$

This means that $T^*\eta = \xi$.

b) Let N be the annihilator of $\operatorname{Im} T^*$ in E. We have $\langle Tx, \eta\rangle = 0$ for all $\eta \in F^*$ if and only if $x \in N$. Thus $N = \operatorname{Ker} T$. If we assume that $\operatorname{Im} T^*$ is weak* closed then $N^\circ = \operatorname{Im} T^*$, by Theorem 2.6.26. Let T_1 be the map from E/N to F induced by T, thus $T = T_1 q$ where q is the quotient map $E \to E/N$. Then $T^* = q^* T_1^*$ where T_1^* maps F^* to the dual space $N^\circ = \operatorname{Im} T^*$ of E/N and q^* is the injection of N° in E^*. (Compare with Theorem 2.6.7 which immediately carries over to the present situation.) Thus T_1^* is surjective.

Changing notation so that T_1 is called T it is thus sufficient to prove the statement when $\operatorname{Im} T^* = E^*$, which implies that T is injective. We claim that T^{-1} is then continuous, that is, $Tx_j \to 0$ implies $x_j \to 0$. This will prove that the range of T is closed and complete the proof. We need two lemmas.

Lemma 2.6.36. *A sequence x_j in a metrizable locally convex topological vector space E converges to 0 if and only if there exists a sequence of positive reals $\varepsilon_j \to 0$ such that x_j/ε_j is bounded.*

Proof. If x_j/ε_j is bounded, we have $p(x_j/\varepsilon_j) \le C_p$ for every continuous semi-norm p on E. Hence $p(x_j) \to 0$ if $\varepsilon_j \to 0$, that is, $x_j \to 0$ when $j \to \infty$. On the other hand, assume that $x_j \to 0$. Let $p_1 \le p_2 \le \dots$ be a sequence of continuous semi-norms such that the sets $\{x \in E; p_j(x) < 1\}$ form a fundamental system of neighborhoods of 0. By hypothesis we have $p_j(x_k) < 1/j$ when $k \ge k_j$, where k_j is an increasing sequence. If we define $\varepsilon_k = 1/j$ when $k_j \le k < k_{j+1}$, it follows that $\varepsilon_k \to 0$ when $k \to \infty$ and that $p_j(x_k/\varepsilon_k) < 1$ when $k_j \le k < k_{j+1}$. Hence $p_j(x_k/\varepsilon_k) < 1$ for $k \ge k_j$ since p_j increases with j. Thus x_k/ε_k is a bounded sequence.

Lemma 2.6.37. *A sequence x_j in a locally convex topological vector space E is bounded if and only if the sequence $\langle x_j, \xi\rangle$ is bounded for every $\xi \in E^*$.*

Proof. If the sequence is bounded we have a bound for $|\langle x_j, \xi\rangle|$ for every $\xi \in E^*$ because $x \mapsto |\langle x, \xi\rangle|$ is a continuous semi-norm. On the other hand, assume that all these scalar products form bounded sequences. Let p be a continuous semi-norm on E and let E_p^* be the set of all $\xi \in E^*$ which are continuous with respect to p, that is,

$$\|\xi\|_p = \sup_{x \in E, p(x) \neq 0} |\langle x, \xi \rangle| / p(x) < \infty.$$

This is a Banach space. We have

$$|\langle x_j, \xi \rangle| \leq p(x_j) \|\xi\|_p, \quad \xi \in E_p^*,$$

so that $\xi \mapsto \langle x_j, \xi \rangle$ is a continuous linear form on E_p^*. By hypothesis it is pointwise bounded, so Theorem 2.4.8 proves that there is a uniform bound, that is,

$$|\langle x_j, \xi \rangle| \leq C_p \|\xi\|_p, \quad \xi \in E_p^*, \ j = 1, 2, \dots.$$

In view of the Hahn-Banach theorem (see Theorem 2.6.2 or Theorem 2.6.26) it follows that $p(x_j) \leq C$ for every j. Hence the sequence x_j is bounded.

End of proof of Theorem 2.6.35. We can now continue the discussion in part b) of the proof. Let $Tx_j \to 0$. By Lemma 2.6.36 there is a sequence $\varepsilon_j \to 0$ such that Tx_j / ε_j is bounded. Hence

$$\langle Tx_j / \varepsilon_j, \eta \rangle = \langle x_j / \varepsilon_j, T^* \eta \rangle$$

is bounded for every $\eta \in E^*$. Since every $\xi \in E^*$ is of the form $\xi = T^* \eta$, it follows from Lemma 2.6.37 that the sequence x_j / ε_j is bounded, hence that $x_j \to 0$ by Lemma 2.6.36. The proof is complete.

Example 2.6.38. (Malgrange) If $P(D)$ is a partial differential operator with constant coefficients in $\mathbf{R}^n$ then $P(D) C^\infty(\mathbf{R}^n) = C^\infty(\mathbf{R}^n)$.

To prove this we observe that the adjoint is the operator $P(-D) : \mathcal{E}'(\mathbf{R}^n) \to \mathcal{E}'(\mathbf{R}^n)$ where $\mathcal{E}'(\mathbf{R}^n)$ is the dual space of $C^\infty(\mathbf{R}^n)$, that is, the space of distributions of compact support. As usual D stands for $-i\partial/\partial x$. The adjoint is injective, for if $\mu \in \mathcal{E}'(\mathbf{R}^n)$ and $P(-D)\mu = 0$, taking Laplace transforms gives $P(-\zeta)\hat{\mu}(\zeta) = 0$. Here $\hat{\mu}$ is an entire function so it must vanish identically, which implies that $\mu = 0$. By Theorems 2.6.34 and 2.6.35 it remains only to show that if q is a continuous semi-norm in $C^\infty(\mathbf{R}^n)$, then $P(-D)\mathcal{E}' \cap M$ is weakly closed if

$$M = \{\mu \in \mathcal{E}'; |\mu(u)| \leq q(u) \text{ when } u \in C^\infty(\mathbf{R}^n)\}.$$

We may assume that for some constants C and A

$$q(u) = \sum_{|\alpha| \leq N} \sup_K |D^\alpha u|, \quad u \in C^\infty(\mathbf{R}^n); \ K = \{x \in \mathbf{R}^n; |x| \leq A\}.$$

M is a set of distributions of order N with support in K, and the Paley-Wiener theorem gives

$$|\hat{\mu}(\zeta)| \leq C(1 + |\zeta|)^N e^{A|\operatorname{Im} \zeta|}, \quad \zeta \in \mathbf{C}^n, \ \mu \in M.$$

Since the exponential functions are in $C^\infty(\mathbf{R}^n)$, the maps $M \ni \mu \mapsto \hat{\mu}(\zeta)$ are continuous for the weak topology $\sigma(\mathcal{E}', C^\infty)$. This is metrizable on M since C^∞

has a dense countable subset, say the set of all polynomials with rational coefficients. If a sequence $\mu_j \in M$ converges in the weak topology, it follows that $\hat{\mu}_j(\zeta)$ converges uniformly on every compact set in $\mathbf{C}^n$, for the functions $\hat{\mu}_j(\zeta)$ and their derivatives are uniformly bounded on every compact set in $\mathbf{C}^n$.

Let us now consider $M \cap P(-D)\mathcal{E}'$, that is, the set of all $\mu \in M$ which can be written $\mu = P(-D)\nu$ with $\nu \in \mathcal{E}'$. Then we have

$$P(-\zeta)\hat{\nu}(\zeta) = \hat{\mu}(\zeta).$$

Assume the coordinates chosen so that $P(-\zeta) = c\zeta_1^m + \dots$ where c is a constant $\neq 0$ and the dots indicate terms of lower order with respect to ζ_1. Then we have

$$P(-\zeta_1 - t, -\zeta_2, \dots) = c\prod_1^m (t - t_j),$$

and since there are only m zeros we can choose a circle $|t| = r$ with $0 < r < 1$ such that $\prod |t - t_j| \geq (2m)^{-m}$ when $|t| = r$. By the maximum principle it follows that

$$|\hat{\nu}(\zeta)| \leq \sup_{|t|=r} |\hat{\nu}(t + \zeta_1, \zeta_2, \dots)| \leq (2m)^m |c|^{-1} \sup_{|t|=r} |\hat{\mu}(t + \zeta_1, \zeta_2, \dots)|.$$

Hence there is a constant C_1 such that

$$|\hat{\nu}(\zeta)| \leq C_1 \sup_{|\zeta'|<1} |\hat{\mu}(\zeta + \zeta')|, \quad \text{if } P(-\zeta)\hat{\nu}(\zeta) = \hat{\mu}(\zeta).$$

If now $\mu_j = P(-D)\nu_j$ and $\mu_j \in M$ converges in the weak topology to a limit $\mu \in M$, it follows that $\hat{\mu}_j - \hat{\mu}_k \to 0$ uniformly on compact sets. Hence this is true for $\hat{\nu}_j - \hat{\nu}_k$ too, so $\hat{\nu}_j(\zeta)$ converges uniformly on compact sets to an entire analytic function f such that

$$|f(\zeta)| \leq C_2(1 + |\zeta|)^N e^{A|\operatorname{Im}\zeta|}.$$

By the Paley-Wiener theorem $f = \hat{\nu}$ for some $\nu \in \mathcal{E}'$, and $P(-D)\nu = \mu$ since

$$P(-\zeta)\hat{\nu}(\zeta) = \lim \hat{\mu}_j(\zeta) = \hat{\mu}(\zeta).$$

It follows that $M \cap P(-D)\mathcal{E}'$ is weakly closed, and the assertion is therefore proved.

Chapter III

Spectral theory for self-adjoint operators

3.1. Some basic facts on Hilbert spaces. In the preceding chapter we have mainly discussed properties shared by at least general Banach spaces. Before passing to the main theme of this chapter we shall call attention to some of the special features of a Hilbert space. First we expand Corollary 2.3.6.

Theorem 3.1.1. *Let H be a Hilbert space, and let G_i, $i \in I$, be closed, mutually orthogonal subspaces, that is,*

$$(x, y) = 0, \text{ if } x \in G_i,\ y \in G_j, \text{ and } i \neq j.$$

Set

$$G^{\perp} = \{x \in H; (x, y) = 0\ \forall y \in \bigcup_{i \in I} G_i\}.$$

Then every $x \in H$ can be written in a unique way as a sum

$$x = \sum_{i \in I} x_i + y, \qquad x_i \in G_i,\ y \in G^{\perp}, \tag{3.1.1}$$

where the series is absolutely convergent: For every $\varepsilon > 0$ there is a finite subset I_ε of I such that for every finite set J with $I_\varepsilon \subset J \subset I$ we have

$$\|x - \sum_{j \in J} x_j - y\| < \varepsilon.$$

We have

$$\|x\|^2 = \sum_{i \in I} \|x_i\|^2 + \|y\|^2. \tag{3.1.2}$$

In particular, $\{i \in I; x_i \neq 0\}$ is countable.

Proof. By Corollary 2.3.6 there is a unique $x_i \in G_i$ such that $x - x_i \in G_i^{\perp}$. Since $G_i^{\perp} \supset G_j$ if $j \neq i$, it follows for every finite $J \subset I$ that $y_J = x - \sum_{j \in J} x_j \in G_i^{\perp}$, for all $i \in J$. We have

$$\|x\|^2 = \sum_{j \in J} \|x_j\|^2 + \|y_J\|^2 \geq \sum_{j \in J} \|x_j\|^2,$$

which proves that $\sum_{i \in I} \|x_i\|^2 \leq \|x\|^2$. Thus $\{i \in I; x_i \neq 0\}$ is countable, and $X = \sum_{i \in I} x_i$ exists, for if the finite set I_ε is chosen so that $\sum_{I \setminus I_\varepsilon} \|x_i\|^2 < \varepsilon^2$, then

$$\|\sum_{j \in J} x_j - \sum_{i \in I_\varepsilon} x_i\| = \big(\sum_{i \in J \setminus I_\varepsilon} \|x_i\|^2\big)^{\frac{1}{2}} < \varepsilon, \text{ if } J \supset I_\varepsilon \text{ is finite.}$$

By Cauchy's convergence principle this proves the existence of X. Since $y = x - X = x - x_i + x_i - X$ is orthogonal to G_i for every i, it follows that $y \in G^{\perp}$, hence y is orthogonal to X. For any decomposition (3.1.1) we have (3.1.2), which proves the uniqueness.

One denotes by $\oplus_{i\in I} G_i$ the set of all convergent sums $x = \sum_{i\in I} x_i$ where $x_i \in G_i$; this is the orthogonal space of $G^\perp$. The notation is also used when we are just given a family of unrelated Hilbert spaces H_i, $i \in I$; one then *defines* $H = \oplus_{i\in I} H_i$ as the set of all $x = \{x_i\}_{i\in I}$ where $x_i \in H_i$ and $\sum_{i\in I} \|x_i\|^2 < \infty$. Then the scalar product

$$(x, y) = \sum_{i\in I} (x_i, y_i)_{H_i}, \qquad x = \{x_i\}_{i\in I},\ y = \{y_i\}_{i\in I},$$

is well defined. H is complete, for suppose that $x^\nu = \{x_i^\nu\}_{i\in I}$ is a Cauchy sequence. For fixed i the completeness of H_i proves that $x_i^\nu \to x_i \in H_i$ as $\nu \to \infty$. For arbitrary $\varepsilon > 0$ we have by hypothesis

$$\sum_{i\in I} \|x_i^\nu - x_i^\mu\|^2 < \varepsilon^2, \text{ if } \nu, \mu > N(\varepsilon).$$

Let J be a finite subset of I, restrict the summation to J, and let $\mu \to \infty$. Then we obtain

$$\sum_{i\in J} \|x_i^\nu - x_i\|^2 \le \varepsilon^2, \text{ if } \nu > N(\varepsilon),$$

and since J is arbitrary we can replace J by I and conclude that $\sum_{i\in I} |x_i|^2 < \infty$ and that $\{x_i^\nu\}_{i\in I} \to \{x_i\}_{i\in I}$, which proves the completeness.

An example is the space l^2 of sequences $x = (x_1, x_2, \dots)$ with $x_j \in \mathbf{C}$ and $\|x\|^2 = \sum |x_j|^2 < \infty$. This is a separable Hilbert space, for elements with rational coordinates and only finitely many $\neq 0$ are dense. This is more than an example:

Theorem 3.1.2. *In every separable Hilbert space over* $\mathbf{C}$ *there exists an orthonormal basis* $e_1, e_2, \dots,$ *that is, elements with*

$$(e_j, e_j) = 1, \quad (e_j, e_k) = 0 \textit{ if } j \neq k,$$

such that $H = \oplus_1^\infty (\mathbf{C}e_j)$. *This means that every* $x \in H$ *can be uniquely written* $x = \sum_1^\infty x_j e_j$ *where* $x_j \in \mathbf{C}$ *and* $\|x\|^2 = \sum_1^\infty |x_j|^2$.

Proof. Let $y_1, y_2, \dots$ be a countable subset of H such that their finite linear combinations are dense in H. We may assume that y_j are linearly independent, for otherwise we can drop elements which are linear combinations of the preceding ones. We can then get an orthonormal sequence by the Gram-Schmidt orthogonalisation procedure, already mentioned in a remark after Lemma 2.5.9,

$$e_1 = y_1/\|y_1\|, \dots, e_j = Y_j/\|Y_j\|, \quad Y_j = y_j - \sum_{k<j} e_k (y_j, e_k).$$

$Y_j \neq 0$ because of the linear independence assumed. By induction we obtain $(e_j, e_k) = 0$ for $k < j$, and $(e_j, e_j) = 1$. The linear combinations of $y_1, \dots, y_j$ are also linear combinations of $e_1, \dots, e_j$, so these are dense in H. Hence $x = 0$ if $(x, e_j) = 0$ for every j. By Theorem 3.1.1 it follows that $H = \oplus_1^\infty (\mathbf{C}e_j)$.

Remark. In every Hilbert space Zorn's lemma shows that there exists a maximal set of orthonormal vectors; by Theorem 3.1.1 it must be a basis in the Hilbert space sense. There is of course an analogue of Theorem 3.1.2 for Hilbert spaces over $\mathbf{R}$. For the sake of brevity we shall usually just make statements for complex Hilbert spaces and leave for the reader to decide if they remain true in the real case.

Definition 3.1.3. If H_1 and H_2 are Hilbert spaces and T is a linear map with domain $\mathcal{D}_T \subset H_1$ and range $\operatorname{Im} T \subset H_2$, then T is called an *isometry* if

$$\|Tx\|_2 = \|x\|_1, \quad x \in \mathcal{D}_T; \text{ hence } (Tx, Ty)_2 = (x, y)_1, \quad x, y \in \mathcal{D}_T.$$

If in addition $\mathcal{D}_T = H_1$ and $\operatorname{Im} T = H_2$, then T is called *unitary.*

With this terminology Theorem 3.1.2 states that for every separable Hilbert space H of infinite dimension there is a unitary map $U : l^2 \to H$. (The term Hilbert space is often reserved for the infinite dimensional case. However, this restriction tends to complicate statements.) More generally, we have

Theorem 3.1.4. *Let H_1, H_2 be separable Hilbert spaces. The closure $\overline{T}$ of a linear isometric map T with domain $\mathcal{D}_T \subset H_1$ and range* $\operatorname{Im} T \subset H_2$ *is an isometry with closed domain and closed range. It can be extended to a unitary map U from H_1 to H_2 if and only if* $\operatorname{codim} \mathcal{D}_{\overline{T}}$ *and* $\operatorname{codim} \operatorname{Im} \overline{T}$ *are equal.*

Proof. If $u_n \in \mathcal{D}_T$ and $f_n = Tu_n$, then $\|u_n - u_m\| = \|f_n - f_m\|$, so u_n converges to a limit u if and only if f_n converges to a limit f, and then we have $\|u\| = \|f\|$. This proves the first statement. If U is an isometric extension of T, then

$$(Ux, Ty)_2 = (Ux, Uy)_2 = (x, y)_1 = 0, \text{ if } x \in (\mathcal{D}_T)^\perp, \ y \in \mathcal{D}_T,$$

so U maps $(\mathcal{D}_T)^\perp$ to $(\operatorname{Im} T)^\perp$. If U is unitary then this restriction must be unitary, which implies that the dimensions are the same. Conversely, if the dimensions are the same it follows from Theorem 3.1.2 that we can find a unitary map $U_1 : (\mathcal{D}_T)^\perp \to (\operatorname{Im} T)^\perp$, which gives a unitary map

$$H_1 = \mathcal{D}_{\overline{T}} \oplus (\mathcal{D}_T)^\perp \xrightarrow{\overline{T} \oplus U_1} \operatorname{Im} \overline{T} \oplus (\operatorname{Im} T)^\perp = H_2.$$

Remark. The proof shows that the isometric map T can always be extended to an isometric map with either the domain equal to H_1 or the range equal to H_2.

3.2. Symmetric and self-adjoint operators. Let H be a separable Hilbert space over $\mathbf{C}$.

Definition 3.2.1. A linear operator A with domain $\mathcal{D}_A$ and range $\operatorname{Im} A$ contained in H is called *symmetric* if

$$(Au, v) = (u, Av), \qquad u, v \in \mathcal{D}_A. \tag{3.2.1}$$

If A is densely defined this means that the adjoint A^* is an extension of A. If $A^* = A$ then A is called *self-adjoint.*

Remark. In the literature one often requires symmetric operators to be densely defined.

To clarify the difference between the notions of self-adjointness and symmetry we shall discuss an elementary example (see also Exercise 17):

Example 3.2.2. Let $H = L^2(I)$, $I = (-1, 1)$, and define $Au = u''$ when $u \in C_0^2(I)$. Then A is obviously symmetric and densely defined. If $u_n \in C_0^2(I)$ and $u_n \to u$, $u_n'' \to f$ in $L^2(I)$, then

$$u_n'(t) = \int_{-1}^{t} u_n''(s)\, ds \to \int_{-1}^{t} f(s)\, ds,$$

uniformly in I, so $u \in C^1(\overline{I})$, $u = u' = 0$ at ± 1, and $u'' = f \in L^2(I)$ in the sense of distribution theory. Conversely, these conditions imply that (u, f) is in the closure of the graph of A, for if u, f are defined as 0 outside I, then $u'' = f$ on $\mathbf{R}$ in the sense of distribution theory, and

$$\begin{aligned} u_n(t) &= n \int_{|s|<1/2n} u(n(t-s)/(n-1))\, ds \in C_0^2(I), \\ u_n''(t) &= n^3(n-1)^{-2} \int_{|s|<1/2n} f(n(t-s)/(n-1))\, ds, \end{aligned}$$

which shows that $u_n \to u$, $u_n'' \to f$ in $L^2(I)$ as $n \to \infty$.

That $A^*u = f$ means on the other hand that $u, f \in L^2(I)$ and that

$$(u, Av) = (f, v), \qquad v \in C_0^2(I),$$

that is, $u'' = f$ in I in the sense of distribution theory. This implies $u \in C^1(\overline{I})$. The difference between A and A^* is that the domain of A is restricted by *boundary conditions*. If $u, v \in \mathcal{D}_{A^*}$, then

$$(A^*u, v) - (u, A^*v) = [u'\bar{v} - u\bar{v}']_{-1}^{1},$$

which confirms that A^* is not symmetric. (The adjoint of A^* is of course the closure of A.) The restriction $\widetilde{A}$ of A^* to the set of $u \in \mathcal{D}_{A^*}$ satisfying boundary conditions

$$a_- u(-1) + b_- u'(-1) = 0, \quad a_+ u(1) + b_+ u'(1) = 0,$$

with real a_-, b_-, a_+, b_+ with $a_-^2 + b_-^2 \neq 0$, $a_+^2 + b_+^2 \neq 0$, is self-adjoint though. It is clear that $\widetilde{A}$ is symmetric, so $\widetilde{A}^*$ must be an extension of $\widetilde{A}$ and a restriction of A^*. It is equal to $\widetilde{A}$ for if $[u'v - uv']_{-1}^{1} = 0$ for all smooth v satisfying the boundary conditions it is clear that u also satisfies them.

We shall now discuss the extension of symmetric densely defined operators A in general, which is a great deal more complicated since the dimension of $\mathcal{D}_{A^*}/\mathcal{D}_A$ is usually infinite. However, this will give an abstract frame for discussion of boundary conditions for partial differential operators. Since A^* is a closed extension of A, we can always close A to obtain a closed symmetric operator $\overline{A}$ which is a restriction of every self-adjoint extension of A. In what follows we may therefore assume that A is closed.

Proposition 3.2.3. *Let A be a closed symmetric operator. If $z \in \mathbf{C} \setminus \mathbf{R}$ then $A - zI$ is injective, the range is closed, and*

$$|\operatorname{Im} z| \|u\| \leq \|(A - zI)u\|, \quad u \in \mathcal{D}_A. \tag{3.2.2}$$

The codimension of $\operatorname{Im}(A - zI)$ *only depends on the sign of* $\operatorname{Im} z$.

Proof. If $z = x + iy$ we obtain using the symmetry of A

$$\|(A - zI)u\|^2 = \|(A - xI)u\|^2 + y^2\|u\|^2 \geq y^2\|u\|^2, \tag{3.2.3}$$

which proves (3.2.2). If $u_n \in \mathcal{D}_A$ and $(A-zI)u_n \to f$, it follows from this inequality applied to $u_n - u_m$ that $u_n \to u$, hence $u \in \mathcal{D}_A$ and $(A - zI)u = f$. This proves that the range is closed. If $\zeta \in \mathbf{C}$ then

$$T : G_A \ni (u, Au) \mapsto (A - zI)u \in H, \qquad S : G_A \ni (u, Au) \mapsto \zeta u \in H$$

satisfy the hypotheses of Theorem 2.5.3, with $\operatorname{Ker} T = \{0\}$, so it follows that the range of $A-(z+\zeta)I$ has the same codimension as that of $A-zI$ if $|\zeta|$ is sufficiently small. The proof is complete since the half planes $\{z; \pm \operatorname{Im} z > 0\}$ are connected.

Definition 3.2.4. The codimensions $n_\pm$ of $\overline{\operatorname{Im}(A - zI)}$ when $\pm \operatorname{Im} z > 0$ are called the *defect indices* of the symmetric operator A.

Since $(u, (A - zI)v) = 0$, $v \in \mathcal{D}_A$, is equivalent to $u \in \mathcal{D}_{A^*}$ and $A^*u - \bar{z}u = 0$, if A is densely defined, we also have

$$n_\pm = \dim \operatorname{Ker}(A^* - zI), \quad \pm \operatorname{Im} z < 0.$$

Hence $n_+ = n_- = 0$ if A is self-adjoint. Conversely, *if A is closed, symmetric, densely defined, and the defect indices $n_\pm$ are 0, then A is self-adjoint.* In fact, if $u \in \mathcal{D}_{A^*}$ and $\operatorname{Im} z > 0$, we can find $v \in \mathcal{D}_A$ with $(A - zI)v = (A^* - zI)u$ since $n_+ = 0$. Hence $(A^* - zI)(u - v) = 0$, for A^* is an extension of A, so $u - v = 0$ since $n_- = 0$. Thus $u \in \mathcal{D}_A$ and A is self-adjoint. For self-adjoint operator the resolvent is well behaved off the real axis:

Theorem 3.2.5. *If A is self-adjoint and $z \in \mathbf{C} \setminus \mathbf{R}$ then the inverse $R(z) = (A - zI)^{-1}$ is defined in all of H, and $\|R(z)\| \leq 1/|\operatorname{Im} z|$. We have $R(z)^* = R(\bar{z})$, and*

$$R(z)R(w)(z - w) = R(z) - R(w) = R(w)R(z)(z - w) \quad \textit{if } \operatorname{Im} z \operatorname{Im} w \neq 0. \tag{3.2.4}$$

Proof. It only remains to prove (3.2.4). Since the range of $R(w)$ is in the domain of A, we have

$$((A - wI) - (A - zI))R(w) = (z - w)R(w),$$

and (3.2.4) follows if we multiply to the left by $R(z)$.

The extension problem for symmetric operators can be reduced to that for isometric operators already discussed in Theorem 3.1.4:

Proposition 3.2.6. *If A is symmetric and densely defined then*

$$T : (A + iI)u \mapsto (A - iI)u, \qquad u \in \mathcal{D}_A, \tag{3.2.5}$$

is isometric, $\operatorname{Ker}(I - T) = \{0\}$, $\operatorname{Im}(I - T)$ *is dense, and*

$$\mathcal{D}_A = \{v - Tv; v \in \mathcal{D}_T\}, \quad A(v - Tv) = i(v + Tv). \tag{3.2.6}$$

Conversely, if T is a given isometric operator with $\operatorname{Im}(I-T)$ *dense, then* $\operatorname{Ker}(I-T)=\{0\}$ *and (3.2.6) defines a symmetric operator with dense domain. T is closed if and only if A is closed.*

Proof. That (3.2.5) defines an isometric operator when A is symmetric is a consequence of (3.2.3). With $w = Au + iu$ and $Tw = Au - iu$ we obtain $2iu = w - Tw$ and $2Au = w + Tw$, or $u = v - Tv$ and $Au = i(v + Tv)$ if we set $w = 2iv$. We have $v - Tv \neq 0$ if $v \neq 0$, and the range of $I - T$ is equal to the domain of A. Now assume instead that an isometric operator T with $\operatorname{Im}(I-T)$ dense is given. If $v \in \mathcal{D}_T$ and $Tv = v$, we have for every $w \in \mathcal{D}_T$

$$(w - Tw, v) = (w, v) - (Tw, v) = (Tw, Tv) - (Tw, v) = (Tw, Tv - v) = 0,$$

which proves that $v = 0$. Hence $v - Tv$ determines v, so we can define A by (3.2.6). Then A is symmetric, for if $v, w \in \mathcal{D}_T$ then

$$(i(v + Tv), w - Tw) = (v - Tv, i(w + Tw)), \text{ since } (v, w) = (Tv, Tw),$$

and A is densely defined, which completes the proof since the graphs of A and of T are related by a continuous linear isomorphism.

Theorem 3.2.7. *A closed densely defined symmetric operator A has a self-adjoint extension if and only if the defect indices are equal.*

Proof. By Proposition 3.2.6 the existence of a self-adjoint extension of A is equivalent to the existence of a unitary extension of the corresponding isometric operator T. Since the defect indices are the codimensions of $\operatorname{Im}(A \mp iI)$, that is, of $\mathcal{D}_T$ and $\operatorname{Im} T$, the theorem follows from Theorem 3.1.4.

We shall finally give some sufficient conditions for the existence of a self-adjoint extension.

Theorem 3.2.8. *Let A be a densely defined symmetric operator such that for some $\lambda \in \mathbf{R}$ and constant C*

$$\|u\| \le C\|(A - \lambda I)u\|, \quad u \in \mathcal{D}_A.$$

Then the defect indices of A are equal, so A has a self-adjoint extension.

Proof. Without restriction we may assume that A is closed. The proof of Proposition 3.2.3 shows that the codimension of $\operatorname{Im}(A - zI)$ is constant when $|z - \lambda|$ is small, which in particular means that the defect indices are equal.

Corollary 3.2.9. *If A is a densely defined symmetric operator which is bounded from below, that is,*

$$(Au, u) \ge C(u, u), \quad u \in \mathcal{D}_A,$$

then A has a self-adjoint extension.

Proof. Replacing A by $A + (1 - C)I$ we may assume that $C = 1$. Then

$$\|u\|^2 \le (Au, u) \le \|Au\|\|u\|, \quad u \in \mathcal{D}_A,$$

hence $\|u\| \le \|Au\|$, $u \in \mathcal{D}_A$, and the assertion follows from Theorem 3.2.8.

If $(Au, u) \ge 0$, $u \in \mathcal{D}_A$, one says that A is a positive operator and writes $A \ge 0$. With an obvious extension of this notation the hypothesis in Corollary 3.2.9 is that $A \ge CI$, and one calls A *semi-bounded.* For such operators there is in fact a natural self-adjoint extension (the Friedrichs extension) with the same lower bound as A:

Theorem 3.2.10. *Let A be a densely defined symmetric operator such that*

$$C = \inf_{u\in\mathcal{D}_A, \|u\|=1} (Au, u) > -\infty.$$

Then there is a self-adjoint extension A_D with $(A_D u, u) \geq C\|u\|^2$, $u \in \mathcal{D}_{A_D}$.

Proof. As in the proof of Corollary 3.2.9 we may assume that $C = 1$. Set

$$(u, v)_D = (Au, v), \quad \|u\|_D^2 = (u, u)_D, \qquad u, v \in \mathcal{D}_A.$$

Then $\mathcal{D}_A$ becomes a prehilbert space. Let D be the completion of $\mathcal{D}_A$ with the norm $\|\cdot\|_D$. Since $\|u\| \leq \|u\|_D$, $u \in \mathcal{D}_A$, we have a natural map of norm ≤ 1 from D to H. To prove that it is injective, we let $u_j \in \mathcal{D}_A$ be a Cauchy sequence in D which converges to 0 in H, that is,

$$(A(u_j - u_k), u_j - u_k) \to 0 \text{ as } j, k \to \infty; \qquad \|u_j\| \to 0 \text{ as } j \to \infty.$$

We must prove that $(Au_j, u_j) \to 0$ as $j \to \infty$. Now

$$\begin{aligned}(A(u_j - u_k), u_j - u_k) &= (Au_j, u_j) + (Au_k, u_k) - 2\operatorname{Re}(Au_j, u_k)\\ &\geq (Au_j, u_j) - 2\operatorname{Re}(Au_j, u_k) \to (Au_j, u_j), \text{ as } k \to \infty.\end{aligned}$$

Hence $(Au_j, u_j) \to 0$ when $j \to \infty$, as claimed. We may therefore consider D as a subset of H.

Next we prove that

$$(u, v)_D = (A^*u, v), \text{ if } u, v \in D \text{ and } u \in \mathcal{D}_{A^*}. \tag{3.2.7}$$

To prove this we take sequences $u_j, v_k \in \mathcal{D}_A$ with $u_j \to u$ and $v_k \to v$ in D. Then

$$(A^*u, v) = \lim_{k\to\infty} (A^*u, v_k) = \lim_{k\to\infty} (u, Av_k) = \lim_{k\to\infty} \lim_{j\to\infty} (u_j, Av_k) = (u, v)_D.$$

If A_D is the restriction of A^* to $\mathcal{D}_{A^*} \cap D$ it follows that A_D is symmetric. The equation $A_D u = f$ has a unique solution $u \in \mathcal{D}_{A_D} = \mathcal{D}_{A^*} \cap D$ for every $f \in H$. For by (3.2.7) this equation implies

$$(u, v)_D = (f, v), \quad v \in D. \tag{3.2.8}$$

Conversely, if $u \in D$ then (3.2.8) implies that

$$(u, Av) = (f, v), \quad v \in \mathcal{D}_A,$$

so $u \in \mathcal{D}_{A^*}$ and $A^*u = f$, that is, $A_D u = f$. Now

$$|(f, v)| \leq \|f\|\|v\| \leq \|f\|\|v\|_D,$$

so it follows from Corollary 2.3.7 that (3.2.8) is valid for a unique $u \in D$, and that $\|u\| \leq \|u\|_D \leq \|f\|$. But this implies that A_D is self-adjoint. For assume that $u, f \in H$ and that

$$(u, A_D v) = (f, v), \quad v \in \mathcal{D}_{A_D}.$$

Choose $U \in \mathcal{D}_{A_D}$ with $A_D U = f$. Then

$$(U, A_D v) = (A_D U, v) = (f, v), \quad v \in \mathcal{D}_{A_D},$$

since A_D is symmetric, so

$$(u - U, A_D v) = 0, v \in \mathcal{D}_{A_D}.$$

Since the range of A_D is equal to H it follows that $U = u$, hence that $A_D u = f$.

The Friedrichs extension constructed here is in fact independent of how the positive lower bound was chosen, and it is in a sense the largest possible self-adjoint extension. We shall not discuss the possible extensions systematically here but content ourselves with an elementary example which shows the classical origin of the construction (Dirichlet's principle).

Example 3.2.11. The closure of the operator A in Example 3.2.2 is defined for all $u \in C^1(\overline{I})$ with $u = u' = 0$ at ± 1 and $u'' \in L^2(I)$ in the sense of distribution theory. We have

$$(-Au, u) = \int_I |u'|^2 \, dt, u \in \mathcal{D}_A,$$

and since for $u \in \mathcal{D}_A$

$$\int_I |u|^2 \, dt = -2 \int_I \operatorname{Re} u u' t \, dt \leq 2\|u\| \|u'\|, \text{ hence } \|u\| \leq 2\|u'\|,$$

it follows that $-4A$ satisfies the hypotheses in the proof of Theorem 3.2.10. The space D consists of all $u \in C(\overline{I})$ vanishing at ± 1 with $u' \in L^2$, so the Friedrichs extension corresponds to the Dirichlet boundary condition $u = 0$ at the boundary ∂I. The analogue for the Laplacian is the classical Dirichlet principle.

Theorem 3.2.12. *(Kato) Let A be a self-adjoint operator in H, and let V be a symmetric operator with $\mathcal{D}_V \supset \mathcal{D}_A$ and*

$$\|Vu\| \leq a\|Au\| + b\|u\|, \quad u \in \mathcal{D}_A. \tag{3.2.9}$$

If $a < 1$ it follows that the operator $A + V$ with domain $\mathcal{D}_A$ is also self-adjoint.

Proof. It is clear that $A + V$ is symmetric. When $u \in \mathcal{D}_A$ we have

$$\|(Au + Vu)\| \geq \|Au\| - \|Vu\| \geq (1 - a)\|Au\| - b\|u\|.$$

This implies that the graph norms of A and $A + V$ are equivalent so $A + V$ is closed. We must prove that the defect indices are 0. Since all the maps $A + tV$, $0 \leq t \leq 1$, are closed and symmetric, it follows from Theorem 2.5.3 as in the proof of Proposition 3.2.3 that the codimension of $\operatorname{Im}(A + tV \mp iI)$ is then independent of t. Thus the defect indices of $A + tV$ remain 0 since this is true when $t = 0$, which proves the theorem.

We shall give an example using Theorem 3.2.12 later on (Example 3.3.9).

3.3. The spectral theorem. Let us first as an introduction show how the diagonalization of A is obtained from Theorem 3.2.5 if H is finite dimensional. In that case we can represent $R(z)$ by a matrix with rational functions as entries, the denominator being given by $\det(A - zI)$. If z is a pole it follows from Theorem 3.2.5 that z must be real and that the pole is simple. Since $R(z) \to 0$ when $z \to \infty$, we conclude that

$$R(z) = \sum (\lambda_j - z)^{-1} E_j,$$

where λ_j is real, the sum is finite, and $E_j \in \mathcal{L}(H, H)$ is independent of z. Now we obtain from (3.2.4)

$$\sum (\lambda_j - z)^{-1} (\lambda_k - w)^{-1} E_j E_k = \sum (\lambda_j - z)^{-1} (\lambda_j - w)^{-1} E_j,$$

which means that

$$E_j E_k = E_j \quad \text{if } j = k; \quad E_j E_k = 0 \quad \text{if } j \neq k. \tag{3.3.1}$$

In particular, E_j is a projection on a space $H_j = E_j H$, annihilated by E_k when $k \neq j$. Furthermore, letting $z \to \infty$ we see that $I = \sum E_j$. Thus the whole space H is the direct sum of the spaces H_j, and since

$$(A - \lambda_j I) E_j = \lim_{z \to \lambda_j} (\lambda_j - z)(A - \lambda_j I) R(z) = \lim_{z \to \lambda_j} (\lambda_j - z)(I + (z - \lambda_j) R(z))$$

we have $(A - \lambda_j I) E_j = 0$. Thus the elements of H_j are eigenvectors belonging to the eigenvalue λ_j. Since $R(z)^* = R(\bar{z})$ we have $E_j^* = E_j$, hence

$$(E_j u, (I - E_j) v) = (u, (E_j - E_j^2) v) = 0,$$

so H_j is orthogonal to the kernel of E_j and therefore to H_k for $k \neq j$. Thus H is the orthogonal direct sum of the eigenspaces H_j corresponding to real eigenvalues λ_j.

Next we shall extend the preceding argument to compact self-adjoint operators in a Hilbert space. The following fact is an immediate consequence of the results of section 2.5 but also easy to prove directly.

Lemma 3.3.1. *Let A be compact and self-adjoint, $0 \neq \lambda \in \mathbf{R}$. Then* $\operatorname{Ker}(A - \lambda I)$ *is finite dimensional and $(A - \lambda I)$ restricted to the orthogonal complement is a bijection.*

Proof. If $\operatorname{Ker}(A - \lambda I)$ were not finite dimensional it would contain an infinite orthonormal sequence $x_1, x_2, \dots$. Thus $A x_j = \lambda x_j$ so that $\|A x_j - A x_k\| = |\lambda| \sqrt{2}$ when $j \neq k$, which excludes the existence of a convergent subsequence. Since

$$(Ax - \lambda x, y) = (x, Ay - \lambda y) = 0, \quad \forall x \iff y \in \operatorname{Ker}(A - \lambda I),$$

the closure G of $\operatorname{Im}(A - \lambda I)$ is the orthogonal space of $\operatorname{Ker}(A - \lambda I)$. Hence $A - \lambda I$ maps G injectively into G, and the range is dense. A standard compactness argument now shows that the range is closed, so it is equal to G.

As in Section 2.5 (see (2.5.6)) we conclude that for z sufficiently close to λ we have if $\lambda \neq 0$

$$R(z) = (A - zI)^{-1} = (\lambda - z)^{-1} E + R_1(z)$$

where $R_1(z)$ is analytic at λ and $E = \lim_{z\to\lambda}(\lambda - z)R(z)$ is the orthogonal projection on the kernel of $A - \lambda I$. Hence there are at most countably many eigenvalues $\neq 0$ of A, say $\lambda_1, \lambda_2, \dots$. The orthogonal projections E_j on the corresponding finite dimensional eigenspaces are mutually orthogonal, that is, (3.3.1) is valid. This follows as before from (3.2.4). Hence

$$Eu = \sum E_j u$$

exists for every $u \in H$ (Theorem 3.1.1). E is an orthogonal projection commuting with all E_j, $EE_j = E_jE = E_j$. Writing $E_0 = I - E$ and $\lambda_0 = 0$, we claim that

$$R(z)u = (A - zI)^{-1}u = \sum_0^\infty (\lambda_j - z)^{-1}E_j u, \quad \operatorname{Im} z \neq 0.$$

By Theorem 3.1.1 the right-hand side is well defined and has norm $\leq \|u\|/|\operatorname{Im} z|$. Thus

$$F(z) = (R(z)u, v) - \sum_0^\infty ((\lambda_j - z)^{-1}E_j u, v)$$

is an analytic function when $z \neq 0$, and since $\|(A - zI)^{-1}u + z^{-1}u\| = O(|z|^{-2})$ we have

$$|F(z)| \leq 2\|u\|\|v\|/|\operatorname{Im} z|, \quad F(z) = O(|z|^{-2}) \text{ as } z \to \infty.$$

These conditions imply that F vanishes identically. In fact, it suffices to prove that the coefficients in the Laurent series expansion vanish, that is, that

$$\int_{|z|=R} F(z)z^k\,dz = 0$$

for all k. When $k \leq 0$ this follows by letting $R \to \infty$. If $k > 0$ and the assertion is already proved with k replaced by $k - 2$, then

$$\int_{|z|=R} F(z)z^k\,dz = \int_{|z|=R} F(z)z^{k-2}(z^2 - R^2)\,dz.$$

The second integral can be estimated by

$$2\|u\|\|v\|R^k \int_{|z|=1} |z^2 - 1|/|\operatorname{Im} z|\,|dz| \to 0, \quad R \to 0.$$

Hence $F = 0$.

As already observed, we have $(A - \lambda_j I)E_j = 0$, $j \neq 0$, so

$$u = \sum_1^\infty E_j u - (A - zI)z^{-1}E_0 u = u - z^{-1}AE_0 u,$$

which shows that $AE_0 = 0$. Thus, with orthogonal direct sum,

$$H = \bigoplus_0^\infty H_j, \quad H_j = E_j H;\ (A - \lambda_j)H_j = \{0\}.$$

Again the whole space is therefore the orthogonal direct sum of all the eigenspaces.

We shall now pass to a study of general self-adjoint operators. This requires somewhat deeper tools from analytic function theory concerning functions analytic outside the real axis. These shall be applied to the functions $f(z) = (R(z)u, u)$, $u \in H$. We note that

$$\overline{f(z)} = (u, R(z)u) = (R(\bar{z})u, u), \quad \operatorname{Im} z \neq 0,$$

so we have in view of (3.2.4)

$$2i \operatorname{Im} (R(z)u, u) = (R(z)u - R(\bar{z})u, u) = 2i \operatorname{Im} z(R(z)u, R(z)u) = 2i \operatorname{Im} z \|R(z)u\|^2.$$

Thus

$$\operatorname{Im} f(z) \geq 0, \quad \operatorname{Im} z > 0; \qquad |f(z)| \leq C/|\operatorname{Im} z|; \tag{3.3.2}$$

where $C = \|u\|^2$. The analyticity of $f(z)$ for $\operatorname{Im} z \neq 0$ follows immediately from Theorem 3.2.5 which gives first that $\|R(z) - R(w)\|^2 \to 0$ when $w \to z$, then that

$$(R(z) - R(w))/(z - w) \to R(z)^2 \text{ when } w \to z.$$

Lemma 3.3.2. *Let f be an analytic function in the upper half plane satisfying (3.3.2). Then it follows that*

$$\int \operatorname{Im} f(x + iy)\, dx \leq C\pi, \quad y > 0,$$

and that there is a positive measure $d\mu$ of finite mass on $\mathbf{R}$ such that

$$\int \varphi(x)\pi^{-1} \operatorname{Im} f(x + iy)\, dx \to \int \varphi(x)\, d\mu(x), \quad y \to +0, \tag{3.3.3}$$

for any $\varphi \in C_B = C(\mathbf{R}) \cap L^\infty(\mathbf{R})$. We have

$$f(z) = \int d\mu(\xi)/(\xi - z), \quad \operatorname{Im} z > 0, \tag{3.3.4}$$

$$\int d\mu(\xi) = \lim_{y \to +\infty} y \operatorname{Im} f(iy) = \lim_{z \to \infty} -zf(z), \tag{3.3.5}$$

where $z \to \infty$ with the argument bounded away from 0 and π. If $d\mu$ is a bounded measure and f is defined by (3.3.4) then (3.3.3) and (3.3.5) hold.

Proof. Let L_c denote the line $\operatorname{Im} z = c$, oriented in the direction of increasing $\operatorname{Re} z$. If $0 < c < \operatorname{Im} z$, the reflection z^* of z in L_c is below L_c, and $(z^* + z)/2 = ic + (z + \bar{z})/2$, so $z - z^* = z - \bar{z} - 2ic$ and

$$\zeta - z^* = \zeta^* - z^* = \bar{\zeta} - \bar{z}, \quad \text{if } \zeta \in L_c.$$

If we apply Cauchy's integral formula to a large half disc in the upper half plane with the straight boundary on L_c, we obtain when the radius $\to \infty$

$$\begin{aligned} f(z) &= (2\pi i)^{-1} \int_{L_c} f(\zeta)(1/(\zeta - z) - 1/(\zeta - z^*))\, d\zeta \\ &= (\operatorname{Im} z - c)/\pi \int_{L_c} f(\zeta)|\zeta - z|^{-2}\, d\zeta, \quad \text{hence} \end{aligned}$$

$$\operatorname{Im} f(z) = (\operatorname{Im} z - c)/\pi \int_{L_c} \operatorname{Im} f(\zeta)|\zeta - z|^{-2}\, d\zeta.$$

If we multiply by $\operatorname{Im} z$ and let $\operatorname{Im} z \to \infty$ keeping $\operatorname{Re} z$ fixed, it follows from (3.3.2) and Fatou's lemma that

$$\pi^{-1} \int_{-\infty}^{\infty} \operatorname{Im} f(\xi + ic)\, d\xi \le C.$$

Hence, by Theorem 2.6.24, we can choose a sequence $c_n \to 0$ such that

$$\lim \pi^{-1} \operatorname{Im} f(\xi + ic_n)\, d\xi = d\mu(\xi)$$

exists weakly, that is, in the weak* topology in the dual space of the space of continuous functions vanishing at infinity, with the maximum norm. Passing to the limit in the preceding formulas we obtain

$$\operatorname{Im} f(z) = \operatorname{Im} z \int |\xi - z|^{-2}\, d\mu(\xi).$$

The difference between the two sides in (3.3.4) is a constant since the imaginary part is 0, and letting $z \to \infty$ along the imaginary axis we conclude that the constant is 0, so (3.3.4) is valid.

Whenever (3.3.4) holds, multiplication by $\varphi \in C_B$ and integration gives

$$\pi^{-1} \int \varphi(x) \operatorname{Im} f(x+iy)\, dx = \int d\mu(\xi) y/\pi \int |\xi - x - iy|^{-2} \varphi(x)\, dx \to \int \varphi(\xi)\, d\mu(\xi).$$

To justify this we note that

$$y/\pi \int |\xi - x - iy|^{-2} \varphi(x)\, dx = \pi^{-1} \int (1+t^2)^{-1} \varphi(\xi + ty)\, dt$$

is bounded by $\sup |\varphi|$ and converges to $\varphi(\xi)$ when $y \to 0$ by the dominated convergence theorem. In the same way we justify the limit of the repeated integral. Thus

$$\int \varphi(x) \operatorname{Im} \pi^{-1} f(x+iy)\, dx \to \int \varphi(x)\, d\mu(x), \;\; y \to +0, \quad \text{if } \varphi \in C_B.$$

Since (3.3.5) follows at once from the representation formulas for f and $\operatorname{Im} f$, the lemma is proved.

If we apply Lemma 3.3.2 to $f(z) = (R(z)u, u)$, $u \in H$, which satisfies (3.3.2) with $C = \|u\|^2$, we conclude that for $u = v$ the limit

$$\begin{aligned} e(u,v,\varphi) &= \lim_{y \to +0} (2\pi i)^{-1} \int (R(x+iy)u - R(x-iy)u, v)\varphi(x)\, dx \\ &= \lim_{y \to +0} y/\pi \int (R(x+iy)u, R(x+iy)v)\varphi(x)\, dx \end{aligned} \tag{3.3.6}$$

exists if $\varphi \in C_B$ and is a measure with norm $\le \|u\|^2$. (The second equality follows from (3.2.4).) Thus

$$|e(u,u,\varphi)| \le \|u\|^2 \sup|\varphi|, \quad e(u,u,\varphi) \ge 0 \quad \text{if } \varphi \ge 0, \tag{3.3.7}$$

$$e(u,v,\varphi_j) \to e(u,v,\varphi) \quad \text{if } \varphi_j \to \varphi \text{ pointwise and boundedly,} \tag{3.3.8}$$

by which we shall mean that there is a constant C such that $|\varphi_j| \le C$ for every j, and that $\varphi_j(x) \to \varphi(x)$ for every x when $j \to \infty$. Here $\varphi, \varphi_j \in C_B$. We obtain (3.3.8) from (3.3.7) and the dominated convergence theorem when $u = v$. The statement for arbitrary u, v follows, for if $S(u,v)$ is linear in u, antilinear in v, then

$$4S(u,v) = S(u+v,u+v) - S(u-v,u-v) + iS(u+iv,u+iv) - iS(u-iv,u-iv).$$

(This is called *polarization.*) Instead of (3.3.7) we only obtain by first considering the case where $\|u\| = \|v\| = 1$

$$(3.3.7)' \qquad |e(u,v,\varphi)| \le 2\|u\|\|v\| \sup|\varphi|; \quad u,v \in H,\ \varphi \in C_B.$$

The factor 2 will be removed later on.

Next we shall prove that

$$(3.3.9) \qquad e(u,v,1) = (u,v).$$

To do so we may assume that $u, v \in \mathcal{D}_A$, for this is a dense set and $e(u,v,1)$ is a continuous function of u and v. But then we have

$$(A - zI)u = Au - zu, \quad \text{hence } u = R(z)Au - R(z)zu,$$

so it follows that

$$R(z)u = -u/z + R(z)Au/z = -u/z + O(|z|^{-2}), \quad \text{if } z \to \infty, \quad \text{with } \operatorname{Re} z \text{ fixed.}$$

In view of (3.3.5) this proves (3.3.9) when $u = v \in \mathcal{D}_A$. By polarization we obtain (3.3.9) for arbitrary $u, v \in \mathcal{D}_A$.

If $u \in \mathcal{D}_A$ we have already observed that $R(z)Au = u + zR(z)u$. Taking $z = x \pm iy$ with x real and $y > 0$, we obtain

$$(R(x+iy) - R(x-iy))Au = x(R(x+iy) - R(x-iy))u + iy(R(x+iy) + R(x-iy))u,$$

which implies that when $\varphi \in C_B$ and $\varphi_1(\lambda) = \lambda\varphi(\lambda) \in C_B$

$$e(Au,v,\varphi) = e(u,v,\varphi_1) + \lim_{y \to +0} \int \frac{y}{2\pi}((R(x+iy) + R(x-iy))u, v)\varphi(x)\,dx.$$

Now we have

$$\pi^{-1}|y| \int \|R(x+iy)u\|^2\,dx \to e(u,u,1) = \|u\|^2, \quad y \to 0,$$

and φ is square integrable, so it follows by Cauchy-Schwarz' inequality that the integral converges to 0. Hence

(3.3.10)

$$e(Au,v,\varphi) = e(u,v,\varphi_1), \quad \text{if } u \in \mathcal{D}_A,\ v \in H,\ \varphi, \varphi_1 \in C_B, \text{ and } \varphi_1(\lambda) = \lambda\varphi(\lambda).$$

If $0 \le \varphi \in C_B$ has compact support, two applications of this result give

$$e(Au,Au,\varphi) = e(u,u,\lambda^2\varphi), \quad u \in \mathcal{D}_A.$$

Letting $\varphi \nearrow 1$ we obtain if e also denotes the extension of the positive measure $\varphi \mapsto e(u,u,\varphi)$

$$e(u,u,\lambda^2) = \|Au\|^2 < \infty, \quad u \in \mathcal{D}_A. \tag{3.3.11}$$

The converse of (3.3.11) stating that $u \in \mathcal{D}_A$ if $e(u,u,\lambda^2) < \infty$ will be proved later.

Before proceeding we sum up what has been proved so far concerning the trilinear form $e(u,v,\varphi)$; $u,v \in H$, $\varphi \in C_B$; defined by (3.3.6):

(i) $e(u,v,\varphi)$ is linear in u and in φ, antilinear in v, and has the hermitian symmetry property

$$e(u,v,\varphi) = \overline{e(v,u,\bar{\varphi})}; \quad u,v \in H,\ \varphi \in C_B. \tag{3.3.12}$$

(ii) The continuity properties (3.3.7), (3.3.7)$'$, (3.3.8) are valid.

(iii)
$$e(u,v,1) = (u,v);\ u,v \in H.$$

(iv) If $u \in \mathcal{D}_A$, $v \in H$, $\varphi, \varphi_1 \in C_B$, $\varphi_1(\lambda) = \lambda\varphi(\lambda)$, then

$$e(Au,v,\varphi) = e(u,v,\varphi_1).$$

From these properties we have already derived (3.3.11). Our next step is to extend the definition of $e(u,v,\varphi)$ to a wider class of functions φ, namely the smallest class B of bounded functions containing C_B which has the property

$$\begin{aligned} &\text{If } \varphi_1, \varphi_2, \dots \in B,\ |\varphi_j| \le C \text{ for some } C \text{ and all } j, \\ &\qquad \varphi_j(\lambda) \to \varphi(\lambda) \text{ for every } \lambda \text{ as } j \to \infty, \text{ then } \varphi \in B. \end{aligned} \tag{B}$$

This is the algebra of bounded *Borel measurable* functions. Clearly B is a Banach space with the sup norm, for it is a closed linear subspace of $l^\infty(\mathbf{R})$. Every $\varphi \in B$ is measurable with respect to every measure on $\mathbf{R}$, for the set of functions in B having this property satisfies (B) and contains C_B, and B is minimal. In the same way we conclude that B is an algebra and that all properties (i)–(iv) are valid if $\varphi \in B$.

For fixed $\varphi \in C_B$ and $u \in H$, the map

$$v \mapsto e(u,v,\varphi)$$

is an antilinear form with norm $\le 2\|u\| \sup|\varphi|$, so there is a unique element $A_\varphi u \in H$ such that

$$e(u,v,\varphi) = (A_\varphi u, v); \quad u,v \in H,\ \varphi \in B. \tag{3.3.13}$$

A_φ is obviously a linear operator and

$$\|A_\varphi\| \le 2\sup|\varphi|. \tag{3.3.7$''$}$$

In view of (3.3.12) it is also clear that $A_\varphi^* = A_{\bar{\varphi}}$. From (3.3.8) it follows that $A_{\varphi_j} u \to A_\varphi u$ in the weak topology for every $u \in H$ if $\varphi_j \to \varphi$ pointwise and boundedly; this will be improved below.

Let φ be a continuous function with compact support, and define φ_k by $\varphi_k(\lambda) = \lambda^k \varphi(\lambda)$. It follows from (iv) that

$$(A_\varphi u, Av) = e(u, Av, \varphi) = e(u, v, \varphi_1) = (A_{\varphi_1} u, v); \quad v \in \mathcal{D}_A,\ u \in H.$$

Hence $A_\varphi u$ is in the domain of A and $AA_\varphi = A_{\varphi_1}$. Iteration gives $A^k A_\varphi = A_{\varphi_k}$ for every k. For every polynomial p, the product $p(A)A_\varphi$ is therefore well defined and equal to $A_{p\varphi}$. From this we shall deduce

$$A_{\psi\varphi} = A_\psi A_\varphi; \quad \psi, \varphi \in B. \tag{3.3.14}$$

To do so we first assume that φ and ψ are continuous functions with compact support, and we choose another such function χ which is real valued and equal to 1 on the supports of ψ and of φ. If p is a polynomial, we have

$$(A_\varphi u, A_{p\chi} v) = (A_\varphi u, p(A)A_\chi v) = (\bar{p}(A)A_\varphi u, A_\chi v) = (A_{\bar{p}\varphi} u, A_\chi v).$$

Now choose a sequence of polynomials p_j such that $\bar{p}_j \chi \to \psi$ uniformly, hence $\bar{p}_j \varphi \to \psi\varphi$ uniformly. Then we conclude that

$$(A_\varphi u, A_\psi^* v) = (A_{\psi\varphi} u, A_\chi v).$$

Letting $\chi \to 1$ boundedly and using that $A_1 = I$, we conclude that

$$(A_\psi A_\varphi u, v) = (A_{\psi\varphi} u, v),$$

which proves (3.3.14) for continuous functions with compact support.

Now the set of all $\psi \in B$ such that (3.3.14) is valid for every continuous φ with compact support satisfies (B) and contains C_B, so it must be equal to B. Repeating the same argument with the roles of φ and ψ reversed, we obtain (3.3.14) for all $\varphi, \psi \in B$.

An immediate consequence is that

$$\|A_\varphi\| \le \sup |\varphi|, \quad \varphi \in B. \tag{3.3.15}$$

In fact, if $M = \sup |\varphi|$ then

$$(A_\varphi u, A_\varphi u) = (A_{|\varphi|^2} u, u) = e(u, u, |\varphi|^2) \le e(u, u, M^2) = M^2(u, u).$$

The factor 2 has thereby been removed from (3.3.7)$'$ and (3.3.7)$''$. Next we note that

$$\|A_{\varphi_j} u - A_\varphi u\| \to 0 \quad \text{if } \varphi_j \to \varphi \text{ pointwise and boundedly.} \tag{3.3.16}$$

This follows from (3.3.8) since the square of the norm on the left-hand side is

$$(A_{|\varphi_j - \varphi|^2} u, u) = e(u, u, |\varphi_j - \varphi|^2).$$

We can also prove a converse of (3.3.11) promised above. Thus assume that $e(u, u, \lambda^2)$ is finite. Let $0 \le \varphi \le 1$ be continuous with compact support. Then $AA_\varphi u = A_{\varphi_1} u$ if $\varphi_1(\lambda) = \lambda\varphi(\lambda)$, hence

$$\|AA_\varphi u\|^2 = e(u, u, \varphi_1^2).$$

If we take a sequence of such functions φ^ν increasing to 1, it follows that

$$\|AA_{\varphi^\nu} u - AA_{\varphi^\mu} u\|^2 = e(u, u, (\varphi^\nu - \varphi^\mu)^2 \lambda^2) \to 0, \quad \text{as } \nu, \mu \to \infty.$$

Hence $f = \lim_{\nu\to\infty} AA_{\varphi^\nu} u$ exists, and since $A_{\varphi^\nu} u \to u$, it follows that $u \in \mathcal{D}_A$ and that $Au = f$. Thus we have proved the converse of (3.3.11) and in fact also that

$$\begin{gathered}\text{if } e(u, u, \lambda^2) < \infty, \text{ then } u \in \mathcal{D}_A \text{ and } Au = \lim AA_{\varphi^\nu} u, \text{ if } B \ni \varphi^\nu \to 1 \\ \text{pointwise and boundedly, and } \lambda\varphi^\nu(\lambda) \text{ is bounded for every } \nu.\end{gathered} \tag{3.3.17}$$

We are now ready to give a first version of the *spectral theorem*.

Theorem 3.3.3. *Given a self-adjoint operator A in H, there exists a unique homomorphism α from the algebra B of bounded Borel measurable functions to $\mathcal{L}(H,H)$ such that*

(i) $\alpha(\bar\varphi) = \alpha(\varphi)^*$, $\varphi \in B$;

(ii)$'$ $\|\alpha(\varphi)\| \le \sup|\varphi|$, *thus* $\|\alpha(\varphi_j) - \alpha(\varphi)\| \to 0$ *if* $\varphi_j \to \varphi$ *uniformly;*

(ii)$''$ *If* $B \ni \varphi_j \to \varphi$ *pointwise and boundedly, then* $\|\alpha(\varphi_j)u - \alpha(\varphi)u\| \to 0$ *for every* $u \in H$;

(iii) $\alpha(\varphi_1) = A\alpha(\varphi)$ *if* $\varphi, \varphi_1 \in B$ *and* $\varphi_1(\lambda) = \lambda\varphi(\lambda)$.

An element $u \in H$ is in the domain $\mathcal{D}_A$ of A if and only if there is an upper bound for $\|\alpha(\varphi_1)u\|$ when $\varphi_1(\lambda) = \lambda\varphi(\lambda)$ and $0 \le \varphi \le 1$ has compact support. In that case Au is the limit of $\alpha(\varphi_1^\nu)u$ if we take a sequence φ^ν converging to 1 everywhere, such that φ^ν has compact support and $0 \le \varphi^\nu \le 1$.

Proof. It only remains to prove that (i)–(iii) determine α uniquely. Let $\operatorname{Im} z \ne 0$ and set $R_z(\lambda) = (\lambda - z)^{-1}$. Then we have $\alpha(R_z) = R(z)$. In fact, since $\lambda R_z(\lambda) = 1 + zR_z(\lambda)$ we obtain from (iii)

$$A\alpha(R_z) = I + z\alpha(R_z),$$

that is, $(A - zI)\alpha(R_z) = I$. Hence $\alpha(R_z) = R(z)$. If $d\mu$ is the measure $\varphi \mapsto (\alpha(\varphi)u, u)$ for some fixed $u \in H$, this means that

$$\int (\lambda - z)^{-1}\, d\mu(\lambda) = (R(z)u, u).$$

But then the proof of Lemma 3.3.2 shows that $d\mu$ is the limit as $y \to +0$ of $\operatorname{Im}(R(x+iy)u, u)\, dx/\pi$, which proves the uniqueness.

Definition 3.3.4. A homomorphism $B \to \mathcal{L}(H,H)$ with the properties (i)–(ii) in Theorem 3.3.3 is called a *spectral measure* on $\mathbf{R}$.

The conditions (i)–(ii) are somewhat redundant. In particular, by (i)

$$\alpha(|\varphi|^2) = \alpha(\varphi)\alpha(\bar\varphi) = \alpha(\varphi)\alpha(\varphi)^*$$

is a positive operator, thus $\alpha(\varphi)$ is positive for every $\varphi \ge 0$ in B. Since $\alpha(M) = MI$ if M is a constant, the bound $\|\alpha(\varphi)\| \le \sup|\varphi|$ follows immediately.

Given any spectral measure we shall now see how to obtain a corresponding self-adjoint operator. More generally, we shall discuss how to define $\alpha(\varphi)$ as a possibly unbounded operator if φ is any Borel measurable function which is finite everywhere. We introduce as before for $\varphi \in B$

$$e(u, v, \varphi) = (\alpha(\varphi)u, v),$$

which is a positive measure when $u = v$. Its extension to arbitrary positive Borel measurable functions is denoted by $\tilde e(u, u, \varphi)$. Note that if $\varphi \in B$ we have

$$\|\alpha(\varphi)u\|^2 = (\alpha(|\varphi|^2)u, u) = e(u, u, |\varphi|^2).$$

For a general finite Borel measurable φ it is therefore natural to define the domain of $\alpha(\varphi)$ as the set of all $u \in H$ such that

$$\tilde e(u, u, |\varphi|^2) < \infty. \tag{3.3.18}$$

When this condition is fulfilled we take any sequence $\varphi_j \in B$ such that $\varphi_j \to \varphi$ pointwise and $|\varphi_j| \le \Phi$ for some Φ with $\tilde{e}(u,u,\Phi^2) < \infty$. For example, we can define $\varphi_j(\lambda) = \varphi(\lambda)$ if $|\varphi(\lambda)| \le j$ and $\varphi_j(\lambda) = 0$ otherwise, taking $\Phi = |\varphi|$. Then $\alpha(\varphi_j)u$ has a limit, for

$$\|(\alpha(\varphi_j) - \alpha(\varphi_k))u\|^2 = \tilde{e}(u,u,|\varphi_j - \varphi_k|^2) \to 0$$

by the dominated convergence theorem. The limit is clearly independent of the chosen sequence, for two sequences can be mixed alternatingly and we still have convergence. We can therefore define $\alpha(\varphi)u$ to be the limit of $\alpha(\varphi_j)u$. Since the domain of $\alpha(\varphi)$ defined by (3.3.18) is obviously linear and $\alpha(\varphi_j)$ is linear, it follows that $\alpha(\varphi)$ is a linear operator.

The domain of $\alpha(\varphi)$ is dense. For if χ_n is the characteristic function of the set where $|\varphi| \le n$, then $0 \le \chi_n \nearrow 1$ as $n \to \infty$, so $\alpha(\chi_n)u \to u$ for every $u \in H$. On the other hand, the definition of $\alpha(\varphi)$ shows at once that the range of $\alpha(\chi_n)$ is in the domain of $\alpha(\varphi)$ and that $\alpha(\varphi)\alpha(\chi_n)$ is the bounded operator $\alpha(\varphi\chi_n)$.

The adjoint of $\alpha(\varphi)$ is equal to $\alpha(\bar{\varphi})$. In fact,

$$(\alpha(\varphi\chi_n)u, v) = (u, \alpha(\bar{\varphi}\chi_n)v); \quad u, v \in H;$$

since $\varphi\chi_n$ is bounded. If u and v are in the domains of $\alpha(\varphi)$ and $\alpha(\bar{\varphi})$, it follows when $n \to \infty$ that $(\alpha(\varphi)u, v) = (u, \alpha(\bar{\varphi})v)$. On the other hand, assume that

$$(\alpha(\varphi)u, v) = (u, f), \quad \forall\, u \in \mathcal{D}_{\alpha(\varphi)},$$

while *a priori* we only know that $v, f \in H$. Then we obtain

$$(\alpha(\varphi)\alpha(\chi_n)u, v) = (\alpha(\chi_n)u, f), \quad \forall\, u \in H.$$

Since the left-hand side is equal to $(\alpha(\varphi\chi_n)u, v)$ and $\varphi\chi_n$ is bounded, it follows that $\alpha(\bar{\varphi}\chi_n)v = \alpha(\chi_n)f$. Hence

$$e(v,v,|\varphi\chi_n|^2) = e(f,f,|\chi_n|^2) \le \|f\|^2.$$

When $n \to \infty$ we obtain $e(v,v,|\varphi|^2) < \infty$. Thus v is in the domain of $\alpha(\bar{\varphi})$. We have therefore proved that $\alpha(\varphi)$ is a closed densely defined operator with adjoint $\alpha(\bar{\varphi})$. In particular, $\alpha(\varphi)$ *is self-adjoint if* φ *is real valued.*

It is immediately verified that

$$\alpha(\varphi)\alpha(\psi)u = \alpha(\varphi\psi)u, \quad \text{if } u \in \mathcal{D}_{\alpha(\psi)} \cap \mathcal{D}_{\alpha(\varphi\psi)}.$$

Hence condition (iii) in Theorem 3.3.3 is fulfilled if we take $A = \alpha(\varphi)$, where $\varphi(\lambda) = \lambda$. We have therefore established a bijective correspondence between self-adjoint operators and spectral measures.

Just as a positive measure on the real line can be described completely by means of an increasing function, we can also describe a spectral measure completely in terms of the operators

$$E_\lambda = \alpha(\varphi_\lambda); \quad \varphi_\lambda(t) = \begin{cases} 1, & \text{when } t < \lambda, \\ 0, & \text{when } t \ge \lambda. \end{cases}$$

We have

$$E_\lambda \text{ is an orthogonal projection, } E_\lambda E_\mu = E_{\min(\lambda,\mu)}, \tag{3.3.19}$$

$$E_\lambda f \to E_\mu f \quad \text{if } \lambda \nearrow \mu; \quad f \in H, \tag{3.3.20}$$

$$E_\lambda f \to 0 \quad \text{if } \lambda \to -\infty; \quad E_\lambda f \to f \quad \text{if } \lambda \to +\infty; \quad f \in H. \tag{3.3.21}$$

(3.3.19) shows that the spaces $E_\lambda H$ increase with λ. Conversely, given projections E_λ with these properties we can define $\alpha(\varphi)$ first for step functions which are continuous to the right, and prove that the properties of a spectral measure are valid then. The definition is then extended as in the definition of the Riemann integral to all continuous functions with compact support. Since $e(u,u,\varphi) = (\alpha(\varphi)u, u)$ is then defined for $u \in H$ and all continuous φ with compact support, and is a measure with total mass $\|u\|^2$, we can extend the definition as before to all Borel measurable bounded functions. It is immediately verified using (3.3.21) that the complete spectral measure thus obtained gives back the projections we started from. This justifies the use of the formal notation

$$\alpha(\varphi) = \int \varphi(\lambda)\, dE_\lambda,$$

analogous to the notation for the Stieltjes integral. In particular, the operator A is written in the form

$$A = \int \lambda\, dE_\lambda.$$

The domain of $\alpha(\varphi)$ consists of all u with

$$\int |\varphi(\lambda)|^2\, d\|E_\lambda u\|^2 < \infty. \tag{3.3.18$'$}$$

This integral is also equal to $\|\alpha(\varphi)u\|^2$.

Example 3.3.5. Let M be a locally compact topological space and $d\mu$ a positive measure on M. Take $H = L^2(d\mu)$. If a is any real valued function on M which is measurable with respect to $d\mu$, we define an operator A so that

$$Au = f \text{ means that } u, f \in L^2(d\mu), \text{ and } au = f.$$

It is clear that A is self-adjoint. The corresponding spectral measure α assigns to the Borel function φ the operator corresponding to multiplication by $\varphi(a)$. In particular, the projection E_λ is multiplication by the characteristic function of the set where $a < \lambda$. If $M = \mathbf{R}$ and $a(\xi) = \xi$, the projection E_λ is multiplication by the characteristic function of $(-\infty, \lambda)$. Note that the eigenvalues correspond to points where $d\mu$ has positive mass — or in the general case level surfaces of a which have a positive measure for $d\mu$. There may of course be no eigenvalues at all.

Example 3.3.6. Let $P(D)$, $D = -i\partial/\partial x$, be a partial differential operator in $\mathbf{R}^n$ with constant coefficients such that $P(\xi)$ is real for $\xi \in \mathbf{R}^n$. We define $Au = f$ if $u, f \in H = L^2(\mathbf{R}^n)$ and $f = P(D)u$ in the sense of distribution theory. The Fourier transformation $F : H \to H$ is a *unitary* operator when properly normalized, and the transformed operator FAF^{-1} is multiplication by $P(\xi)$ as in Example 3.3.5. This shows that the projections E_λ can be written as convolution with the inverse Fourier transform of the characteristic function of the set where $P(\xi) < \lambda$, multiplied by a constant factor.

For differential operators with variable coefficients or operators in subsets of $\mathbf{R}^n$, spectral theory consists in developing a substitute for the Fourier transformation which will transform the operator to the special form discussed in Example 3.3.5. We shall study two elementary examples.

Example 3.3.7. Let $0 \le p \in C_0(\mathbf{R})$. With $H = L^2(\mathbf{R}_+)$ we denote by A the operator defined by $Au = -u'' + pu$ when u is in

$$\mathcal{D}_A = \{u \in C^1(\overline{\mathbf{R}}_+) \cap H; u(0) = 0, u'' \in H \text{ in the sense of distribution theory}\}.$$

(That $u'' \in L^2$ in the sense of distribution theory means precisely that u' is absolutely continuous and that u'', defined pointwise, is in L^2.) We leave as an exercise for the reader to verify that A is self-adjoint and positive if $p = 0$ (cf. Example 3.2.2). Since $u \mapsto pu$ is positive, bounded and self-adjoint, it follows that this remains true when $0 \le p \in C_0(\mathbf{R})$. There is no eigenfunction with eigenvalue 0, for if $u \in \mathcal{D}_A$, $Au = -u'' + pu = 0$, then u is a linear function for large t, which must be 0 since $u \in H$. Hence u is identically 0 by the uniqueness theorem for the Cauchy problem. In the same way we conclude that there is no positive eigenvalue either.

To determine the spectral measure we can therefore concentrate on studying $e(u, v, \varphi)$ when φ is continuous and has compact support in $(0, \infty)$. To do so we shall examine the resolvent of A. If $f \in C_0(\mathbf{R}_+)$ and $\operatorname{Im} z > 0$, then $u = R(z)f$ means that

$$-u'' + pu - zu = f, \quad u(0) = 0, \tag{3.3.22}$$

so this equation must have a unique solution in H. When t is large, u is a solution of the homogeneous equation $-u'' - zu = 0$, so u must be a linear combination of $\exp(it\sqrt{z})$ and $\exp(-it\sqrt{z})$ then. But one of these is exponentially increasing so it is ruled out since $u \in L^2$. If $\sqrt{z}$ is defined so that the imaginary part is positive, it follows that $R(z)f$ is the unique solution of (3.3.22) which for large t is of the form $C\exp(it\sqrt{z})$. We shall now show that $R(x + iy)f$ has a limit when $y \to +0$, although it will not belong to H.

The solution of the Cauchy problem

$$-v'' + pv - zv = f, \quad v(0) = v'(0) = 0 \tag{3.3.23}$$

depends continuously on z, and for large t it is of the form

$$C_+(z)\exp(it\sqrt{z}) + C_-(z)\exp(-it\sqrt{z}).$$

Here C_+ and C_- are continuous functions of z when $\operatorname{Im} z \ge 0$, if $\sqrt{z}$ is in the first quadrant. The solution of the Cauchy problem for the homogeneous equation

$$-w'' + pw - zw = 0, \quad w(0) = 0,\ w'(0) = 1, \tag{3.3.24}$$

is for large t of the form

$$A_+\exp(it\sqrt{z}) + A_-\exp(-it\sqrt{z})$$

where A_+ and A_- are also continuous functions of z. (We assume that $z \neq 0$.) When $z > 0$ the coefficients A_+ and A_- are complex conjugates since w is real, and since w is not identically 0 neither one can be 0 then. Hence $A_- \neq 0$ in a neighborhood of $\mathbf{R}_+$. When z is in this neighborhood and $\operatorname{Im} z > 0$, it follows that

$$u = R(z)f = v - C_- w/A_-.$$

When $\operatorname{Im} z > 0$ and $z \to \lambda > 0$ it follows that $R(z)f$ converges to $v - C_- w/A_-$, which is proportional to $e^{it\sqrt{\lambda}}$ for large t. We shall now determine this solution fairly explicitly.

We want to write the desired solution of (3.3.22) with $z = \lambda > 0$ in the form

$$u(t) = \int_0^\infty K_\lambda(t,s) f(s)\, ds.$$

This requires that

$$K_\lambda(0,s) = 0; \quad K_\lambda(t,s) \text{ is proportional to } e^{it\sqrt{\lambda}} \text{ for large } t, \tag{3.3.25}$$

$$\begin{aligned} &(-\frac{d^2}{dt^2} + p - \lambda) K_\lambda(t,s) = 0 \text{ when } t \neq s; \\ &K_\lambda(s-0,s) = K_\lambda(s+0,s), \ \ \frac{dK_\lambda(s-0,s)}{dt} = 1 + \frac{dK_\lambda(s+0,s)}{dt}. \end{aligned} \tag{3.3.26}$$

((3.3.26) means of course that $(-d^2/dt^2 + p - \lambda)K_\lambda(t,s) = \delta(t-s)$ in the sense of distribution theory.) Let $w_\lambda(t)$ be the solution of $-u'' + pu - \lambda u = 0$ which is equal to $\exp(it\sqrt{\lambda})$ for large t. Clearly w_λ and $\bar{w}_\lambda$ are linearly independent so they form a basis for the solutions of this linear differential equation. We can now rewrite (3.3.25), (3.3.26) as follows:

$$K_\lambda(t,s) = a(s) w_\lambda(t) + b(s)\overline{w_\lambda(t)}, \quad t < s; \ K_\lambda(t,s) = c(s) w_\lambda(t), \quad t > s;$$
$$a w_\lambda(0) + b\overline{w_\lambda(0)} = 0, \quad a w_\lambda(s) + b\overline{w_\lambda(s)} - c w_\lambda(s) = 0,$$
$$a w'_\lambda(s) + b\overline{w'_\lambda(s)} - c w'_\lambda(s) = 1.$$

To solve these equations for $a(s), b(s), c(s)$ we recall that the Wronsky determinant is a constant, thus

$$\begin{vmatrix} \overline{w_\lambda(s)} & w_\lambda(s) \\ \overline{w'_\lambda(s)} & w'_\lambda(s) \end{vmatrix} = 2i\sqrt{\lambda}$$

since this is true for large s. (That the determinant is constant follows if one differentiates each row and takes the differential equation into account.) This gives

$$c(s) = (w_\lambda(s)\overline{w_\lambda(0)} - w_\lambda(0)\overline{w_\lambda(s)})/2i\sqrt{\lambda} w_\lambda(0)$$

which is the only coefficient that we shall need.

Since $R(\bar{z})f = \overline{R(z)\bar{f}}$, because A is real, the limit of $R(z)f$ as $z \to \lambda > 0$ from the lower half plane is $\int \overline{K_\lambda(t,s)} f(s)\, ds$. Hence (see (3.3.6))

$$\begin{aligned} e(u,v,\varphi) &= \lim_{y \to +0} (2\pi i)^{-1} \int ((R(\lambda + iy) - R(\lambda - iy))u, v)\varphi(\lambda)\, d\lambda \\ &= \pi^{-1} \iiint \big(\operatorname{Im} K_\lambda(t,s)\big) u(s)\overline{v(t)} \varphi(\lambda)\, d\lambda\, ds\, dt, \end{aligned}$$

if $u, v, \varphi \in C_0(\mathbf{R}_+)$. Here

$$\operatorname{Im} K_\lambda(t,s) = (c(s)w_\lambda(t) - \overline{c(s)w_\lambda(t)})/2i,$$

when $t > s$. But $\operatorname{Im} K(t,s)$ and $d \operatorname{Im} K(t,s)/dt$ are continuous when $t = s$, so this is true for all s and t. Hence

$$\begin{aligned}\operatorname{Im} K_\lambda(s,t) &= (c(s)w_\lambda(t) - \overline{c(s)w_\lambda(t)})/2i \\ &= c(s)(w_\lambda(t) - w_\lambda(0)\overline{w_\lambda(t)}/\overline{w_\lambda(0)})/2i = e(t,\lambda)\overline{e(s,\lambda)}/4\sqrt{\lambda},\end{aligned}$$

where

$$e(s,\lambda) = w_\lambda(s) - w_\lambda(0)\overline{w_\lambda(s)}/\overline{w_\lambda(0)}.$$

This is the unique solution of the differential equation $-u'' + pu - \lambda u = 0$ with $u(0) = 0$ which for large s is of the form $e^{it\sqrt{\lambda}} + C_\lambda e^{-it\sqrt{\lambda}}$, where C_λ is a constant (of absolute value 1). Summing up, we have found that

$$e(u,v,\varphi) = \int_0^\infty u(s)\overline{e(s,\lambda)}\,\overline{v(t)}e(t,\lambda)\varphi(\lambda)\,ds\,dt\,d\mu(\lambda), \tag{3.3.27}$$

where $d\mu(\lambda) = d\lambda/4\pi\sqrt{\lambda} = d(\sqrt{\lambda}/2\pi)$. Here it is assumed that $u, v, \varphi \in C_0(\mathbf{R}_+)$.

(3.3.27) suggests that one should introduce

$$\tilde{u}(\lambda) = \int_0^\infty u(s)\overline{e(s,\lambda)}\,ds, \quad \lambda > 0; \qquad u \in C_0(\mathbf{R}_+). \tag{3.3.28}$$

This can be regarded as a generalized Fourier (or rather sine) transform. We can now write (3.3.27) in the form

$$e(u,v,\varphi) = \int_0^\infty \tilde{u}(\lambda)\overline{\tilde{v}(\lambda)}\varphi(\lambda)\,d\mu(\lambda). \tag{3.3.27$'$}$$

Taking $u = v$ and letting $\varphi \nearrow 1$ on $\mathbf{R}_+$, we obtain

$$e(u,u,\varphi) = \int_0^\infty |\tilde{u}(\lambda)|^2\,d\mu(\lambda),$$

if φ is the characteristic function of $\mathbf{R}_+$. Since $A \geq 0$ the left-hand side is 0 if φ is replaced by the characteristic function of $\mathbf{R}_-$, and since A has no eigenfunction corresponding to the eigenvalue 0, the same is true if φ is replaced by the characteristic function of the origin. Hence

$$\|u\|^2 = e(u,u,1) = \int_0^\infty |\tilde{u}(\lambda)|^2\,d\mu(\lambda).$$

The map $u \mapsto \tilde{u}$ can therefore be extended by continuity to an isometric linear map $U : H \to L^2(\mathbf{R}_+, d\mu)$. It remains to prove that it is surjective and that it transforms A to multiplication by λ.

If $\varphi \in B$ we have

$$(A_\varphi u, v) = e(u,v,\varphi) = \int_0^\infty \tilde{u}(\lambda)\overline{\tilde{v}(\lambda)}\varphi(\lambda)\,d\mu(\lambda); \quad u, v \in H.$$

On the other hand, the isometry of U gives

$$\int_0^\infty f(t)\overline{v(t)}\,dt = \int_0^\infty \tilde{f}(\lambda)\overline{\tilde{v}(\lambda)}\,d\mu(\lambda); \quad f, v \in H;$$

so $\widetilde{A_\varphi u}$ must be the projection in $L^2(\mathbf{R}_+, d\mu)$ of $\varphi\tilde{u}$ on the range of U. Since

$$\|\widetilde{A_\varphi u}\|^2 = \|A_\varphi u\|^2 = e(u, u, |\varphi|^2) = \int |\tilde{u}|^2|\varphi|^2\,d\mu = \int |\varphi\tilde{u}|^2\,d\mu,$$

it follows that $\widetilde{A_\varphi u}$ is equal to $\varphi\tilde{u}$, for the norm would otherwise be smaller. This implies that the range of U is invariant under multiplication by functions in B. For every $\lambda > 0$ we can choose $u \in C_0(\mathbf{R})$ with $\tilde{u}(\lambda) \neq 0$. Using a partition of unity we conclude that the range of U contains $C_0(\mathbf{R}_+)$, so it must be equal to $L^2(\mathbf{R}_+, d\mu)$.

Summing up, we have therefore proved that the closure of the map $u \mapsto \tilde{u}$ defined by (3.3.28) is a unitary operator $U : H \to L^2(\mathbf{R}_+, d\mu)$, and that UAU^{-1} is multiplication by λ as in Example 3.3.5.

The same method can be used in principle to study the Schrödinger equation for many variables when the potential is sufficiently small at infinity. However, the technicalities are of course more difficult then. We shall instead give an example showing how the method can be used with strikingly different conclusions when the potential is periodic.

Example 3.3.8. Let $p \in C(\mathbf{R})$ be real valued and periodic with period 1. With $H = L^2(\mathbf{R})$ define $Au = -u'' + pu$ with the domain

$$\mathcal{D}_A = \{u \in C^1(\mathbf{R}) \cap H; u'' \in H \text{ in the sense of distribution theory}\}.$$

Since $u \mapsto pu$ is bounded and self-adjoint, it is clear by Example 3.3.6 that A is self-adjoint. To study the resolvent we take $f \in C_0(\mathbf{R})$ and look when $\operatorname{Im} z > 0$ for the solution of

$$-u'' + pu - zu = f, \qquad u \in L^2(\mathbf{R}). \tag{3.3.29}$$

Outside a compact set we have a solution of the homogeneous differential equation, and we introduce a basis for such solutions,

$$-u_j'' + pu_j - zu_j = 0 \text{ on } \mathbf{R}; \quad \begin{pmatrix} u_0 & u_1 \\ u_0' & u_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ at } 0. \tag{3.3.30}$$

The solutions $u_j(t, z)$ are analytic functions of $z \in \mathbf{C}$, and the determinant of the Wronsky matrix

$$W(t, z) = \begin{pmatrix} u_0(t, z) & u_1(t, z) \\ u_0'(t, z) & u_1'(t, z) \end{pmatrix}$$

is independent of t, hence equal to 1. If u is a solution of the homogeneous equation $-u'' + pu - zu = 0$, then

$$\begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} = W(t, z)\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix},$$

so the periodicity of p gives for every integer n

$$W(t+n,z) = W(t,z)W(n,z) = W(t,z)W(1,z)^n. \tag{3.3.31}$$

Since the product of the eigenvalues of $W(1,z)$ is equal to 1, there is an eigenvalue τ with $|\tau| \le 1$. We shall show that $|\tau| < 1$ if $\operatorname{Im} z \ne 0$. To do so we choose $(a,b) \in \mathbf{C}^2$ with $|a|^2+|b|^2 = 1$ and $W(1,z)\begin{pmatrix} a \\ b \end{pmatrix} = \tau \begin{pmatrix} a \\ b \end{pmatrix}$, and let $u(t) = au_0(t)+bu_1(t)$. Since $u(t+n) = \tau^n u(t)$ for integers n, it follows if we multiply the equation $-u''+pu-zu = 0$ by $\bar u$ and integrate from 0 to a positive integer N that

$$0 = u'(0)\overline{u(0)} - u'(N)\overline{u(N)} + \int_0^N (p-z)|u|^2\,dt + \int_0^N |u'(t)|^2\,dt,$$

$$\operatorname{Im}\,(u'(0)\overline{u(0)} - u'(N)\overline{u(N)}) = \operatorname{Im} z \int_0^N |u|^2\,dt = \operatorname{Im} z \int_0^1 |u|^2\,dt \sum_0^{N-1} |\tau|^{2n}.$$

When $N \to \infty$ we conclude that $|\tau| < 1$ and that

$$|\operatorname{Im} z|/(1-|\tau|^2) \le C$$

when $|z|$ is bounded. Hence the difference between the eigenvalues of W can be bounded from below,

$$|\tau - 1/\tau| = |(\tau^2-1)/\tau| \ge (1-|\tau^2|)/|\tau| \ge |\operatorname{Im} z|/(C|\tau|). \tag{3.3.32}$$

The characteristic root $\tau(z)$ in the unit disc is therefore an analytic function in the upper half plane. The characteristic equation $\det\,(W(1,z)-\tau I) = 0$ can be written

$$\tau^2 - 2\gamma(z)\tau + 1 = 0, \qquad 2\gamma(z) = u_0(1,z) + u_1'(1,z)$$

with roots $\gamma(z) \pm \sqrt{\gamma^2-1}$. It follows that $\gamma(z) \ne \pm 1$ when $\operatorname{Im} z \ne 0$, and (3.3.32) shows that $1-\gamma^2$ can at most have double zeros on the real axis. When $\lambda \in \mathbf{R}$ and $\gamma(\lambda) = 1$, then the equation $-u''+pu-\lambda u = 0$ has a non-trivial solution with period 1, and when $\gamma(\lambda) = -1$ there is one with $u(t+1) = -u(t)$, hence $u(t+2) = u(t)$. There is no eigenfunction $\in L^2$, for if $\gamma(\lambda) = \pm 1$ then the equation $-u''+pu-\lambda u = 0$ has either two linearly independent periodic solutions or else one which is periodic and one for which $u(t+1) \mp u(t)$ is periodic and not identically 0, which implies that u cannot be in L^2.

When $\operatorname{Im} z > 0$ we denote by $u_\pm(t,z)$ solutions of the equation $-u''+pu-zu = 0$ with

$$u_\pm(t+1,z) = \tau(z)^{\pm 1} u_\pm(t,z), \tag{3.3.33}$$

normalized so that the Wronsky determinant is 1,

$$\begin{vmatrix} u_+ & u_- \\ u_+' & u_-' \end{vmatrix} = 1. \tag{3.3.34}$$

This can be done continuously when $\operatorname{Im} z \geq 0$ and $\tau(z)^2 \neq 1$. When $\operatorname{Im} z > 0$ the solution of (3.3.29) is given by

$$u(t) = \int_{-\infty}^{\infty} K_z(t,s) f(s)\, ds, \tag{3.3.35}$$

$$K_z(t,s) = \begin{cases} u_-(s,z)u_+(t,z), & \text{if } t > s, \\ u_+(s,z)u_-(t,z), \text{if } t < s. \end{cases} \tag{3.3.36}$$

In fact, $K_z(t,s)$ decreases exponentially when $t \to \pm\infty$, $(-d^2/dt^2 + p - z)K_z(t,s) = 0$ when $t \neq s$, $K(t,s)$ is continuous also when $t = s$, and

$$dK_z(s-0,s)/dt - dK_z(s+0,s)/dt = u_+(s,z)u_-'(s,z) - u_-(s,z)u_+'(s,z) = 1$$

by (3.3.34). This shows that u satisfies (3.3.29) and there is no other solution since every solution $au_+ + bu_- \not\equiv 0$ of the homogeneous equation is exponentially increasing either at $+\infty$ or at $-\infty$.

If $\varphi \in C_0(\mathbf{R})$ and $\gamma^2(\lambda) \neq 1$ when $\lambda \in \operatorname{supp} \varphi$, we obtain for $u, v \in C_0(\mathbf{R})$ (see (3.3.6))

$$\begin{aligned} e(u,v,\varphi) &= \lim_{y \to +0} \frac{1}{2\pi i} \int \big((R(\lambda + iy) - R(\lambda - iy))u, v\big)\varphi(\lambda)\, d\lambda \\ &= \frac{1}{\pi} \iiint \operatorname{Im} K_{\lambda+i0}(t,s) u(s)\overline{v(t)}\varphi(\lambda)\, d\lambda\, ds\, dt. \end{aligned} \tag{3.3.37}$$

Now

$$\operatorname{Im} K_{\lambda+i0}(t,s) = (u_-(s,\lambda)u_+(t,\lambda) - \overline{u_-(s,\lambda)u_+(t,\lambda)})/2i,$$

for this is true when $t > s$ and $t \mapsto \operatorname{Im} K_{\lambda+i0}(t,s)$ is a solution of the homogeneous differential equation on all of $\mathbf{R}$. At a point with $\gamma(\lambda)^2 > 1$, the characteristic roots are real so $u_\pm$ can be chosen real, hence $\operatorname{Im} K_{\lambda+i0}(t,s) = 0$ then. This means that there is no spectrum in such intervals, which we could also see right away from the fact that $K_\lambda(t,s)$ is then the kernel of a bounded operator in L^2, so the resolvent exists also at λ. In intervals where $\gamma(\tau)^2 < 1$ the eigenvalues of $W(1,\lambda)$ are complex conjugates, so

$$\overline{u}_-(s,\lambda) = cu_+(s,\lambda).$$

Hence u_+ and $\overline{u}_+$ are a basis for the solutions of $-u'' + pu - \lambda u = 0$. The normalization condition (3.3.34) becomes

$$\bar{c}\begin{vmatrix} u_+ & \bar{u}_+ \\ u_+' & \bar{u}_+' \end{vmatrix} = 1, \quad -c^{-1}\begin{vmatrix} u_- & \bar{u}_- \\ u_-' & \bar{u}_-' \end{vmatrix} = 1,$$

which shows that c is purely imaginary and that

$$2i \operatorname{Im} K_{\lambda+i0}(t,s) = \frac{u_+(t,\lambda)\overline{u}_+(s,\lambda)}{\begin{vmatrix} u_+ & \bar{u}_+ \\ u_+' & \bar{u}_+' \end{vmatrix}} - \frac{u_-(t,\lambda)\overline{u}_-(s,\lambda)}{\begin{vmatrix} u_- & \bar{u}_- \\ u_-' & \bar{u}_-' \end{vmatrix}}.$$

When $\varphi(\lambda)\, d\lambda$ tends to the Dirac measure at λ it follows from (3.3.37) that

$$\iint \operatorname{Im} K_{\lambda+i0}(t,s)v(s)\overline{v}(t)\, ds\, dt \geq 0,$$

and choosing $v \in C_0(\mathbf{R})$ with $\int u_-(t,\lambda)\overline{v(t)}\,dt = 0$ or $\int u_+(t,\lambda)\overline{v(t)}\,dt = 0$, we conclude that

$$2i\begin{vmatrix} u_+ & \bar u_+ \\ u'_+ & \bar u'_+ \end{vmatrix} > 0, \quad 2i\begin{vmatrix} u_- & \bar u_- \\ u'_- & \bar u'_- \end{vmatrix} < 0.$$

We can therefore change the normalization of $u_\pm$ so that instead of (3.3.34)

$$2i\begin{vmatrix} u_+ & \bar u_+ \\ u'_+ & \bar u'_+ \end{vmatrix} = 1, \quad 2i\begin{vmatrix} u_- & \bar u_- \\ u'_- & \bar u'_- \end{vmatrix} = -1. \tag{3.3.38}$$

With that convention we can write (3.3.37) in the form

$$e(u,v,\varphi) = \int_\Sigma \tilde u_+(\lambda)\overline{\tilde v_+(\lambda)}\varphi(\lambda)\,d\lambda/\pi + \int_\Sigma \tilde u_-(\lambda)\overline{\tilde v_-(\lambda)}\varphi(\lambda)\,d\lambda/\pi,$$

where

$$\Sigma = \{\lambda \in \mathbf{R}; \gamma(\lambda)^2 < 1\}, \qquad \tilde u_\pm(\lambda) = \int u(t)\overline{u_\pm(t,\lambda)}\,dt$$

provided that $\varphi \in C_0(\mathbf{R})$ and $\gamma(\lambda)^2 \neq 1$ in the support. Since we know that the isolated zeros of $\gamma(\lambda)^2 - 1$ carry no spectral measure, it follows when we let $\varphi(\lambda) \nearrow 1$ that

$$\|u\|^2 = \int_\Sigma (|\tilde u_+(\lambda)|^2 + |\tilde u_-(\lambda)|^2)\,d\lambda/\pi.$$

As in the preceding example we conclude that the closure of the map $C_0(\mathbf{R}) \ni u \mapsto (\tilde u_+, \tilde u_-)$ is a unitary map $U : L^2(\mathbf{R}) \to L^2(\Sigma, d\lambda/\pi) \oplus L^2(\Sigma, d\lambda/\pi)$, and that UAU^{-1} is multiplication by λ. We leave for the reader to prove this by recalling the arguments used in the preceding example.

It is instructive to consider the trivial case $p = 0$. Then

$$\begin{gathered} u_0(t) = \cos(t\sqrt{z}),\ u_1(t) = \sin(t\sqrt{z})/\sqrt{z}, \\ W_0(1,z) = \begin{pmatrix} \cos(\sqrt{z}) & \sin(\sqrt{z})/\sqrt{z} \\ -\sqrt{z}\sin(\sqrt{z}) & \cos(\sqrt{z}) \end{pmatrix}, \\ \gamma(z) = \cos(\sqrt{z}), \quad \tau(z) = e^{i\sqrt{z}}, \quad u_\pm(t,\lambda) = e^{\pm it\sqrt{\lambda}}/(2\sqrt[4]{\lambda}). \end{gathered}$$

Hence $\tilde u_\pm(\lambda) = \hat u(\pm\sqrt{\lambda})/(2\sqrt[4]{\lambda})$ and

$$\int (|\tilde u_+|^2 + |\tilde u_-|^2)\,d\lambda/\pi = \int_{-\infty}^{\infty} |\hat u(\tau)|^2\,d\tau/2\pi,$$

where $\hat u$ is the Fourier transform of u, so we have just obtained Parseval's formula. In this example the equation $\gamma(\lambda) = 1$ has a simple root $\lambda = 0$ and double roots $\lambda = (2k\pi)^2$, where k is a positive integer. The equation $\gamma(\lambda) = -1$ has the double roots $\lambda = ((2k-1)\pi)^2$. These alternate and are all the real zeros of $\det(W(2,\lambda)-I)$. By a continuity argument using the fact that the zeros of $\gamma(\lambda)^2 - 1$ can be at most double, one obtains in general: The zeros of $\det(W(2,\lambda)-I)$, that is, the eigenvalues of $-u'' + pu$ on the interval $[0,2]$ with periodic boundary conditions are

$$\lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \ldots;$$

$\gamma = -1$ at $\lambda_{1+4k}, \lambda_{2+4k}$ and $\gamma < -1$ in between; $\gamma = 1$ at $\lambda_{3+4k}, \lambda_{4+4k}$ and $\gamma > 1$ in between. In $(\lambda_{4k}, \lambda_{4k+1})$ we have $\operatorname{Im} \tau > 0$, and in $(\lambda_{4k+2}, \lambda_{4k+3})$ we have $\operatorname{Im} \tau < 0$. The spectrum consists of the closure of the union Σ of these intervals.

Finally, we give an example showing how one can often prove self-adjointness for the Schrödinger equation with many variables with the singularities which are natural in the many-body problem.

Example 3.3.9. For the quantum mechanical two body problem with Coulomb force, the Schrödinger operator in $L^2(\mathbf{R}^3)$ is the sum of a kinetic energy part

$$H_0 = \sum_1^3 D_j^2, \quad D_j = -i\partial/\partial x_j,$$

and a potential part

$$V = \gamma |x|^{-1}.$$

We have seen in Example 3.3.6 that H_0 defines a self-adjoint operator in $L^2(\mathbf{R}^3)$, and claim that $H_0 + V$ is self-adjoint with the same domain. This will follow from Theorem 3.2.12 when γ is small, if we prove that

$$\int |u|^2 |x|^{-2}\, dx \le C(\|H_0 u\| + \|u\|)^2, \quad u \in \mathcal{D}_{H_0}. \tag{3.3.39}$$

Changing scales one can then remove the restriction that γ should be small. To prove (3.3.39) we first observe that taking Fourier transforms one obtains

$$\int |u'|^2\, dx \le \|H_0 u\| \|u\|, \quad u \in \mathcal{D}_{H_0}.$$

If $\chi \in C_0^2$ is equal to 1 in the unit ball and 0 outside the concentric ball with radius 2, we conclude that

$$\|\Delta(\chi u)\| \le C(\|\Delta u\| + \|u\|).$$

Now $\chi u = E * \Delta(\chi u)$, $E(x) = -(4\pi|x|)^{-1}$, so we have

$$|\chi u|^2 \le \int_{|x|<4} |E(x)|^2\, dx \|\Delta(\chi u)\|^2 \le C(\|H_0 u\| + \|u\|)^2,$$

hence

$$\int |\chi u|^2 |x|^{-2}\, dx \le C'(\|H_0 u\| + \|u\|)^2,$$

which immediately gives (3.3.39).

It is not hard to extend the argument above to the Schrödinger operator of a general many-body system, but we leave that for the reader.

3.4. The complete spectral theorem. We shall now analyze the spectral measure further and show that every self-adjoint operator is unitarily equivalent to a multiplication operator as described in Example 3.3.5:

Definition 3.4.1. A self-adjoint operator A_1 in the Hilbert space H_1 is said to be unitarily equivalent to a self-adjoint operator A_2 in the Hilbert space H_2, if there is a unitary map $U : H_1 \to H_2$, such that $A_2 U = U A_1$.

Examples 3.3.6, 3.3.7 and 3.3.8 were two concrete cases of unitary equivalence. We want to emphasize that the abstract results to be proved do not eliminate the interest of concrete solutions for concrete operators. Indeed, scattering theory is devoted to such results.

Let A be a self-adjoint operator in H, and let $u \in H$. If φ is a Borel measurable function with $e(u, u, |\varphi|^2) < \infty$, we have seen that $\varphi(A)u$ can be defined. (We use the notation $\varphi(A)$ instead of A_φ or $\alpha(\varphi)$ which occurred in Section 3.3.) Moreover,

$$\|\varphi(A)u\|^2 = e(u, u, |\varphi|^2).$$

If L_u^2 is the space of square integrable functions with respect to the measure $\varphi \mapsto e(u, u, \varphi)$, we obtain an isometric map

$$L_u^2 \ni \varphi \mapsto \varphi(A)u \in H.$$

The range is a closed subspace H_u of H. A dense subset is of course obtained by taking only functions $\varphi \in C_0$. If v is orthogonal to H_u it follows that H_v is orthogonal to H_u. In fact,

$$(\varphi(A)u, \psi(A)v) = ((\varphi\bar{\psi})(A)u, v) = 0; \quad \varphi, \psi \in C_0;$$

since $(\varphi\bar{\psi})(A)u \in H_u$.

Lemma 3.4.2. *Let H be a separable Hilbert space and A a self-adjoint operator in H. Then there exist finitely or countably many elements $u_1, u_2, \dots$ in H such that*

$$H = \bigoplus_1^N H_{u_j},$$

the orthogonal direct sum taken in the sense of Theorem 3.1.1.

Proof. Let $v_1, v_2, \dots$ be a dense sequence of elements in H. Set $u_1 = v_1$, $u_2 =$ orthogonal projection of v_2 on the orthogonal complement of H_{u_1}, ... , $u_k =$ orthogonal projection of v_k on the orthogonal complement of $H_{u_1} \oplus \cdots \oplus H_{u_{k-1}}$. The spaces then obtained are orthogonal to each other, and their orthogonal direct sum contains all v_j, so it must be equal to H. The lemma is proved.

By the lemma we have a unitary map

$$U : \bigoplus L_{u_j}^2 \ni (\varphi_1, \varphi_2, \dots) \mapsto \sum \varphi_j(A)u_j \in H.$$

If $\varphi \in B$ and $\dot{\varphi}$ denotes multiplication by φ in $L_{u_j}^2$, then the fact that $(\varphi\varphi_j)(A)$ is an extension of $\varphi(A)\varphi_j(A)$ shows that

$$U\dot{\varphi} = \varphi(A)U.$$

Since a spectral measure determines the corresponding operator uniquely, it follows that $U^{-1}AU$ is multiplication by λ in $\bigoplus_1^N L_{u_j}^2$. This is essentially the spectral theorem we are aiming for, but we shall study $\bigoplus_1^N L_{u_j}^2$ further in order to obtain an essentially unique representation for A. However, considering $\bigoplus L_{u_j}^2$ as the L^2

space on $\mathbf{R}\times\{j; 1\le j\le N\}$ with respect to the measure which is $\varphi\mapsto e(u_j,u_j,\varphi)$ when restricted to $\mathbf{R}\times\{j\}$, we already have the situation in Example 3.3.5.

A better understanding is obtained if we use the Radon-Nikodym theorem to find a measure $d\mu$ on $\mathbf{R}$ such that all the measures $\mu_j(\varphi)=e(u_j,u_j,\varphi)$ are absolutely continuous with respect to $d\mu$. We may for example choose

$$\mu(\varphi)=\sum_1^\infty 2^{-j}\mu_j(\varphi)/\|u_j\|^2,$$

which has total mass one. There exist non-negative Borel functions $\varrho_j\in L^2_{\text{loc}}(d\mu)$ such that $d\mu_j=\varrho_j^2\,d\mu$, $j=1,2,\dots$. If $(\varphi_1,\varphi_2,\dots)\in\bigoplus L^2_{u_j}$ we have then

$$\sum_j\|\varphi_j\|^2_{L^2_{u_j}}=\sum_j\int|\varrho_j\varphi_j|^2\,d\mu.$$

We shall now remove components which never play any role.

For every λ let $N(\lambda)$ be the number of integers $j=1,\dots,N$ such that $\varrho_j(\lambda)\ne 0$. Then we have $0\le N(\lambda)\le\infty$, and $N(\lambda)$ is Borel measurable. Since the integrals above do not change if $d\mu$ is replaced by the product with the characteristic function of the set where $N(\lambda)>0$, it is no restriction to assume that $N(\lambda)>0$ almost everywhere with respect to $d\mu$. Now we set

$$M_N=\{(\lambda,j);\lambda\in\mathbf{R},\ j\text{ positive integer}\le N(\lambda)\}. \tag{3.4.1}$$

This is a Borel set. On $\mathbf{R}\times\{1,2,\dots\}$ we consider the measure $d\nu$ which is the direct product of $d\mu$ and the counting measure on $\{1,2,\dots\}$; abusing notation slightly we denote the restriction to M_N also by $d\nu$. Now we obtain a unitary map

$$U_1:\bigoplus L^2_{u_j}\to L^2(M_N,d\nu)$$

as follows: If $(\varphi_1,\varphi_2,\dots)\in\bigoplus L^2_{u_j}$, then $U_1(\varphi_1,\varphi_2,\dots)$ is for the points in M_N over λ the sequence obtained from $(\varrho_1(\lambda)\varphi_1(\lambda),\varrho_2(\lambda)\varphi_2(\lambda),\dots)$ by dropping the places where $\varrho_j(\lambda)=0$. It is an easy exercise to verify that this is a unitary map. The map UU_1^{-1} is then a unitary equivalence between A and multiplication by λ in $L^2(M_N,d\nu)$.

Theorem 3.4.3. *Let A be a self-adjoint operator in a separable Hilbert space H. Then A is unitarily equivalent to an operator Λ defined as follows: $d\mu$ is a positive measure on $\mathbf{R}$, N is a Borel measurable function on $\mathbf{R}$ whose values are positive integers or $+\infty$, and M_N is defined by (3.4.1). By $d\nu$ we denote the direct product of $d\mu$ and the counting measure on the integers, restricted to M_N. Then Λ is multiplication by λ on $L^2(M_N,d\nu)$. An operator Λ_1 defined in the same way with another choice of measure $d\mu_1$ and dimension function N_1 is unitarily equivalent to Λ if and only if the measures $d\mu$ and $d\mu_1$ are equivalent and $N_1=N$ almost everywhere with respect to them.*

Proof. Only the uniqueness statement remains to be proved. First note that if $\varphi\in B$, then $\varphi(\Lambda)=0$ if and only if $\varphi=0$ almost everywhere with respect to $d\mu$. If Λ and Λ_1 are unitarily equivalent it follows that the null functions for $d\mu$ and $d\mu_1$

are the same, so the measures are equivalent. Thus $d\mu_1 = \varrho^2\, d\mu$, where ϱ is a Borel function $\neq 0$ almost everywhere and locally square integrable with respect to $d\mu$. The map $L^2(M_{N_1}, d\nu_1) \to L^2(M_{N_1}, d\nu)$ defined by multiplication with ϱ is then a unitary map transforming Λ_1 to the operator defined with the measure $d\mu$ and the dimension function N_1. Thus we may assume that $d\mu = d\mu_1$ in what follows.

Assume that $U : L^2(M_N, d\nu) \to L^2(M_{N_1}, d\nu)$ is a unitary equivalence, that is, $U\Lambda = \Lambda_1 U$. With a positive integer k we denote by χ the characteristic function of the set where $N(\lambda) \geq k$, and set for $j = 1, \dots, k$

$$u_j(\lambda, i) = \begin{cases} \chi(\lambda), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then we have for $\varphi \in C_0$

$$(\varphi(\Lambda)u_i, u_j) = \delta_{ij} \int \varphi(\lambda)\chi(\lambda)\, d\mu(\lambda).$$

If $v_i = Uu_i \in L^2(M_{N_1}, d\nu)$, then $(\varphi(\Lambda)u_i, u_j) = (\varphi(\Lambda_1)v_i, v_j)$, so by Fubini's theorem

$$\int \sum_{p=1}^{N_1(\lambda)} v_i(\lambda, p)\overline{v_j(\lambda, p)}\varphi(\lambda)\, d\mu(\lambda) = \delta_{ij} \int \varphi(\lambda)\chi(\lambda)\, d\mu(\lambda), \quad i, j = 1, \dots, k.$$

Hence we have

$$\sum_{p=1}^{N_1(\lambda)} v_i(\lambda, p)\overline{v_j(\lambda, p)} = \delta_{ij}\chi(\lambda)$$

for almost all λ with respect to $d\mu$. For such λ with $\chi(\lambda) = 1$ it follows that the vectors

$$(v_i(\lambda, 1), v_i(\lambda, 2), \dots), \quad i = 1, \dots, k,$$

are orthonormal, which implies that $N_1(\lambda) \geq k$. Thus $N_1(\lambda) \geq k$ almost everywhere in the set where $N(\lambda) \geq k$, so $N_1 \geq N$ almost everywhere. The proof is now complete, for the roles of N and N_1 may be interchanged.

Note that for the operator Λ in Theorem 3.4.3 the eigenvalues are the atomic part of the measure $d\mu$. At a point where $d\mu$ carries positive mass, the value of N is uniquely determined and is of course the dimension of the space of eigenvectors, that is, the multiplicity of the eigenvalue. It is therefore natural to call $N(\lambda)$ the *spectral multiplicity* also in the general situation. However, one should keep in mind that $N(\lambda)$ is not defined uniquely at any individual point outside the point spectrum, but only determined almost everywhere with respect to $d\lambda$.

Example 3.3.7 is clearly a case where the spectral multiplicity is 1 on $\mathbf{R}_+$ and 0 on $\overline{\mathbf{R}}_-$. What we obtained was an explicit version of Theorem 3.4.3. In Example 3.3.8 the spectral multiplicity is 2 in Σ and 0 elsewhere. For self-adjoint operators defined by ordinary differential operators the spectral multiplicity never exceeds the order of the operator — simply because a homogeneous ordinary differential equation has at most that many linearly independent solutions. For partial differential operators the spectral multiplicities usually become infinite (see Example 3.3.6). For this reason the rather arbitrary choice of an orthogonal decomposition in Lemma 3.4.2 does not give a satisfactory spectral representation for concrete partial differential operators.

Ordered sets. A set E is called (partially) ordered if there is given a subset $\mathcal{P} \subset E \times E$ such that, with the notation $x \prec y$ if $(x, y) \in \mathcal{P}$ we have

i) $x \prec y$ and $y \prec z \implies x \prec z$;
ii) if $x \prec y$ and $y \prec x$ then $x = y$.

An example is $\mathbf{R}^n$; we define $(x_1, \dots, x_n) \prec (y_1, \dots, y_n)$ if $x_j \le y_j$ for every j. Another example is the set of subsets of another set M.

An ordered set E is called linearly (or completely or totally) ordered if in addition

iii) for arbitrary $x, y \in E$ either $x \prec y$ or $y \prec x$ or $x = y$.

In every linearly ordered *finite* set there is a largest (and a smallest) element; we just have to compare the elements two by two to find one.

Definition. An ordered set E is called *inductively* ordered if every linearly ordered subset E_0 has an upper bound in E, that is, there is an element $x \in E$ with $y \prec x$ for every $y \in E_0$.

With this definition we have:

Zorn's lemma. *Every inductively ordered set has at least one maximal element, that is, an element x such that $x \prec y$, $y \in E$ implies $y = x$.*

In the applications E is often a set consisting of subsets of another set M. Then we have the following special case:

Let E be a set of subsets of another set M, ordered by inclusion, such that if F is a linearly ordered subset of E the union of the sets in F is also in E. Then E contains a maximal set, that is, a set contained in no strictly larger one belonging to E.

The Jordan normal form. In this section we shall complete the discussion of the spectral decomposition of a compact operator $T \in \mathcal{L}(B, B)$ given in Section 2.5. What remains is to study the decomposition of a nilpotent linear operator T in a finite dimensional vector space.

Theorem. *Let V be a finite dimensional vector space over $\mathbf{C}$ and let $T : V \to V$ be a nilpotent linear map, that is, $T^N = 0$ for some N. Then V can be written $V = \bigoplus V_j$ where $TV_j \subset V_j$ for each j, and for every V_j there is an element $x_j \in V_j$ and an integer $k_j > 0$ such that x_j, Tx_j, $\dots$, $T^{k_j - 1}x_j$ is a basis in V_j while $T^{k_j}x_j = 0$.*

Proof. The theorem is trivial if the dimension of V is equal to 1 or, more generally, if $\operatorname{Ker} T = V$, for then we just have to write V as a direct sum of one dimensional subspaces. By induction we may thus assume that it has already been proved for spaces of lower dimension and that $\operatorname{Ker} T \ne V$. Let $q : V \to W = V/\operatorname{Ker} T$ be the quotient map. T induces a map $W \to V$, hence a map $\widetilde{T} : W \to W$ such that $qTx = \widetilde{T}qx$, $x \in V$. The inductive hypothesis applied to W shows that $W = \bigoplus_1^J W_j$ where $\widetilde{T}W_j \subset W_j$ and W_j has a basis $\widetilde{T}^\nu y_j$, $0 \le \nu < k_j$, where $\widetilde{T}^{k_j} y_j = 0$. Choose x_j so that $qx_j = y_j$. Then $T^\nu x_j$ are linearly independent for

$1 \le j \le J$ and $0 \le \nu \le k_j$, but $T^{k_j+1}x_j = 0$ since $qT^{k_j}x_j = \widetilde{T}^{k_j}y_j = 0$. To prove the linear independence assume that we have a linear relation

$$\sum_1^J \sum_0^{k_j} a_{j\nu} T^\nu x_j = 0.$$

If we apply q it follows that

$$\sum_1^J \sum_0^{k_j-1} a_{j\nu} \widetilde{T}^\nu y_j = 0,$$

which proves that $a_{j\nu} = 0$ if $\nu < k_j$. Thus

$$\sum_1^J a_{jk_j} T^{k_j} x_j = 0,$$

which means that $\sum a_{jk_j} T^{k_j-1} x_j \in \operatorname{Ker} T$, that is,

$$\sum_1^J a_{jk_j} \widetilde{T}^{k_j-1} y_j = 0.$$

Hence all the coefficients vanish and the linear independenced is proved. Set

$$\widehat{V} = \bigoplus\nolimits_1^J V_j.$$

Since $q\widehat{V} = W$ we have $V = \widehat{V} + \operatorname{Ker} T$, so we can choose a subspace $V_0 \subset \operatorname{Ker} T$ such that $V = \widehat{V} \oplus V_0$. This completes the proof.

If with the notation in the theorem $T^{k_j-1}x_j, \dots, x_j$ are taken as basis vectors in V_j, then the matrix of T restricted to V_j takes the form

$$T_j = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Hence a nilpotent operator always has a matrix consisting of such diagonal blocks and zeros elsewhere. If $(T - \lambda I)^N = 0$ for some $\lambda \ne 0$ the only difference is that the diagonal elements are equal to λ. Repeated use of Theorem 2.5.15 specialized to the finite dimensional case now proves that every linear transformation T in a finite dimensional complex vector space V has a matrix built up by such Jordan blocks for a suitable choice of the basis. Theorem 2.5.15 also allows one to apply the result to arbitrary compact operators.

EXERCISES TO CHAPTER I

1. Determine $\dim \operatorname{Ker} T$ and $\dim \operatorname{Coker} T$ when $T = d/dx : V_1 \to V_2$ and
a) $V_1 = V_2$=the space of all polynomials on $\mathbf{R}$.
b) V_1 is the space of continuously differentiable functions on $\mathbf{R}$ and V_2 is the space of continuous functions on $\mathbf{R}$.
c) V_1 and V_2 consist of the functions with period 1 in the preceding case.
d) V_1 consists of the twice continuously differentiable functions on $\mathbf{R}$ and V_2 is the space of continuous functions on $\mathbf{R}$

2. Let V_1 and V_2 be linear subspaces of the vector space V. Prove that $\dim V_2 \le \operatorname{codim} V_1$ if $V_1 \cap V_2 = \{0\}$. Prove more generally that $\dim V_2 \le \dim(V_1 \cap V_2) + \operatorname{codim} V_1$ is always true.

3. Let (1.2.4) be a complex with all V_j finite dimensional. Prove that

$$\sum (-1)^j \dim V_j = \sum (-1)^j \dim (\operatorname{Ker} T_j / \operatorname{Im} T_{j-1}).$$

4. Show that if

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3,$$

is exact then

$$\dim V_1 + \dim V_3 = \dim V_2 + \dim \operatorname{Ker} T_1 + \dim \operatorname{Coker} T_2 \ge \dim V_2,$$

with no assumption on finite dimensionality.

5. Let V be the space of polynomials on $\mathbf{R}$, and let p be a fixed element $\neq 0$ in V. Determine the index of the multiplication operator $V \to V : q \mapsto pq$.

6. Let $T : V_1 \to V_2$ be a linear map such that $\operatorname{ind} T$ is defined. Show that one can find a linear map $S : V_1 \to V_2$ of finite rank such that either $\dim \operatorname{Ker}(T + S)$ or $\dim \operatorname{Coker}(T + S)$ is equal to 0. What is the other dimension then? Can one choose S of finite rank so that $\dim \operatorname{Ker}(T+S)$ and $\dim \operatorname{Coker}(T+S)$ have arbitrary non-negative integer values or $+\infty$ with difference $\operatorname{ind} T$?

7. Let V be a vector space over K and let $T : V \to V$ be a linear map of finite rank. Prove that one can choose $a_j \in V$ and K valued linear forms L_j on V, $j = 1, \dots, k$, such that

$$Tx = \sum_1^k L_j(x) a_j, \quad x \in V,$$

and that the rank of T is the smallest possible value for the integer k. Show that

$$\sum_1^k L_j(a_j)$$

is independent of the choice of representation; it is called the *trace* of T and denoted $\operatorname{Tr} T$.

Exercises to Chapter II

1. Let $\mathbf{T} = \{z \in \mathbf{C}; |z| = 1\}$ be the unit circle and let $\mathbf{T}^{\mathbf{R}}$ be the set of functions $\mathbf{R} \to \mathbf{T}$ with the product topology. Denote by M the subset consisting of the functions $f_\lambda : \mathbf{R} \ni x \mapsto e^{i\lambda x} \in \mathbf{T}$ where $\lambda \in \mathbf{R}$. Show that:
a) the closure of M in $\mathbf{T}^{\mathbf{R}}$ consists of all functions $\chi : \mathbf{R} \to \mathbf{T}$ such that $\chi(x+y) = \chi(x)\chi(y)$ when $x, y \in \mathbf{R}$.
b) that if $\lambda_j \in \mathbf{R}$ is a *sequence* such that f_{λ_j} converges in $\mathbf{T}^{\mathbf{R}}$ then λ_j converges to a limit in $\mathbf{R}$, so M is sequentially closed.
c) Show that a function χ defined on a subspace V of $\mathbf{R}$ with respect to the rational number field $\mathbf{Q}$ with values in $\mathbf{T}$ and satisfying the condition $\chi(x+y) = \chi(x)\chi(y)$ for all $x, y \in V$ can be extended to $\mathbf{R}$ so that these properties remain valid.

2. Show that if M is a subset of a metric space E then M is closed if and only if the limit of every convergent sequence of points in M also belongs to M.

3. Show that a metric space E is compact if and only if every sequence $x_1, x_2, \dots$ in E has a convergent subsequence. (Hint: To prove the converse show first that there is a countable basis for neighborhoods in E.)

4. $V_1 \subset V_2$ are linear subspaces of the locally convex topological vector space W such that V_2/V_1 is finite dimensional. Show that V_2 is closed if V_1 is closed. Is the converse true?

5. Show that if F is a closed and W is a finite dimensional linear subspace of a normed space N, and $W \cap F = \{0\}$, then there is a constant C such that

$$\|x\| + \|y\| \le C\|x - y\|, \quad x \in F,\ y \in W.$$

Use this to prove that if q is the quotient map $N \to N/W$, then qF is closed in N/W.

6. Show that there is a hyperplane H in $C([0,1])$ containing all functions $f \in C([0,1])$ such that $f'(0)$ exists. What is the closure of H.

7. Show that a linear map $T : E_1 \to E_2$ where E_1 and E_2 are metrizable locally convex topological vector spaces is continuous if and only if T maps every bounded sequence in E_1 to a bounded sequence in E_2.

8. A locally convex topological vector space is called a *Montel* space if every bounded closed subset is compact. Which of the following spaces is a Montel space?
a) $C^\infty(\Omega)$ where Ω is an open subset of $\mathbf{R}^n$.
b) $C^m(\Omega)$ where Ω is an open subset of $\mathbf{R}^n$ and m is an integer ≥ 0.
c) $A(\Omega)$, the subspace of $C(\Omega)$, Ω open in $\mathbf{C}$, consisting of analytic functions.
d) $A(D)$, the space of continuous functions in the closed unit disc which are analytic in the interior, with the maximum norm.

9. Let M be a measurable subset of $\mathbf{R}^n$ with finite positive measure. Prove that $L^q(M)$ is of the first category in $L^p(M)$ if $1 \le p < q \le \infty$.

10. A function $f \in C([0,1])$ is called Hölder continuous of order $\delta > 0$ if there is a constant C such that $|f(x) - f(y)| \le C|x-y|^\delta$, $x, y \in [0,1]$. Show that the Hölder continuous functions form a set of the first category in $C([0,1])$.

11. Let V_1, V_2 be closed linear subspaces of a Banach space B, and assume that $B = V_1 + V_2$. Show that there is a constant C such that every $x \in B$ can be written $x = x_1 + x_2$ with $x_j \in V_j$ and

$$\|x_1\| + \|x_2\| \le C\|x\|.$$

12. a) Let L be a continuous linear form on $l^\infty(M)$, $M = \{1, 2, \ldots, \}$. Define $(\varphi L)(u) = L(\varphi u)$ if $\varphi, u \in l^\infty(M)$ and show that

$$\|L\| = \sum_1^k \|\varphi_j L\|$$

if the functions φ_j, $j = 1, \ldots, k$, are characteristic functions of subsets of M with $\sum \varphi_j = 1$.
b) Show that if L_n is a sequence of continuous linear forms on $l^\infty(M)$ with $\|L_n\| \le C$, then one can find characteristic functions $\varphi_1 \ge \varphi_2 \ge \ldots$ of infinite subsets of M such that

$$\|\varphi_n L_n\| < 1/n, \quad \text{hence } \|\varphi_j L_n\| < 1/n, \ j \ge n.$$

c) Show that if L_n is a sequence of continuous linear forms on $l^\infty(M)$ with $L_n(f) = f(n)$ if $f \in c(M)$ (the set of sequences converging to 0) then $L_n(\varphi) \not\to 0$ for some $\varphi \in l^\infty(M)$. (Hint: Assume this is false, apply b) (legitimate?) and take a sequence n_k with $\varphi_k(n_k) = 1$; let φ be the characteristic function of the sequence.)
d) Conclude that $c(M)$ does not have a topological supplement in $l^\infty(M)$.
13. Let B be a complex Banach space and F a function $\Omega \to \mathcal{L}(B)$, where Ω is an open set in $\mathbf{C}$. Assume that the function $z \mapsto \langle F(z)u, v\rangle$ is analytic in Ω for arbitrary $u \in B$ and $v \in B^*$. Prove that $F'(z) = \lim_{w\to 0}(F(z+w) - F(z))/w$ exists in the operator norm for every $z \in \Omega$ and that

$$\langle F'(z)u, v\rangle = \frac{d}{dz}\langle F(z)u, v\rangle, \quad z \in \Omega, \ u, v \in H.$$

Prove that if Ω is the disc $\{z \in \mathbf{C}; |z| < R\}$ then $F(z)$ can be expanded in a power series which converges in operator norm when $|z| < R$. (Assume that B is a Hilbert space if you have not yet read Section 2.6.)
14. B is a complex Banach space, and $T \in \mathcal{L}(B, B)$. Prove
a) that there is a compact set $\sigma(T) \subset \mathbf{C}$ (called the *spectrum of* T, such that the resolvent $R(z) = (T - zI)^{-1}$ exists if and only if $z \notin \sigma(T)$;
b) that $R(z)$ is analytic in the complement of $\sigma(T)$ with the various equivalent definitions in exercise 13;
c) that $|z| \le \|T\|$ if $z \in \sigma(T)$;
d) that

$$\sup_{z\in\sigma(T)} |z| = \varlimsup_{n\to\infty} \|T^n\|^{1/n}.$$

e) that the limit actually exists; it is called the *spectral radius*. (Hint: Show that for any sequence a_n of positive numbers with $a_{n+m} \le a_n a_m$ the sequence $a_n^{1/n}$ has a limit as $n \to \infty$.)
f) that M is larger than the spectral radius if and only if there is an equivalent norm in B such that the norm of T is smaller than M with respect to this norm.
15. Let B be a Banach space and let $T \in \mathcal{L}(B, B)$. Show that if the resolvent exists for all z with $|z| = 1$, then

$$P_0 = -\frac{1}{2\pi i}\int_{|z|=1} R(z)\,dz, \quad P_1 = \frac{T}{2\pi i}\int_{|z|=1} R(z)\,dz/z$$

are bounded operators with $P_0 + P_1 = I$, $P_0P_1 = P_1P_0 = 0$, and therefore projections on closed subspaces B_0, B_1 of B with direct topological sum equal to B. Prove that $TB_j \subset B_j$, $j = 0, 1$, and show that

$$T^nP_0 = -\frac{1}{2\pi i}\int_{|z|=1} R(z)z^n\,dz,\ n \geq 0; \quad P_1 = \frac{T^n}{2\pi i}\int_{|z|=1} R(z)z^{-n}\,dz,\ n \geq 1.$$

Deduce that T restricted to B_1 has an inverse S, and that T restricted to B_0 and S both have spectral radius < 1.

16. Let $a_n, n \in \mathbf{Z}$ be a sequence of complex numbers such that $a_nb_n, n \in \mathbf{Z}$ is the sequence of Fourier coefficients of a continuous function on $\mathbf{R}/(2\pi\mathbf{Z})$ when this is true for the sequence $b_n, n \in \mathbf{Z}$. Prove that there is a measure with Fourier coefficients $a_n, n \in \mathbf{Z}$.
17. Let B be a Banach space and let L be a linear form, defined on a linear subspace $\mathcal{D}_L$ of B. What is the closure of the graph of L if L is not continuous?
18. a) Let B_1, B_2, B_3 be Banach spaces. Prove that if T is a closed linear map with domain $\mathcal{D}_T \subset B_2$ and range $\subset B_3$, then TS is closed if $S \in \mathcal{L}(B_1, B_2)$. b) Prove that if T is a linear map with domain $\mathcal{D}_T \subset B_1$ and range in B_2, and if ST is closed for some $S \in \mathcal{L}(B_2, B_3)$, then T is also closed if T is preclosed.
19. If H_i, $i \in I$, are Hilbert spaces, we define $H = \bigoplus_{i\in I} H_i$ as the set of all sequences $x = \{x_i\}_{i\in I}$ such that
$$\|x\|^2 = \sum_{i\in I} \|x_i\|^2 < \infty.$$
Show that H is a Hilbert space with the natural scalar product. When is H separable?
20. Let H_1, H_2 be Hilbert spaces and let T be a closed linear operator with domain dense in H_1 and values in H_2. Prove that the projection on the graph of T in $H_1 \oplus H_2$ has the block matrix form
$$\begin{pmatrix} (I_1 + T^*T)^{-1} & T^*(I_2 + TT^*)^{-1} \\ T(I_1 + T^*T)^{-1} & TT^*(I_2 + TT^*)^{-1} \end{pmatrix},$$
where I_j is the identity operator in H_j. In particular, the operator in the (j, k) block is in $\mathcal{L}(H_k, H_j)$.
21. Prove that if H is a Hilbert space and B is a Banach space, then $\mathcal{L}_c(B, H)$ is the closure of the set of operators in $\mathcal{L}(B, H)$ which are of finite rank.
22. Let B be a Banach space and let $D = \{z \in \mathbf{C}; |z| < 1\}$. Assume that for every $z \in \overline{D}$ we are given an operator $T(z) \in \mathcal{L}_c(B, B)$ which depends continuously on $z \in \overline{D}$ in the operator norm and is analytic in D as in Exercise 13 above. Set
$$\Sigma = \{z \in \overline{D}; \dim \operatorname{Ker}(I + T(z)) \neq 0\}.$$
Show that Σ is closed and that if $\Sigma \neq \overline{D}$ then $\Sigma \cap D$ is discrete and $\{\theta \in \mathbf{R}; e^{i\theta} \in \Sigma\}$ has measure 0.
23. Let B_1 and B_2 be Banach spaces and let $T \in \mathcal{L}(B_1, B_2)$. Prove that if T is compact, then $\lim_{n\to\infty} \|Tu_n\|_2 = 0$ for every sequence $u_n \in B_1$ such that $u_n \to 0$ in the weak topology $\sigma(B_1, B_1^*)$. Prove the converse when B_1 is reflexive and B_1^* is separable.
24. Prove that a Hausdorff topological vector space is a normed space if and only if it contains a bounded, open, convex set.

25. Let B be a Banach space and let $T \in \mathcal{L}(B, B)$ satisfy an equation $p(T) = 0$ where p is a polynomial with only simple zeros $\lambda_1, \ldots, \lambda_m$. Prove that

$$B = \bigoplus_1^m B_j, \qquad B_j = \operatorname{Ker}(T - \lambda_j I),$$

and that $p_j(T)$ is the projection on B_j along $\bigoplus_{k \neq j} B_k$ if

$$p_j(z) = p(z)/((z - \lambda_j)p'(\lambda_j)).$$

26. Let E be a locally convex topological vector space over $\mathbf{R}$. If X is a non-empty subset of E, then the *supporting function* H of X is defined by

$$H(\xi) = \sup_{x \in X} \langle x, \xi \rangle, \qquad \xi \in E'.$$

Prove
1) that H is convex and positively homogeneous, that is, with the convention $0 \cdot \infty = 0$,

$$H(s\xi + t\eta) \leq sH(\xi) + tH(\eta), \qquad s, t \geq 0, \quad \xi, \eta \in E',$$

and that H is lower semi-continuous for the topology $\sigma(E', E)$.
2) that $\{x \in E; \langle x, \xi \rangle \leq H(\xi),\ \forall \xi \in E'\}$ is the smallest closed convex set $\subset E$ containing X, and that its supporting function is equal to H.
3) that for every H satisfying the conditions in 1) there is exactly one closed convex set $X \subset E$ with supporting function H.

27. Let E be a locally convex topological vector space over $\mathbf{R}$, and let f be a function defined in E with values in $[0, +\infty]$. The *Legendre transform* $\tilde{f}$ is defined by

$$\tilde{f}(\xi) = \sup_{x \in E} (\langle x, \xi \rangle - f(x)), \qquad \xi \in E'.$$

In analogy to Exercise 26 try to give conditions characterizing Legendre transforms $\tilde{f}$, and necessary and sufficient conditions on f in order that

$$f(x) = \sup_{\xi \in E'} (\langle x, \xi \rangle - \tilde{f}(\xi)), \qquad x \in E.$$

28. What is the Legendre transform of the norm in a Banach space?
29. Determine the closed convex hull in $l^p(\mathbf{N})$, $1 \leq p \leq \infty$, of the elements with one coordinate equal to 1 and the others equal to 0.
30. Let B be a Banach space such that the norm is the sum of two norms p_1 and p_2; let B_j be the completion of B with the norm p_j. Prove that $\{x \in B; p_1(x) = 1\}$ is closed in B_2 if and only if $B = B_2$.
31. Let Ω be a convex subset of a locally convex topological vector space E with dual E'. Show that the closure in the weak topology $\sigma(E, E')$ of the boundary $\partial\Omega$ in the original topology is either equal to $\partial\Omega$ or the closure of Ω in the original topology. Apply the result when Ω is the unit ball in a Banach space.
32. Show that if B is a Banach space then $\mathcal{L}(B, B) \ni T \mapsto \operatorname{rank} T$ is a lower semi-continuous function with values in $[0, \infty]$.
33. Let B be a Banach space. Prove that the set of operators $T \in \mathcal{L}_c(B, B)$ such that $\dim \operatorname{Ker}(T - \lambda I)^k > 1$ for some $\lambda \neq 0$ and some integer $k > 0$ is of the first category. (Hint: Examine Jordan canonical forms in finite dimensions first.)

Exercises to Chapter III

1. Let A be a bounded self-adjoint operator ≥ 0 on a Hilbert space H. Prove without using the spectral theorem that

$$\|A\| = \sup_{\|u\|=1} (Au, u), \qquad \|Au\|^2 \leq \|A\|(Au, u),\ u \in H.$$

2. Let $A_1 \geq A_2 \geq A_3 \geq \cdots \geq 0$ be bounded self-adjoint operators in a Hilbert space H. Prove that there is a bounded self-adjoint operator A in H such that $A_n \to A$ in the strong topology, that is, $\|A_n u - Au\| \to 0$ as $n \to \infty$, for every $u \in H$.
3. Let A be a bounded self-adjoint operator in the Hilbert space H, and set

$$a = \inf_{\|u\|=1} (Au, u), \qquad b = \sup_{\|u\|=1} (Au, u).$$

Prove that the spectrum of A is contained in the interval $[a, b]$ and contains the end points.
4. Prove that if H is a Hilbert space and $T \in \mathcal{L}(H, H)$, then each of the following conditions is necessary and sufficient for T to be isometric:
a) $(Tu, Tv) = (u, v)$ for all $u, v \in H$.
b) $T^*T = I$.
5. Show that if U is a unitary map in the Hilbert space H, then the resolvent of U is defined and analytic outside the unit circle.
6. Prove that if H is a complex Hilbert space and $T \in \mathcal{L}(H, H)$, then there is a unique decomposition $T = A + iB$ where A and B are self-adjoint. Prove that

$$\|Nu\| = \|N^*u\|,\ u \in H \iff AB = BA \iff T^*T = TT^*.$$

The operator T is then called *normal*.
7. Prove that every bounded self-adjoint operator in a Hilbert space can be written in one and only one way as a difference $A = B - C$ with B, C self-adjoint, positive and bounded, and $BC = CB = 0$.
8. Let H_1, H_2 be Hilbert spaces and let $T \in \mathcal{L}(H_1, H_2)$. Set $A = (T^*T)^{\frac{1}{2}}$. Prove that $\operatorname{Ker} T = \operatorname{Ker} A$, that $\operatorname{Im} A$ is dense in the orthogonal complement of $\operatorname{Ker} A$, and that there is an isometric operator $U : (\operatorname{Ker} T)^\perp \to H_2$ such that $T = UA$ (the polar decomposition).
9. Prove that if H_1, H_2 are Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$, then

$$\sum \|Tx_j\|_2^2 = \sum \|T^*y_k\|_1^2$$

if $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are orthogonal bases in H_1 and in H_2. The square root $\|T\|_{HS}$ of the sum is called the Hilbert-Schmidt norm if it is finite. Prove that such operators form a Hilbert space with the Hilbert-Schmidt norm. Prove that $\|TS\|_{HS} \leq \|T\|_{HS}\|S\|$ and that $\|TS\|_{HS} \leq \|T\|\|S\|_{HS}$.
10. Let X and Y be locally compact spaces with positive measures $d\mu$, $d\nu$, and let K be a function in $L^2(X \times Y, d\mu \otimes d\nu)$. Prove that if $f \in L^2(Y, d\nu)$, then

$$(\mathcal{K}f)(x) = \int K(x, y) f(y)\, d\nu(y)$$

exists for almost every $x \in X$ with respect to $d\mu$, and that $\mathcal{K}$ is a Hilbert-Schmidt operator from $L^2(Y, d\nu)$ to $L^2(X, d\mu)$ with Hilbert-Schmidt norm equal to the norm of K in $L^2(X \times Y, d\mu \otimes d\nu)$. Prove that every Hilbert-Schmidt operator in these L^2 spaces is of this form.

11. Let $T_1 \in \mathcal{L}(H_1, H_2)$ and $T_2 \in \mathcal{L}(H_2, H_1)$, and set $T = T_2T_1 \in \mathcal{L}(H_1, H_1)$. Prove that if T_1 and T_2 are Hilbert-Schmidt operators then
$$\operatorname{Tr} T = \sum (Te_j, e_j)$$
exists if e_j is an orthonormal basis for H_1, and prove that the sum is independent of the choice of basis.
12. Let H be a Hilbert space and let A be a self-adjoint operator in H. A point $\lambda \in \mathbf{R}$ is said to be in the essential spectrum of A if it is in the spectrum and is not an isolated eigenvalue of finite multiplicity. Prove that this is equivalent to the existence of an orthonormal sequence u_n with $\|(A - \lambda I)u_n\| \to 0$ as $n \to \infty$, and conclude that if $K \in \mathcal{L}_c(H, H)$ then λ is also in the essential spectrum of $A + K$.
13. For $z \in \mathbf{C}$, $|z| < 1$ define $f_z = (1, z, z^2, \dots) \in l^2$. Prove that all these vectors are linearly independent. Prove that if Z is a subset of the open unit disc then the linear hull of $\{f_z; z \in Z\}$ is dense in l^2 if and only if
$$\sum_{z \in Z} (1 - |z|) = \infty.$$
14. Let H be a Hilbert space and A a self-adjoint operator in H. Prove that if $0 \le A \le I$ then the limit $Px = \lim_{n\to\infty} A^n x$ exists for every $x \in H$, and show that P is the orthogonal projection on the subspace of fixed points of A. Also prove that for every $A \in \mathcal{L}(H, H)$ with $\|A\| \le 1$ the limit $Qx = \lim_{n\to\infty}(I - (I - A^*A)^n)x$ exists for every $x \in H$. Show that Q is an orthogonal projection and describe the range.
15. Let H be a Hilbert space and let P and Q be two orthogonal projections. Prove that $Rx = \lim_{n\to\infty}(PQ)^n x$ exists for every $x \in H$ and show that R is an orthogonal projection. What is the range?
16. In the Hilbert space $l^2(\{1, 2, \dots\})$ let
$$Tx = (\sum_{j>1} x_j, 0, 0, \dots), \qquad \text{when } x = (x_1, x_2, \dots), x_1 = 0, \text{ and } x_j = 0 \text{ for large } j.$$
Show that T is symmetric (although not densely defined), and determine the closure of the graph. Is T preclosed?
17. In $L^2(I)$, $I = (0, \infty)$, let $Au = iu'$ with domain $\mathcal{D}_A = C_0^\infty(I)$. Show that A is symmetric, and determine the closure and the defect indices. Does A have a self-adjoint extension? What are the answers if I is a finite interval or if $I = \mathbf{R}$?
18. Let H be the Hilbert space $L^2(I)$, $I = [0, 1]$, and let for $z \in \mathbf{C}$ the operator A_z be the closure of the operator $u \mapsto iu'$ with domain consisting of all $u \in C^1(I)$ with $u(1) = zu(0)$. Determine the values of z such that A_z is self-adjoint. What is the spectrum then?
19. Let A be the operator in $L^2(\mathbf{R})$ with domain $\mathcal{D}_A = C_0^\infty(\mathbf{R})$ defined by $(Au)(x) = i(x^2u'(x) + xu(x))$. Prove that A is symmetric and determine the defect indices. Does A have a self-adjoint extension?
20. Let H be a Hilbert space and let V be a dense linear subspace. Prove that if G is a closed subspace of finite codimension, then $V \cap G$ is dense in G. Prove that for

arbitrary $x, y \in H$ with $(x, y) = 0$ there are sequences $x_n \in V$, $y_m \in V$ such that $x_n \to x$, $y_m \to y$, and $(x_n, y_m) = 0$ for all n and m.

21. Let A be a closed densely defined symmetric operator in the complex Hilbert space H. Prove that

$$G(A^*) = G(A) \oplus W_+ \oplus W_-, \quad W_\pm = \{(x, \pm ix); x \in H\} \cap G(A^*),$$

where the sums are orthogonal and G denotes the graphs.

22. Let H be a real Hilbert space and let $\widehat{H}$ be the orthogonal direct sum $H \oplus H$, viewed as a complex Hilbert space through $i(x, y) = (-y, x)$. What is the scalar product in this space then? Show that a linear (symmetric) operator on H gives rise to a linear (symmetric) operator on $\widehat{H}$ with equal defect indices.
23. Let I be an open interval on $\mathbf{R}$, and let $p \in C^1(I)$, $q \in C^0(I)$, be real valued functions. Show that the operator $Au = -(pu')' + qu$ with domain $C_0^2(I)$ has a self-adjoint extension in $L^2(I)$.
24. Prove that if H is a Hilbert space and $x_n \in H$ is a sequence which converges weakly to $x \in H$, then $\|x - x_n\| \to 0$ if and only if $\|x_n\| \to \|x\|$ as $n \to \infty$.
25. Let $1 < p < \infty$ and define $q \in (1, \infty)$ by $1/p + 1/q = 1$. Show that there is a positive constant C_p such that if $g \in L^p(X, d\mu)$ and $\tilde{g} = |g|^p/g$, then $\tilde{g} \in L^q(X, d\mu)$ and

$$\|g + f\|_p^p \geq \|g\|_p^p + p\,\mathrm{Re}\langle f, \tilde{g}\rangle + C_p \int_X |f|^2(|f| + |g|)^{p-2}\, d\mu, \quad f \in L^p(X, d\mu).$$

Use this to extend the result in exercise 24 to L^p, $1 < p < \infty$. Would it be possible to extend it also to $p = \infty$?

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