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Partially obseved optimal control problem for SDEs of Mckean-Vlasov type and Applications

Presented by : Hakima *Miloudi*

Members of the jury:

Djabrane Yahia	Pr,	University of Biskra	**President**.
Mokhtar Hafayed	Pr,	University of Biskra	**Supervisor.**
Imad Eddine Lakhdari	Dr,	University of Biskra	**Co-supervisor.**
Boulakhras Gherbal	Pr,	University of Biskra	**Examiner.**
Youcef Djnaihi	Dr,	University of Setif	**Examiner.**

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♡

Hakima

Résumé

Les problèmes de contrôle partiellement observé ont reçu beaucoup d'attention et sont devenus un outil puissant dans de nombreux domaines, tels que la finance mathématique, le contrôle optimal, etc. Du point de vue de la réalité, de nombreuses situations, l'information complète n'est pas toujours disponible pour les contrôleurs, mais l'information partielle avec bruit. De plus, les travaux récents de Buckdahn, R. [7] et Hafayed, M. [24] sur les équations différentielles stochastiques de type Mckean-Vlasov et leur contrôle optimal ouvre une nouvelle voie pour l'étude des problèmes de contrôle optimal en général.
L'objectif de cette thèse est d'étendre ces résultats de [7] et [24] au cas d'un problème de contrôle optimal partiellement observé. Plus précisément, on s'intéresse par l'étude des problèmes de contrôle optimal partiellement observés pour des équations différentielles stochastique de type McKean-Vlasov, où les coefficients dépendent de l'état du processus de solution également sa loi de probabilité et de la variable de contrôle. En appliquant le théorème de Girsanov avec une méthode variationnelle convexe standard, nous développons le principe du maximum stochastique à nos problèmes de contrôle partiellement observés où le domaine de contrôle est convexe. Ainsi, dans cette thèse, nous prouvons un nouveau principe du maximum stochastique pour une classe de problèmes de contrôle optimal partiellement observés de type Mckean-Vlasov avec sauts. Le système stochastique considéré est dépendent par une équation différentielle stochastique gouvernée par une mesure aléatoire de Poisson et un mouvement brownien indépendant. Alors, pour obtenir nos principaux résultats nous avons basés sur les dérivés par rapport à la mesure

de probabilité et on appliquant la formule d'Itô associés. Et comme an application, en appliquant notre principe du maximum au problème de contrôle quadratique linéaire de type McKean-Vlasov avec saut, où le contrôle optimal partiellement observé est obtenu explicitement sous forme de feedback.

Mots Clés. Contrôle optimal partiellement observé, Principe du maximum stochastique, Dérivées par rapport à la mesure, équations différentielles de McKean-Vlasov, Système stochastique McKean-Vlasov avec sauts, Mesure de probabilité, Théorème de Girsanov.

Abstract

Partially observed control problems have received much attention and became a powerful tool in many fields, such as mathematical finance, optimal control, etc. From the viewpoint of reality, many situations, full information is not always available to controllers, but the partial one with noise. Furthermore, the recent work of Buckdahn, R. [7] and Hafayed, M. [24] on Mckean-Vlasov type stochastic differential equations and their optimal control opens a new avenue for the study of optimal control problems in general.

The objective of this thesis is to extend these results of [7] and [24] to the case of a partially observed optimal control problem. More precisely, we study partially observed optimal control problems of general McKean-Vlasov differential equations, in which the coefficients depend on the state of the solution process as well as of its law and the control variable. By applying Girsanov's theorem with a standard convex variational technique, we develop the stochastic maximum principle for our partially observed control problem where the control domain is convex. Also, in this thesis, we prove a new stochastic maximum principle for a class of partially observed optimal control problems of Mckean-Vlasov type with jumps. The stochastic system under consideration is governed by a stochastic differential equation driven by Poisson random measure and an independent Brownian motion. The derivatives with respect to probability measure and the associate Itô-formula are applied to prove our main results. And as an illustration, by applying our maximum principle, McKean-Vlasov type linear quadratic control problem with jump is discussed, where the partially observed optimal control is obtained explicitly in feedback form.

Key words. Partially observed optimal control, Stochastic maximum principle, Derivatives with respect to the measure, McKean-Vlasov differential equations, McKean-Vlasov stochastic system with jumps, Probability measure, Girsanov's theorem.

Contents

Symbols and Acronyms

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space.
- $\{\mathcal{F}_t\}_{t\geq 0}$: filtration.
- $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$: filtered probability space.
- $\mathbb{R}$: Real numbers.
- $\mathbb{N}$: Natural numbers.
- $L^2(r, s; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued deterministic function $\eta(t)$, such that
$$\int_r^s |\eta(t)|^2 \, dt < +\infty.$$
- $L^2(\mathcal{F}_t; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable φ, such that
$$\mathbb{E}\, |\varphi|^2 < +\infty.$$
- $L^2_{\mathcal{F}}(r, s; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\psi(\cdot)$, such that
$$\mathbb{E} \int_r^s |\psi(t)|^2 \, dt < +\infty.$$
- $\mathbb{M}^2([0,T]; \mathbb{R})$ the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted measurable process $g(\cdot)$, such that
$$\mathbb{E} \int_0^T \int_\Theta |g(t,\theta)|^2 \, m(d\theta) \, dt < +\infty.$$
- $\mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ is the Hilbert space.
- $Q_2\left(\mathbb{R}^d\right)$ the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$.
- *a.e.,:* almost everywhere.

- *a.s.,:* almost surely.
- *e.g.*: for example (abbreviation of Latin exempli gratia).
- *i.e,.* that is (abbreviation of Latin id est).
- *SDE*: Stochastic differential equations.
- *BSDE:* Backward stochastic differential equation.
- *PDE*: Partial differential equation.
- *ODE:* Ordinary differential equation.
- $\dfrac{\partial f}{\partial x}, f_x$: The derivatives with respect to x.
- $\mathbb{P}\otimes dt$: The product measure of $\mathbb{P}$ with the Lebesgue measure dt on $[0,T]$.
- $\mathbb{P}_X$ the law of the random variable $X(\cdot)$.
- $\mathbb{E}(\cdot)$: Expectation.
- $\mathbb{E}(\cdot|\, F_t)$: Conditional expectation.
- $\sigma(A)$: $\sigma-$algebra generated by A.
- 1_A : Indicator function of the set A.
- $\mathbb{E}^v$ denotes expectation on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$.
- $k(\cdot)$ be a stationary $\mathcal{F}_t$-Poisson point process with the characteristic measure $m(d\theta)$.
- $N(d\theta, dt)$ the counting measure or Poisson measure induced by $k(\cdot)$.
- Θ is a fixed nonempty subset of $\mathbb{R}$.
- $\mathcal{F}^X$: The filtration generated by the process X.

- $W(\cdot)$: Brownian motions.

- $\mathcal{F}_t^W$: the natural filtration generated by the brownian motion $W(\cdot)$.

- $\mathcal{F}_1 \vee \mathcal{F}_2$ denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$.

- $\partial_\mu f$: the derivatives with respect to measure μ.

- $\mathcal{D}_\zeta f(\mu_0)$: the *Fréchet-derivative* of f at μ_0 in the direction ξ.

Introduction

The main objective of this thesis is to study the maximum principle for the partially observed optimal control problem, where the stochastic system is driven by McKean-Vlasov stochastic differential equation (SDE). In practice, the controllers usually cannot be able to observe the full information, but the partial one with noise. This makes partially observed optimal control problems receive extensive attentions. Stochastic optimal control of partially observed diffusions has been established by many authors, see for example [2, 20, 91]. For diffusions of mean-field type, partial observed optimal control problem was given by [18, 70, 76]. Stochastic optimal control for partially observed of forward-backward stochastic differential equations have been studied by [69, 76, 83]. Wang et al. [78] extended the stochastic maximum for partially observable optimal control of diffusions for risk-neutral performance functionals of mean-field type. The partially observed time-inconsistency problems have been discussed by [77]. Recently, the partially observed time-inconsistent stochastic linear-quadratic control problem with random jumps has been studied by Wu and Zhuang [84].

The stochastic differential equations of McKean-Vlasov is very general, in the meaning that the dependence of the coefficient on the law of the solution $\mathbb{P}_{x^v(t)}$ could be genuinely nonlinear as an element of the space of probability measures. This kind of equations was discussed by Kac [45] as a stochastic model for the Vlasov-kinetic equation of plasma and the study of which was initiated by McKean [55] to provide a rigorous treatment of special nonlinear partial differential equations.

The stochastic maximum principle is based on the use of adjoint processes, which are defined as solutions to adjoint backward stochastic differential equations (BSDEs). In this case the derivatives of the Hamiltonian function are respect to the state variable. In the case of McKean-Vlasov SDEs, the derivatives of the Hamiltonian function are also respect to the probability measures of the state variable. Carmona and Delarue [14] developed a rigorous probabilistic analysis of the optimal control of McKean-Vlasov type nonlinear stochastic dynamical systems. A stochastic maximum principle for general mean-field systems has been extended by using the tool of the second-order derivatives with respect to measures in the work of [7]. Hafayed et al. [24] established the necessary and sufficient optimality conditions of optimal singular control problem for general Mckean-Vlasov differential equations. In [23], optimal mixed regular-singular control problems for nonlinear stochastic systems with Poisson jump processes of McKean-Vlasov type is studied. For general McKean-Vlasov–type forward-backward differential equations driven by Teugels martingales associated with some Lévy process, we refer to Meherrem and Hafayed [56]. Necessary and sufficient conditions of optimality for system driven by Brownian motions and Poisson random measure where states and observations are correlated have been discussed by Xiao [85] . Partially observed optimal control problem for forward–backward stochastic systems with jump has been discussed by Wang, Shi, & Meng [82] . Stochastic maximum principle for partially observed forward-backward stochastic system with jumps and regime switching has been investigated by Zhang, Xiong, & Liu, [89] . Partially observed time-inconsistent stochastic linear-quadratic control problem with random jumps has been established by Wu & Zhuang [84] .
Maximum principle for optimal control of McKean-Vlasov forward-backward stochastic differential equations (FBSDEs) with Lévy process via the differentiability with respect to probability law has been established by Meherrem & Hafayed [56] . A general necessary optimality conditions for stochastic continuous-singular control of McKean-Vlasov type equations, where the control domain is not assumed convex have been studied by Guenane, Hafayed, Meherrem, & Abbas, [22] . Stochastic maximum principle for partially observed optimal control problems of Mckean-Vlasov type has been proved by Lakhdari, Miloudi, & Hafayed, 2020 [52] .

In this thesis, we prove a stochastic maximum principle for a class of partially observed optimal control problems of stochastic differential equation (SDE) of McKean-Vlasov type. Where the first part of study, the dynamics of the controlled system take the following

$$\begin{cases} dx^{v}(t) = f\left(t, x^{v}(t), \mathbb{P}_{x^{v}(t)}, v(t)\right) dt + \sigma\left(t, x^{v}(t), \mathbb{P}_{x^{v}(t)}, v(t)\right) dW(t) \\ \qquad\qquad + \alpha\left(t, x^{v}(t), \mathbb{P}_{x^{v}(t)}, v(t)\right) d\widetilde{W}(t), \quad t \in [0, T], \\ x^{v}(0) = x_0, \end{cases}$$

where $W(\cdot)$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\widetilde{W}(\cdot)$ denotes a stochastic process depending on the control variable $v(\cdot)$ and $\mathbb{P}_X$ denotes the law of the random variable X. The coefficients $f : [0, T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^n$ and $\sigma, \alpha : [0, T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^{n\times d}$ are given deterministic functions, where $Q_2\left(\mathbb{R}^d\right)$ is the space of all probability measures μ on $\mathbb{R}^d$, endowed with 2-Wasserstein metric.

The cost functional to be minimized over the class of admissible controls is also of McKean-Vlasov type, which has the following form

$$J(v(\cdot)) = \mathbb{E}^{v}\left[\int_0^T l\left(t, x^{v}(t), \mathbb{P}_{x^{v}(t)}, v(t)\right) dt + \psi(x^{v}(T), \mathbb{P}_{x^{v}(T)})\right],$$

where $l : [0, T] \times \mathbb{R}^n \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$, $\psi : \mathbb{R}^n \times Q_2(\mathbb{R}) \to \mathbb{R}$ are deterministic functions, and $\mathbb{E}^{v}$ denotes expectation on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{v})$.

The purpose of this study is to establish a stochastic maximum principle for partially observed control problem of general Mckean-Vlasov differential equations, in which the coefficients depend, nonlinearly, on both the state process as well as of its law. The control domain is assumed to be convex.

In other part of this study we prove a new stochastic maximum principle for a class of partially observed optimal control problems of Mckean-Vlasov type with jumps. The stochastic system under consideration is governed by a stochastic differential equation driven by Poisson random measure and an independent Brownian motion. We define it by the following form

$$
\begin{cases}
dx^{v}(t) = & f\left(t, x^{v}(t), \mathbb{P}_{x^{v}(t)}, v(t)\right) dt + \sigma\left(t, x^{v}(t), \mathbb{P}_{x^{v}(t)}, v(t)\right) dW(t) \\
& +\alpha\left(t, x^{v}(t), \mathbb{P}_{x^{v}(t)}, v(t)\right) d\widetilde{W}(t) + \int_{\Theta} g\left(t, x^{v}(t_{-}), \mathbb{P}_{x^{v}(t_{-})}, v(t), \theta\right) \widetilde{N}(d\theta, dt), \\
x^{v}(0) = & x_0, \quad t \in [0, T],
\end{cases}
$$

where $\mathbb{P}_X = \mathbb{P} \circ X^{-1}$ denotes the law of the random variable X. The maps $f : [0,T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^{n\times d}$, $\alpha : [0,T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^{n\times d}$, $g : [0,T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \times \Theta \to \mathbb{R}^{n\times d}$ are given deterministic functions.

The cost functional to be minimized over the class of admissible controls is also of McKean-Vlasov type, which has the following form

$$
J(v(\cdot)) = \mathbb{E}^{v}\left[\int_0^T l\left(t, x^{v}(t), \mathbb{P}_{x^{v}(t)}, v(t)\right) dt + \psi(x^{v}(T), \mathbb{P}_{x^{v}(T)})\right],
$$

where, $l : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$, $\psi : \mathbb{R}^n \times Q_2(\mathbb{R}) \to \mathbb{R}$ and $\mathbb{E}^{v}$ stands for the mathematical expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^{v})$.

In this work, the derivatives with respect to probability measure and the associate Itô-formula are applied to prove our main results. Noting that the our general McKean-Vlasov partially observed control problem occur naturally in the probabilistic analysis of financial optimization problems. Our class of partially observed control problem is strongly motivated by the recent study of the McKean-Vlasov games and recently play an important role in different fields of economics and finance. And as an illustration, by applying our maximum principle, McKean-Vlasov type linear quadratic control problem with jump is discussed, where the partially observed optimal control is obtained explicitly in feedback form.

This thesis is structured around four chapters:

In **Chapter 1,** we reviewed some processes and some classes of stochastic control with particularly interesting properties in our study (stochastic processes, naturel filtration, admissible control, feedback controls, relaxed controls...etc), we presented strong and weak formulations of stochastic optimal control problems, then, we used the stochastic maximum principle in the classical case where the system is governed by Brownian motion for solving stochastic control problems, and also we discussed the partially observed control problem.

In **Chapter 2,** we use the stochastic maximum principle for solving partially observed optimal control problems of stochastic differential equations (SDE).

In **Chapter 3,** Includes the prove of the necessary condition of the optimal control for partially observed problems of general McKean-Vlasov differential equations. we use Girsanov's theorem as well as standard variational technique to transform our optimal control problem to completely observable problem. And as an application, partially observed linear-quadratic control problem is discussed.

In **Chapter 4,** under the domain of control is convex, we prove a new stochastic maximum principle for a class of partially observed optimal control problems of general McKean-Vlasov stochastic differential equations type with jumps. The stochastic system under consideration is governed by a stochastic differential equation (SDE) driven by Poisson random measure and an independent Brownian motion. The coefficients of our McKean-Vlasov dynamic depend nonlinearly on both the state process as well as of its probability law.

Published Author Papers

The content of this thesis was the subject of the following papers:

1. I.E. Lakhdari, & H. Miloudi, & M. Hafayed, Stochastic maximum principle for partially observed optimal control problems of general McKean–Vlasov differential equations, Bull. Iran. Math. Soc, DOI 10.1007/s41980-020-00426-1, (2020).

2. H. Miloudi, & S. Meherrem, & I.E. Lakhdari, & M. Hafayed: Necessary conditions for partially observed optimal control of general McKean–Vlasov stochastic differential equations with jumps, International Journal of Control, DOI: 10.1080/00207179.2021.1961020., (2021).

Chapter 1

Stochastic optimal control problems

1.1 Stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T be a nonempty index set. A stochastic process is a set of random variables $\{X(t) : t \in T\}$ from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}^n$. For any $w \in \Omega$ the map $t \mapsto X(t, w)$ is called a sample path.

1.2 Natural fitration

Consider the stochastic process $X = (X_t, t \geq 0)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. denoted by $\mathcal{F}_t^X$ for the natural filtration of X which is defined by $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$. Also, we called the filtaration generated by X.

1.3 Brownian motion

A stochastic process $(W(t), t \geq 0)$ is called a standard Brownian motion if:

- $\mathbb{P}[W(0) = 0] = 1$.
- $t \to W(t, w)$ is continuous. $\mathbb{P}-p.s.$
- $\forall s \leq t$, $W(t) - W(s)$ is normally distributed; center with variation $(t - s)$ i.e $W(t) - W(s) \sim \mathcal{N}(0, t - s)$.
- $\forall n$, $\forall\ 0 \leq t_0 \leq t_1 \leq ... \leq t_n$, the variables $\left(W_{t_n} - W_{t_{n-1}}, ..., W_{t_1 -} W_{t_0}, W_{t_0}\right)$ are independents.

1.4 Integration by parts formula

Assume that the processes $x_i(t)$ are given by: for $i = 1, 2$, $t \in [0, T]$:

$$\begin{cases} dx_i(t) = f\left(t, x_i(t), v(t)\right) dt + \sigma\left(t, x_i(t), v(t)\right) dW(t) \\ x_i(0) = 0. \end{cases}$$

Then we get

$$\begin{aligned} \mathbb{E}\left(x_1(T)x_2(T)\right) &= \mathbb{E}\left[\int_0^T x_1(t)dx_2(t) + \int_0^T x_2(t)dx_1(t)\right] \\ &\quad + \mathbb{E}\int_0^T \sigma^{\intercal}\left(t, x_1(t), v(t)\right)\sigma\left(t, x_2(t), v(t)\right) dt. \end{aligned}$$

1.5 Strong formulation

Let be a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ on which we define an m-dimensional standard Brownian motion $W\ (\cdot)$. We consider the following stochastic differential equation:

$$\begin{cases} dx\left(t\right) = f\left(t, x\left(t\right), v\left(t\right)\right) dt + \sigma\left(t, x\left(t\right), v\left(t\right)\right) dW\left(t\right), \\ x\left(0\right) = x_0, \end{cases} \tag{1.1}$$

where $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$, and U a separable metric space, $T \in [0, +\infty[$ fixed. $x\left(\cdot\right)$ is the state variable, $v\left(\cdot\right)$ is called the control and represents the decision made by the controller. At every one time, the controller is aware of certain information (as specified by the information filed $\{\mathcal{F}_t\}_{t\geq 0}$) of what has happened up to that moment, but not able to foretell what is going to happen afterwards due to the uncertainty of the system (because of this, for any t the controller cannot exercise his/her decision $v(t)$ before the time t really comes) which can be expressed in mathematical term as " $v(\cdot)$ is $\{\mathcal{F}_t\}_{t\geq 0}$ adapted", the control v is taken from the set $\mathcal{U}\left[0, T\right] \triangleq \left\{v : [0, T] \times \Omega \longrightarrow U \mid v\left(\cdot\right) \text{ is } \{\mathcal{F}_t\}_{t\geq 0} \text{ adapted}\right\}$. We define the criterion to be optimized, the cost function by:

$$J\left(v\left(\cdot\right)\right) = \mathbb{E}\left(\int_0^T l(t, x(t), v(t))dt + \psi(x(T))\right). \tag{1.2}$$

A control $v(\cdot)$ called an admissible control, and $(x(\cdot), v(\cdot))$ an admissible pair, if

- $v(\cdot) \in \mathcal{U}\,[0,T]$.
- $x(\cdot)$ is the unique solution of equation (1.1)
- $l(\cdot, x(\cdot), v(\cdot)) \in L^1_{\mathcal{F}}\,(0,T,\mathbb{R})$ and $\psi(x(T)) \in L^1_{\mathcal{F}}\,(\Omega,\mathbb{R})$.

We denote by $\mathcal{U}_{ad}\,[0,T]$ the set of all admissible controls.
The stochastic control problem is to find the best possible control $u(\cdot) \in \mathcal{U}_{ad}\,[0,T]$ (if it ever exists) for to minimize the cost function J, i.e

$$J\,(u\,(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{ad}[0,T]} J\,(v\,(\cdot))\,, \tag{1.3}$$

where $u\,(\cdot)$ is called an optimal control.

1.6 Weak formulation

In the weak formulation the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ on which we define the Brownian motion W are not fixed, where we consider them as a parts of the control. This is the difference between it and the strong formulation.

Definition 1.1

$\pi = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W\,(\cdot)\,, v(\cdot)\right)$ is called a weak-admissible control, and $x\,(\cdot)\,, v\,(\cdot)$ is called a weak-admissible pair if

1. *$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ is a filtered probability space satisfying the usual conditions;*
2. *$W\,(\cdot)$ is an m-dimensional standard Brownian motion defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$;*
3. *$v\,(\cdot)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in U;*
4. *$x\,(\cdot)$ is the unique solution of equation (1.1)*
5. *$l(\cdot, x(\cdot), v(\cdot)) \in L^1_{\mathcal{F}}\,(0,T,\mathbb{R})$ and $\psi(x(T)) \in L^1_{\mathcal{F}}\,(\Omega,\mathbb{R})$.*

$\mathcal{U}^w_{ad}\,[0,T]$ denotes the set of all weak admissible controls.
The stochastic optimal control problem under weak formulation is to find an optimal control $\pi^*(\cdot) \in \mathcal{U}^w_{ad}\,[0,T]$ (if it ever exists), such that

$$J\,(\pi^*\,(\cdot)) = \inf_{\pi(\cdot)\in\mathcal{U}^w_{ad}[0,T]} J\,(\pi\,(\cdot))\,. \tag{1.4}$$

1.7 Stochastic maximum principle (SMP)

In this part, we'll utilize an method for to solve stochastic control problem, we called it a stochastic maximum principle. The first version of the stochastic maximum principle (SMP) developed in the 1970s by Bismut [4], Kushner [47], and Haussmann [44], under the condition that there is no control on the diffusion coefficient. The basic idea is to derive a set of necessary and sufficient conditions that must be satisfied by any optimal control. In [43] Haussman used Girsanov's transformation to create a powerful form of the Stochastic Maximum Principle for the feedback class of controls, and applied it to solve some problems in stochastic control.
The stochastic maximum principle is being considered for use in fiance. Cadenillas and Karatzas [11] are said to be the first to employ the stochastic maximum principle in finance. The stochastic maximum principle has been used to solve mean-variance portfolio selection problems (e.g., Yong and Zhou [88] and Zhou and Yin [92])), where the problem was stated as a stochastic linear-quadratic problem. Optimal stochastic control has been studied by Kushner; see [46]. On the stochastic maximum principle, control time is fixed. This has been studied by Kushner [48]. A general stochastic maximum principle for optimal control problems has been established by Peng [62].

1.7.1 Problem formulation

Now, we will begin this work by represent a formulation of the stochastic control problem. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P}\right)$ be a filtered probability space. We assume that $\{\mathcal{F}_t\}_{t\leq T}$ is generated by a d-dimensional standard Brownian motion W. Let's now define the stochastic controlled system as following

$$\begin{cases} dx(t) = f(t, x(t), v(t))\, dt + \sigma(t, x(t), v(t))\, dW(t), \\ x(0) = x_0, \end{cases} \tag{1.5}$$

where $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$, are deterministic functions. the cost function $J(v)$ to be minimized shoud be as follows:

$$J(v(\cdot)) = \mathbb{E}\left(\int_0^T l(t, x(t), v(t))dt + \psi(x(T))\right). \tag{1.6}$$

where $l : [0, T] \times \mathbb{R}^n \times U_1 \to \mathbb{R}, \quad \psi : \mathbb{R}^n \to \mathbb{R}$. The stochastic control problem is to find an optimal control $u \in \mathcal{U}$ such that

$$J(u) = \inf_{v \in \mathcal{U}} J(v), \tag{1.7}$$

Throughout this section, we need to make the following assumptions about the coefficients f, σ, l and ψ.

(A1) The functions f, σ, and l are continuously differentiable with respect to (x, v), and ψ is continuously differentiable in x.

(A2) The derivatives $f_x, f_v, \sigma_x, \sigma_v, l_x, l_v$, and ψ_x are continuous in (x, v) and uniformly bounded.

(A3) f, σ, l are bounded by $K_1(1 + |x| + |v|)$, and ψ is bounded by $K_1(1 + |x|)$, for some $K_1 > 0$.

1.7.2 The stochastic maximum principle

Let us begin by defining $H : [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as a Hamiltonian function. We define it by the formula as follows:

$$H(t, x, v, \Phi, Q) = l(t, x, v) + f(t, x, v)\Phi + \sigma(t, x, v) Q. \tag{1.8}$$

Let u be an optimal control and $\hat{x}$ denote the corresponding optimal trajectory. Then, we consider a pair (Φ, Q) of square integrable adapted processes associated to u, with values in $\mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} d\Phi(t) = & -H_x(t, \widehat{x}(t), u(t), \Phi(t), Q(t))dt + Q(t) dW(t), \\ \Phi(T) = & \psi_x(\widehat{x}(T)). \end{cases} \tag{1.9}$$

1.7.3 Necessary conditions of optimality

This subsection's goal is to discover the optimality necessary conditions satisfied by an optimal control, assuming that one exists. The idea is to use convex perturbation for optimal control, along with some state trajectory and performance functional estimations, and then send the perturbations to zero to obtain some inequality, which is then completed with martingale representation theorems to express the maximum principle in terms of an adjoint process. We can state the stochastic maximum principle in a stronger form.

Theorem 1.1

We assume that the control domain is convex.

If u is an optimal control that minimizes the performance functional J over $\mathcal{U}$, and $\widehat{x}$ is the corresponding optimal trajectory, then there exists an adapted processes $(\Phi, Q) \in \mathbb{L}^2\left(\left([0,T];\mathbb{R}^n\right)\right) \times \mathbb{L}^2\left(\left([0,T];\mathbb{R}^{n\times d}\right)\right)$ which is the unique solution of the BSDE (1.9), such that for all $v \in U$

$$\mathbb{E}\int_0^T H_v\left(t, \widehat{x}\left(t\right), u\left(t\right), \Phi\left(t\right), Q\left(t\right)\right)\left(v_t - u\left(t\right)\right) \geq 0, \quad t \in [0,T].$$

This result, it has been introduced by Bensoussan 1983 [3]. In order to give the proof of theorem 1.1, it is convenient to present the following

1.7.4 Variational equation

Let $v \in \mathcal{U}$ be such that $(u+v) \in \mathcal{U}$, the convexity condition of the control domain ensure that, for $\varepsilon \in (0,1)$ the control $(u+\varepsilon v)$ is also in $\mathcal{U}$. Let x^ε denote the solution of the SDE (1.5) correspond to the control $(u+\varepsilon v)$, then by standard arguments from stochastic calculus, it is easy to check the following convergence result.

Lemma 1.1

Under assumption $(A1)$ we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}\left|x^\varepsilon\left(t\right) - \widehat{x}\left(t\right)\right|^2\right] = 0. \tag{1.10}$$

Proof: According to assumption $(A1)$, we get by using the Burkholder-Davis-Gundy inequality (Appendix)

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|x^\varepsilon\left(t\right) - \widehat{x}\left(t\right)\right|^2\right] \leq K\int_0^t \mathbb{E}\left[\sup_{\tau\in[0,s]}\left|x^\varepsilon\left(r\right) - \widehat{x}\left(r\right)\right|^2\right] ds$$
$$+ K\varepsilon^2\left(\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]}\left|v\left(r\right)\right|^2\right] ds\right). \tag{1.11}$$

From Gronwall's lemma (Appendix), the result follows immediately by letting ε go to zero.

We define the process $\phi(t) = \phi^{u,v}(t)$ by

$$\begin{cases} d\phi(t) &= \{f_x(t, \widehat{x}(t), v(t))\phi(t) + f_v(t, \widehat{x}(t), u(t))v(t)\} dt \\ & \quad + \{\sigma_x(t, \widehat{x}(t), u(t))\phi(t) + \sigma_v(t, \widehat{x}(t), u(t))v(t)\} dW(t), \\ \phi(0) &= 0. \end{cases} \tag{1.12}$$

According to $(A2)$, one can find a unique solution ϕ which solves the variational equation (1.12), and the following estimation holds.

Lemma 1.2

Under assumption $(A1)$, *it holds that*

$$\lim_{\varepsilon \to 0} \mathbb{E} \left| \frac{x^{\varepsilon}(t) - \widehat{x}(t)}{\varepsilon} - \phi(t) \right|^2 = 0. \tag{1.13}$$

Proof: Let

$$\eta^{\varepsilon}(t) = \frac{x^{\varepsilon}(t) - \widehat{x}(t)}{\varepsilon} - \phi(t),$$

and

$$x^{\lambda,\varepsilon}(t) = \widehat{x}(t) + \lambda\varepsilon(\eta^{\varepsilon}(t) + \phi(t)),$$

$$v^{\lambda,\varepsilon}(t) = u(t) + \lambda\varepsilon v(t),$$

just for notational convenience.

$$\begin{aligned} d\eta^{\varepsilon}(t) = &\left\{ \frac{1}{\varepsilon} \left(f\left(t, x^{\lambda,\varepsilon}(t), v^{\lambda,\varepsilon}(t)\right) - f(t, \widehat{x}(t), u(t)) \right) \right. \\ &- \left(f_x(t, \widehat{x}(t), u(t))\phi(t) + f_v(t, \widehat{x}(t), u(t))v(t) \right) \} dt \\ &+ \left\{ \frac{1}{\varepsilon} \left(\sigma\left(t, x^{\lambda,\varepsilon}(t), u^{\lambda,\varepsilon}(t)\right) - \sigma(t, \widehat{x}(t), u(t)) \right) \right. \\ &- \left(\sigma_x(t, \widehat{x}(t), u(t))\phi(t) + \sigma_u(t, \widehat{x}(t), u(t))v(t) \right) \} dW(t) \end{aligned}$$

Since the derivatives of the coefficients are bounded, it is easy to verify by Gronwall's inequality that

$$\begin{aligned} \mathbb{E}|\eta^{\varepsilon}(t)|^2 \leq & K\mathbb{E}\int_0^t \left| \int_0^1 f_x\left(s, x^{\lambda,\varepsilon}(s), v^{\lambda,\varepsilon}(s)\right) \eta^{\varepsilon}(s) d\lambda \right|^2 ds + K\mathbb{E}|\rho^{\varepsilon}(t)|^2 \\ &+ K\mathbb{E}\int_0^t \left| \int_0^1 \sigma_x\left(s, x^{\lambda,\varepsilon}(s), v^{\lambda,\varepsilon}(s)\right) \eta^{\varepsilon}(s) d\lambda \right|^2 ds, \end{aligned}$$

where $\rho^{\varepsilon}(t)$ is equal to

$$\begin{aligned}
\rho^{\varepsilon}(t) = & -\int_0^t f_x(s,\widehat{x}(s),u(s))\,\phi(s)\,ds \\
& -\int_0^t \sigma_x(s,\widehat{x}(s),u(s))\,\phi(s)\,dW(s) \\
& -\int_0^t f_v(s,\widehat{x}(s),u(s))\,v(s)\,ds \\
& -\int_0^t \sigma_v(s,\widehat{x}(s),u(s))\,v(s)\,dW(s) \\
& +\int_0^t\int_0^1 f_x\left(s,x^{\lambda,\varepsilon}(s),u^{\lambda,\varepsilon}(s)\right)\phi(s)\,d\lambda ds \\
& +\int_0^t\int_0^1 f_v\left(s,x^{\lambda,\varepsilon}(s),u^{\lambda,\varepsilon}(s)\right)v(s)\,d\lambda ds \\
& +\int_0^t\int_0^1 \sigma_x\left(s,x^{\lambda,\varepsilon}(s),u^{\lambda,\varepsilon}(s)\right)\phi(s)\,d\lambda dW(s) \\
& +\int_0^t\int_0^1 \sigma_v\left(s,x^{\lambda,\varepsilon}(s),u^{\lambda,\varepsilon}(s)\right)v(s)\,d\lambda dW(s).
\end{aligned}$$

Since f_x, σ_x are bounded, then

$$\mathbb{E}\left|\eta^{\varepsilon}(t)\right|^2 \leq M\mathbb{E}\int_0^t \left|\eta^{\varepsilon}(s)\right|^2 ds + M\mathbb{E}\left|\rho^{\varepsilon}(t)\right|^2,$$

where M is a generic constant depending on the constant K and T. We conclude from lemma 1.2 that $\lim\limits_{\varepsilon\to 0}\rho^{\varepsilon}(t) = 0$. Hence (1.13) follows from Gronwall lemma and by letting ε go to 0.

1.7.5 Variational inequality

Let Λ be the fundamental solution of the linear equation, for $0 < t \leq T$

$$\begin{cases} d\Lambda_t & = f_x(t,\widehat{x}(t),u(t))\,\Lambda_t dt + \sigma_x(t,\widehat{x}(t),u(t))\,\Lambda_t dW(t), \\ \Lambda_0 & = 1, \end{cases}$$

this equation is linear with bounded coefficients, then it admits a unique strong solution. From Itô's formula we can easily check that $d(\Lambda_t\Psi_t) = 0$, and $\Lambda_s\Psi_s = 1$, where Ψ is the solution of the following equation

$$\begin{cases} d\Psi_t & = -\Psi_t\left\{f_x(t,\widehat{x}(t),u(t)) - \sigma_x(t,\widehat{x}(t),u(t))\right\}dt \\ & \quad - \Psi_t\sigma_x(t,\widehat{x}(t),u(t))\,dW(t), \\ \Psi_0 & = 1, \end{cases}$$

so $\Psi = \Lambda^{-1}$. By integrating by part formula we can see that, the solution of (1.12) is given by $\phi(t) = \Lambda_t \Upsilon_t$, where Υ_t is the solution of the stochastic differential equation

$$\begin{cases} d\Upsilon_t & = \Psi_t \left\{ f_v\left(t, \widehat{x}(t), u(t)\right) v(t) - \sigma_x\left(t, \widehat{x}(t), u(t)\right) \sigma_v\left(t, \widehat{x}(t), u(t)\right) v(t) \right\} dt \\ & \quad + \Psi_t \sigma_v\left(t, x_t^\star, u_t^\star\right) v(t)\, dW(t), \\ \Upsilon_0 & = 0. \end{cases}$$

Let us introduce the following convex perturbation of the optimal control u by

$$v^\varepsilon = u + \varepsilon v, \tag{1.14}$$

for any $v \in \mathcal{U}$, and $\varepsilon \in (0,1)$. Since u is an optimal control, then $\varepsilon^{-1}\left(J\left(v^\varepsilon\right) - J(u)\right) \geq 0$. Thus a necessary condition for optimality is that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\left(J\left(v^\varepsilon\right) - J(u)\right) \geq 0. \tag{1.15}$$

The rest is devoted to the computation of the above limit. We shall see that the expression (1.15) leads to a precise description of the optimal control u in terms of the adjoint process. First, it is easy to prove the following lemma

Lemma 1.3

Under assumptions $(A1)$, *we have*

$$\begin{aligned} I &= \lim_{\varepsilon \to 0} \varepsilon^{-1}\left(J\left(v^\varepsilon\right) - J(u)\right) \\ &= \mathbb{E}\left[\int_0^T \left\{ l_x\left(s, \widehat{x}(s), u(s)\right) \phi(s) + l_v\left(s, \widehat{x}(s), u(s)\right) v(s) \right\} ds + \psi_x\left(\widehat{x}(T)\right) \phi(T)\right]. \end{aligned} \tag{1.16}$$

Proof: We use the same notations as in the proof of lemma 1.2. First, we have

$$\begin{aligned} &\varepsilon^{-1}\left(J\left(v^\varepsilon\right) - J(u)\right) \\ &= \mathbb{E}\left[\int_0^T \int_0^1 \left\{ l_x\left(s, x^{\mu,\varepsilon}(s), v^{\mu,\varepsilon}(s)\right) \phi(s) + l_v\left(s, x^{\mu,\varepsilon}(s), v^{\mu,\varepsilon}(s)\right) v(s) \right\} d\mu ds \right. \\ &\qquad \left. + \int_0^1 \psi_x\left(x^{\mu,\varepsilon}(T)\right) \phi(T)\, d\mu\right] + \beta^\varepsilon(t), \end{aligned}$$

where

$$\beta^{\varepsilon}(t)=\mathbb{E}\left[\int_0^T\int_0^1 l_x\left(s,x^{\mu,\varepsilon}(s),v^{\mu,\varepsilon}(s)\right)\eta^{\varepsilon}(s)\,d\mu ds+\int_0^1\psi_x\left(x^{\mu,\varepsilon}(T)\right)\eta^{\varepsilon}(T)\,d\mu\right].$$

By using the lemma 1.2, and since the derivatives l_x, l_u, and ψ_x are bounded, we have $\lim_{\varepsilon\to 0}\beta^{\varepsilon}(t)=0$. Then, the result follows by letting ε go to 0 in the above equality.
Substituting by $\phi(t)=\Lambda_t\Upsilon_t$ in (1.16), this leads to

$$I=\mathbb{E}\left[\int_0^T\left\{l_x\left(s,\widehat{x}(s),u(s)\right)\Lambda_s\Upsilon_s+l_v\left(s,\widehat{x}(s),u(s)\right)v(s)\right\}ds+\psi_x\left(\widehat{x}(T)\right)\Lambda_T\Upsilon_T\right].$$

Consider the right continuous version of the square integrable martingale

$$M(t):=\mathbb{E}\left[\int_0^T l_x\left(s,\widehat{x}(s),u(s)\right)\Lambda_s ds+\psi_x\left(\widehat{x}(T)\right)\Lambda_T\,|\mathcal{F}_t\right].$$

By the representation theorem, there exist q where $q\in\mathbb{L}^2$

$$M(t)=\mathbb{E}\left[\int_0^T l_x\left(s,\widehat{x}(s),u(s)\right)\Lambda_s ds+\psi_x\left(\widehat{x}(T)\right)\Lambda_T\right]+\int_0^t q(s)\,dW(s).$$

We introduce some more notation, write $\widehat{x}(t)=M(t)-\int_0^t l_x\left(s,\widehat{x}(s),u(s)\right)\Lambda_s ds$. The adjoint variable is the processes defined by

$$\begin{cases}\Phi(t) & =\widehat{x}(t)\,\Psi_t,\\ Q(t) & =q(t)\,\Psi_t-\Phi(t)\,\sigma_x\left(t,\widehat{x}(t),u(t)\right).\end{cases}\tag{1.17}$$

Theorem 1.2

Under assumptions $(A1)$, *we have*

$$I=\mathbb{E}\left[\int_0^T\left\{l_v\left(s,\widehat{x}(s),u(s)\right)+f_v\left(s,\widehat{x}(s),u(s)\right)\Phi(s)+\sigma_v\left(s,\widehat{x}(s),u(s)\right)Q(s)\right\}v(t)\,dt\right].$$

Proof: From the integration by part formula, and by using the definition of $\Phi(t), Q(t)$, we easily check that

$$\begin{aligned}E\left[\widehat{x}(T)\Upsilon(T)\right] &=\mathbb{E}\left[\int_0^T\left\{\Phi(t)\,f_v\left(s,\widehat{x}(s),u(s)\right)+Q(s)\,\sigma_v\left(s,\widehat{x}(s),u(s)\right)\right\}v(t)\,dt\right.\\ &\quad-\int_0^T l_x\left(s,\widehat{x}(s),u(s)\right)\Upsilon_t\Lambda_t dt.\end{aligned}\tag{1.18}$$

Also we have

$$I = \mathbb{E}\left[\widehat{x}(T)\Upsilon(T) + \int_0^T l_x(s,\widehat{x}(s),u(s))\Lambda_t\Upsilon_t dt + \int_0^T l_v(s,\widehat{x}(s),u(s))v(t)dt\right], \tag{1.19}$$

substituting (1.18) in (1.19), This completes the proof.

1.8 Partial observation control problem

So far, it's been assumed that the controller completely observed on the state system. In many real applications, he is only able to observe partially the state via other variables and there is noise in the observation system. As exemple in financial models, one may observe the asset price but not completely its rate of return and/or its volatility, and the portfolio investment is based only on the asset price information. We are facing a partial observation control problem. This may be formulated this problem in a general form as follows:

We consider a controlled signal (unobserved) process governed by the following SDE:

$$\begin{cases} dx^v(t) = f(t,x^v(t),v(t))dt + \sigma(t,x^v(t),v(t))dW(t), \quad t\in[0,T], \\ x^v(0) = x_0, \end{cases}$$

and

$$\begin{cases} dY(t) = h(t,x^v(t),v(t))dt + d\widetilde{W}(t) \\ Y(0) = 0, \end{cases}$$

The objective of the problem is to choose an admissible control such that the following cost functional is minimized:

$$J(v(\cdot)) = \mathbb{E}^v\left[\int_0^T l(t,x^v(t),v(t))dt + \psi(x^v(T))\right],$$

1.8.1 Assumptions and Problem Formulation

Throughout this section, we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{R}^{n\times d}$ the collection of $n\times d$ matrices. For a given Euclidean space, We denote by $|\cdot|$ the norm and by $\langle\cdot,\cdot\rangle$ the scalar product. And we denotes by superscript $\top$ to the transpose of matrices or vectors. We have $W(\cdot), Y(\cdot)$ two independent standard Brownian motions valued in

$\mathbb{R}^d$ and $\mathbb{R}^r$, respectively. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a complete filtered probability space equipped with a natural filtration

$$\mathcal{F}_t = \sigma \{W(s), Y(s); 0 \leq s \leq t\},$$

Let $\mathcal{F} := \mathcal{F}_t$, and let $T > 0$ be the finite time duration. $\mathbb{E}$ defines the expectation on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$. Moreover, we denote by

– $L^2(r, s; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued deterministic function $\eta(t)$, such that $\int_r^s |\eta(t)|^2 dt < +\infty$,

– $L^2(\mathcal{F}_t; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable φ, such that $\mathbb{E}|\varphi|^2 < +\infty$,

and by

– $L^2_{\mathcal{F}}(r, s; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\psi(\cdot)$, such that $\mathbb{E}\int_r^s |\psi(t)|^2 dt < +\infty$.

We define

$$\mathcal{F}_t^Y := \sigma \{Y(s); 0 \leq s \leq t\},$$

We have U a nonempty convex subset of $\mathbb{R}^k$.

Definition 1.2

An admissible control variable $\nu : [0, T] \times \Omega \to U$ is a control variable $\mathcal{F}_t^Y$−adapted and satisfies $\sup_{t\in[0,T]} \mathbb{E}|\nu|^m < \infty$, $m = 2, 3, \cdots$.

Denote by U_{ad} the set of the admissible control variables.
For given $\nu(\cdot) \in U_{ad}$, we consider the following class of stochastic control problems of the type

$$\begin{cases} dx^v(t) = f(t, x^v(t), v(t))\, dt + \sigma(t, x^v(t), v(t))\, dW(t), \quad t \in [0, T], \\ x^v(0) = x_0, \end{cases} \tag{1.20}$$

where x_0 is the initial path of $x(\cdot)$ and f and σ are given deterministic functions such as

$$f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n,$$
$$\sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$$

We suppose that the state processes $x^v(\cdot)$ is not completely observable, instead, it is partially observed through the related process $Y(\cdot)$, which is described by the following equation

$$\begin{cases} dY(t) = h(t, x^v(t), v(t))dt + d\widetilde{W}(t) \\ Y(0) = 0, \end{cases} \tag{1.21}$$

where $h : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^r$ is a function, and $\widetilde{W}(\cdot)$ is a stochastic process depending on the control $v(\cdot)$.

The cost functional is

$$J(v(\cdot)) = \mathbb{E}^v \left[\int_0^T l(t, x^v(t), v(t))dt + \psi(x^v(T)) \right], \tag{1.22}$$

Here, $\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$ and

$$l : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}, \;\; \psi : \mathbb{R}^n \to \mathbb{R}.$$

Throughout this section, we need to make the following hypotheses.

hypothesis (**H1**) We suppose that the coefficients f, σ are continuously differentiable in x, and their partial derivatives are uniformly bounded; they are uniformly Lipschitz in v and there exists a constant $C > 0$ such that both f and σ are bounded by $C(1 + |x| + |v|)$; the function h is continuously differentiable in x and continuous in v, its derivatives and h are all uniformly bounded.

For any $v(\cdot) \in U_{ad}$, the hypothesis (**H1**) implies that (1.20) admits a unique $\mathcal{F}_t$-adapted solution. Define $d\mathbb{P}^v = \rho^v(t)d\mathbb{P}$ with

$$\rho^v(t) = \exp\left\{ \int_0^t h(s, x^v(s), , v(s))dY(s) - \frac{1}{2}\int_0^t |h(s, x^v(s), v(s))|^2 \, ds \right\},$$

and $\rho(\cdot)$ is the unique $\mathcal{F}_t^Y$-adapted solution of the following linear SDE

$$\begin{cases} d\rho^v(t) = \rho^v(t)h\left(t, x^v(t), v(t)\right)dY(t), \\ \rho^v(0) = 1. \end{cases} \tag{1.23}$$

By Itô's formula, we can prove that $\sup_{t\in[0,T]} \mathbb{E}\left|\rho_t^v\right|^m < \infty$, $m = 2, 3, \cdots$. Hence, by Girsanov's theorem and hypothesis (**H1**), $\mathbb{P}^v$ is a new probability measure and $(W(\cdot), \widetilde{W}(\cdot))$ is a two-dimensional standard Brownian motion defined on the new probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$.

hypothesis (H2)

(i) l is a function continuous in v, continuously differentiable in x, and its partial derivatives are continuous in (x, v) and bounded by $C(1 + |x| + |v|)$;

(ii) ψ is a function continuously differentiable and ψ_x is bounded by $C(1 + |x|)$.

Our partially observed optimal control problem is to minimize the cost functional (1.22) over $v(\cdot) \in U_{ad}$ subject to (1.20) and (1.21), i.e., to find $u(\cdot) \in U_{ad}$ satisfying

$$J(u(\cdot)) = \inf_{v(\cdot)\in U_{ad}} J(v(\cdot)). \tag{1.24}$$

Clearly, cost functional (1.22) can be rewritten as following

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T \rho^v(t)l(t, x^v(t), v(t))dt + \rho^v(T)\psi(x^v(T))\right]. \tag{1.25}$$

Then the original problem (1.24) is equivalent to minimize (1.25) over $v(\cdot) \in U_{ad}$ subject to (1.20) and (1.23).

Our aim is to establish a set of necessary conditions of the partially observed optimal control $u(\cdot)$ in the form of stochastic maximum principle.

1.8.2 Stochastic maximum principle for partially observed optimal control problem

In this section, we are based on Girsanov's theorem with a standard convex variational technique to developed the stochastic maximum principle for our partially observed control problem.

Let x be the optimal trajectory corresponding to the optimal control $u(\cdot)$. Then for any

$0 \leq \varepsilon \leq 1$ and $v(\cdot) \in \mathcal{U}_{ad}([0,T])$, we define the variational control by $v^{\varepsilon}(\cdot) = u(\cdot)+\varepsilon v(\cdot) \in \mathcal{U}_{ad}([0,T])$. With clear notation, we denote by $x^{\varepsilon}(\cdot), x(\cdot), \rho^{\varepsilon}(\cdot), \rho(\cdot)$ the state trajectories of (1.20) and (1.23) corresponding to $v^{\varepsilon}(\cdot)$ and $u(\cdot)$.

We now introduce the following SDEs

$$\begin{cases} d\phi(t) = \left\{ f_x(t, x(t), u(t))\, \phi(t) + f_v(t, x(t), u(t)) v(t) \right\} dt \\ \qquad\qquad + \left\{ \sigma_x(t, x(t), u(t)) \phi(t) + \sigma_v(t, x(t), u(t)) v(t) \right\} dW(t) \\ \phi(0) = 0, \end{cases} \tag{1.26}$$

and

$$\begin{cases} d\rho_1(t) = \left\{ \rho_1(t) h(t, x(t), u(t)) + \rho(t) h_x(t, x(t), u(t)) \phi(t) \right. \\ \qquad\qquad \left. + \rho(t) h_v(t, x(t), u(t)) v(t) \right\} dY(t), \\ \rho_1(0) = 0. \end{cases} \tag{1.27}$$

By hypothesis (**H1**), it is obvious to infer that the stochastic differential equations (SDEs) (1.26) and (1.27) admit unique adapted solutions $\phi(\cdot)$ and $\rho_1(\cdot)$, respectively. □

Lemma 1.4

Let hypothesis (**H1**) *hold. Then, we have*

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{x^{\varepsilon}(t) - x(t)}{\varepsilon} - \phi(t) \right|^2 = 0. \tag{1.28}$$

We also need to get some ε-order estimations of the difference between the perturbed observed process $\rho^{\varepsilon}(\cdot)$ with the sum of the optimal observed process $\rho(\cdot)$ and the variational observed $\rho_1(\cdot)$. The following lemma play an important role when we derive the variational inequality.

Lemma 1.5

Let hypothesis (**H1**) *hold. Then, we have*

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{\rho^{\varepsilon}(t) - \rho(t)}{\varepsilon} - \rho_1(t) \right|^2 = 0. \tag{1.29}$$

Proof. According to the definition of $\rho(\cdot)$ and $\rho_1(\cdot)$, we have inequality.

$$\begin{aligned}\rho(t)+\varepsilon\rho_1(t) &= 1+\int_0^t \rho(s)h(s,x(s),u(s))dY(s)\\ &+\varepsilon\int_0^t [\rho_1(s)\,h(s,x(s),u(s))+\rho(s)h_x(s,x(s),u(s))\,\phi(s)\\ &+\rho(s)h_v(s,x(s),u(s))\,v(s)]\,dY(s)\\ &= 1+\varepsilon\int_0^t \rho_1(s)h(s,x(s),u(s))dY(s)\\ &+\int_0^t \rho(s)\,h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s)\,dY(s)\\ &-\varepsilon\int_0^t \rho(s)\,[A^\varepsilon(s)]\,dY(s),\end{aligned}$$

where

$$\begin{aligned}A^\varepsilon(s) = &\int_0^1 [h_x(s,x(s)+\lambda\varepsilon\phi(s),u(s)+\lambda\varepsilon v(s))\\ &-h_x(s,x(s),u(s))]\,d\lambda\phi(s)\\ &+\int_0^1 [h_v(s,x(s)+\lambda\varepsilon\phi(s),u(s)+\lambda\varepsilon v(s))\\ &-h_v(s,x(s),u(s))]\,d\lambda v(s).\end{aligned}$$

Then, we have

$$\begin{aligned}\rho^\varepsilon(t)-\rho(t)-\varepsilon\rho_1(t) &= \int_0^t \rho^\varepsilon(s)\,h(s,x^\varepsilon(s),v^\varepsilon(s))\,dY(s)\\ &-\varepsilon\int_0^t \rho_1(s)h(s,x(s),u(s))dY(s)\\ &-\int_0^t \rho(s)h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))\,dY(s)\\ &+\varepsilon\int_0^t \rho(s)\,[A^\varepsilon(s)]\,dY(s)\\ &=\int_0^t (\rho^\varepsilon(s)-\rho(s)-\varepsilon\rho_1(s))\,h(s,x^\varepsilon(s),v^\varepsilon(s))\,dY(s)\\ &+\int_0^t (\rho(s)+\varepsilon\rho_1(s))\,[h(s,x^\varepsilon(s),v^\varepsilon(s))\\ &-h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))]\,dY(s)\\ &+\varepsilon\int_0^t \rho_1(s)\,h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))dY(s)\\ &-\varepsilon\int_0^t \rho_1(s)\,h(s,x(s),u(s))dY(s)+\varepsilon\int_0^t \rho(s)\,[A^\varepsilon(s)]\,dY(s)\\ &=\int_0^t (\rho^\varepsilon(s)-\rho(s)-\varepsilon\rho_1(s))\,h(s,x^\varepsilon(s),v^\varepsilon(s))\,dY(s)\\ &+\int_0^t (\rho(s)+\varepsilon\rho_1(s))\,[\Lambda_1^\varepsilon(s)]\,dY(s)+\varepsilon\int_0^t \rho_1(s)\,[\Lambda_2^\varepsilon(s)]\,dY(s)\\ &+\varepsilon\int_0^t \rho(s)\,[A^\varepsilon(s)]\,dY(s),\end{aligned}$$

where

$$\Lambda_1^\varepsilon(s) = h\left(s, x^\varepsilon\left(s\right), v^\varepsilon\left(s\right)\right) - h\left(s, x\left(s\right) + \varepsilon\phi\left(s\right), u\left(s\right) + \varepsilon v\left(s\right)\right),$$
$$\Lambda_2^\varepsilon(s) = h(s, x\left(s\right) + \varepsilon\phi\left(s\right), u\left(s\right) + \varepsilon v\left(s\right)) - h(s, x\left(s\right), u\left(s\right)).$$

Note that

$$\Lambda_1^\varepsilon(s) = \int_0^1 [h_x(s, x\left(s\right) + \varepsilon\phi\left(s\right) + \lambda(x^\varepsilon\left(s\right) - x\left(s\right) - \varepsilon\phi\left(s\right)), v^\varepsilon\left(s\right))]d\lambda(x^\varepsilon\left(s\right) - x\left(s\right) - \varepsilon\phi\left(s\right))$$

By Lemma 1.4, we know that

$$\mathbb{E}\int_0^t \left|(\rho(s) + \varepsilon\rho_1(s))\Lambda_1^\varepsilon(s)\right|^2 ds \leq C_\varepsilon \varepsilon^2, \tag{1.30}$$

hereafter C_ε denotes some nonnegative constant such that $C_\varepsilon \to 0$ as $\varepsilon \to 0$.
Also, it's not difficult to see that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\varepsilon\int_0^t \rho(s)A^\varepsilon(s)dY(s)\right]^2 \leq C_\varepsilon \varepsilon^2, \tag{1.31}$$

and

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\varepsilon\int_0^t \rho_1(s)\Lambda_2^\varepsilon(s)dY(s)\right]^2 \leq C_\varepsilon \varepsilon^2. \tag{1.32}$$

By (1.30), (1.31) and (1.32), we have

$$\begin{aligned}
&\mathbb{E}\left|(\rho^\varepsilon(t) - \rho(t)) - \varepsilon\rho_1(t)\right|^2 \\
&\leq C\left[\int_0^t \mathbb{E}\left|(\rho^\varepsilon\left(s\right) - \rho\left(s\right)) - \varepsilon\rho_1(s)\right|^2 + \mathbb{E}\int_0^t \left|(\rho\left(s\right) + \varepsilon\rho_1\left(s\right))\Lambda_1^\varepsilon(s)\right|^2 ds \right.\\
&\quad \left. + \sup_{0\leq s\leq t}\mathbb{E}\left(\varepsilon\int_0^t \rho(s)A^\varepsilon(s)dY(s)\right)^2 + \sup_{0\leq s\leq t}\mathbb{E}\left(\varepsilon\int_0^t \rho_1(s)\Lambda_2^\varepsilon(s)dY(s)\right)^2\right] \\
&\leq C\int_0^t \mathbb{E}\left|\rho^\varepsilon(s) - \rho(s) - \varepsilon\rho_1(s)\right|^2 ds + C_\varepsilon \varepsilon^2.
\end{aligned}$$

Finally, by using Gronwall's inequality, we get the desired result. □

Lemma 1.6

Under hypothesis (**H1**), *one has*

$$\begin{aligned}0 \leq \mathbb{E}\int_0^T &\{\rho_1(t)\, l(t, x(t), u(t)) + \rho(t)\, l_x(t, x(t), u(t))\phi(t) \\ &+ \rho(t) l_v(t, x(t), u(t)) v(t)\}\, dt \\ &+ \mathbb{E}\left[\rho_1(T)\, \psi(x(T))\right] + \mathbb{E}\left[\rho(T)\, \psi_x(x(T))\phi(T)\right] \end{aligned} \tag{1.33}$$

Proof. Using Lemmas 1.4 and 1.5, Taylor expansion, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{E}\left[\rho^{\varepsilon}(T)\, \psi(x^{\varepsilon}(T)) - \rho(T)\, \psi(x(T))\right] \\ &= \mathbb{E}\left[\rho_1(T)\psi(x(T)) + \rho(T)\psi_x(x(T))\phi(T)\right],\end{aligned}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{E}\int_0^T \{\rho^{\varepsilon}(t) l(t, x^{\varepsilon}(t), u^{\varepsilon}(t)) - \rho(t) l(t, x(t), u(t))\}\, dt \\ &= \mathbb{E}\int_0^T \{\rho_1(t) l(t, x(t), u(t)) + \rho(t) l_x(t, x(t), u(t))\phi(t) \\ &+ \rho(t) l_v(t, x(t), u(t)) v(t)\}\, dt.\end{aligned}$$

Then, by the optimality of $u(\cdot)$, we draw the desired conclusion.

Now, we define the Hamiltonian $H : [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, by

$$H(t, x, v, \Phi, Q, K) = l(t, x, v) + f(t, x, v)\Phi + \sigma(t, x, v)Q + h(t, x, v)\, K. \tag{1.34}$$

Then, we introduce the adjoint equations involved in the stochastic maximum principle:

$$\begin{cases} -dy(t) = l(t, x(t), u(t))dt - z(t)\, dW(t) - K(t)\, d\widetilde{W}(t), \\ y(T) = \psi(x(T)), \end{cases} \tag{1.35}$$

and

$$\begin{cases} -d\Phi(t) = \{f_x(t, x(t), u(t))\, \Phi(t) + \sigma_x(t, x(t), u(t))\, Q(t) \\ \qquad + l_x(t, x(t), u(t)) + h_x(t, x(t), u(t))\, K(t)\Big\}\, dt \\ \qquad - Q(t) dW(t) - \overline{Q}(t) d\widetilde{W}(t), \\ \Phi(T) = \psi_x(x(T)). \end{cases} \tag{1.36}$$

Set $\widetilde{\rho}(t) = \rho^{-1}(t)\rho_1(t)$, by using Itô's formula, we have

$$\begin{cases} d\widetilde{\rho}(t) = \{h_x(t,x(t),u(t))\phi(t) + h_v(t,x(t),u(t))v(t)\}\, d\widetilde{W}(t), \\ \widetilde{\rho}(0) = 0, \end{cases} \tag{1.37}$$

Then, applying Itô's formula to $\Phi(t)\,\phi(t)$, $y(t)\,\widetilde{\rho}(t)$ and taking expectation respectively, we obtain

$$\begin{aligned} \mathbb{E}^u\left[\Phi(T)\,\phi(T)\right] &= \mathbb{E}^u\int_0^T \Phi(t)\,d\phi(t) + \mathbb{E}^u\int_0^T \phi(t)\,d\Phi(t) \\ &+ \mathbb{E}^u\int_0^T Q(t)\left\{\sigma_x(t,x(t),u(t))\phi(t) + \sigma_v(t,x(t),u(t))v(t)\right\} dt \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{1.38}$$

where

$$\begin{aligned} I_1 &= \mathbb{E}^u\int_0^T \Phi(t)\,d\phi(t) \\ &= \mathbb{E}^u\int_0^T \Phi(t)\left\{f_x(t,x(t),u(t))\phi(t) + f_v(t,x(t),u(t))v(t)\right\} dt \\ &= \mathbb{E}^u\int_0^T \Phi(t)\, f_x(t,x(t),u(t))\phi(t)dt + \mathbb{E}^u\int_0^T \Phi(t)\, f_v(t,x(t),u(t))v(t)dt \end{aligned}$$

Consequently,

$$\begin{aligned} I_2 &= \mathbb{E}^u\int_0^T \phi(t)\,d\Phi(t) \\ &= -\mathbb{E}^u\int_0^T \phi(t)\left\{f_x(t,x(t),u(t))\,\Phi(t) + \sigma_x(t,x(t),u(t))\,Q(t)\right. \\ &\left. + l_x(t,x(t),u(t)) + h_x(t,x(t),u(t))\,K(t)\right\} dt. \end{aligned}$$

By simple computation, we deduce

$$\begin{aligned} I_2 = &-\mathbb{E}^u\int_0^T \phi(t)\, f_x(t,x(t),u(t))\,\Phi(t)\,dt \\ &- \mathbb{E}^u\int_0^T \phi(t)\,\sigma_x(t,x(t),u(t))\,Q(t)\,dt \\ &- \mathbb{E}^u\int_0^T \phi(t)\,h_x(t,x(t),u(t))\,K(t)dt \\ &- \mathbb{E}^u\int_0^T \phi(t)\,l_x(t,x(t),u(t))\,dt \end{aligned}$$

Similarly, we obtain

$$I_3 = \mathbb{E}^u\int_0^T Q(t)\sigma_x(t,x(t),u(t))\phi(t)dt + \mathbb{E}^u\int_0^T Q(t)\sigma_v(t,x(t),u(t))v(t)dt,$$

and

$$\begin{aligned}\mathbb{E}^u\left[y\left(T\right)\widetilde{\rho}\left(T\right)\right] &= \mathbb{E}^u\int_0^T y\left(t\right)d\widetilde{\rho}\left(t\right)+\mathbb{E}^u\int_0^T \widetilde{\rho}\left(t\right)dy\left(t\right) \\ &+\mathbb{E}^u\int_0^T K\left(t\right)\left\{h_x(t,x\left(t\right),u\left(t\right))\phi\left(t\right)\right. \\ &\left.+h_v(t,x\left(t\right),u\left(t\right))v(t)\right\}dt \\ &= J_1+J_2+I_3,\end{aligned} \tag{1.39}$$

where $J_1=\mathbb{E}^u\int_0^T y\left(t\right)d\widetilde{\rho}\left(t\right)$ is a martingale with zero expectation, and

$$\begin{aligned}J_2 &= \mathbb{E}^u\int_0^T \widetilde{\rho}\left(t\right)dy\left(t\right) \\ &= -\mathbb{E}^u\int_0^T \widetilde{\rho}\left(t\right)l(t,x(t),u(t))dt.\end{aligned}$$

Similarly, we can obtain

$$J_3=\mathbb{E}^u\int_0^T K\left(t\right)\left\{h_x(t,x(t),u(t))\phi(t)\ +h_v(t,x(t),u(t))v(t)\right\}dt$$

Finally, substituting (1.38) and (1.39) into (1.33), we get

$$\mathbb{E}^u\left[H_v(t,x(t),u(t),\Phi(t),Q(t),K(t))v\left(t\right)\right]\geq 0. \tag{1.40}$$

Using the similar method developed in [24], our main result of this part is the following Theorem. □

Theorem 1.3

Let hypothesis (**H1**) *hold. Let* $u(\cdot)$ *be optimal. Then, the maximum principle*

$$\mathbb{E}^u\left[H_v(t,x(t),u(t),\Phi(t),Q(t),K(t))\left(v\left(t\right)-u\left(t\right)\right)\mid\mathcal{F}_t^Y\right]\geq 0,\ \forall v\in U,\qquad a.e., a.s.,$$

holds, where the Hamiltonian function H *is defined by* (1.34).

1.9 Some classes of stochastic controls

Let $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},P)$ be a complete filtred probability space.

1.9.1 Optimal control

The goal of the optimal control problem is to minimize a cost function $J(v)$ over the set of admissible control $\mathcal{U}$. We say that the control $u(\cdot)$ is an optimal control if

$$J(u(t)) \leq J(v(t)), \text{ for all } v(\cdot) \in \mathcal{U}.$$

1.9.2 Admissible control

$\mathcal{F}_t$-adapted process $v(t)$ with values in a borelian $A \subset \mathbb{R}^n$ is An admissible control

$$\mathcal{U} := \left\{ v(\cdot) : [0,T] \times \Omega \to A : v(t) \text{ is } \mathcal{F}_t\text{-adapted} \right\}.$$

1.9.3 Near-optimal control

Let $\varepsilon > 0$, a control is a near-optimal control (or ε-optimal) if for any control $v(\cdot) \in \mathcal{U}$ we have

$$J(v^\varepsilon(t)) \leq J(v(t)) + \varepsilon.$$

1.9.4 Feedback control

We say that $v(\cdot)$ is a feedback control if the control $v(\cdot)$ depends on the state variable $X(\cdot)$. If $\mathcal{F}_t^X$ the natural filtration generated by the process X, then $v(\cdot)$ is a feedback control if $v(\cdot)$ is $\mathcal{F}_t^X-$adapted.

1.9.5 Random horizon

The time horizon in a classical problem is fixed until a deterministic terminal time T. Because the temporal horizon in some real-world applications may be random, the cost functional is as follows:

$$J(v(\cdot)) = \mathbb{E}\left[h(x(\tau)) + \int_0^\tau h(t, x(t), y(t), v(t))\, dt \right],$$

where τ s a finite random time.

1.9.6 Relaxed control

The basic idea is then to compact the space of controls $\mathcal{U}$ by extending the definition of controls to include the space of probability measures on U. The set of relaxed controls $\mu_t(du)\,dt$, where μ_t is a probability measure, is the closure under weak* topology of the measures $\delta_{u(t)}(du)dt$ corresponding to usual, or strict, controls. Young introduces the concept of relaxed control for deterministic optimal control problems (Young, L.C. Lectures on the Calculus of Variations and Optimal Control Theory, W.B. Saunders Co., 1969). (For more information, see Borkar [5].)

Chapter 2

Partially-Observed Optimal Control Problems for SDEs

2.1 Formulation of the Problem

In this section we would like to give a formulation of our problem. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{R}^{n\times d}$ the collection of $n \times d$ matrices. For a given Euclidean space, We denote by $|\cdot|$ the norm and by $\langle \cdot, \cdot \rangle$ the scalar product. And we denotes by superscript $\intercal$ to the transpose of matrices or vectors. $W(\cdot), Y(\cdot)$ are two independent standard Brownian motions valued in $\mathbb{R}^d$ and $\mathbb{R}^r$, respectively. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a complete filtered probability space equipped with a natural filtration

$$\mathcal{F}_t = \sigma\{W(s), Y(s); 0 \leq s \leq t\},$$

Let $\mathcal{F} := \mathcal{F}_t$, and let $T > 0$ be the finite time duration. $\mathbb{E}$ defines the expectation on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$. And let be

- $L^2(r, s; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued deterministic function $\eta(t)$, such that $\int_r^s |\eta(t)|^2 dt < +\infty$,
- $L^2(\mathcal{F}_t; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable φ, such that $\mathbb{E}|\varphi|^2 < +\infty$,
- $L^2_{\mathcal{F}}(r, s; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\psi(\cdot)$, such that $\mathbb{E}\int_r^s |\psi(t)|^2 dt < +\infty$.

We define

$$\mathcal{F}_t^Y := \sigma\{Y(s); 0 \leq s \leq t\},$$

Let U be a nonempty convex subset of $\mathbb{R}^k$.

Definition 2.1

An admissible control variable $\nu : [0,T] \times \Omega \rightarrow U$ *is a control variable* $\mathcal{F}_t^Y-$*adapted and satisfies* $\sup_{t\in[0,T]} \mathbb{E}\left|\nu\right|^m < \infty$, $m = 2, 3, \cdots$.

Denote by U_{ad} the set of the admissible control variables. For given $\nu(\cdot) \in U_{ad}$, we study a class of stochastic control problems of the type

$$\begin{cases} dx^v(t) = f(t, x^v(t), v(t))\, dt + \sigma(t, x^v(t), v(t))\, dW(t) \\ \qquad\qquad + \alpha(t, x^v(t), v(t))\, d\widetilde{W}(t), \quad t \in [0,T], \\ x^v(0) = x_0, \end{cases} \tag{2.1}$$

where x_0 is the initial path of $x(\cdot)$ and f, σ and α are given deterministic functions such as

$$f : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n,$$
$$\sigma : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n\times d}$$
$$\alpha : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n\times d}.$$

We assume that the state processes $x^v(\cdot)$ is not completely observable, instead, it is partially observed through the related process $Y(\cdot)$, which is described by the following equation

$$\begin{cases} dY(t) = h(t, x^v(t), v(t))dt + d\widetilde{W}(t) \\ Y(0) = 0, \end{cases} \tag{2.2}$$

where $h : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^r$, and $\widetilde{W}(\cdot)$ is a stochastic process depending on the control $v(\cdot)$. The cost functional is

$$J(v(\cdot)) = \mathbb{E}^v \left[\int_0^T l(t, x^v(t), v(t))dt + \psi(x^v(T)) \right], \tag{2.3}$$

Here, $\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$ and

$$l : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R},\ \psi : \mathbb{R}^n \rightarrow \mathbb{R}.$$

Throughout this chapter, we need to make the following hypothesis.

hypothesis (**A1**) We assume that the coefficients $f, \sigma, \alpha, l : [0,T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ and the function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable in all variables. also, $f(\cdot,\cdot,v), \sigma(\cdot,\cdot,v), \alpha(\cdot,\cdot,v), l(\cdot,\cdot,v) \in$

$\mathbb{C}_b^{1,1}\left(\mathbb{R}^d,\mathbb{R}\right)$ and $\psi(\cdot,\cdot) \in \mathbb{C}_b^{1,1}\left(\mathbb{R}^d,\mathbb{R}\right)$ for all $v \in U$. More precisely, the functions $f(x,v), \sigma(x,v), \alpha(x,v), l(x,v), \psi(x)$ satisfies the following properties:

1. all the derivatives $\partial_x f$, $\partial_x \sigma$, $\partial_x \alpha$, $\partial_x l$, $\partial_x \psi$ are bounded and Lipschitz continuous, with Lipschitz constants independent of $v \in U$;

2. the functions f, σ, α and l are continuously differentiable with respect to control variable v, and all their derivatives $\partial_v f, \partial_v \sigma, \partial_v \alpha$ and $\partial_v l$ are continuous and bounded.

3. h is a uniformly bounded function that is continuously differentiable in x and continuous in v, and its derivatives are also uniformly bounded.

For any $v(\cdot) \in U_{ad}$, the hypothesis (**A1**) implies that (2.1) admits a unique $\mathcal{F}_t$-adapted solution. Define $d\mathbb{P}^v = \rho^v(t)d\mathbb{P}$ with

$$\rho^v(t) = \exp\left\{\int_0^t h(s,x^v(s),v(s))dY(s) - \frac{1}{2}\int_0^t |h(s,x^v(s),v(s))|^2 ds\right\},$$

and $\rho(\cdot)$ is the unique $\mathcal{F}_t^Y$-adapted solution of the linear SDE

$$\begin{cases} d\rho^v(t) = \rho^v(t)h\left(t,x^v(t),v(t)\right)dY(t), \\ \rho^v(0) = 1. \end{cases} \tag{2.4}$$

By Itô's formula, we can prove that $\sup_{t\in[0,T]} \mathbb{E}\left|\rho_t^v\right|^m < \infty$, $m = 2,3,\cdots$. Hence, by Girsanov's theorem and hypothesis (**A1**), $\mathbb{P}^v$ is a new probability measure and $(W(\cdot), \widetilde{W}(\cdot))$ is a two-dimensional standard Brownian motion defined in the new probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$.

Our partially observed optimal control problem is to minimize the cost functional (2.3) over $v(\cdot) \in U_{ad}$ subject to (2.1) and (2.2), i.e., to find $u(\cdot) \in U_{ad}$ satisfying

$$J(u(\cdot)) = \inf_{v(\cdot)\in U_{ad}} J(v(\cdot)). \tag{2.5}$$

Clearly, cost functional (2.3) can be rewritten as

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T \rho^v(t)l(t,x^v(t),v(t))dt + \rho^v(T)\psi(x^v(T))\right]. \tag{2.6}$$

Then the original problem (2.5) is equivalent to minimize (2.6) over $v(\cdot) \in U_{ad}$ subject to (2.1) and (2.4). Our aim is to seek the necessary condition of the partially observed optimal control $u(\cdot)$ in the form of stochastic maximum principle.

2.2 Stochastic Maximum Principle for Partially Observed Optimal Control Problems

In this section, we develop the stochastic maximum principle for our partially observed control problem, throughout this, we are based on Girsanov's theorem with a standard convex variational technique.

Let x be the optimal trajectory corresponding to the optimal control $u(\cdot)$. For given $v(\cdot) \in U_{ad}$ and for any $\varepsilon \in (0,1)$, we define the variational control as following: $v^{\varepsilon}(\cdot) = u(\cdot) + \varepsilon v(\cdot)$, where $v^{\varepsilon}(\cdot) \in U_{ad}$.

With obvious notation, we denote by $x^{\varepsilon}(\cdot), x(\cdot), \rho^{\varepsilon}(\cdot), \rho(\cdot)$ the state trajectories of (2.1) and (2.4) corresponding to $v^{\varepsilon}(\cdot)$ and $u(\cdot)$. We now introduce the following SDEs

$$\begin{cases} d\phi(t) = \left\{ f_x\left(t, x(t), u(t)\right)\phi(t) + f_v(t, x(t), u(t))v(t) \right\} dt \\ \qquad\qquad + \left\{ \sigma_x(t, x(t), u(t))\phi(t) + \sigma_v(t, x(t), u(t))v(t) \right\} dW(t) \\ \qquad\qquad + \left\{ \alpha_x(t, x(t), u(t))\phi(t) + \alpha_v(t, x(t), u(t))v(t) \right\} d\widetilde{W}(t) \\ \phi(0) = 0, \end{cases} \tag{2.7}$$

and

$$\begin{cases} d\rho_1(t) = \left\{ \rho_1(t)h(t, x(t), u(t)) + \rho(t)h_x(t, x(t), u(t))\phi(t) \right. \\ \qquad\qquad \left. + \rho(t)h_v(t, x(t), u(t))v(t) \right\} dY(t), \\ \rho_1(0) = 0. \end{cases} \tag{2.8}$$

By hypothesis (**A1**), it is clear to infer that the stochastic differential equations (SDEs) (2.7) and (2.8) admit unique adapted solutions $\phi(\cdot)$ and $\rho_1(\cdot)$, respectively.

Lemma 2.1

Let hypothesis (**A1**) *hold. Then, we have*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|x^{\varepsilon}(t) - x(t)\right|^2\right] = 0.$$

Proof. From standard estimates, we get by using the Burkholder-Davis-Gundy inequality

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t}|x^{\varepsilon}(s)-x(s)|^{2}\right] &\leq \mathbb{E}\int_{0}^{t}|f(s,x^{\varepsilon}(s),v^{\varepsilon}(s))-f(s,x(s),u(s))|^{2}\,ds \\ &+\mathbb{E}\int_{0}^{t}|\sigma(s,x^{\varepsilon}(s),v^{\varepsilon}(s))-\sigma(s,x(s),u(s))|^{2}\,ds \\ &+\mathbb{E}\int_{0}^{t}|\alpha(s,x^{\varepsilon}(s),v^{\varepsilon}(s))-\alpha(s,x(s),u(s))|^{2}\,ds\end{aligned}$$

We use Lipschitz conditions on the functions f,σ and α with respect to x and v, we find

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}|x^{\varepsilon}(t)-x(t)|^{2}\right] &\leq C_{T}\mathbb{E}\int_{0}^{t}\left[|x^{\varepsilon}(s)-x(s)|^{2}\right]ds \\ &+C_{T}\varepsilon^{2}\mathbb{E}\int_{0}^{t}|v(s)|^{2}\,ds.\end{aligned} \tag{2.9}$$

according to (2.9) we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|x^{\varepsilon}(t)-x(t)|^{2}\right]\leq C_{T}\mathbb{E}\int_{0}^{t}\sup_{r\in[0,s]}|x^{\varepsilon}(r)-x(r)|^{2}\,ds+M_{T}\varepsilon^{2}.$$

By applying Gronwall's lemma, the result follows immediately by letting ε go to zero. □

Lemma 2.2

Let hypothesis (**A1**) *hold. Then, we have*

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}\mathbb{E}\left|\frac{x^{\varepsilon}(t)-x(t)}{\varepsilon}-\phi(t)\right|^{2}=0. \tag{2.10}$$

Proof. Let

$$\eta^{\varepsilon}(t)=\frac{x^{\varepsilon}(t)-x(t)}{\varepsilon}-\phi(t),\quad t\in[0,T].$$

For simplicity, We want to put the following notations

$$\begin{aligned}x^{\lambda,\varepsilon}(s)&=x(s)+\lambda\varepsilon\left(\eta^{\varepsilon}(s)+\phi(s)\right),\\ \widehat{x}^{\lambda,\varepsilon}(s)&=x(s)+\lambda\varepsilon\left(\widehat{\eta}^{\varepsilon}(s)+\widehat{\phi}(s)\right),\\ v^{\lambda,\varepsilon}(s)&=u(s)+\lambda\varepsilon v(s).\end{aligned}$$

So,

$$\begin{aligned}\eta^{\varepsilon}(t) &= \frac{1}{\varepsilon}\int_0^t \left[f(s,x^{\varepsilon}(s),v^{\varepsilon}(s)) - f(s,x(s),u(s))\right] ds \\ &+ \frac{1}{\varepsilon}\int_0^t \left[\sigma(s,x^{\varepsilon}(s),v^{\varepsilon}(s)) - \sigma(s,x(s),u(s))\right] dW(s) \\ &+ \frac{1}{\varepsilon}\int_0^t \left[\alpha(s,x^{\varepsilon}(s),v^{\varepsilon}(s)) - \alpha(s,x(s),u(s))\right] d\widetilde{W}(s) \\ &- \int_0^t \left\{f_x(s,x(s),u(s))\phi(s) + f_v(s,x(s),u(s))v(s)\right\} ds \\ &- \int_0^t \left\{\sigma_x(s,x(s),u(s))\phi(s) + \sigma_v(s,x(s),u(s))v(s)\right\} dW(s) \\ &- \int_0^t \left\{\alpha_x(s,x(s),u(s))\phi(s) + \alpha_v(s,x(s),u(s))v(s)\right\} d\widetilde{W}(s).\end{aligned}$$

Now, we decompose $\frac{1}{\varepsilon}\int_0^t \left(f(s,x^{\varepsilon}(s),v^{\varepsilon}(s)) - f(s,x(s),u(s))\right) ds$ into the following parts

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^t \left(f(s,x^{\varepsilon}(s),v^{\varepsilon}(s)) - f(s,x(s),u(s))\right) ds \\ &= \frac{1}{\varepsilon}\int_0^t \left(f(s,x^{\varepsilon}(s),v^{\varepsilon}(s)) - f(s,x(s),v^{\varepsilon}(s))\right) ds \\ &\quad + \frac{1}{\varepsilon}\int_0^t \left(f(s,x(s),v^{\varepsilon}(s)) - f(s,x(s),u(s))\right) ds.\end{aligned}$$

Noting that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^t \left(f(s,x^{\varepsilon}(s),v^{\varepsilon}(s)) - f(s,x(s),v^{\varepsilon}(s))\right) ds \\ &= \int_0^t \int_0^1 \left[f_x\left(s,x^{\lambda,\varepsilon}(s),v^{\varepsilon}(s)\right)(\eta^{\varepsilon}(s)+\phi(s))\right] d\lambda ds,\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^t \left(f(s,x(s),v^{\varepsilon}(s)) - f(s,x(s),u(s))\right) ds \\ &= \int_0^t \int_0^1 \left[f_v\left(s,x(s),v^{\lambda,\varepsilon}(s)\right)v(s)\right] d\lambda ds.\end{aligned}$$

With the same work, we can obtain a similar decomposition for σ and α. Therefore, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}|\eta^{\varepsilon}(s)|^2\right] &= C(t)\left[\mathbb{E}\int_0^t\int_0^1 \left|f_x\left(s,x^{\lambda,\varepsilon}(s),v^{\varepsilon}(s)\right)\eta^{\varepsilon}(s)\right|^2 d\lambda ds\right. \\ &+ \mathbb{E}\int_0^t\int_0^1 \left|\sigma_x\left(s,x^{\lambda,\varepsilon}(s),v^{\varepsilon}(s)\right)\eta^{\varepsilon}(s)\right|^2 d\lambda ds \\ &+ \mathbb{E}\int_0^t\int_0^1 \left|\alpha_x\left(s,x^{\lambda,\varepsilon}(s),v^{\varepsilon}(s)\right)\eta^{\varepsilon}(s)\right|^2 d\lambda ds \\ &\left.+ \mathbb{E}\left[\sup_{s\in[0,t]}|\gamma^{\varepsilon}(s)|^2\right]\right],\end{aligned}$$

where

$$\begin{aligned}
&\gamma^{\varepsilon}(t) \\
&= \int_0^t \int_0^1 \left[f_x \left(s, x^{\lambda,\varepsilon}(s), v^{\varepsilon}(s) \right) - f_x \left(s, x(s), u(s) \right) \right] \phi(s) d\lambda ds \\
&\quad + \int_0^t \int_0^1 \left[f_v \left(s, x(s), v^{\lambda,\varepsilon}(s) \right) - f_v \left(s, x(s), u(s) \right) \right] v(s) d\lambda ds \\
&\quad + \int_0^t \int_0^1 \left[\sigma_x \left(s, x^{\lambda,\varepsilon}(s), v^{\varepsilon}(s) \right) - \sigma_x \left(s, x(s), u(s) \right) \right] \phi(s) d\lambda dW(s) \\
&\quad + \int_0^t \int_0^1 \left[\sigma_v \left(s, x(s), v^{\lambda,\varepsilon}(s) \right) - \sigma_v \left(s, x(s), u(s) \right) \right] v(s) d\lambda dW(s) \\
&\quad + \int_0^t \int_0^1 \left[\alpha_x \left(s, x^{\lambda,\varepsilon}(s), v^{\varepsilon}(s) \right) - \alpha_x \left(s, x(s), u(s) \right) \right] \phi(s) d\lambda d\widetilde{W}(s) \\
&\quad + \int_0^t \int_0^1 \left[\alpha_v \left(s, x(s), v^{\lambda,\varepsilon}(s) \right) - \alpha_v \left(s, x(s), u(s) \right) \right] v(s) d\lambda d\widetilde{W}(s)
\end{aligned}$$

Now, we have the Property of Lipschitz continuous in (x, v) for the derivatives of the functions f, σ and α with respect to (x, v), Therefore, we get

$$\lim_{\varepsilon \to 0} \mathbb{E} \left[\sup_{s \in [0,T]} |\gamma^{\varepsilon}(s)|^2 \right] = 0.$$

Since the derivatives of f, σ and α with respect to (x, v) are bounded, we have $\forall t \in [0, T]$:

$$\mathbb{E} \left[\sup_{s \in [0,t]} |\eta^{\varepsilon}(s)|^2 \right] \leq c(t) \left\{ \mathbb{E} \int_0^t |\eta^{\varepsilon}(s)|^2 ds + \mathbb{E} \left[\sup_{s \in [0,t]} |\gamma^{\varepsilon}(s)|^2 \right] \right\}.$$

by applying Gronwall's lemma, we find $\forall t \in [0, T]$

$$\mathbb{E} \left[\sup_{s \in [0,t]} |\eta^{\varepsilon}(s)|^2 \right] \leq c(t) \left\{ \mathbb{E} \left[\sup_{s \in [0,t]} |\gamma^{\varepsilon}(s)|^2 \right] \exp \left\{ \int_0^t c(s) ds \right\} \right\}.$$

Finally, putting $t = T$ and letting ε go to zero, this completes the proof. □

We also to obtain some ε-order estimations of the difference between the perturbed observed process $\rho^{\varepsilon}(\cdot)$ with the sum of the optimal observed process $\rho(\cdot)$ and the variational observed $\rho_1(\cdot)$. The following lemma play an important role in computing the variational inequality for the cost functional (2.6) subject to (2.1) and (2.4).

Lemma 2.3

Let hypothesis (**A1**) *hold. Then, we have*

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{\rho^{\varepsilon}(t) - \rho(t)}{\varepsilon} - \rho_1(t) \right|^2 = 0. \tag{2.11}$$

Proof. According to the definition of $\rho(\cdot)$ and $\rho_1(\cdot)$, we have

$$\begin{aligned}
&\rho(t)+\varepsilon\rho_1(t)\\
&=1+\int_0^t \rho(s)h(s,x(s),u(s))dY(s)\\
&\quad+\varepsilon\int_0^t \left[\rho_1(s)\,h(s,x(s),u(s))+\rho(s)h_x(s,x(s),u(s))\,\phi(s)+\rho(s)h_v(s,x(s),u(s))\,v(s)\right]dY(s)\\
&=1+\varepsilon\int_0^t \rho_1(s)h(s,x(s),u(s))dY(s)+\int_0^t \rho(s)\,h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s)\,dY(s)\\
&\quad-\varepsilon\int_0^t \rho(s)\left[A^\varepsilon(s)\right]dY(s),
\end{aligned}$$

where

$$\begin{aligned}
A^\varepsilon(s)&=\int_0^1\left[h_x(s,x(s)+\lambda\varepsilon\phi(s),u(s)+\lambda\varepsilon v(s))-h_x(s,x(s),u(s))\right]d\lambda\phi(s)\\
&\quad+\int_0^1\left[h_v(s,x(s)+\lambda\varepsilon\phi(s),u(s)+\lambda\varepsilon v(s))-h_v(s,x(s),u(s))\right]d\lambda v(s).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\rho^\varepsilon(t)-\rho(t)-\varepsilon\rho_1(t)\\
&=\int_0^t \rho^\varepsilon(s)\,h(s,x^\varepsilon(s),v^\varepsilon(s))\,dY(s)-\varepsilon\int_0^t \rho_1(s)h(s,x(s),u(s))dY(s)\\
&\quad-\int_0^t \rho(s)h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))\,dY(s)+\varepsilon\int_0^t \rho(s)\left[A^\varepsilon(s)\right]dY(s)\\
&=\int_0^t (\rho^\varepsilon(s)-\rho(s)-\varepsilon\rho_1(s))\,h(s,x^\varepsilon(s),v^\varepsilon(s))\,dY(s)\\
&\quad+\int_0^t (\rho(s)+\varepsilon\rho_1(s))\left[h(s,x^\varepsilon(s),v^\varepsilon(s))-h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))\right]dY(s)\\
&\quad+\varepsilon\int_0^t \rho_1(s)\,h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))dY(s)\\
&\quad-\varepsilon\int_0^t \rho_1(s)\,h(s,x(s),u(s))dY(s)+\varepsilon\int_0^t \rho(s)\left[A^\varepsilon(s)\right]dY(s)\\
&=\int_0^t (\rho^\varepsilon(s)-\rho(s)-\varepsilon\rho_1(s))\,h(s,x^\varepsilon(s),v^\varepsilon(s))\,dY(s)\\
&\quad+\int_0^t (\rho(s)+\varepsilon\rho_1(s))\left[\Lambda_1^\varepsilon(s)\right]dY(s)+\varepsilon\int_0^t \rho_1(s)\left[\Lambda_2^\varepsilon(s)\right]dY(s)\\
&\quad+\varepsilon\int_0^t \rho(s)\left[A^\varepsilon(s)\right]dY(s),
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1^\varepsilon(s)&=h(s,x^\varepsilon(s),v^\varepsilon(s))-h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s)),\\
\Lambda_2^\varepsilon(s)&=h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))-h(s,x(s),u(s)).
\end{aligned}$$

Note that

$$\Lambda_1^\varepsilon(s) = \int_0^1 \left[h_x(s, x(s) + \varepsilon\phi(s) + \lambda(x^\varepsilon(s) - x(s) - \varepsilon\phi(s)), v^\varepsilon(s))\right] d\lambda(x^\varepsilon(s) - x(s) - \varepsilon\phi(s)).$$

By Lemma 2.2, we know that

$$\mathbb{E}\int_0^t \left|(\rho(s) + \varepsilon\rho_1(s))\Lambda_1^\varepsilon(s)\right|^2 ds \leq C_\varepsilon \varepsilon^2, \tag{2.12}$$

hereafter C_ε denotes some nonnegative constant such that $C_\varepsilon \to 0$ as $\varepsilon \to 0$. Also, it is easy to see that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\varepsilon \int_0^t \rho(s) A^\varepsilon(s) dY(s)\right]^2 \leq C_\varepsilon \varepsilon^2, \tag{2.13}$$

and

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\varepsilon \int_0^t \rho_1(s) \Lambda_2^\varepsilon(s) dY(s)\right]^2 \leq C_\varepsilon \varepsilon^2. \tag{2.14}$$

By (2.12), (2.13) and (2.14), we have

$$\begin{aligned}
&\mathbb{E}\left|(\rho^\varepsilon(t) - \rho(t)) - \varepsilon\rho_1(t)\right|^2 \\
&\leq C\Bigg[\int_0^t \mathbb{E}\left|(\rho^\varepsilon(s) - \rho(s)) - \varepsilon\rho_1(s)\right|^2 + \mathbb{E}\int_0^t \left|(\rho(s) + \varepsilon\rho_1(s))\Lambda_1^\varepsilon(s)\right|^2 ds \\
&\quad + \sup_{0\leq s\leq t} \mathbb{E}\left(\varepsilon \int_0^t \rho(s) A^\varepsilon(s) dY(s)\right)^2 + \sup_{0\leq s\leq t} \mathbb{E}\left(\varepsilon \int_0^t \rho_1(s) \Lambda_2^\varepsilon(s) dY(s)\right)^2\Bigg] \\
&\leq C \int_0^t \mathbb{E}\left|\rho^\varepsilon(s) - \rho(s) - \varepsilon\rho_1(s)\right|^2 ds + C_\varepsilon \varepsilon^2.
\end{aligned}$$

Finally, by using Gronwall's inequality, we obtain the desired result. □

Lemma 2.4

Under hypothesis (**A1**), *one has*

$$\begin{aligned}
0 \leq \mathbb{E}\int_0^T &\{\rho_1(t)\, l(t, x(t), u(t)) + \rho(t)\, l_x(t, x(t), u(t))\phi(t) \\
&+ \rho(t) l_v(t, x(t), u(t)) v(t)\}\, dt \\
&+ \mathbb{E}\left[\rho_1(T)\, \psi(x(T))\right] + \mathbb{E}\left[\rho(T)\, \psi_x(x(T))\phi(T)\right]
\end{aligned} \tag{2.15}$$

Proof. Using Lemmas 2.2 and 2.3, Taylor expansion, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{-1} \mathbb{E}\left[\rho^\varepsilon(T)\, \psi(x^\varepsilon(T)) - \rho(T)\, \psi(x(T))\right] \\
&= \mathbb{E}\left[\rho_1(T)\psi(x(T)) + \rho(T)\psi_x(x(T))\phi(T)\right],
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \varepsilon^{-1} \mathbb{E} \int_0^T \left\{ \rho^\varepsilon(t) l(t, x^\varepsilon(t), u^\varepsilon(t)) - \rho(t) l(t, x(t), u(t)) \right\} dt \\
&= \mathbb{E} \int_0^T \left\{ \rho_1(t) l(t, x(t), u(t)) + \rho(t) l_x(t, x(t), u(t)) \phi(t) \right. \\
&\left. + \rho(t) l_v(t, x(t), u(t)) v(t) \right\} dt.
\end{aligned}$$

Then, by the optimality of $u(\cdot)$, we draw the desired conclusion. Define the Hamiltonian $H : [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, by

$$H(t, x, v, \Phi, Q, \overline{Q}, K) = l(t, x, v) + f(t, x, v)\Phi + \sigma(t, x, v) Q + \alpha(t, x, v) \overline{Q} + h(t, x, v) K. \tag{2.16}$$

Now, we introduce the adjoint equations involved in the stochastic maximum principle:

$$\begin{cases} -dy(t) = l(t, x(t), u(t)) dt - z(t)\, dW(t) - K(t)\, d\widetilde{W}(t), \\ y(T) = \psi(x(T)), \end{cases} \tag{2.17}$$

and

$$\begin{cases} -d\Phi(t) = \left\{ f_x(t, x(t), u(t)) \Phi(t) + \sigma_x(t, x(t), u(t)) Q(t) \right. \\ \qquad\qquad + \alpha_x(t, x(t), u(t)) \overline{Q}(t) + l_x(t, x(t), u(t)) \\ \qquad\qquad \left. + h_x(t, x(t), u(t)) K(t) \right\} dt - Q(t) dW(t) - \overline{Q}(t) d\widetilde{W}(t), \\ \Phi(T) = \psi_x(x(T)). \end{cases} \tag{2.18}$$

Set $\widetilde{\rho}(t) = \rho^{-1}(t) \rho_1(t)$, by using Itô's formula, we have

$$\begin{cases} d\widetilde{\rho}(t) = \left\{ h_x(t, x(t), u(t)) \phi(t) + h_v(t, x(t), u(t)) v(t) \right\} d\widetilde{W}(t), \\ \widetilde{\rho}(0) = 0, \end{cases} \tag{2.19}$$

Then, applying Itô's formula to $\Phi(t)\phi(t)$, $y(t)\widetilde{\rho}(t)$ and taking expectation respectively, we get

$$\begin{aligned}
\mathbb{E}^u [\Phi(T)\phi(T)] &= \mathbb{E}^u \int_0^T \Phi(t)\, d\phi(t) + \mathbb{E}^u \int_0^T \phi(t)\, d\Phi(t) + \mathbb{E}^u \int_0^T Q(t) \left\{ \sigma_x(t, x(t), u(t)) \phi(t) \right. \\
&\left. + \sigma_v(t, x(t), u(t)) v(t) \right\} dt + + \mathbb{E}^u \int_0^T \overline{Q}(t) \left\{ \alpha_x(t, x(t), u(t)) \phi(t) \right. \\
&\left. + \alpha_v(t, x(t), u(t)) v(t) \right\} dt. \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}
I_1 &= \mathbb{E}^u \int_0^T \Phi(t)\, d\phi(t) \\
&= \mathbb{E}^u \int_0^T \Phi(t) \left\{ f_x(t, x(t), u(t))\phi(t) + f_v(t, x(t), u(t))v(t) \right\} dt \\
&= \mathbb{E}^u \int_0^T \Phi(t)\, f_x(t, x(t), u(t))\phi(t) dt + \mathbb{E}^u \int_0^T \Phi(t)\, f_v(t, x(t), u(t))v(t) dt
\end{aligned}$$

Consequently,

$$\begin{aligned}
I_2 &= \mathbb{E}^u \int_0^T \phi(t)\, d\Phi(t) \\
&= -\mathbb{E}^u \int_0^T \phi(t) \left\{ f_x(t, x(t), u(t))\, \Phi(t) + \sigma_x(t, x(t), u(t))\, Q(t) \right. \\
&\left. + \alpha_x(t, x(t), u(t))\, \overline{Q}(t) + l_x(t, x(t), u(t)) + h_x(t, x(t), u(t))\, K(t) \right\} dt.
\end{aligned}$$

By simple computation, we deduce

$$\begin{aligned}
I_2 = &-\mathbb{E}^u \int_0^T \phi(t)\, f_x(t, x(t), u(t))\, \Phi(t)\, dt - \mathbb{E}^u \int_0^T \phi(t)\, \sigma_x(t, x(t), u(t))\, Q(t) dt \\
&- \mathbb{E}^u \int_0^T \phi(t)\, \alpha_x(t, x(t), u(t))\, \overline{Q}(t) dt - \mathbb{E}^u \int_0^T \phi(t)\, l_x(t, x(t), u(t))\, dt \\
&- \mathbb{E}^u \int_0^T \phi(t)\, h_x(t, x(t), u(t))\, K(t) dt
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
I_3 = &\mathbb{E}^u \int_0^T Q(t)\sigma_x(t, x(t), u(t))\phi(t) dt + \mathbb{E}^u \int_0^T Q(t)\sigma_v(t, x(t), u(t))v(t) dt \\
&+ \mathbb{E}^u \int_0^T \overline{Q}(t)\alpha_x(t, x(t), u(t))\phi(t) dt + \mathbb{E}^u \int_0^T \overline{Q}(t)\alpha_v(t, x(t), u(t))v(t) dt,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^u \left[y(T)\, \widetilde{\rho}(T) \right] &= \mathbb{E}^u \int_0^T y(t)\, d\widetilde{\rho}(t) + \mathbb{E}^u \int_0^T \widetilde{\rho}(t)\, dy(t) \\
&+ \mathbb{E}^u \int_0^T K(t) \left\{ h_x(t, x(t), u(t))\phi(t) + h_v(t, x(t), u(t))v(t) \right\} dt \qquad (2.21) \\
&= J_1 + J_2 + I_3,
\end{aligned}$$

where $J_1 = \mathbb{E}^u \int_0^T y(t)\, d\widetilde{\rho}(t)$ is a martingale with zero expectation, and

$$\begin{aligned}
J_2 &= \mathbb{E}^u \int_0^T \widetilde{\rho}(t)\, dy(t) \\
&= -\mathbb{E}^u \int_0^T \widetilde{\rho}(t)\, l(t, x(t), u(t)) dt.
\end{aligned}$$

Similarly, we can obtain

$$J_3 = \mathbb{E}^u \int_0^T K(t) \{ h_x(t, x(t), u(t)) \phi(t) + h_v(t, x(t), u(t)) v(t) \} \, dt$$

Finally, substituting (2.20) and (2.21) into (2.15), we get

$$\mathbb{E}^u \left[H_v(t, x(t), u(t), \Phi(t), Q(t), \overline{Q}(t), K(t)) v(t) \right] \geq 0. \tag{2.22}$$

Using the similar method developed in [24], our main result of this section is the following Theorem. □

Theorem 2.1

Let hypothesis (**A1**) *hold. Let* $u(\cdot)$ *be optimal. Then, the maximum principle*

$$\mathbb{E}^u \left[H_v(t, x(t), u(t), \Phi(t), Q(t), \overline{Q}(t), K(t)) (v(t) - u(t)) \mid \mathcal{F}_t^Y \right] \geq 0, \ \forall v \in U, \quad a.e., a.s.,$$

holds, where the Hamiltonian function H *is defined by* (2.16).

2.3 Application: Partially observed linear-quadratic control problem

In this section, we apply the results obtained in the previous section and classical filtering theory to study a partially observed linear-quadratic control problem. The optimal control is given in feedback form involving the state of the solution process via the solutions of ordinary differential equations (ODEs). Let us consider the following control problem:

$$\min \{ J(v(\cdot)), \ v \in U_{ad} \}, \ J(v(\cdot)) = \mathbb{E}^u \left[\int_0^T L(t) v^2(t) \, dt + M_T x^2(T) \right], \tag{2.23}$$

subject to

$$\begin{cases} dx(t) = \left\{ A(t) x(t) + C(t) v(t) \right\} dt + D(t) \, dW(t), \\ x(0) = x_0, \end{cases} \tag{2.24}$$

and

$$\begin{cases} dY(t) = G(t) \, dt + d\widetilde{W}(t) \\ Y(0) = 0, \end{cases} \tag{2.25}$$

where

$$\begin{aligned} f\left(t,x^{v}\left(t\right),v\left(t\right)\right) &= A\left(t\right)x\left(t\right)+C\left(t\right)v\left(t\right), \\ \sigma\left(t,x^{v}\left(t\right),v\left(t\right)\right) &= D\left(t\right), \\ h(t,x^{v}\left(t\right),v\left(t\right)) &= G\left(t\right). \end{aligned}$$

Here $L\left(\cdot\right),A\left(\cdot\right),C\left(\cdot\right),D\left(\cdot\right)$ and $G\left(\cdot\right)$ are bounded continuous functions and $M_T \geq 0$. Then for any $v \in U_{ad}$, (2.24) and (2.25) have unique solutions respectively. Let $u\left(\cdot\right)$ be an optimal control of (2.23), the corresponding optimal trajectory denoted by $x\left(\cdot\right)$. Further due to (2.18), the corresponding adjoint equation gets the form

$$\begin{cases} -d\Phi\left(t\right)=A\left(t\right)\Phi\left(t\right)dt-Q\left(t\right)dW\left(t\right)-\overline{Q}(t)d\widetilde{W}(t), \\ \Phi\left(T\right)=2M_T x\left(T\right). \end{cases} \tag{2.26}$$

Obviously, (2.26) admits a unique solution. In this case, the Hamiltonian function is defined as

$$\begin{aligned} H(t,x,v,\Phi,Q,K) &= \left[A\left(t\right)x\left(t\right)+C\left(t\right)v\left(t\right)\right]\Phi\left(t\right) \\ &\quad +D\left(t\right)Q\left(t\right)+G\left(t\right)K+L\left(t\right)v^{2}\left(t\right). \end{aligned} \tag{2.27}$$

If $u\left(\cdot\right)$ is optimal, then it follows from Theorem 2.1 and (2.27) that

$$u\left(t\right)=-\frac{1}{2}L^{-1}\left(t\right)C\left(t\right)\mathbb{E}\left[\Phi\left(t\right)\mid\mathcal{F}_t^Y\right]. \tag{2.28}$$

Next, we will give a more explicit representation of $u\left(\cdot\right)$. Assume that $\widehat{\Phi}\left(t\right)=\mathbb{E}^{u}\left[\Phi\left(t\right)\mid\mathcal{F}_t^Y\right]$, $\widehat{\overline{Q}}\left(t\right)=\mathbb{E}^{u}\left[Q\left(t\right)\mid\mathcal{F}_t^Y\right]$ and $\widehat{x}\left(t\right)=\mathbb{E}^{u}\left[x\left(t\right)\mid\mathcal{F}_t^Y\right]$ are the filtering estimates of adjoint processes $\Phi\left(\cdot\right),Q\left(\cdot\right)$ and the optimal trajectory $x\left(\cdot\right)$ respectively. From Theorems 8.1 in [50], we obtain

$$\begin{cases} d\widehat{x}\left(t\right)=\left\{A\left(t\right)\widehat{x}\left(t\right)-\frac{1}{2}L^{-1}\left(t\right)C^{2}\left(t\right)\widehat{\Phi}\left(t\right)\right\}dt \\ -d\widehat{\Phi}\left(t\right)=A\left(t\right)\widehat{\Phi}\left(t\right)dt-\widehat{\overline{Q}}\left(t\right)d\widetilde{W}\left(t\right), \\ \widehat{x}\left(0\right)=x_0,\ \widehat{\Phi}\left(T\right)=2M_T\widehat{x}\left(T\right),\ \widehat{\overline{Q}}\left(t\right)=0. \end{cases} \tag{2.29}$$

To solve (2.29), set $\widehat{\Phi}\left(t\right)=\varphi\left(t\right)\widehat{x}\left(t\right)$, where $\varphi\left(\cdot\right)$ is deterministic differential function which will be specified below. Then

$$-A\left(t\right)\widehat{\Phi}\left(t\right)=\dot{\varphi}\left(t\right)\widehat{x}\left(t\right)+\varphi\left(t\right)\left\{A\left(t\right)\widehat{x}\left(t\right)-\frac{1}{2}L^{-1}\left(t\right)C^{2}\left(t\right)\widehat{\Phi}\left(t\right)\right\} \tag{2.30}$$

By comparing coefficient of $\widehat{x}(t)$ of the above equation respectively, we get the following ODEs:

$$\begin{cases} \dot{\varphi}(t) + 2A(t)\varphi(t) - \frac{1}{2}L^{-1}(t)C^2(t)\varphi^2(t) = 0, \\ \varphi(T) = 2M_T, \end{cases} \tag{2.31}$$

Note that equation (2.31) is a Bernoulli differential equation of the form:

$$\dot{\varphi}(t) = a(t)\varphi(t) + b(t)\varphi(t)^2, \ \varphi(T) = 2M_T \tag{2.32}$$

where $a(t) = -2A(t)$, $b(t) = \frac{1}{2}L^{-1}(t)C^2(t)$. To solve the differential equation (2.32) we using the transformation $\xi(t) = \frac{1}{\varphi(t)}$ it follows that

$$\dot{\xi}(t) + a(t)\xi(t) = b(t), \ \xi(T) = \frac{1}{2M_T}. \tag{2.33}$$

Now, equation (2.33) is a linear differential equation of first-order. Applying the integrating factor method, we obtain

$$\xi(t) = \frac{1}{\mu(t)}\left[\int \mu(s)\frac{C^2(s)}{2L(s)}ds + c\right],$$

where c is an arbitrary canstant, and $\mu(t)$ the integrating factor given by

$$\mu(t) = \exp\int -2A(t)dt. \tag{2.34}$$

By a simple computation, we have

$$\varphi(t) = \frac{\mu(t)}{\int \mu(s)\frac{C^2(s)}{2L(s)}ds + c}. \tag{2.35}$$

As an illustration we consider the simple case when the functions $A(t) = -\frac{1}{t}$, $C(t) = 4t$, $L(t) = 2t$ with $T = 1$, $M_t = \frac{1}{4} > 0$, then equation (2.33) being

$$\dot{\xi}(t) + \frac{2}{t}\xi(t) = 4t,$$

with integrating factor $\mu(t) = t^2$. A simple computation shows that

$$\xi(t) = t^2 + \frac{c}{t^2}, \ \xi(1) = 2,$$

which implied that $c = 1$. Since $\varphi(t) = \frac{1}{\xi(t)}$ we get

$$\varphi(t) = \frac{t^2}{t^4+1}, \ \varphi(1) = \frac{1}{2}.$$

Moreover, $A(\cdot)$, $C(\cdot)$, $D(\cdot)$, $G(\cdot)$, $L(\cdot)$ and $\varphi(\cdot)$ are a given bounded continuous functions. Hence, the optimal control $u(\cdot) \in U_{ad}$ for the problem (2.23) is given in the feedback form

$$u(t) = -\frac{1}{2} L^{-1}(t) C(t) \varphi(t) \widehat{x}(t),$$

with $\varphi(\cdot)$ determined by (2.31).

CHAPTER 3

Stochastic maximum principle for partially observed optimal control problems of general McKean-Vlasov differential equations

3.1 Introduction

In this chapter, we study a class of partially observed optimal control problems for general McKean-Vlasov differential equations in which the coefficients depend on the state of the solution process as well as on its law and the control variable. By applying Girsanov's theorem with a standard variational technique, we prove a stochastic maximum principle on the assumption that the control domain is convex. We define the system of our partially observed optimal control problems, which we will study in this part, by the following form:

$$\begin{cases} dx^v(t) = f\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) dt + \sigma\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) dW(t) \\ \qquad\qquad + \alpha\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) d\widetilde{W}(t), \quad t \in [0, T], \\ x^v(0) = x_0, \end{cases}$$

where $\mathbb{P}_X$ is the law of the random variable X, $W(\cdot)$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $\widetilde{W}(\cdot)$ denotes a stochastic process depending on the control variable $v(\cdot)$. The coefficients $f : [0, T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^n$ and $\sigma, \alpha : [0, T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^{n \times d}$ are given deterministic functions, where $Q_2\left(\mathbb{R}^d\right)$ is the space of all probability measures μ on $\mathbb{R}^d$, endowed with 2-Wasserstein metric.
Now, we define the cost functional to be minimized over the class of admissible controls is also of McKean-Vlasov type, which has the following form

$$J(v(\cdot)) = \mathbb{E}^v\left[\int_0^T l\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) dt + \psi(x^v(T), \mathbb{P}_{x^v(T)})\right],$$

where $\mathbb{E}^v$ denotes expectation on the space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$ and $l : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$, $\psi : \mathbb{R}^n \times Q_2(\mathbb{R}) \to \mathbb{R}$ are deterministic functions.

Several authors, for example [2, 20, 91], have established stochastic optimal control of partially observed diffusions. A stochastic maximum principle for SDEs of mean-field type has been established by Buckdahn [8], a general maximum principle for stochastic differential equations of mean-field type with jumps has been studied by Hafayed and Abbas [26], and a stochastic maximum principle for systems with jumps, with applications to fiance, has been studied by Cadenillas [12]. The control of McKean–Vlasov dynamics versus mean-filed games has been discussed by Carmona [15]. Shen and Siu [65] established the maximum principle for a jump-diffusion mean-field model and its application to the mean-variance problem. The maximum principle for mean-field jump-diffusions to stochastic delay differential equations and its application to fiance has been studied by Shen, Meng, and Shi [66]. Yong [87] investigated a linear-quadratic optimal control problem for mean-field stochastic differential equations. For continuous time mean-variance portfolio selection using a stochastic LQ framework, see [90].

To begin with, in this chapter, we will formulate our problem as follows: Then, we include the precise definition of the derivatives with respect to the probability measure and give the notations and assumptions that are needed throughout this work. Then, we prove the stochastic maximum principle for our partially observed control problem of general Mckean-Vlasov differential equations. As an application, a partially observed linear-quadratic control problem is discussed.

3.2 Assumptions and Problem Formulation

Throughout this chapter, we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{R}^{n\times d}$ the collection of $n \times d$ matrices. Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which are defined two independent standard one-dimensional Brownian motions $W(\cdot)$ and $Y(\cdot)$. Let $\mathcal{F}_t^W$ and $\mathcal{F}_t^Y$ be the $\mathbb{P}$-completed natural filtration generated by $W(\cdot)$ and $Y(\cdot)$, respectively. Set $\mathcal{F}_t = \mathcal{F}_t^Y \vee \mathcal{F}_t^W$ and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$. For a given Euclidean space, we would denote by $\langle \cdot, \cdot \rangle$ (resp. $|\cdot|$) the scalar product (resp. norm). $\mathbb{E}$ denotes the expectation on the space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Additionally given $r < s$,

- $L^2(r, s; \mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued deterministic function $\eta(t)$, such that
$$\int_r^s |\eta(t)|^2 dt < +\infty.$$
- $L^2(\mathcal{F}_t; \mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable φ, such that
$$\mathbb{E}|\varphi|^2 < +\infty.$$
- $L^2_{\mathcal{F}}(r, s; \mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\psi(\cdot)$, such that
$$\mathbb{E}\int_r^s |\psi(t)|^2 dt < +\infty.$$
- $\mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ is the Hilbert space with inner product $(x, y)_2 = \mathbb{E}[x.y]$, $x, y \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ and the norm $\|x\|_2 = \sqrt{(x, x)_2}$.
- $Q_2\left(\mathbb{R}^d\right)$ is the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ with finite second moment, i.e, $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty$, endowed with the following 2-Wasserstein metric; for $\mu, \nu \in Q_2\left(\mathbb{R}^d\right)$,
$$\mathbb{W}_2(\mu, \nu) = \inf\left\{\left[\int_{\mathbb{R}^d} |x - y|^2 \rho(dx, dy)\right]^{\frac{1}{2}} : \rho \in Q_2\left(\mathbb{R}^{2d}\right), \rho\left(\cdot, \mathbb{R}^d\right) = \mu, \rho\left(\mathbb{R}^d, \cdot\right) = \nu\right\}.$$

Let $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ be a probability space represent a copy of the space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. For any pair of random variable $(\vartheta, \xi) \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right) \times \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$, we let $\left(\widehat{\vartheta}, \widehat{\xi}\right)$ be an independent copy of (ϑ, ξ) defined on $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}}\right)$. We consider the product probability space $\left(\Omega \times \widehat{\Omega}, \mathcal{F} \otimes \widehat{\mathcal{F}}, \mathbb{F} \otimes \widehat{\mathbb{F}}, \mathbb{P} \otimes \widehat{\mathbb{P}}\right)$ and setting $\left(\widehat{\vartheta}, \widehat{\xi}\right)(w, \widehat{w}) = (\vartheta(\widehat{w}), \xi(\widehat{w}))$ for any $(w, \widehat{w}) \in \Omega \times \widehat{\Omega}$. Let $(\widehat{u}(t), \widehat{x}(t))$ be an independent copy of $(u(t), x(t))$ so that $\mathbb{P}_{x(t)} = \widehat{\mathbb{P}}_{\widehat{x}(t)}$. We can denote by $\widehat{\mathbb{E}}$ the expectation under probability measure $\widehat{\mathbb{P}}$ and $\mathbb{P}_X = \mathbb{P} \circ X^{-1}$ denotes the law of the random variable X. The general results of the differentiability with respect to probability measures have been studied by several authors, see e.g. [7, 14, 13, 24]. The main idea is to identify a distribution $\mu \in Q_2\left(\mathbb{R}^d\right)$ with a random variables $\vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ so that $\mu = \mathbb{P}_\vartheta$. To be more precise, we suppose that probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is rich enough in the sense that for every $\mu \in Q_2\left(\mathbb{R}^d\right)$, there is a random variable $\vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ such that $\mu = \mathbb{P}_\vartheta$. It is well-known that the probability space $([0,1], \mathcal{B}[0,1], dx)$, where dx is the Borel measure, has this property. Now, we present the basic notations in the differentiability with respect to probability measures. Return to [7], for any function $f : Q_2\left(\mathbb{R}^d\right) \to \mathbb{R}$, we induce a function $\widetilde{f} : \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right) \to \mathbb{R}$ such that $\widetilde{f}(\vartheta) := f(\mathbb{P}_\vartheta)$, $\vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$. Clearly, the function $\widetilde{f}$ called the lift of f, depends only on the law of $\vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ and is independent of the choice of the representative ϑ.

Definition 3.1

Let $f : Q_2\left(\mathbb{R}^d\right) \to \mathbb{R}$ be a function, f is said to be differentiable at $\mu_0 \in Q_2\left(\mathbb{R}^d\right)$ if there exists $\vartheta_0 \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ with $\mu_0 = \mathbb{P}_{\vartheta_0}$ such that its lift $\widetilde{f}$ is Fréchet differentiable at ϑ_0. More precisely, there exists a continuous linear functional $\mathbf{D}\widetilde{f}(\vartheta_0) : \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right) \to \mathbb{R}$ such that

$$\widetilde{f}(\vartheta_0 + \xi) - \widetilde{f}(\vartheta_0) = \left\langle \mathbf{D}\widetilde{f}(\vartheta_0), \xi \right\rangle + O(\|\xi\|_2) = \mathbf{D}_\xi f(\mu_0) + O(\|\xi\|_2), \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ is the dual product on the space $\mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$, and we will refer to $\mathbf{D}_\xi f(\mu_0)$ as the Fréchet derivative of f at μ_0 in the direction ξ. In this case, we have

$$\mathbf{D}_\xi f(\mu_0) = \left\langle \mathbf{D}\widetilde{f}(\vartheta_0), \xi \right\rangle = \frac{d}{dt}\widetilde{f}(\vartheta_0 + t\xi)\Big|_{t=0}, \quad \text{with } \mu_0 = \mathbb{P}_{\vartheta_0}.$$

According to Riesz' representation theorem, there is a unique random variable $\Theta_0 \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ such that $\left\langle \mathbf{D}\widetilde{f}(\vartheta_0), \xi \right\rangle = (\Theta_0, \xi)_2 = \mathbb{E}\left[(\Theta_0, \xi)_2\right]$, where $\xi \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$. It was shown (see the works of Buckdahn, Li, Ma [7] and Lions, P.L. [51]) that there exists a Boral function $h[\mu_0] : \mathbb{R}^d \to \mathbb{R}^d$, depending only on the law $\mu_0 = \mathbb{P}_{\vartheta_0}$ but not on the particular choice of the representative ϑ_0 such that $\Theta_0 = h[\mu_0](\vartheta_0)$. Thus, we can write (3.1) as

$$f(\mathbb{P}_\vartheta) - f(\mathbb{P}_{\vartheta_0}) = (h[\mu_0](\vartheta_0), \vartheta - \vartheta_0)_2 + O(\|\vartheta - \vartheta_0\|_2), \quad \forall \vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right).$$

And we denote

$$\partial_\mu f(\mathbb{P}_{\vartheta_0}, x) = h[\mu_0](x), \; x \in \mathbb{R}^d.$$

We also have the following identities:

$$\mathbf{D}\widetilde{f}(\vartheta_0) = \Theta_0 = h[\mu_0](\vartheta_0) = \partial_\mu f(\mathbb{P}_{\vartheta_0}, \vartheta_0),$$

and

$$\mathbf{D}_\xi f(\mathbb{P}_{\vartheta_0}) = \langle \partial_\mu f(\mathbb{P}_{\vartheta_0}, \vartheta_0), \xi \rangle,$$

where $\xi = \vartheta - \vartheta_0$.

We now identify an important remark.

Remark 3.1

We note that for each $\mu \in Q_2(\mathbb{R}^d)$, $\partial_\mu f(\mathbb{P}_\vartheta, \cdot) = h[\mathbb{P}_\vartheta](\cdot)$ is only defined in a $\mathbb{P}_\vartheta(dx)$-a.e sense, where $\mu = \mathbb{P}_\vartheta$.

Definition 3.2

(Space of differentiable functions in $Q_2(\mathbb{R}^d)$)
Let $f \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^d))$ be a function.
We say that the function $f \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^d))$ if for all $\vartheta \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^d)$, there exists a $\mathbb{P}_\vartheta$-modification of $\partial_\mu f(\mathbb{P}_\vartheta, \cdot)$ such that $\partial_\mu f : Q_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipchitz continuous. That is for some $C > 0$, it holds that

A1. $|\partial_\mu f(\mu, x)| \leq C, \forall \mu \in Q_2(\mathbb{R}^d), \forall x \in \mathbb{R}^d$;
A2. $|\partial_\mu f(\mu_1, x_1) - \partial_\mu f(\mu_2, x_2)| \leq C(\mathbb{W}_2(\mu_1, \mu_2) + |x_1 - x_2|), \forall \mu_1, \mu_2 \in Q_2(\mathbb{R}^d), \forall x_1, x_2 \in \mathbb{R}^d$.

We would like to make a point out that the version of $\partial_\mu f(\mathbb{P}_\vartheta, \cdot), \vartheta \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^d)$ indicated in the above definition is unique (see Remark 2.2 in [7] for more information).

Our objective of this chapter is to study a class of general stochastic control problems of general McKean-Vlasov differential equations, with partial observation which is an extension of the model considered by Fleming, W.H.[20]. We define the model as follows.

(**i**) An admissible control v is an $\mathcal{F}_t^Y$-adapted process with values in a non-empty convex subset U of $\mathbb{R}^k$ satisfies $\sup_{t \in [0,T]} \mathbb{E}|v_t|^m < \infty, m = 2, 3, \cdots$. The set of the admissible control variables is denoted by U_{ad}.

(**ii**) For given a control process $v(\cdot) \in U_{ad}$, we consider the following controlled system wich take the following stochastic differential equation (SDE) of McKean-Vlasov type

$$\begin{cases} dx^v(t) = f\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) dt + \sigma\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) dW(t) \\ \qquad\qquad + \alpha\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) d\widetilde{W}(t), \quad t \in [0,T], \\ x^v(0) = x_0, \end{cases} \tag{3.2}$$

with $\mathbb{P}_X = \mathbb{P} \circ X^{-1}$ denotes the law of the random variable X, the maps

$$\begin{aligned} f &: [0,T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^n, \\ \sigma &: [0,T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^{n\times d}, \\ \alpha &: [0,T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^d\right) \times U \to \mathbb{R}^{n\times d}, \end{aligned}$$

are given deterministic functions.

(**iii**) Assume that the state processes $x^v(\cdot)$ is not completely observable, instead, it is partially observed through the related process $Y(\cdot)$, which is governed by the following equation

$$\begin{cases} dY(t) = h(t, x^v(t), v(t))dt + d\widetilde{W}(t) \\ Y(0) = 0, \end{cases} \tag{3.3}$$

where $h : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^r$ is function, and $\widetilde{W}(\cdot)$ is a stochastic process depending on the control $v(\cdot)$. We introduce the following cost functional

$$J(v(\cdot)) = \mathbb{E}^v\left[\int_0^T l(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t))dt + \psi(x^v(T), \mathbb{P}_{x^v(T)})\right], \tag{3.4}$$

where $\mathbb{E}^v$ denotes expectation on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$ and

$$l : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}) \times U \to \mathbb{R}, \ \psi : \mathbb{R}^n \times Q_2(\mathbb{R}) \to \mathbb{R}.$$

Throughout this chapter, we will make the following assumption.

Assumption (**A1**) The coefficients $f, \sigma, \alpha, l : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$ and $\psi : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}$ are measurable in all variables. Moreover, $f(\cdot,\cdot,v), \sigma(\cdot,\cdot,v), \alpha(\cdot,\cdot,v), l(\cdot,\cdot,v) \in \mathbb{C}_b^{1,1}\left(\mathbb{R}^d \times Q_2(\mathbb{R}), \mathbb{R}\right)$ and $\psi(\cdot,\cdot) \in \mathbb{C}_b^{1,1}\left(\mathbb{R}^d \times Q_2(\mathbb{R}), \mathbb{R}\right)$ for all $v \in U$. More than that, denoting $\rho(x,\mu) = f(x,\mu,v), \sigma(x,\mu,v), \alpha(x,\mu,v), l(x,\mu,v), \psi(x,\mu)$, the function $\rho(\cdot,\cdot)$

satisfies the following properties:

1. for fixed $x \in \mathbb{R}$, $\rho(x, \cdot) \in \mathbb{C}_b^{1,1} Q_2\left(\mathbb{R}^d\right)$;
2. for fixed $\mu \in Q_2(\mathbb{R})$, $\rho(\cdot, \mu) \in \mathbb{C}_b^1(\mathbb{R})$;
3. all the derivatives of the function ρ, $\partial_x \rho$ and $\partial_\mu \rho$, for $\rho = f, \sigma, \alpha, l, \psi$, are bounded and Lipschitz continuous, with Lipschitz constants independent of $v \in U$;
4. the functions f, σ, α and l are continuously differentiable with respect to control variable v, and all their derivatives $\partial_v f, \partial_v \sigma, \partial_v \alpha$ and $\partial_v l$ are continuous and bounded.
5. the function h is continuously differentiable in x and continuous in v, its derivatives and h are all uniformly bounded.

For any $v(\cdot) \in U_{ad}$, assumption (**A1**) implies that (3.2) admits a unique $\mathcal{F}_t$-adapted solution (see also the work of Buckdahn et al [7]). Define $d\mathbb{P}^v = \rho^v(t) d\mathbb{P}$ with

$$\rho^v(t) = \exp\left\{\int_0^t h(s, x^v(s), v(s)) dY(s) - \frac{1}{2}\int_0^t |h(s, x^v(s), v(s))|^2 ds\right\},$$

and $\rho(\cdot)$ is the unique $\mathcal{F}_t^Y$-adapted solution of the linear stochastic differential equation (SDE)

$$\begin{cases} d\rho^v(t) = \rho^v(t) h\left(t, x^v(t), v(t)\right) dY(t), \\ \rho^v(0) = 1. \end{cases} \tag{3.5}$$

By Itô's formula, we can prove that $\sup_{t \in [0,T]} \mathbb{E}\left|\rho_t^v\right|^m < \infty, m = 2, 3, \cdots$. Hence, by Girsanov's theorem and assumption (**A1**), we have $\mathbb{P}^v$ is a new probability measure and $(W(\cdot), \widetilde{W}(\cdot))$ is a two-dimensional standard Brownian motion defined on the new probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$. Our partially observed optimal control problem is to minimize the cost functional (3.4) over $v(\cdot) \in U_{ad}$ subject to (3.2) and (3.3), i.e., to find $u(\cdot) \in U_{ad}$ satisfying

$$J(u(\cdot)) = \inf_{v(\cdot) \in U_{ad}} J(v(\cdot)). \tag{3.6}$$

Obviously, cost functional (3.4) can be rewritten as

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T \rho^v(t) l(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)) dt + \rho^v(T) \psi(x^v(T), \mathbb{P}_{x^v(T)})\right]. \tag{3.7}$$

Then the original problem (3.6) is equivalent to minimize (3.7) over $v(\cdot) \in U_{ad}$ subject to (3.2) and (3.5). Our aim is to seek the necessary condition of the partially observed optimal control $u(\cdot)$ in the form of stochastic maximum principle.

3.3 Stochastic Maximum Principle

In this section, by using Girsanov's theorem with a standard convex variational technique, we develop the stochastic maximum principle for our partially observed control problem of general Mckean-Vlasov differential equations. Let $u(\cdot)$ be an optimal control and let x be the corresponding optimal trajectory. For any $\varepsilon \in (0,1)$ and $v(\cdot) \in U_{ad}$, we take the variational control $v^{\varepsilon}(\cdot) = u(\cdot) + \varepsilon v(\cdot)$. The convexity condition of the control domain ensures that $v^{\varepsilon}(\cdot)$ is also in U_{ad}. With clearly notation, we denote by $x^{\varepsilon}(\cdot), x(\cdot), \rho^{\varepsilon}(\cdot), \rho(\cdot)$ the state trajectories of (3.2) and (3.5) corresponding to $v^{\varepsilon}(\cdot)$ and $u(\cdot)$. We now introduce the following stochastic differential equations SDEs

$$\begin{cases} d\phi(t) = \Big\{ f_x\left(t, x(t), \mathbb{P}_{x(t)}, u(t)\right)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu f\left(t, x(t), \mathbb{P}_{x(t)}, u(t); \widehat{x}(t)\right)\widehat{\phi}(t)\right] \\ \qquad + f_v(t, x(t), \mathbb{P}_{x(t)}, u(t))v(t)\Big\} dt \\ \qquad + \Big\{\sigma_x(t, x(t), \mathbb{P}_{x(t)}, u(t))\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu \sigma\left(t, x(t), \mathbb{P}_{x(t)}, u(t); \widehat{x}(t)\right)\widehat{\phi}(t)\right] \\ \qquad + \sigma_v(t, x(t), \mathbb{P}_{x(t)}, u(t))v(t)\Big\} dW(t) \\ \qquad + \Big\{\alpha_x(t, x(t), \mathbb{P}_{x(t)}, u(t))\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu \alpha\left(t, x(t), \mathbb{P}_{x(t)}, u(t); \widehat{x}(t)\right)\widehat{\phi}(t)\right] \\ \qquad + \alpha_v(t, x(t), \mathbb{P}_{x(t)}, u(t))v(t)\Big\} d\widetilde{W}(t) \\ \phi(0) = 0, \end{cases} \tag{3.8}$$

and also

$$\begin{cases} d\rho_1(t) = \{\rho_1(t)h(t, x(t), u(t)) + \rho(t)h_x(t, x(t), u(t))\phi(t) \\ \qquad + \rho(t)h_v(t, x(t), u(t))v(t)\}\, dY(t), \\ \rho_1(0) = 0. \end{cases} \tag{3.9}$$

According to assumption (**A1**), it is easy to know that SDEs (3.8) and (3.9) admit unique adapted solutions $\phi(\cdot)$ and $\rho_1(\cdot)$, respectively.

Lemma 3.1

Let assumption (**A1**) *hold. Then, we have*

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{0 \le t \le T} \left|x^{\varepsilon}(t) - x(t)\right|^2\right] = 0.$$

Proof. From standard estimates, we obtain by using the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|x^{\varepsilon}(s)-x(s)|^2\right]\leq \mathbb{E}\int_0^t\left|f(s,x^{\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s))-f(s,x(s),\mathbb{P}_{x(s)},u(s))\right|^2ds$$
$$+\mathbb{E}\int_0^t\left|\sigma(s,x^{\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s))-\sigma(s,x(s),\mathbb{P}_{x(s)},u(s))\right|^2ds$$
$$+\mathbb{E}\int_0^t\left|\alpha(s,x^{\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s))-\alpha(s,x(s),\mathbb{P}_{x(s)},u(s))\right|^2ds.$$

By using the Lipschitz conditions on the coefficients f,σ and α with respect to x,μ and v, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|x^{\varepsilon}(t)-x(t)|^2\right]\leq C_T\mathbb{E}\int_0^t\left[|x^{\varepsilon}(s)-x(s)|^2+\left|\mathbb{W}_2\left(\mathbb{P}_{x^{\varepsilon}(s)},\mathbb{P}_{x(s)}\right)\right|^2\right]ds$$
$$+C_T\varepsilon^2\mathbb{E}\int_0^t|v(s)|^2ds. \tag{3.10}$$

We recall that for the 2-Wasserstein metric $\mathbb{W}_2(\cdot,\cdot)$, we have

$$\mathbb{W}_2\left(\mathbb{P}_{x^{\varepsilon}(s)},\mathbb{P}_{x(s)}\right)=\inf\left\{\left[\mathbb{E}\left|\widetilde{x}^{\varepsilon}(s)-\widetilde{x}(s)\right|^2\right]^{\frac{1}{2}},\text{ for all }\widetilde{x}^{\varepsilon}(\cdot),\widetilde{x}(\cdot)\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right),\right.$$
$$\left.\text{with }\mathbb{P}_{x^{\varepsilon}(s)}=\mathbb{P}_{\widetilde{x}^{\varepsilon}(s)}\text{ and }\mathbb{P}_{x(s)}=\mathbb{P}_{\widetilde{x}(s)}\right\},$$
$$\leq\left[\mathbb{E}\left|x^{\varepsilon}(s)-x(s)\right|^2\right]^{\frac{1}{2}}. \tag{3.11}$$

From (3.10), (3.11), and Definition 3.1, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|x^{\varepsilon}(t)-x(t)|^2\right]\leq C_T\mathbb{E}\int_0^t\sup_{r\in[0,s]}|x^{\varepsilon}(r)-x(r)|^2ds+M_T\varepsilon^2.$$

By using Gronwall's lemma, the result follows immediately by letting ε go to zero. □

Lemma 3.2

Let assumption (**A1**) *hold. Then, we have*

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}\mathbb{E}\left|\frac{x^{\varepsilon}(t)-x(t)}{\varepsilon}-\phi(t)\right|^2=0. \tag{3.12}$$

Proof. Let

$$\eta^{\varepsilon}(t)=\frac{x^{\varepsilon}(t)-x(t)}{\varepsilon}-\phi(t),\quad t\in[0,T].$$

For convenience and facilitate, we will use the following notations

$$\begin{aligned}
x^{\lambda,\varepsilon}(s) &= x(s) + \lambda\varepsilon\left(\eta^{\varepsilon}(s) + \phi(s)\right),\\
\widehat{x}^{\lambda,\varepsilon}(s) &= x(s) + \lambda\varepsilon\left(\widehat{\eta}^{\varepsilon}(s) + \widehat{\phi}(s)\right),\\
v^{\lambda,\varepsilon}(s) &= u(s) + \lambda\varepsilon v(s).
\end{aligned}$$

From Definition 3.1 and (3.2), we have the following simple form of the Taylor expansion

$$f\left(\mathbb{P}_{\vartheta_0+\xi}\right) - f\left(\mathbb{P}_{\vartheta_0}\right) = \mathbf{D}_{\xi} f\left(\mathbb{P}_{\vartheta_0}\right) + \mathcal{R}(\xi),$$

where $\mathcal{R}(\xi)$ is of order $O(\|\xi\|_2)$ with $O(\|\xi\|_2) \to 0$ for $\xi \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$.

$$\begin{aligned}
\eta^{\varepsilon}(t) = \frac{1}{\varepsilon}\int_0^t &\left[f(s,x^{\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)) - f(s,x(s),\mathbb{P}_{x(s)},u(s))\right] ds\\
&+ \frac{1}{\varepsilon}\int_0^t \left[\sigma(s,x^{\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)) - \sigma(s,x(s),\mathbb{P}_{x(s)},u(s))\right] dW(s)\\
&+ \frac{1}{\varepsilon}\int_0^t \left[\alpha(s,x^{\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)) - \alpha(s,x(s),\mathbb{P}_{x(s)},u(s))\right] d\widetilde{W}(s)\\
&- \int_0^t \Big\{f_x(s,x(s),\mathbb{P}_{x(s)},u(s))\phi(s) + \widehat{\mathbb{E}}\left[\partial_\mu f(s,x(s),\mathbb{P}_{x(s)},u(s);\widehat{x}(s))\widehat{\phi}(s)\right]\\
&+ f_v(s,x(s),\mathbb{P}_{x(s)},u(s))v(s)\Big\} ds\\
&- \int_0^t \Big\{\sigma_x(s,x(s),\mathbb{P}_{x(s)},u(s))\phi(s) + \widehat{\mathbb{E}}\left[\partial_\mu \sigma(s,x(s),\mathbb{P}_{x(s)},u(s);\widehat{x}(s))\widehat{\phi}(s)\right]\\
&+ \sigma_v(s,x(s),\mathbb{P}_{x(s)},u(s))v(s)\Big\} dW(s)\\
&- \int_0^t \Big\{\alpha_x(s,x(s),\mathbb{P}_{x(s)},u(s))\phi(s) + \widehat{\mathbb{E}}\left[\partial_\mu \alpha(s,x(s),\mathbb{P}_{x(s)},u(s);\widehat{x}(s))\widehat{\phi}(s)\right]\\
&+ \alpha_v(s,x(s),\mathbb{P}_{x(s)},u(s))v(s)\Big\} d\widetilde{W}(s).
\end{aligned}$$

Now, we decompose $\frac{1}{\varepsilon}\int_0^t \left(f(s,x^{\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)) - f(s,x(s),\mathbb{P}_{x(s)},u(s))\right) ds$ into the following parts

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t \left(f(s,x^{\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)) - f(s,x(s),\mathbb{P}_{x(s)},u(s))\right) ds\\
&= \frac{1}{\varepsilon}\int_0^t \left(f(s,x^{\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)) - f(s,x(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s))\right) ds\\
&\quad + \frac{1}{\varepsilon}\int_0^t \left(f(s,x(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)) - f(s,x(s),\mathbb{P}_{x(s)},v^{\varepsilon}(s))\right) ds\\
&\quad + \frac{1}{\varepsilon}\int_0^t \left(f(s,x(s),\mathbb{P}_{x(s)},v^{\varepsilon}(s)) - f(s,x(s),\mathbb{P}_{x(s)},u(s))\right) ds.
\end{aligned}$$

Noting that

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t \Big(f(s,x^\varepsilon(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s)) - f(s,x(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s))\Big)\,ds \\
&= \int_0^t\int_0^1 \Big[f_x\Big(s,x^{\lambda,\varepsilon}(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s)\Big)(\eta^\varepsilon(s)+\phi(s))\Big]\,d\lambda ds, \\
&\frac{1}{\varepsilon}\int_0^t \Big(f(s,x^\varepsilon(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s)) - f(s,x^\varepsilon(s),\mathbb{P}_{x(s)},v^\varepsilon(s)\Big)\,ds \\
&= \int_0^t\int_0^1 \widehat{\mathbb{E}}\Big[\partial_\mu f(s,x^\varepsilon(s),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(s)},v^\varepsilon(s);\widehat{x}(s))\Big(\widehat{\eta}^\varepsilon(s)+\widehat{\phi}(s)\Big)\Big]\,d\lambda ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t \Big(f(s,x(s),\mathbb{P}_{x(s)},v^\varepsilon(s)) - f(s,x(s),\mathbb{P}_{x(s)},u(s))\Big)\,ds \\
&= \int_0^t\int_0^1 \Big[f_v\Big(s,x(s),\mathbb{P}_{x(s)},v^{\lambda,\varepsilon}(s)\Big)v(s)\Big]\,d\lambda ds.
\end{aligned}
$$

Analogously, we can have a similar decomposition for σ and α. Therefore, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]}|\eta^\varepsilon(s)|^2\right] = C(t)\Bigg[&\mathbb{E}\int_0^t\int_0^1 \Big|f_x\Big(s,x^{\lambda,\varepsilon}(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s)\Big)\eta^\varepsilon(s)\Big|^2 d\lambda ds \\
&+\mathbb{E}\int_0^t\int_0^1 \widehat{\mathbb{E}}\Big|\partial_\mu f(s,x^\varepsilon(s),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(s)},v^\varepsilon(s);\widehat{x}(s))\widehat{\eta}^\varepsilon(s)\Big|^2 d\lambda ds \\
&+\mathbb{E}\int_0^t\int_0^1 \Big|\sigma_x\Big(s,x^{\lambda,\varepsilon}(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s)\Big)\eta^\varepsilon(s)\Big|^2 d\lambda ds \\
&+\mathbb{E}\int_0^t\int_0^1 \widehat{\mathbb{E}}\Big|\partial_\mu \sigma(s,x^\varepsilon(s),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(s)},v^\varepsilon(s);\widehat{x}(s))\widehat{\eta}^\varepsilon(s)\Big|^2 d\lambda ds \\
&+\mathbb{E}\int_0^t\int_0^1 \Big|\alpha_x\Big(s,x^{\lambda,\varepsilon}(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s)\Big)\eta^\varepsilon(s)\Big|^2 d\lambda ds \\
&+\mathbb{E}\int_0^t\int_0^1 \widehat{\mathbb{E}}\Big|\partial_\mu \alpha(s,x^\varepsilon(s),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(s)},v^\varepsilon(s);\widehat{x}(s))\widehat{\eta}^\varepsilon(s))\Big|^2 d\lambda ds \\
&+\mathbb{E}\left[\sup_{s\in[0,t]}|\gamma^\varepsilon(s)|^2\right]\Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
&\gamma^{\varepsilon}(t)\\
&=\int_0^t\int_0^1\left[f_x\left(s,x^{\lambda,\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)\right)-f_x\left(s,x(s),\mathbb{P}_{x(s)},u(s)\right)\right]\phi(s)d\lambda ds\\
&\quad+\int_0^t\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu f(s,x^{\varepsilon}(s),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(s)},v^{\varepsilon}(s);\widehat{x}(s))-\partial_\mu f(s,x(s),\mathbb{P}_{x(s)},u(s);\widehat{x}(s))\right)\widehat{\phi}(s)\right]d\lambda ds\\
&\quad+\int_0^t\int_0^1\left[f_v\left(s,x(s),\mathbb{P}_{x(s)},v^{\lambda,\varepsilon}(s)\right)-f_v\left(s,x(s),\mathbb{P}_{x(s)},u(s)\right)\right]v(s)d\lambda ds\\
&\quad+\int_0^t\int_0^1\left[\sigma_x\left(s,x^{\lambda,\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)\right)-\sigma_x\left(s,x(s),\mathbb{P}_{x(s)},u(s)\right)\right]\phi(s)d\lambda dW(s)\\
&\quad+\int_0^t\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu\sigma(s,x^{\varepsilon}(s),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(s)},v^{\varepsilon}(s);\widehat{x}(s))-\partial_\mu\sigma(s,x(s),\mathbb{P}_{x(s)},u(s);\widehat{x}(s))\right)\widehat{\phi}(s)\right]d\lambda dW(s)\\
&\quad+\int_0^t\int_0^1\left[\sigma_v\left(s,x(s),\mathbb{P}_{x(s)},v^{\lambda,\varepsilon}(s)\right)-\sigma_v\left(s,x(s),\mathbb{P}_{x(s)},u(s)\right)\right]v(s)d\lambda dW(s)\\
&\quad+\int_0^t\int_0^1\left[\alpha_x\left(s,x^{\lambda,\varepsilon}(s),\mathbb{P}_{x^{\varepsilon}(s)},v^{\varepsilon}(s)\right)-\alpha_x\left(s,x(s),\mathbb{P}_{x(s)},u(s)\right)\right]\phi(s)d\lambda d\widetilde{W}(s)\\
&\quad+\int_0^t\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu\alpha(s,x^{\varepsilon}(s),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(s)},v^{\varepsilon}(s);\widehat{x}(s))-\partial_\mu\alpha\left(s,x(s),\mathbb{P}_{x(s)},u(s);\widehat{x}(s)\right)\right)\widehat{\phi}(s)\right]d\lambda d\widetilde{W}(s)\\
&\quad+\int_0^t\int_0^1\left[\alpha_v\left(s,x(s),\mathbb{P}_{x(s)},v^{\lambda,\varepsilon}(s)\right)-\alpha_v\left(s,x(s),\mathbb{P}_{x(s)},u(s)\right)\right]v(s)d\lambda d\widetilde{W}(s)
\end{aligned}
$$

Now, the derivatives of the functions f,σ and α with respect to (x,μ,v) are Lipschitz continuous in (x,μ,v), we get

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{s\in[0,T]}|\gamma^{\varepsilon}(s)|^2\right]=0.$$

Since the derivatives of f,σ and α with respect to (x,μ,v) are bounded, we have $\forall t\in[0,T]$:

$$\mathbb{E}\left[\sup_{s\in[0,t]}|\eta^{\varepsilon}(s)|^2\right]\leq c(t)\left\{\mathbb{E}\int_0^t|\eta^{\varepsilon}(s)|^2ds+\mathbb{E}\left[\sup_{s\in[0,t]}|\gamma^{\varepsilon}(s)|^2\right]\right\}.$$

From Gronwall's lemma, we obtain $\forall t\in[0,T]$

$$\mathbb{E}\left[\sup_{s\in[0,t]}|\eta^{\varepsilon}(s)|^2\right]\leq c(t)\left\{\mathbb{E}\left[\sup_{s\in[0,t]}|\gamma^{\varepsilon}(s)|^2\right]\exp\left\{\int_0^t c(s)\,ds\right\}\right\}.$$

Finally, putting $t=T$ and letting ε go to zero, this completes the proof. □

We also to get some ε-order estimations of the difference between the perturbed observed process $\rho^{\varepsilon}(\cdot)$ with the sum of the optimal observed process $\rho(\cdot)$ and the variational observed $\rho_1(\cdot)$. The following lemma play an important role in computing the variational inequality for the cost functional (3.7) subject to (3.2) and (3.5).

Lemma 3.3

Let assumption (**A1**) *hold. Then, we have*

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le T}\mathbb{E}\left|\frac{\rho^{\varepsilon}(t)-\rho(t)}{\varepsilon}-\rho_1(t)\right|^2=0. \tag{3.13}$$

Proof. Using the definition of $\rho(\cdot)$ and $\rho_1(\cdot)$, we have

$$\begin{aligned}
&\rho(t)+\varepsilon\rho_1(t)\\
&=1+\int_0^t\rho(s)h(s,x(s),u(s))dY(s)\\
&\quad+\varepsilon\int_0^t\left[\rho_1(s)h(s,x(s),u(s))+\rho(s)h_x(s,x(s),u(s))\phi(s)+\rho(s)h_v(s,x(s),u(s))v(s)\right]dY(s)\\
&=1+\varepsilon\int_0^t\rho_1(s)h(s,x(s),u(s))dY(s)+\int_0^t\rho(s)h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s)\,dY(s)\\
&\quad-\varepsilon\int_0^t\rho(s)\left[A^{\varepsilon}(s)\right]dY(s),
\end{aligned}$$

where

$$\begin{aligned}
A^{\varepsilon}(s)=&\int_0^1\left[h_x(s,x(s)+\lambda\varepsilon\phi(s),u(s)+\lambda\varepsilon v(s))-h_x(s,x(s),u(s))\right]d\lambda\phi(s)\\
&+\int_0^1\left[h_v(s,x(s)+\lambda\varepsilon\phi(s),u(s)+\lambda\varepsilon v(s))-h_v(s,x(s),u(s))\right]d\lambda v(s).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\rho^{\varepsilon}(t)-\rho(t)-\varepsilon\rho_1(t)\\
&=\int_0^t\rho^{\varepsilon}(s)h(s,x^{\varepsilon}(s),v^{\varepsilon}(s))dY(s)-\varepsilon\int_0^t\rho_1(s)h(s,x(s),u(s))dY(s)\\
&\quad-\int_0^t\rho(s)h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))dY(s)+\varepsilon\int_0^t\rho(s)\left[A^{\varepsilon}(s)\right]dY(s)\\
&=\int_0^t\left(\rho^{\varepsilon}(s)-\rho(s)-\varepsilon\rho_1(s)\right)h(s,x^{\varepsilon}(s),v^{\varepsilon}(s))dY(s)\\
&\quad+\int_0^t\left(\rho(s)+\varepsilon\rho_1(s)\right)\left[h(s,x^{\varepsilon}(s),v^{\varepsilon}(s))-h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))\right]dY(s)\\
&\quad+\varepsilon\int_0^t\rho_1(s)h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))dY(s)\\
&\quad-\varepsilon\int_0^t\rho_1(s)h(s,x(s),u(s))dY(s)+\varepsilon\int_0^t\rho(s)\left[A^{\varepsilon}(s)\right]dY(s)\\
&=\int_0^t\left(\rho^{\varepsilon}(s)-\rho(s)-\varepsilon\rho_1(s)\right)h(s,x^{\varepsilon}(s),v^{\varepsilon}(s))dY(s)\\
&\quad+\int_0^t\left(\rho(s)+\varepsilon\rho_1(s)\right)\left[\Lambda_1^{\varepsilon}(s)\right]dY(s)+\varepsilon\int_0^t\rho_1(s)\left[\Lambda_2^{\varepsilon}(s)\right]dY(s)\\
&\quad+\varepsilon\int_0^t\rho(s)\left[A^{\varepsilon}(s)\right]dY(s),
\end{aligned}$$

where

$$\Lambda_1^\varepsilon(s) = h\left(s, x^\varepsilon\left(s\right), v^\varepsilon\left(s\right)\right) - h\left(s, x\left(s\right) + \varepsilon\phi\left(s\right), u\left(s\right) + \varepsilon v\left(s\right)\right),$$
$$\Lambda_2^\varepsilon(s) = h(s, x\left(s\right) + \varepsilon\phi\left(s\right), u\left(s\right) + \varepsilon v\left(s\right)) - h(s, x\left(s\right), u\left(s\right)).$$

Note that

$$\Lambda_1^\varepsilon(s) = \int_0^1 \left[h_x(s, x\left(s\right) + \varepsilon\phi\left(s\right) + \lambda(x^\varepsilon\left(s\right) - x\left(s\right) - \varepsilon\phi\left(s\right)), v^\varepsilon\left(s\right))\right] d\lambda(x^\varepsilon\left(s\right) - x\left(s\right) - \varepsilon\phi\left(s\right)).$$

From Lemma 3.2, we know that

$$\mathbb{E}\int_0^t \left|(\rho(s) + \varepsilon\rho_1(s))\Lambda_1^\varepsilon(s)\right|^2 ds \le C_\varepsilon \varepsilon^2, \tag{3.14}$$

hereafter C_ε denotes some nonnegative constant such that $C_\varepsilon \to 0$ as $\varepsilon \to 0$. Also, it is easy to see that

$$\sup_{0\le t\le T} \mathbb{E}\left[\varepsilon \int_0^t \rho(s)A^\varepsilon(s)dY(s)\right]^2 \le C_\varepsilon \varepsilon^2, \tag{3.15}$$

and

$$\sup_{0\le t\le T} \mathbb{E}\left[\varepsilon \int_0^t \rho_1(s)\Lambda_2^\varepsilon(s)dY(s)\right]^2 \le C_\varepsilon \varepsilon^2. \tag{3.16}$$

By (3.14), (3.15) and (3.16), we have

$$\begin{aligned}
&\mathbb{E}\left|(\rho^\varepsilon(t) - \rho(t)) - \varepsilon\rho_1(t)\right|^2 \\
&\le C\left[\int_0^t \mathbb{E}\left|(\rho^\varepsilon\left(s\right) - \rho\left(s\right)) - \varepsilon\rho_1(s)\right|^2 + \mathbb{E}\int_0^t \left|(\rho\left(s\right) + \varepsilon\rho_1\left(s\right))\Lambda_1^\varepsilon(s)\right|^2 ds\right. \\
&\quad \left. + \sup_{0\le s\le t} \mathbb{E}\left(\varepsilon \int_0^t \rho(s)A^\varepsilon(s)dY(s)\right)^2 + \sup_{0\le s\le t} \mathbb{E}\left(\varepsilon \int_0^t \rho_1(s)\Lambda_2^\varepsilon(s)dY(s)\right)^2\right] \\
&\le C\int_0^t \mathbb{E}\left|\rho^\varepsilon(s) - \rho(s) - \varepsilon\rho_1(s)\right|^2 ds + C_\varepsilon \varepsilon^2.
\end{aligned}$$

Finally, by using Gronwall's inequality, we obtain the desired result. □

Lemma 3.4

Under assumption (**A1**), *one has*

$$\begin{aligned}
0 \le \mathbb{E}\int_0^T &\left\{\rho_1\left(t\right) l(t, x\left(t\right), \mathbb{P}_{x(t)}, u\left(t\right)) + \rho\left(t\right) l_x(t, x\left(t\right), \mathbb{P}_{x(t)}, u\left(t\right))\phi\left(t\right)\right. \\
&\left. + \rho(t)\widehat{\mathbb{E}}\left[\partial_\mu l(t, x(t), \mathbb{P}_{x(t)}, u(t); \widehat{x}\left(t\right))\right]\phi(t) + \rho(t)l_v(t, x(t), \mathbb{P}_{x(t)}, u(t))v(t)\right\} dt \\
&+ \mathbb{E}\left[\rho_1\left(T\right)\psi(x\left(T\right), \mathbb{P}_{x(T)})\right] + \mathbb{E}\left[\rho\left(T\right)\psi_x(x\left(T\right), \mathbb{P}_{x(T)})\phi\left(T\right)\right] \\
&+ \mathbb{E}\left[\rho\left(T\right)\widehat{\mathbb{E}}\left[\partial_\mu\psi(x\left(T\right), \mathbb{P}_{x(T)}; \widehat{x}\left(T\right))\right]\phi\left(T\right)\right].
\end{aligned} \tag{3.17}$$

Proof. Using Lemmas 3.2 and 3.3, Taylor expansion, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\left[\rho^{\varepsilon}(T)\,\psi(x^{\varepsilon}(T),\mathbb{P}_{x^{\varepsilon}(T)})-\rho(T)\,\psi(x(T),\mathbb{P}_{x(T)})\right]\\&=\mathbb{E}\left[\rho_1(T)\psi(x(T),\mathbb{P}_{x(T)})+\rho(T)\psi_x(x(T),\mathbb{P}_{x(T)})\phi(T)\right]\\&\quad+\mathbb{E}\left[\rho(T)\,\widehat{\mathbb{E}}\left[\partial_\mu\psi(x(T),\mathbb{P}_{x(T)};\widehat{x}(T))\right]\phi(T)\right],\end{aligned}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\int_0^T\left\{\rho^{\varepsilon}(t)l(t,x^{\varepsilon}(t),\mathbb{P}_{x^{\varepsilon}(t)},u^{\varepsilon}(t))-\rho(t)l(t,x(t),\mathbb{P}_{x(t)},u(t))\right\}dt\\&=\mathbb{E}\int_0^T\left\{\rho_1(t)l(t,x(t),\mathbb{P}_{x(t)},u(t))+\rho(t)l_x(t,x(t),\mathbb{P}_{x(t)},u(t))\phi(t)\right.\\&\left.+\ \rho(t)\widehat{\mathbb{E}}\left[\partial_\mu l(t,x(t),\mathbb{P}_{x(t)},u(t));\widehat{x}(t)\right]\widehat{\phi}(t)+\rho(t)l_v(t,x(t),u(t))v(t)\right\}dt.\end{aligned}$$

Then, by the optimality of $u(\cdot)$, we draw the desired conclusion.

Define the Hamiltonian $H:[0,T]\times\mathbb{R}\times Q_2(\mathbb{R})\times U\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, by

$$H(t,x,\mu,v,\Phi,Q,\overline{Q},K)=l(t,x,\mu,v)+f(t,x,\mu,v)\Phi+\sigma(t,x,\mu,v)Q+\alpha(t,x,\mu,v)\overline{Q}+h(t,x,v)\,K. \tag{3.18}$$

Now, we introduce the adjoint equations involved in the stochastic maximum principle:

$$\begin{cases}-dy(t)=l(t,x(t),\mathbb{P}_{x(t)},u(t))dt-z(t)\,dW(t)-K(t)\,d\widetilde{W}(t),\\ y(T)=\psi(x(T),\mathbb{P}_{x(T)}),\end{cases} \tag{3.19}$$

and

$$\begin{cases}-d\Phi(t)=\Big\{f_x\Big(t,x(t),\mathbb{P}_{x(t)},u(t)\Big)\Phi(t)+\widehat{\mathbb{E}}\left[\partial_\mu f\Big(t,\widehat{x}(t),\mathbb{P}_{x(t)},\widehat{u}(t);x(t)\Big)\widehat{\Phi}(t)\right]\\ \qquad+\sigma_x\Big(t,x(t),\mathbb{P}_{x(t)},u(t)\Big)Q(t)+\widehat{\mathbb{E}}\left[\partial_\mu\sigma\Big(t,\widehat{x}(t),\mathbb{P}_{x(t)},\widehat{u}(t);x(t)\Big)\widehat{Q}(t)\right]\\ \qquad+\alpha_x\Big(t,x(t),\mathbb{P}_{x(t)},u(t)\Big)\overline{Q}(t)+\widehat{\mathbb{E}}\left[\partial_\mu\alpha\Big(t,\widehat{x}(t),\mathbb{P}_{x(t)},\widehat{u}(t);x(t)\Big)\widehat{\overline{Q}}(t)\right]\\ \qquad+l_x\Big(t,x(t),\mathbb{P}_{x(t)},u(t)\Big)+\widehat{\mathbb{E}}\left[\partial_\mu l\Big(t,\widehat{x}(t),\mathbb{P}_{x(t)},\widehat{u}(t);x(t)\Big)\right]\\ \qquad+\ h_x(t,x(t),u(t))\,K(t)\Big\}dt-Q(t)dW(t)-\overline{Q}(t)d\widetilde{W}(t),\\ \Phi(T)=\psi_x(x(T),\mathbb{P}_{x(T)})+\widehat{\mathbb{E}}\left[\partial_\mu\psi(\widehat{x}(T),\mathbb{P}_{x(T)};x(T))\right].\end{cases} \tag{3.20}$$

Set $\widetilde{\rho}(t)=\rho^{-1}(t)\rho_1(t)$, by using Itô's formula, we have

$$\begin{cases}d\widetilde{\rho}(t)=\{h_x(t,x(t),u(t))\phi(t)+h_v(t,x(t),u(t))v(t)\}\,d\widetilde{W}(t),\\ \widetilde{\rho}(0)=0,\end{cases} \tag{3.21}$$

Then, applying Itô's formula to $\Phi(t)\phi(t)$, $y(t)\widetilde{\rho}(t)$ and taking expectation respectively, we obtain

$$\begin{aligned}
\mathbb{E}^u\left[\Phi(T)\phi(T)\right] &= \mathbb{E}^u\int_0^T \Phi(t)\,d\phi(t) + \mathbb{E}^u\int_0^T \phi(t)\,d\Phi(t) + \mathbb{E}^u\int_0^T Q(t)\Big\{\sigma_x(t,x(t),\mathbb{P}_{x(t)},u(t))\phi(t) \\
&+ \widehat{\mathbb{E}}\Big[\partial_\mu\sigma(t,x(t),\mathbb{P}_{x(t)},u(t);\widehat{x}(t))\widehat{\phi}(t)\Big] + \sigma_v(t,x(t),\mathbb{P}_{x(t)},u(t))v(t)\Big\}\,dt \\
&+ \mathbb{E}^u\int_0^T \overline{Q}(t)\Big\{\alpha_x(t,x(t),\mathbb{P}_{x(t)},u(t))\phi(t) + \widehat{\mathbb{E}}\Big[\partial_\mu\alpha(t,x(t),\mathbb{P}_{x(t)},u(t);\widehat{x}(t))\widehat{\phi}(t)\Big] \\
&+ \alpha_v(t,x(t),\mathbb{P}_{x(t)},u(t))v(t)\Big\}\,dt. \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}
I_1 &= \mathbb{E}^u\int_0^T \Phi(t)\,d\phi(t) \\
&= \mathbb{E}^u\int_0^T \Phi(t)\Big\{f_x(t,x(t),\mathbb{P}_{x(t)},u(t))\phi(t) + \widehat{\mathbb{E}}\Big[\partial_\mu f(t,x(t),\mathbb{P}_{x(t)},u(t);\widehat{x}(t))\widehat{\phi}(t)\Big] \\
&+ f_v(t,x(t),\mathbb{P}_{x(t)},u(t))v(t)\Big\}\,dt \\
&= \mathbb{E}^u\int_0^T \Phi(t)\,f_x(t,x(t),\mathbb{P}_{x(t)},u(t))\phi(t)dt \\
&+ \mathbb{E}^u\int_0^T \Phi(t)\,\widehat{\mathbb{E}}\Big[\partial_\mu f(t,x(t),\mathbb{P}_{x(t)},u(t);\widehat{x}(t))\widehat{\phi}(t)\Big]\,dt \\
&+ \mathbb{E}^u\int_0^T \Phi(t)\,f_v(t,x(t),\mathbb{P}_{x(t)},u(t))v(t)dt
\end{aligned}$$

Consequently,

$$\begin{aligned}
I_2 &= \mathbb{E}^u\int_0^T \phi(t)\,d\Phi(t) \\
&= -\mathbb{E}^u\int_0^T \phi(t)\Big\{f_x\Big(t,x(t),\mathbb{P}_{x(t)},u(t)\Big)\Phi(t) + \widehat{\mathbb{E}}\Big[\partial_\mu f\Big(t,\widehat{x}(t),\mathbb{P}_{x(t)},\widehat{u}(t);x(t)\Big)\widehat{\Phi}(t)\Big] \\
&+ \sigma_x\Big(t,x(t),\mathbb{P}_{x(t)},u(t)\Big)Q(t) + \widehat{\mathbb{E}}\Big[\partial_\mu\sigma\Big(t,\widehat{x}(t),\mathbb{P}_{x(t)},\widehat{u}(t);x(t)\Big)\widehat{Q}(t)\Big] \\
&+ \alpha_x\Big(t,x(t),\mathbb{P}_{x(t)},u(t)\Big)\overline{Q}(t) + \widehat{\mathbb{E}}\Big[\partial_\mu\alpha\Big(t,\widehat{x}(t),\mathbb{P}_{x(t)},\widehat{u}(t);x(t)\Big)\widehat{\overline{Q}}(t)\Big] \\
&+ l_x\Big(t,x(t),\mathbb{P}_{x(t)},u(t)\Big) + \widehat{\mathbb{E}}\Big[\partial_\mu l\Big(t,\widehat{x}(t),\mathbb{P}_{x(t)},\widehat{u}(t);x(t)\Big)\Big] \\
&+ h_x(t,x(t),u(t))\,K(t)\Big\}\,dt.
\end{aligned}$$

By simple computation, we deduce

$$\begin{aligned}
I_2 &= -\mathbb{E}^u \int_0^T \phi(t) f_x \left(t, x(t), \mathbb{P}_{x(t)}, u(t)\right) \Phi(t)\, dt \\
&- \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu f \left(t, \widehat{x}(t), \mathbb{P}_{x(t)}, \widehat{u}(t); x(t)\right) \widehat{\Phi}(t)\right] dt \\
&- \mathbb{E}^u \int_0^T \phi(t) \sigma_x \left(t, x(t), \mathbb{P}_{x(t)}, u(t)\right) Q(t) dt \\
&- \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu \sigma \left(t, \widehat{x}(t), \mathbb{P}_{x(t)}, \widehat{u}(t); x(t)\right) \widehat{Q}(t)\right] dt \\
&- \mathbb{E}^u \int_0^T \phi(t) \alpha_x \left(t, x(t), \mathbb{P}_{x(t)}, u(t)\right) \overline{Q}(t) dt \\
&- \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu \alpha \left(t, \widehat{x}(t), \mathbb{P}_{x(t)}, \widehat{u}(t); x(t)\right) \widehat{\overline{Q}}(t)\right] dt \\
&- \mathbb{E}^u \int_0^T \phi(t) l_x \left(t, x(t), \mathbb{P}_{x(t)}, u(t)\right) dt \\
&- \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu l \left(t, \widehat{x}(t), \mathbb{P}_{x(t)}, \widehat{u}(t); x(t)\right)\right] dt \\
&- \mathbb{E}^u \int_0^T \phi(t) h_x(t, x(t), u(t)) K(t) dt
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
I_3 &= \mathbb{E}^u \int_0^T Q(t) \sigma_x(t, x(t), \mathbb{P}_{x(t)}, u(t)) \phi(t) dt \\
&+ \mathbb{E}^u \int_0^T Q(t) \left\{\widehat{\mathbb{E}} \left[\partial_\mu \sigma(t, x(t), \mathbb{P}_{x(t)}, u(t); \widehat{x}(t)) \widehat{\phi}(t)\right] + \sigma_v(t, x(t), \mathbb{P}_{x(t)}, u(t)) v(t)\right\} dt \\
&+ \mathbb{E}^u \int_0^T \overline{Q}(t) \alpha_x(t, x(t), \mathbb{P}_{x(t)}, u(t)) \phi(t) dt \\
&+ \mathbb{E}^u \int_0^T \overline{Q}(t) \left\{\widehat{\mathbb{E}} \left[\partial_\mu \alpha(t, x(t), \mathbb{P}_{x(t)}, u(t); \widehat{x}(t)) \widehat{\phi}(t)\right] + \alpha_v(t, x(t), \mathbb{P}_{x(t)}, u(t)) v(t)\right\} dt,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^u \left[y(T) \widetilde{\rho}(T)\right] &= \mathbb{E}^u \int_0^T y(t)\, d\widetilde{\rho}(t) + \mathbb{E}^u \int_0^T \widetilde{\rho}(t)\, dy(t) \\
&+ \mathbb{E}^u \int_0^T K(t) \left\{h_x(t, x(t), u(t)) \phi(t) + h_v(t, x(t), u(t)) v(t)\right\} dt \qquad (3.23) \\
&= J_1 + J_2 + I_3,
\end{aligned}$$

where $J_1 = \mathbb{E}^u \int_0^T y(t)\, d\widetilde{\rho}(t)$ is a martingale with zero expectation, and

$$\begin{aligned}
J_2 &= \mathbb{E}^u \int_0^T \widetilde{\rho}(t)\, dy(t) \\
&= -\mathbb{E}^u \int_0^T \widetilde{\rho}(t)\, l(t, x(t), \mathbb{P}_{x(t)}, u(t)) dt.
\end{aligned}$$

Similarly, we can obtain

$$J_3 = \mathbb{E}^u \int_0^T K(t) \{h_x(t, x(t), u(t))\phi(t) + h_v(t, x(t), u(t))v(t)\} \, dt$$

Now, using Fubini's theorem, we obtain

$$\begin{aligned} &\mathbb{E}^u \int_0^T \Phi(t) \widehat{\mathbb{E}} \left[\partial_\mu f \left(t, \widehat{x}(t), \mathbb{P}_{x(t)}, \widehat{u}(t); x(t) \right) \widehat{\phi}(t) \right] dt \\ &= \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu f(t, x(t), \mathbb{P}_{x(t)}, u(t); \widehat{x}(t)) \widehat{\Phi}(t) \right] dt, \end{aligned} \tag{3.24}$$

$$\begin{aligned} &\mathbb{E}^u \int_0^T Q(t) \widehat{\mathbb{E}} \left[\partial_\mu \sigma \left(t, \widehat{x}(t), \mathbb{P}_{x(t)}, \widehat{u}(t); x(t) \right) \widehat{\phi}(t) \right] dt \\ &= \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu \sigma(t, x(t), \mathbb{P}_{x(t)}, u(t); x(t)) \widehat{Q}(t) \right] dt, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} &\mathbb{E}^u \int_0^T \overline{Q}(t) \widehat{\mathbb{E}} \left[\partial_\mu \alpha \left(t, \widehat{x}(t), \mathbb{P}_{x(t)}, \widehat{u}(t); x(t) \right) \widehat{\phi}(t) \right] dt \\ &= \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu \alpha(t, x(t), \mathbb{P}_{x(t)}, u(t); \widehat{x}(t)) \widehat{\overline{Q}}(t) \right] dt, \end{aligned} \tag{3.26}$$

Finally, substituting (3.22), (3.23), (3.24), (3.25) and (3.26) into (3.17), we get

$$\mathbb{E}^u \left[H_v(t, x(t), \mathbb{P}_{x(t)}, u(t), \Phi(t), Q(t), \overline{Q}(t), K(t)) v(t) \right] \geq 0. \tag{3.27}$$

Using the similar method developed in [24], our main result of this chapter is the following Theorem. □

Theorem 3.1

Let assumption (**A1**) *hold. Let* $u(\cdot)$ *be optimal. Then, the maximum principle*

$$\mathbb{E}^u \left[H_v(t, x(t), \mathbb{P}_{x(t)}, u(t), \Phi(t), Q(t), \overline{Q}(t), K(t)) (v(t) - u(t)) \mid \mathcal{F}_t^Y \right] \geq 0, \ \forall v \in U, \quad a.e., a.s.,$$

holds, where the Hamiltonian function H *is defined by* (3.18).

3.4 Application:Partially observed linear-quadratic control problem

In this section, we would applying the results obtained in Sect 3 and classical filtering theory to study a partially observed linear-quadratic control problem of Mckean-Vlasov type. The optimal control is given in feedback form involving both the state of the solution process as well as of its law represented by expectation via the solutions of ordinary differential equations (ODEs).

We consider the following control problem:

$$\min\left\{J\left(v\left(\cdot\right)\right),\ v\in U_{ad}\right\},\ J\left(v\left(\cdot\right)\right)=\mathbb{E}^{u}\left[\int_{0}^{T}L\left(t\right)v^{2}\left(t\right)dt+M_{T}x^{2}\left(T\right)\right],\tag{3.28}$$

subject to

$$\begin{cases} dx\left(t\right)=\left\{A\left(t\right)x\left(t\right)+B\left(t\right)\mathbb{E}\left[x\left(t\right)\right]+C\left(t\right)v\left(t\right)\right\}dt+D\left(t\right)dW\left(t\right), \\ x\left(0\right)=x_{0}, \end{cases}\tag{3.29}$$

with an observation

$$\begin{cases} dY(t)=G\left(t\right)dt+d\widetilde{W}\left(t\right) \\ Y(0)=0, \end{cases}\tag{3.30}$$

where

$$\begin{aligned} f\left(t,x^{v}\left(t\right),\mathbb{P}_{x^{v}(t)},v\left(t\right)\right)&=A\left(t\right)x\left(t\right)+B\left(t\right)\mathbb{E}\left[x\left(t\right)\right]+C\left(t\right)v\left(t\right), \\ \sigma\left(t,x^{v}\left(t\right),\mathbb{P}_{x^{v}(t)},v\left(t\right)\right)&=D\left(t\right), \\ \alpha\left(t,x^{v}\left(t\right),\mathbb{P}_{x^{v}(t)},v\left(t\right)\right)&=0, \\ h(t,x^{v}\left(t\right),v\left(t\right))&=G\left(t\right). \end{aligned}$$

With $L\left(\cdot\right),A\left(\cdot\right),B\left(\cdot\right),C\left(\cdot\right),D\left(\cdot\right)$ and $G\left(\cdot\right)$ are bounded continuous functions and $M_{T}\geq 0$. Then for any $v\in U_{ad}$, (3.29) and (3.30) have unique solutions respectively. Let $u\left(\cdot\right)$ be an optimal control of (3.28), the corresponding optimal trajectory denoted by $x\left(\cdot\right)$.

Further due to (3.20), the corresponding adjoint equation gets the form

$$\begin{cases} -d\Phi\left(t\right)=\left\{A\left(t\right)\Phi\left(t\right)+B\left(t\right)\mathbb{E}\left[\Phi\left(t\right)\right]\right\}dt-Q\left(t\right)dW\left(t\right)-\overline{Q}(t)d\widetilde{W}(t), \\ \Phi\left(T\right)=2M_{T}x\left(T\right). \end{cases}\tag{3.31}$$

Obviously, (3.31) admits a unique solution. In this case, the Hamiltonian function is defined as

$$\begin{aligned} H(t,x,v,\Phi,Q,\overline{Q},K)=&\left[A\left(t\right)x\left(t\right)+B\left(t\right)\mathbb{E}\left[x\left(t\right)\right]+C\left(t\right)v\left(t\right)\right]\Phi\left(t\right) \\ &+D\left(t\right)Q\left(t\right)+G\left(t\right)K+L\left(t\right)v^{2}\left(t\right). \end{aligned}\tag{3.32}$$

If $u(\cdot)$ is optimal, then it follows from Theorem 3.1 and (3.32) that

$$u(t) = -\frac{1}{2} L^{-1}(t) C(t) \mathbb{E}\left[\Phi(t) \mid \mathcal{F}_t^Y\right]. \tag{3.33}$$

Next, we will give a more explicit representation of $u(\cdot)$.

Assume that $\widehat{\Phi}(t) = \mathbb{E}^u\left[\Phi(t) \mid \mathcal{F}_t^Y\right]$, $\widehat{\overline{Q}}(t) = \mathbb{E}^u\left[Q(t) \mid \mathcal{F}_t^Y\right]$ and $\widehat{x}(t) = \mathbb{E}^u\left[x(t) \mid \mathcal{F}_t^Y\right]$ are the filtering estimates of adjoint processes $\Phi(\cdot), Q(\cdot)$ and the optimal trajectory $x(\cdot)$ respectively. From Theorems 8.1 in [50], we obtain

$$\begin{cases} d\widehat{x}(t) = \left\{A(t)\widehat{x}(t) + B(t)\mathbb{E}[\widehat{x}(t)] - \frac{1}{2}L^{-1}(t)C^2(t)\widehat{\Phi}(t)\right\}dt \\ -d\widehat{\Phi}(t) = \left\{A(t)\widehat{\Phi}(t) + B(t)\mathbb{E}\left[\widehat{\Phi}(t)\right]\right\}dt - \widehat{\overline{Q}}(t)\,d\widetilde{W}(t), \\ \widehat{x}(0) = x_0,\ \widehat{\Phi}(T) = 2M_T\widehat{x}(T),\ \widehat{\overline{Q}}(t) = 0. \end{cases} \tag{3.34}$$

To solve (3.34), set $\widehat{\Phi}(t) = \varphi(t)\widehat{x}(t) + \psi(t)\mathbb{E}[\widehat{x}(t)]$, where $\varphi(\cdot), \psi(\cdot)$ are deterministic differential functions which will be specified below. Then

$$\begin{aligned} &-\left\{A(t)\widehat{\Phi}(t) + B(t)\mathbb{E}\left[\widehat{\Phi}(t)\right]\right\} \\ &= \dot{\varphi}(t)\widehat{x}(t) + \dot{\psi}(t)\mathbb{E}[\widehat{x}(t)] \\ &+ \varphi(t)\left\{A(t)\widehat{x}(t) + B(t)\mathbb{E}[\widehat{x}(t)] - \frac{1}{2}L^{-1}(t)C^2(t)\widehat{\Phi}(t)\right\} \\ &+ \psi(t)\left\{(A(t)+B(t))\mathbb{E}[\widehat{x}(t)] - \frac{1}{2}L^{-1}(t)C^2(t)\mathbb{E}\left[\widehat{\Phi}(t)\right]\right\}. \end{aligned} \tag{3.35}$$

By comparing coefficients of $\widehat{x}(t)$ and $\mathbb{E}[\widehat{x}(t)]$ of the above equation respectively, we obtain the following ODEs:

$$\begin{cases} \dot{\varphi}(t) + 2A(t)\varphi(t) - \frac{1}{2}L^{-1}(t)C^2(t)\varphi^2(t) = 0, \\ \varphi(T) = 2M_T, \end{cases} \tag{3.36}$$

and also

$$\begin{cases} \dot{\psi}(t) + 2(A(t)+B(t))\psi(t) + 2B(t)\varphi(t) \\ \qquad - L^{-1}(t)C^2(t)\varphi(t)\psi(t) - \frac{1}{2}L^{-1}(t)C^2(t)\psi^2(t) = 0, \\ \psi(T) = 0. \end{cases} \tag{3.37}$$

We note that equation (3.36) is a Bernoulli differential equation of the form:

$$\dot{\varphi}(t) = a(t)\varphi(t) + b(t)\varphi(t)^2,\ \varphi(T) = 2M_T \tag{3.38}$$

where $a(t) = -2A(t)$, $b(t) = \dfrac{1}{2}L^{-1}(t)\,C^2(t)$. To solve the differential equation (3.38) we using the transformation $\xi(t) = \dfrac{1}{\varphi(t)}$ it follows that

$$\dot{\xi}(t) + a(t)\xi(t) = b(t),\ \ \xi(T) = \frac{1}{2M_T}. \tag{3.39}$$

Now, equation (3.39) is a linear differential equation of first-order. Using the integrating factor method, we obtain

$$\xi(t) = \frac{1}{\mu(t)}\left[\int \mu(s)\frac{C^2(s)}{2L(s)}ds + c\right],$$

where c is an arbitrary canstant, and $\mu(t)$ the integrating factor given by

$$\mu(t) = \exp\int -2A(t)dt. \tag{3.40}$$

We make a simple computation, we have

$$\varphi(t) = \frac{\mu(t)}{\int \mu(s)\frac{C^2(s)}{2L(s)}ds + c}. \tag{3.41}$$

As an illustration we consider the simple case when the functions $A(t) = -\dfrac{1}{t}$, $C(t) = 4t$, $L(t) = 2t$ with $T = 1$, $M_t = \dfrac{1}{4} > 0$, then equation (3.39) being

$$\dot{\xi}(t) + \frac{2}{t}\xi(t) = 4t,$$

with integrating factor $\mu(t) = t^2$. By a simple computation shows that

$$\xi(t) = t^2 + \frac{c}{t^2},\ \ \xi(1) = 2,$$

which implied that $c = 1$. Since $\varphi(t) = \dfrac{1}{\xi(t)}$ we get

$$\varphi(t) = \frac{t^2}{t^4+1},\ \ \varphi(1) = \frac{1}{2}.$$

We turn to second equation (3.37). Noting that equation (3.37) is a Riccati differential equation of the form:

$$\dot{\psi}(t) = \alpha_1(t)\psi^2(t) + \alpha_2(t)\psi(t) + \alpha_3(t),\ \ \psi(T) = 0, \tag{3.42}$$

where $\alpha_1(t) = \left[\dfrac{1}{2}L^{-1}(t)\,C^2(t)\right]$, $\alpha_2(t) = \left[L^{-1}(t)\,C^2(t)\,\varphi(t) - 2\left(A(t) + B(t)\right)\right]$ and $\alpha_3(t) = -2B(t)\,\varphi(t)$. Furthermore, $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $G(\cdot)$, $L(\cdot)$ and $\varphi(\cdot)$

are a given bounded continuous functions.

The method illustrated in the preceding equation can used after we find a particular solution which allows us to convert the previous equation (3.42) into Bernoulli differential equation. We refer to Boyce and DiPrima [6] and the references cited therein for the recent developments of ordinary differential equations (ODEs). Hence, the optimal control $u(\cdot) \in U_{ad}$ for the problem (3.28) is given in the feedback form

$$u(t) = -\frac{1}{2} L^{-1}(t) C(t) [\varphi(t) \widehat{x}(t) + \psi(t) \mathbb{E}[\widehat{x}(t)]],$$

with $\varphi(\cdot), \psi(\cdot)$ determined by (3.36) and (3.37) respectively.

CHAPTER 4

Necessary Conditions for Partially Observed Optimal Control of General McKean-Vlasov Stochastic Differential Equations with Jumps

4.1 Introduction

In this chapter, we establish the necessary conditions of optimality for partially observed optimal control problems of the Mckean-Vlasov type. The system is described by a controlled stochastic differential equation governed by Poisson's random measure and an independent Brownian motion. The coefficients of the McKean-Vlasov system depend on the state of the solution process as well as on its probability law and the control variable. This may be formulated in a general form as follows:

$$\begin{cases} dx^v(t) = & f\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) dt + \sigma\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) dW(t) \\ & +\alpha\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) d\widetilde{W}(t) + \int_\Theta g\left(t, x^v(t_-), \mathbb{P}_{x^v(t_-)}, v(t), \theta\right) \widetilde{N}(d\theta, dt), \\ x^v(0) = & x_0, \quad t \in [0, T], \end{cases}$$

The cost function to be minimized over the class of admissible controls is also of the McKean Vlasov type, which has the form

$$J(v(\cdot)) = \mathbb{E}^v\left[\int_0^T l\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) dt + \psi(x^v(T), \mathbb{P}_{x^v(T)})\right].$$

To prove our result, we use Girsanov's theorem, the variational equations, and the derivatives with respect to probability measure under convexity assumption.

The partially observed control problems have received much attention and became a powerful tool in many fields, such as mathematical finance, optimal control, etc. From the viewpoint of reality, many situations, full information is not always available to controllers, but the partial one with noise, see for example Djehiche & Tempine [18], Flemming [20], Lakhdari et al. [52], Tang & Meng [70], Wang & Wu [76], Wang el al. [78] and the references therein. Necessary and sufficient conditions of optimality for system driven by Brownian motions and Poisson random measure where states and observations are correlated have been discussed by Xiao [85]. Partially observed optimal control problem for forward–backward stochastic systems with jump has been obtained by Wang et al. [82]. Stochastic maximum principle for partially observed forward-backward stochastic system with jumps and regime switching has been established by Zhang et al. [89]. Partially observed time-inconsistent stochastic linear-quadratic control problem with random jumps has been established by Wu & Zhuang [84]. The necessary conditions of optimality for forward-backward stochastic control systems with correlated state and observation noise have been investigated by Wang et al [80]. A class of linear-quadratic optimal control problem of forward-backward stochastic differential equations with partial information has been discussed by Wang et al. [81]. Recently, maximum principle for mean-field optimal stochastic control with partial-information has been studied in Wang et al. [79]. McKean-Vlasov stochastic differential equations (SDEs) are Itô's stochastic differential equations, where the coefficients of the state equation depend on the state of the solution process as well as of its probability law. This kind of equations was established by Kac [45] as a stochastic model for the Vlasov-Kinetic equation of plasma and the study of which was initiated by McKean [55] to provide a rigorous treatment of special nonlinear partial differential equations. Optimal control problems for McKean-Vlasov stochastic differential equations SDEs has been investigated by many authors, for example, Buckdahn et al. [7] proved the necessary conditions for general mean-field systems by using the tool of the second-order derivatives with respect to measures. Maximum principle for optimal control of McKean-Vlasov forward-backward stochastic differential equations (FBSDEs) with Lévy process via the differentiability with respect to probability law has been studied by Meherrem & Hafayed [56]. Necessary and sufficient optimality conditions of optimal singular control problem for general Mckean-Vlasov differential equations has been discussed by Hafayed et al. [24]. A general necessary optimality conditions for stochastic continuous-singular control of McKean-Vlasov type equations where the control domain is not assumed convex has been proved by Guenane et al. Ahmed NU investigated nonlinear diffusion governed by the McKean-Vlasov equation in Hilbert space and optimal control [1]. [22]. Lakhdari et al. [52] proved stochastic maximum principle for partially observed optimal control problems of Mckean-Vlasov type.

We will organize this chapter as follows. Firstly, we will begin with a formulation of the partially observed control problem of general Mckean-Vlasov differential equations with jump processes. And we give the notations and definitions of the derivatives with respect to the probability measures and assumptions used throughout the chapter. Then, we prove the necessary conditions of optimality, which are our main results. A linear quadratic control problem of this kind of partially observed control problem is also given as an application.

4.2 Formulation of the problem and preliminaries

Let us denote by T a fixed terminal time and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete filtered probability space on which are defined two independent standard one-dimensional Brownian motions $W(\cdot)$ and $Y(\cdot)$. Let $\mathbb{R}^n$ is a n-dimensional Euclidean space, $\mathbb{R}^{n\times d}$ the collection of $n \times d$ matrices. Let $k(\cdot)$ be a stationary $\mathcal{F}_t$-Poisson point process with the characteristic measure $m\,(d\theta)$, And let $N\,(d\theta, dt)$ be the counting measure or Poisson measure induced by $k(\cdot)$, defined on $\Theta \times \mathbb{R}_+$, where Θ is a fixed nonempty subset of $\mathbb{R}$ with its Borel σ-field $\mathcal{B}\,(\Theta)$ and set $\widetilde{N}\,(d\theta, dt) = N\,(d\theta, dt) - m\,(d\theta)\, dt$ satisfying $\int_\Theta (1 \wedge |\theta|^2) m\,(d\theta) < \infty$ and $m\,(\Theta) < +\infty$. Let $\mathcal{F}_t^W$, $\mathcal{F}_t^Y$ and $\mathcal{F}_t^N$ be the natural filtration generated by $W(\cdot)$, $Y(\cdot)$ and $N(\cdot)$, respectively. We suppose that

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{F}_t^N \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets. We denote by $\langle \cdot, \cdot \rangle$ (resp. $|\cdot|$) the scalar product (resp., norm), $\mathbb{E}$ denotes the expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Moreover, we denote by

1. $L^2\,(r, s; \mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued deterministic function $\beta\,(\cdot)$, such that $\int_r^s |\beta(t)|^2\, dt < +\infty$.

2. $L^2(\mathcal{F}_t;\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable ϕ, such that $\mathbb{E}|\phi|^2 < +\infty$.

3. $L^2_{\mathcal{F}}(r,s;\mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\psi(\cdot)$, such that $\mathbb{E}\int_r^s |\psi(t)|^2 dt < +\infty$.

4. $\mathbb{M}^2([0,T];\mathbb{R})$ the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted measurable process $g(\cdot)$, such that

$$\mathbb{E}\int_0^T \int_\Theta |g(t,\theta)|^2 m(d\theta)\, dt < +\infty.$$

5. $\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ is the Hilbert space with inner product $(x,y)_2 = \mathbb{E}[x.y]$, $x,y \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ and the norm $\|x\|_2 = \sqrt{(x,x)_2}$.

6. $Q_2\left(\mathbb{R}^d\right)$ is the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ with finite second moment, i.e, $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < +\infty$, endowed with the following 2-Wasserstein metric; for $\mu,\nu \in Q_2\left(\mathbb{R}^d\right)$,

$$\mathbb{W}_2(\mu,\nu) = \inf\left\{\left[\int_{\mathbb{R}^d} |x-y|^2 \rho(dx,dy)\right]^{\frac{1}{2}} : \rho \in Q_2\left(\mathbb{R}^{2d}\right), \rho\left(\cdot,\mathbb{R}^d\right) = \mu, \rho\left(\mathbb{R}^d,\cdot\right) = \nu\right\}.$$

Now, we would like to briefly recall the main results of the differentiability with respect to probability measures was studied by Lions 2013 to derive our main result. The main idea is to identify a distribution $\mu \in Q_2\left(\mathbb{R}^d\right)$ with a random variables $\vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ so that $\mu = \mathbb{P}_\vartheta$. To be more precise, we suppose that probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ is rich enough in the sense that for every $\mu \in Q_2\left(\mathbb{R}^d\right)$, there is a random variable $\vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ such that $\mu = \mathbb{P}_\vartheta$. It is well-known that the probability space $([0,1], \mathcal{B}[0,1], dx)$, where dx is the Borel measure has this property, see Buckdahn, Li, & Ma 2016 .

Definition 4.1

(Lift function) Let $f : Q_2\left(\mathbb{R}^d\right) \rightarrow \mathbb{R}$ be a function . We define the lift function $\widetilde{f} : \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right) \rightarrow \mathbb{R}$ such that

$$\widetilde{f}(Z) := f(\mathbb{P}_Z),\ Z \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right).$$

Evidently, the lift function $\widetilde{f}$ of f, depends only on the law of $Z \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ and is independent of the choice of the representative Z.

Definition 4.2

A function $f : Q_2\left(\mathbb{R}^d\right) \rightarrow \mathbb{R}$ is said to be differentiable at $\mu_0 \in Q_2\left(\mathbb{R}^d\right)$ if there exists $\vartheta_0 \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ with $\mu_0 = \mathbb{P}_{\vartheta_0}$ such that its lift function $\widetilde{f}$ is Fréchet differentiable at ϑ_0. More then that, there exists a continuous linear functional $D\widetilde{f}(\vartheta_0) : \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right) \rightarrow \mathbb{R}$ such that

$$\widetilde{f}(\vartheta_0+\xi) - \widetilde{f}(\vartheta_0) = \left\langle D\widetilde{f}(\vartheta_0),\xi\right\rangle + O(\|\xi\|_2) = D_\xi f(\mu_0) + O(\|\xi\|_2), \tag{4.1}$$

where $\langle\cdot,\cdot\rangle$ is the dual product on the space $\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$, and we will refer to $D_\xi f(\mu_0)$ as the Fréchet derivative of f at μ_0 in the direction ξ. In this case, we have

$$D_\xi f(\mu_0) = \left\langle D\widetilde{f}(\vartheta_0),\xi\right\rangle = \frac{d}{dt}\widetilde{f}(\vartheta_0+t\xi)\Big|_{t=0},\ \text{with } \mu_0 = \mathbb{P}_{\vartheta_0}.$$

By using the Riesz' representation theorem, there is a unique random variable $z_0 \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ such that $\left\langle D\widetilde{f}(\vartheta_0),\xi\right\rangle = (z_0,\xi)_2 = \mathbb{E}\left[(z_0,\xi)_2\right]$, where $\xi \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$. It was shown, see the works of Buckdahn, Li, & Ma 2016 and Lions 2013 that there exists a Boral function $h[\mu_0] : \mathbb{R}^d \rightarrow \mathbb{R}^d$, depending only on the law $\mu_0 = \mathbb{P}_{\vartheta_0}$ but not on the particular choice of the representative ϑ_0 such that $z_0 = h[\mu_0](\vartheta_0)$.
Thus, we can write (4.1) as following

$$f(\mathbb{P}_\vartheta) - f(\mathbb{P}_{\vartheta_0}) = (h[\mu_0](\vartheta_0),\vartheta-\vartheta_0)_2 + O(\|\vartheta-\vartheta_0\|_2),\ \forall\vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d).$$

We shall denote

$$\partial_\mu f(\mathbb{P}_{\vartheta_0},x) = h[\mu_0](x),\ x \in \mathbb{R}^d.$$

Moreover, we have the following identities

$$D\widetilde{f}(\vartheta_0) = z_0 = h[\mu_0](\vartheta_0) = \partial_\mu f(\mathbb{P}_{\vartheta_0}, \vartheta_0),$$

and also

$$D_\xi f(\mathbb{P}_{\vartheta_0}) = \langle \partial_\mu f(\mathbb{P}_{\vartheta_0}, \vartheta_0), \xi \rangle,$$

where $\xi = \vartheta - \vartheta_0$.

Remark 4.1

Let us $\mu \in Q_2\left(\mathbb{R}^d\right)$.
We note that for each $\mu \in Q_2\left(\mathbb{R}^d\right)$, $\partial_\mu f(\mathbb{P}_\vartheta, \cdot) = h[\mathbb{P}_\vartheta](\cdot)$ is only defined in a $\mathbb{P}_\vartheta(dx)$-a.e sense, where $\mu = \mathbb{P}_\vartheta$.

Definition 4.3

(Space of differentiable functions in $Q_2\left(\mathbb{R}^d\right)$)
Let us $f \in \mathbb{C}_b^{1,1}\left(Q_2\left(\mathbb{R}^d\right)\right)$ be a function.
We say that the function $f \in \mathbb{C}_b^{1,1}\left(Q_2\left(\mathbb{R}^d\right)\right)$ if for all $\vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$, there exists a $\mathbb{P}_\vartheta$-modification of $\partial_\mu f(\mathbb{P}_\vartheta, \cdot)$ such that $\partial_\mu f : Q_2\left(\mathbb{R}^d\right) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipchitz continuous. That is for some $C > 0$, it holds that

(a) $|\partial_\mu f(\mu, x)| \leq C, \forall \mu \in Q_2\left(\mathbb{R}^d\right), \forall x \in \mathbb{R}^d$;
(b) $|\partial_\mu f(\mu_1, x_1) - \partial_\mu f(\mu_2, x_2)| \leq C(\mathbb{W}_2(\mu_1, \mu_2) + |x_1 - x_2|), \forall \mu_1, \mu_2 \in Q_2\left(\mathbb{R}^d\right), \forall x_1, x_2 \in \mathbb{R}^d$.

We would like to make a point out that the version of $\partial_\mu f(\mathbb{P}_\vartheta, \cdot), \vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ indicated in the above definition is unique (see Remark 2.2 in Buckdahn, Li, & Ma, 2016 for more information).

We denote by $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{F}}_t, \widehat{\mathbb{P}})$ a copy of the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. For any pair of random variable $(\vartheta, \xi) \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right) \times \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$, we let $(\widehat{\vartheta}, \widehat{\xi})$ be an independent copy of (ϑ, ξ) defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{F}}_t, \widehat{\mathbb{P}})$. We consider the product probability space $(\Omega \times \widehat{\Omega}, \mathcal{F} \otimes \widehat{\mathcal{F}}, \mathcal{F}_t \otimes \widehat{\mathcal{F}}_t, \mathbb{P} \otimes \widehat{\mathbb{P}})$ and setting $(\widehat{\vartheta}, \widehat{\xi})(w, \widehat{w}) = (\vartheta(\widehat{w}), \xi(\widehat{w}))$ for any $(w, \widehat{w}) \in \Omega \times \widehat{\Omega}$. And let $(\widehat{u}(t), \widehat{x}(t))$ be an independent copy of $(u(t), x(t))$ so that $\mathbb{P}_{x(t)} = \widehat{\mathbb{P}}_{\widehat{x}(t)}$. We denote by $\widehat{\mathbb{E}}$ the expectation under probability measure $\widehat{\mathbb{P}}$ and $\mathbb{P}_X = \mathbb{P} \circ X^{-1}$ denotes the law of the random variable X.

We denote by U a nonempty convex subset of $\mathbb{R}^k$. An admissible control v is an $\mathcal{F}_t^Y$-adapted process with values in U satisfies $\sup_{t\in[0,T]} \mathbb{E}\left|v_t\right|^n < \infty$, $n = 2, 3, \ldots$. We denote by $\mathcal{U}_{ad}\left([0,T]\right)$ the set of the admissible control variables.

For given control process $v\left(\cdot\right) \in \mathcal{U}_{ad}\left([0,T]\right)$, the dynamics of the controlled system take the following form

$$\begin{cases} dx^v\left(t\right) = & f\left(t, x^v\left(t\right), \mathbb{P}_{x^v(t)}, v\left(t\right)\right) dt + \sigma\left(t, x^v\left(t\right), \mathbb{P}_{x^v(t)}, v\left(t\right)\right) dW\left(t\right) \\ & +\alpha\left(t, x^v\left(t\right), \mathbb{P}_{x^v(t)}, v\left(t\right)\right) d\widetilde{W}\left(t\right) + \int_\Theta g\left(t, x^v\left(t_-\right), \mathbb{P}_{x^v(t_-)}, v\left(t\right), \theta\right) \widetilde{N}\left(d\theta, dt\right), \\ x^v\left(0\right) = & x_0, \quad t \in [0,T], \end{cases} \tag{4.2}$$

where $\mathbb{P}_X = \mathbb{P}\circ X^{-1}$ is defined the law of the random variable X. The coefficients $f : [0,T]\times\mathbb{R}^n\times Q_2\left(\mathbb{R}^d\right)\times U \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n\times Q_2\left(\mathbb{R}^d\right)\times U \to \mathbb{R}^{n\times d}$, $\alpha : [0,T]\times\mathbb{R}^n\times Q_2\left(\mathbb{R}^d\right)\times U \to \mathbb{R}^{n\times d}$, $g : [0,T]\times\mathbb{R}^n\times Q_2\left(\mathbb{R}^d\right)\times U\times\Theta \to \mathbb{R}^{n\times d}$ are given deterministic functions.

We consider the state processes $x^v\left(\cdot\right)$ cannot be observed directly, but the controllers can observe a related noisy process $Y(\cdot)$, which is governed by the following equation

$$\begin{cases} dY(t) & = h(t, x^v\left(t\right), v\left(t\right))dt + d\widetilde{W}\left(t\right) \\ Y(0) & = 0, \end{cases} \tag{4.3}$$

with $h : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^r$, and $\widetilde{W}\left(\cdot\right)$ is a stochastic process depending on the control $v(\cdot)$.

Remark 4.2

Note that if the diffusion term $\alpha \neq 0$ in equation (4.2), then there exist the correlated noise $\widetilde{W}\left(\cdot\right)$ between the state and observation.

Now, we define the cost functional

$$J(v\left(\cdot\right)) = \mathbb{E}^v\left[\int_0^T l(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t))dt + \psi(x^v(T), \mathbb{P}_{x^v(T)})\right]. \tag{4.4}$$

Here, $l : [0,T]\times\mathbb{R}^n\times Q_2\left(\mathbb{R}\right)\times U \to \mathbb{R}$, $\psi : \mathbb{R}^n\times Q_2\left(\mathbb{R}\right) \to \mathbb{R}$ and $\mathbb{E}^v$ stands for the mathematical expectation on the probability space $\left(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^v\right)$.

In this chapter, we shall make use of the following standing assumption.

Assumption (H1) The functions $f, \sigma, \alpha, l : [0, T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$ and $\psi : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}$ are measurable in all variables. Moreover, $f(t, \cdot, \cdot, v)$, $\sigma(t, \cdot, \cdot, v)$, $\alpha(t, \cdot, \cdot, v)$, $l(t, \cdot, \cdot, v)$, $g(t, \cdot, \cdot, v, \theta) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}), \mathbb{R})$ and $\psi(\cdot, \cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}), \mathbb{R})$ for all $v \in U$.

Assumption (H2) Denoting $\varphi(x, \mu) = f(t, x, \mu, v)$, $\sigma(t, x, \mu, v)$, $\alpha(t, x, \mu, v)$, $l(t, x, \mu, v)$, $g(t, x, \mu, v, \theta)$, $\psi(x, \mu)$, we assume that the function $\varphi(\cdot, \cdot)$ satisfies the following properties.

(**i**) For fixed $x \in \mathbb{R}$ and $\mu \in Q_2(\mathbb{R})$, the function $\varphi(\cdot, \mu) \in \mathbb{C}_b^1(\mathbb{R})$ and $\varphi(x, \cdot) \in \mathbb{C}_b^{1,1}(Q_2\left(\mathbb{R}^d\right), \mathbb{R})$.

(**ii**) All the derivatives of φ (φ_x and $\partial_\mu \varphi$), for $\varphi = f, \sigma, \alpha, l, \psi$ are bounded and Lipschitz continuous, with Lipschitz constants independent of $v \in U$. Furthermore, there exists a constants $C(T, m(\Theta)) > 0$ independent to v and Θ such that

$$\sup_{\theta \in \Theta} |\partial_x g(t, x, \mu, u, \theta)| + \sup_{\theta \in \Theta} |\partial_\mu g(t, x, \mu, u, \theta)| \leq C.$$

$$\sup_{\theta \in \Theta} |g_x(t, x, \mu, u, \theta) - g_x(t, x', \mu', u, \theta)| + \sup_{\theta \in \Theta} |\partial_\mu g(t, x, \mu, u, \theta) - \partial_\mu g(t, x', \mu', u, \theta)|$$
$$\leq C[|x - x'| + \mathbb{W}_2(\mu, \mu')]$$

(**iii**) The maps f, σ, α, g and l are continuously differentiable with respect to control variable v, and all their derivatives are continuous and bounded. Furthermore, there exists a constants $C = C(T, m(\Theta)) > 0$ such that

$$\sup_{\theta \in \Theta} |g_u(t, x, \mu, u, \theta)| \leq C.$$

(**iv**) The function h is continuously differentiable in x and continuous in v, its derivatives and h are all uniformly bounded.

Obviously, under assumptions (H1) and (H2), for any $v(\cdot) \in \mathcal{U}_{ad}([0, T])$ the McKean-Vlasov SDE-(4.2) admits a unique strong solution $x^v(t)$ given by

$$x^v(t) = x_0 + \int_0^t f\left(s, x^v(s), \mathbb{P}_{x^v(s)}, v(s)\right) ds + \sigma\left(s, x^v(s), \mathbb{P}_{x^v(s)}, v(s)\right) dW(s)$$
$$+ \alpha\left(s, x^v(s), \mathbb{P}_{x^v(s)}, v(s)\right) d\widetilde{W}(s) + \int_0^t \int_\Theta g\left(s, x^v(s_-), \mathbb{P}_{x^v(s_-)}, v(s), \theta\right) \widetilde{N}(d\theta, ds).$$

Now, we define $d\mathbb{P}^v = \rho^v(t)d\mathbb{P}$ with

$$\rho^v(t) = \exp\left\{\int_0^t h(s, x^v(s), v(s))dY(s) - \frac{1}{2}\int_0^t |h(s, x^v(s), v(s))|^2 ds\right\},$$

where $\rho^v(\cdot)$ is the unique $\mathcal{F}_t^Y$-adapted solution of the linear stochastic differential equation

$$\begin{cases} d\rho^v(t) = \rho^v(t)h\left(t, x^v(t), v(t)\right)dY(t), \\ \rho^v(0) = 1. \end{cases} \tag{4.5}$$

By applying Itô's formula, we can prove that $\sup_{t\in[0,T]} \mathbb{E}\left|\rho_t^v\right|^m < \infty$, $m = 2, 3, \ldots$. Hence, by Girsanov's theorem and under assumptions (H1), (H2), $\mathbb{P}^v$ is a new probability measure and $\left(W(\cdot), \widetilde{W}(\cdot)\right)$ is two-dimensional standard Brownian motion defined on the new probability space $\left(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^v\right)$.

Our partially observed optimal control problem becomes the following minimization problem: to minimize the cost functional in (4.4) over $v(\cdot) \in \mathcal{U}_{ad}([0,T])$ subject to equations (4.2)-(4.3), such that

$$J(u(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{ad}([0,T])} J(v(\cdot)). \tag{4.6}$$

Clearly, we can rewritten the cost functional (4.4) as

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T \rho^v(t)l(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t))dt + \rho^v(T)\psi(x^v(T), \mathbb{P}_{x^v(T)})\right]. \tag{4.7}$$

So the original optimization problem is equivalent to minimizing (4.7) over $v(\cdot) \in \mathcal{U}_{ad}([0,T])$, subject to (4.2)-(4.5).

The main purpose of this chapter is to prove stochastic maximum principle, also called necessary optimality conditions for the partially observed optimal control of Mckean-Vlasov SDE with jumps.

4.3 Necessary Conditions of Optimality

In this section, we would to prove the necessary conditions of optimality for our partially observed optimal control problem of general Mckean-Vlasov stochstic differential equations with jumps. The proof is based on Girsanov's theorem, the derivatives with respect to probability measure and on introducing the variational equations with some estimates of their solutions.

Hamiltonian function. We define the Hamiltonian

$$H:[0,T]\times\mathbb{R}\times Q_2(\mathbb{R})\times U\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},$$

associated with our control problem by

$$\begin{aligned}H(t,x,\mu,v,\Phi,Q,\overline{Q},K,R)&=l(t,x,\mu,v)+f(t,x,\mu,v)\Phi+\sigma(t,x,\mu,v)Q\\&+\alpha(t,x,\mu,v)\overline{Q}+h\left(t,x,v\right)K+\int_{\Theta}g\left(t,x,\mu,v,\theta\right)R\left(\theta\right)m\left(d\theta\right).\end{aligned}\tag{4.8}$$

Let us consider that $(u(\cdot),x(\cdot))$ is the optimal solution of the control problem (4.2)-(4.6). Then for any $0\leq\varepsilon\leq1$ and $v\left(\cdot\right)\in\mathcal{U}_{ad}\left([0,T]\right)$, we define the variational control by $v^{\varepsilon}\left(\cdot\right)=u(\cdot)+\varepsilon v(\cdot)\in\mathcal{U}_{ad}\left([0,T]\right)$. We denote by $x^{\varepsilon}(\cdot),x(\cdot),\rho^{\varepsilon}(\cdot),\rho(\cdot)$ the state trajectories of (4.2) and (4.5) corresponding respectively to $v^{\varepsilon}(\cdot)$ and $u(\cdot)$.
For simplification, we use the short-hand notation

$$\begin{aligned}\varphi\left(t\right)&=\varphi\left(t,x(t),\mathbb{P}_{x(t)},u(t)\right),\\\varphi^{\varepsilon}\left(t\right)&=\varphi(t,x^{\varepsilon}(t),\mathbb{P}_{x^{\varepsilon}(t)},v^{\varepsilon}(t)),\end{aligned}$$

and

$$g\left(t,\theta\right)=g(t,x(t_{-}),\mathbb{P}_{x(t_{-})},u(t),\theta),\qquad h\left(t\right)=h\left(t,x(t),u(t)\right),$$

$$g^{\varepsilon}\left(t,\theta\right)=g(t,x^{\varepsilon}(t_{-}),\mathbb{P}_{x^{\varepsilon}(t_{-})},v^{\varepsilon}(t),\theta),\quad h^{\varepsilon}\left(t\right)=h\left(t,x^{\varepsilon}(t),v^{\varepsilon}(t)\right),$$

with g,h and $\varphi=f,\sigma,\alpha,l$ as well as their partial derivatives with respect to x and v. Also, we will use the following notations $\varphi=f,\sigma,\alpha,l$ and g :

$$\begin{aligned}\partial_{\mu}\varphi\left(t\right)&=\partial_{\mu}\varphi\left(t,x(t),\mathbb{P}_{x(t)},u(t);\widehat{x}(t)\right),\\\partial_{\mu}\widehat{\varphi}\left(t\right)&=\partial_{\mu}\varphi\left(t,\widehat{x}\left(t\right),\mathbb{P}_{x(t)},\widehat{u}\left(t\right);x(t)\right),\end{aligned}$$

and

$$\begin{aligned}\partial_{\mu}g\left(t,\theta\right)&=\partial_{\mu}g\left(t,x(t_{-}),\mathbb{P}_{x(t_{-})},u(t),\theta;\widehat{x}(t)\right),\\\partial_{\mu}\widehat{g}\left(t,\theta\right)&=\partial_{\mu}g\left(t,\widehat{x}\left(t\right),\mathbb{P}_{x(t)},\widehat{u}\left(t\right),\theta;x\left(t\right)\right).\end{aligned}$$

Now, we give the following variational equations

$$\begin{cases} d\phi(t) &= \left[f_x(t)\,\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu f(t)\,\widehat{\phi}(t)\right] + f_v(t)v(t)\right]dt \\ &\quad + \left[\sigma_x(t)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu \sigma(t)\,\widehat{\phi}(t)\right] + \sigma_v(t)v(t)\right]dW(t) \\ &\quad + \left[\alpha_x(t)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu \alpha(t)\,\widehat{\phi}(t)\right] + \alpha_v(t)\,v(t)\right]d\widetilde{W}(t) \\ &\quad + \int_\Theta \left[g_x(t,\theta)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu g(t,\theta)\,\widehat{\phi}(t)\right] + g_v(t,\theta)v(t)\right]\widetilde{N}(d\theta,dt), \\ \phi(0) &= 0, \end{cases} \tag{4.9}$$

and also

$$\begin{cases} d\rho_1(t) = & [\rho_1(t)h(t) + \rho(t)h_x(t)\phi(t) + \rho(t)h_v(t)v(t)]\,dY(t), \\ \rho_1(0) = & 0. \end{cases} \tag{4.10}$$

From assumptions (H1) and (H2), équations (4.9) and (4.10) admits a unique adapted solutions $\phi(\cdot)$ and $\rho_1(\cdot)$, respectively.

Adjoint equation. We are now ready to introduce two new adjoint equations that will be the building blocks of the stochastic maximum principle.

$$\begin{cases} -dy(t) &= l(t)dt - z(t)\,dW(t) - K(t)\,d\widetilde{W}(t) - \int_\Theta R(t,\theta)\,\widetilde{N}(d\theta,dt), \\ y(T) &= \psi(x(T),\mathbb{P}_{x(T)}), \end{cases} \tag{4.11}$$

and also

$$\begin{cases} -d\Phi(t) &= \Big[f_x(t)\,\Phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{f}(t)\,\widehat{\Phi}(t)\right] + \sigma_x(t)\,Q(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\sigma}(t)\,\widehat{Q}(t)\right] \\ &\quad + \alpha_x(t)\,\overline{Q}(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\alpha}(t)\,\widehat{\overline{Q}}(t)\right] + l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{l}(t)\right] \\ &\quad + \int_\Theta \left[g_x(t,\theta)\,R(t,\theta) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{g}(t,\theta)\,\widehat{R}(t,\theta)\right]\right]m(d\theta) + h_x(t)\,K(t)\Big]dt \\ &\quad - Q(t)dW(t) - \overline{Q}(t)d\widetilde{W}(t) - \int_\Theta R(t,\theta)\,\widetilde{N}(d\theta,dt), \\ \Phi(T) &= \psi_x(x(T),\mathbb{P}_{x(T)}) + \widehat{\mathbb{E}}\left[\partial_\mu \psi(\widehat{x}(T),\mathbb{P}_{x(T)};x(T))\right]. \end{cases} \tag{4.12}$$

Obviously, under assumption (H1) and (H2), it is easy to prove that BSDEs (4.11) and (4.12) admits a unique strong solution, given by

$$y(t) = \psi(x(T),\mathbb{P}_{x(T)}) - \int_t^T l(s)ds + \int_t^T z(s)\,dW(s) + \int_t^T K(s)\,d\widetilde{W}(s) + \int_t^T \int_\Theta R(s,\theta)\,\widetilde{N}(d\theta,ds),$$

and also

$$
\begin{aligned}
\Phi(t) &= \psi_x(x(T), \mathbb{P}_{x(T)}) + \widehat{\mathbb{E}}\left[\partial_\mu \psi(\widehat{x}(T), \mathbb{P}_{x(T)}; x(T))\right]. \\
&- \int_t^T \left[f_x(s)\Phi(s) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{f}(s)\widehat{\Phi}(s)\right] + \sigma_x(s) Q(s) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\sigma}(s)\widehat{Q}(s)\right] \right. \\
&+ \alpha_x(s)\overline{Q}(s) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\alpha}(s)\widehat{\overline{Q}}(s)\right] + l_x(s) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{l}(s)\right] \\
&+ \left. \int_\Theta \left[g_x(s,\theta) R(s,\theta) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{g}(s,\theta)\widehat{R}(s,\theta)\right]\right] m(d\theta) + h_x(s) K(s) \right] ds \\
&+ \int_t^T Q(s) dW(s) + \int_t^T \overline{Q}(s) d\widetilde{W}(s) + \int_t^T \int_\Theta R(s,\theta) \widetilde{N}(d\theta, ds),
\end{aligned}
$$

The main result of this chapter is stated in the following theorem.

Theorem 4.1

Let assumption (H1) and assumption (H2) hold. Let $(u(\cdot), x(\cdot))$ *be the optimal solution of the control problem (4.2)-(4.6). Then there exists* $(\Phi(\cdot), Q(\cdot), \overline{Q}(\cdot), K(\cdot), R(\cdot,\theta))$ *solution of (4.12), such that for any* $v \in U$*, we have*

$$
\mathbb{E}^u\left[H_v(t, x(t), \mathbb{P}_{x(t)}, u(t), \Phi(t), Q(t), \overline{Q}(t), K(t), R(t,\theta))(v(t) - u(t)) \mid \mathcal{F}_t^Y\right] \geq 0, \ a.s., \ a.e.,
$$

where the Hamiltonian function H *is defined by (4.8).*

In order to prove our main result in Theorem 4.1 , we present some auxiliary results.

Lemma 4.1

Assume that assumptions (H1) and (H2) hold. Then, we have

$$
\lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{0 \leq t \leq T} |x^\varepsilon(t) - x(t)|^2\right] = 0.
$$

Proof Applying standard estimates, the Burkholder-Davis-Gundy inequality, and Proposition A1 (Appendix) we have

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0 \leq t \leq T} |x^\varepsilon(t) - x(t)|^2\right] &\leq \mathbb{E}\int_0^t |f^\varepsilon(s) - f(s)|^2 ds + \mathbb{E}\int_0^t |\sigma^\varepsilon(s) - \sigma(s)|^2 ds \\
&+ \mathbb{E}\int_0^t |\alpha^\varepsilon(s) - \alpha(s)|^2 ds + \mathbb{E}\int_0^t \int_\Theta |g^\varepsilon(s,\theta) - g(s,\theta)|^2 m(d\theta) ds.
\end{aligned}
$$

From the Lipschitz conditions on the coefficients f, σ, α and g with respect to x, μ and u, (assumptions (H2)-(ii)), we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|x^{\varepsilon}(t)-x(t)|^{2}\right]\leq C_{T}\mathbb{E}\int_{0}^{t}\left[|x^{\varepsilon}(s)-x(s)|^{2}+\left|\mathbb{W}_{2}\left(\mathbb{P}_{x^{\varepsilon}(s)},\mathbb{P}_{x(s)}\right)\right|^{2}\right]ds$$
$$+C_{T}\varepsilon^{2}\mathbb{E}\int_{0}^{t}|v(s)|^{2}\,ds. \tag{4.13}$$

According to the definition of Wasserstein metric $\mathbb{W}_2(\cdot,\cdot)$, we have

$$\mathbb{W}_{2}\left(\mathbb{P}_{x^{\varepsilon}(s)},\mathbb{P}_{x(s)}\right)=\inf\left\{\left[\mathbb{E}\left|\widetilde{x}^{\varepsilon}(s)-\widetilde{x}(s)\right|^{2}\right]^{\frac{1}{2}},\text{ for all }\widetilde{x}^{\varepsilon}(\cdot),\widetilde{x}(\cdot)\in\mathbb{L}^{2}\left(\mathcal{F};\mathbb{R}^{d}\right),\right.$$
$$\left.\text{with }\mathbb{P}_{x^{\varepsilon}(s)}=\mathbb{P}_{\widetilde{x}^{\varepsilon}(s)}\text{ and }\mathbb{P}_{x(s)}=\mathbb{P}_{\widetilde{x}(s)}\right\}$$
$$\leq\left[\mathbb{E}\left|x^{\varepsilon}(s)-x(s)\right|^{2}\right]^{\frac{1}{2}}. \tag{4.14}$$

By Definition 4.2 and from (4.13) and (4.14), we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|x^{\varepsilon}(t)-x(t)|^{2}\right]\leq C_{T}\mathbb{E}\int_{0}^{t}\sup_{r\in[0,s]}|x^{\varepsilon}(r)-x(r)|^{2}\,ds+M_{T}\varepsilon^{2}.$$

By applying Gronwall's inequality, the desired result follows immediately by letting ε go to zero. □

Lemma 4.2

Assume that assumptions (H1) and (H2) hold. Then, we have

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}\mathbb{E}\left|\frac{x^{\varepsilon}(t)-x(t)}{\varepsilon}-\phi(t)\right|^{2}=0. \tag{4.15}$$

Proof We put

$$\eta^{\varepsilon}(t)=\frac{x^{\varepsilon}(t)-x(t)}{\varepsilon}-\phi(t),\ t\in[0,T].$$

To simplify, we can use the following notations, for $\varphi=f,\sigma,\alpha,l$ and g :

$$\varphi_{x}^{\lambda,\varepsilon}(t)=\varphi_{x}\left(t,x^{\lambda,\varepsilon}(t),\mathbb{P}_{x^{\varepsilon}(t)},v^{\varepsilon}(t)\right),\qquad g_{x}^{\lambda,\varepsilon}(t,\theta)=g_{x}\left(t,x^{\lambda,\varepsilon}(t),\mathbb{P}_{x^{\varepsilon}(t)},v^{\varepsilon}(t),\theta\right),$$
$$\partial_{\mu}^{\lambda,\varepsilon}\varphi(t)=\partial_{\mu}\varphi(s,x^{\varepsilon}(t),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(t)},v^{\varepsilon}(t);\widehat{x}(t)),\quad\partial_{\mu}^{\lambda,\varepsilon}g(t,\theta)=\partial_{\mu}g(t,x^{\varepsilon}(t),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(t)},v^{\varepsilon}(t),\theta;\widehat{x}(t)),$$

and also

$$x^{\lambda,\varepsilon}(s)=x(s)+\lambda\varepsilon\left(\eta^{\varepsilon}(s)+\phi(s)\right),$$
$$\widehat{x}^{\lambda,\varepsilon}(s)=x(s)+\lambda\varepsilon(\widehat{\eta}^{\varepsilon}(s)+\widehat{\phi}(s)),$$
$$v^{\lambda,\varepsilon}(s)=u(s)+\lambda\varepsilon v(s).$$

Since $D_{\xi} f\left(\mu_{0}\right)=\left\langle D \widetilde{f}\left(\vartheta_{0}\right), \xi\right\rangle=\left.\frac{d}{d t} \widetilde{f}\left(\vartheta_{0}+t \xi\right)\right|_{t=0}$, from of the Taylor expansion we have the following

$$f\left(\mathbb{P}_{\vartheta_{0}+\xi}\right)-f\left(\mathbb{P}_{\vartheta_{0}}\right)=D_{\xi} f\left(\mathbb{P}_{\vartheta_{0}}\right)+\mathcal{R}(\xi),$$

with $\mathcal{R}(\xi)$ is of order $O\left(\|\xi\|_{2}\right)$ with $O\left(\|\xi\|_{2}\right) \rightarrow 0$ for $\xi \in \mathbb{L}^{2}\left(\mathcal{F} ; \mathbb{R}^{d}\right)$.

$$\begin{aligned}
\eta^{\varepsilon}(t)= & \frac{1}{\varepsilon} \int_{0}^{t}\left[f^{\varepsilon}(s)-f(s)\right] d s+\frac{1}{\varepsilon} \int_{0}^{t}\left[\sigma^{\varepsilon}(s)-\sigma(s)\right] d W(s) \\
& +\frac{1}{\varepsilon} \int_{0}^{t}\left[\alpha^{\varepsilon}(s)-\alpha(s)\right] d \widetilde{W}(s)+\frac{1}{\varepsilon} \int_{0}^{t} \int_{\Theta}\left[g^{\varepsilon}(s, \theta)-g(s, \theta)\right] \widetilde{N}(d \theta, d s) \\
& -\int_{0}^{t}\left[f_{x}(s) \phi(s)+\widehat{\mathbb{E}}\left[\partial_{\mu} f(s) \widehat{\phi}(s)\right]+f_{v}(s) v(s)\right] d s \\
& -\int_{0}^{t}\left[\sigma_{x}(s) \phi(s)+\widehat{\mathbb{E}}\left[\partial_{\mu} \sigma(s) \widehat{\phi}(s)\right]+\sigma_{v}(s) v(s)\right] d W(s) \\
& -\int_{0}^{t}\left[\alpha_{x}(s) \phi(s)+\widehat{\mathbb{E}}\left[\partial_{\mu} \alpha(s) \widehat{\phi}(s)\right]+\alpha_{v}(s) v(s)\right] d \widetilde{W}(s) \\
& -\int_{0}^{t} \int_{\Theta}\left[g_{x}(s, \theta) \phi(s)+\widehat{\mathbb{E}}\left[\partial_{\mu} g(s, \theta) \widehat{\phi}(s)\right]+g_{v}(s, \theta) v(s)\right] \widetilde{N}(d \theta, d s) .
\end{aligned}$$

Now,we decompose $\frac{1}{\varepsilon} \int_{0}^{t}\left[f^{\varepsilon}(s)-f(s)\right] d s$ into the following three parts

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_{0}^{t}\left[f^{\varepsilon}(s)-f(s)\right] d s \\
& =\frac{1}{\varepsilon} \int_{0}^{t}\left[f^{\varepsilon}(s)-f\left(s, x(s), \mathbb{P}_{x^{\varepsilon}(s)}, v^{\varepsilon}(s)\right)\right] d s \\
& \quad+\frac{1}{\varepsilon} \int_{0}^{t}\left[f\left(s, x(s), \mathbb{P}_{x^{\varepsilon}(s)}, v^{\varepsilon}(s)\right)-f\left(s, x(s), \mathbb{P}_{x(s)}, v^{\varepsilon}(s)\right)\right] d s \\
& \quad+\frac{1}{\varepsilon} \int_{0}^{t}\left[f\left(s, x(s), \mathbb{P}_{x(s)}, v^{\varepsilon}(s)\right)-f(s)\right] d s .
\end{aligned}$$

We are note that

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_{0}^{t}\left[f^{\varepsilon}(s)-f\left(s, x(s), \mathbb{P}_{x^{\varepsilon}(s)}, v^{\varepsilon}(s)\right)\right] d s=\int_{0}^{t} \int_{0}^{1}\left[f_{x}^{\lambda, \varepsilon}(s)\left(\eta^{\varepsilon}(s)+\phi(s)\right)\right] d \lambda d s, \\
& \frac{1}{\varepsilon} \int_{0}^{t}\left[f^{\varepsilon}(s)-f\left(s, x^{\varepsilon}(s), \mathbb{P}_{x(s)}, v^{\varepsilon}(s)\right] d s=\int_{0}^{t} \int_{0}^{1} \widehat{\mathbb{E}}\left[\partial_{\mu}^{\lambda, \varepsilon} f(s)\left(\widehat{\eta}^{\varepsilon}(s)+\widehat{\phi}(s)\right)\right] d \lambda d s,\right.
\end{aligned}$$

and

$$\frac{1}{\varepsilon} \int_{0}^{t}\left[f\left(s, x(s), \mathbb{P}_{x(s)}, v^{\varepsilon}(s)\right)-f(s)\right] d s=\int_{0}^{t} \int_{0}^{1}\left[f_{v}\left(s, x(s), \mathbb{P}_{x(s)}, v^{\lambda, \varepsilon}(s)\right) v(s)\right] d \lambda d s .$$

The analogue relations hold for σ, α and g. Therefore, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} |\eta^{\varepsilon}(s)|^2\right] &= C\,(t)\,\mathbb{E}\left[\int_0^t \int_0^1 \left|f_x^{\lambda,\varepsilon}\,(s)\,\eta^{\varepsilon}\,(s)\right|^2 d\lambda ds\right. \\
&+ \int_0^t \int_0^1 \widehat{\mathbb{E}}\left|\partial_\mu^{\lambda,\varepsilon} f\,(s)\,\widehat{\eta}^{\varepsilon}\,(s)\right|^2 d\lambda ds \\
&+ \int_0^t \int_0^1 \left|\sigma_x^{\lambda,\varepsilon}\,(s)\,\eta^{\varepsilon}\,(s)\right|^2 d\lambda ds \\
&+ \int_0^t \int_0^1 \widehat{\mathbb{E}}\left|\partial_\mu^{\lambda,\varepsilon}\sigma(s)\widehat{\eta}^{\varepsilon}\,(s)\right|^2 d\lambda ds \\
&+ \int_0^t \int_0^1 \left|\alpha_x^{\lambda,\varepsilon}\,(s)\,\eta^{\varepsilon}\,(s)\right|^2 d\lambda ds \\
&+ \int_0^t \int_0^1 \widehat{\mathbb{E}}\left|\partial_\mu^{\lambda,\varepsilon}\alpha(s)\widehat{\eta}^{\varepsilon}\,(s))\right|^2 d\lambda ds \\
&+ \int_0^t \int_\Theta \int_0^1 \left|g_x^{\lambda,\varepsilon}\,(s,\theta)\,\eta^{\varepsilon}\,(s)\right|^2 d\lambda m\,(d\theta)\,ds \\
&\left.+ \int_0^t \int_\Theta \int_0^1 \widehat{\mathbb{E}}\left|\partial_\mu^{\lambda,\varepsilon} g\,(s,\theta)\,\widehat{\eta}^{\varepsilon}\,(s))\right|^2 d\lambda m\,(d\theta)\,ds\right] \\
&+ C\,(t)\,\mathbb{E}\left[\sup_{s\in[0,t]} |\gamma^{\varepsilon}(s)|^2\right],
\end{aligned}$$

such that we have

$$\begin{aligned}
\gamma^{\varepsilon}(t) &= \int_0^t \int_0^1 \left[f_x^{\lambda,\varepsilon}\,(s) - f_x\,(s)\right]\phi(s) d\lambda ds \\
&+ \int_0^t \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon} f\,(s) - \partial_\mu f(s)\right)\widehat{\phi}(s)\right] d\lambda ds \\
&+ \int_0^t \int_0^1 \left[f_v\left(s, x(s), \mathbb{P}_{x(s)}, v^{\lambda,\varepsilon}\,(s)\right) - f_v\,(s)\right] v(s) d\lambda ds \\
&+ \int_0^t \int_0^1 \left[\sigma_x^{\lambda,\varepsilon}\,(s) - \sigma_x\,(s)\right]\phi(s) d\lambda dW(s) \\
&+ \int_0^t \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon}\sigma(s) - \partial_\mu\sigma(s)\right)\widehat{\phi}(s)\right] d\lambda dW(s) \\
&+ \int_0^t \int_0^1 \left[\sigma_v\left(s, x(s), \mathbb{P}_{x(s)}, v^{\lambda,\varepsilon}\,(s)\right) - \sigma_v\,(s)\right] v(s) d\lambda dW(s)
\end{aligned}$$

$$\begin{aligned}
&+ \int_0^t \int_0^1 \left[\alpha_x^{\lambda,\varepsilon}\,(s) - \alpha_x\,(s)\right]\phi(s) d\lambda d\widetilde{W}(s) \\
&+ \int_0^t \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon}\alpha(s) - \partial_\mu\alpha\,(s)\right)\widehat{\phi}(s)\right] d\lambda d\widetilde{W}(s) \\
&+ \int_0^t \int_0^1 \left[\alpha_v\left(s, x\,(s)\,, \mathbb{P}_{x(s)}, v^{\lambda,\varepsilon}\,(s)\right) - \alpha_v\,(s)\right] v(s) d\lambda d\widetilde{W}(s) \\
&+ \int_0^t \int_\Theta \int_0^1 \left[g_x^{\lambda,\varepsilon}\,(s,\theta) - g_x\,(s,\theta)\right]\phi(s_-) d\lambda \widetilde{N}\,(d\theta, ds) \\
&+ \int_0^t \int_\Theta \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon} g\,(s,\theta) - \partial_\mu g\,(s,\theta)\right)\widehat{\phi}(s_-)\right] d\lambda \widetilde{N}\,(d\theta, ds) \\
&+ \int_0^t \int_\Theta \int_0^1 \left[g_v\left(s, x(s), \mathbb{P}_{x(s)}, v^{\lambda,\varepsilon}\,(s)\,, \theta\right) - g_v\,(s,\theta)\right] v(s) d\lambda \widetilde{N}\,(d\theta, ds)\,.
\end{aligned}$$

Now, the derivatives of the functions f, σ, α and g with respect to (x, μ, v) are Lipschitz continuous in (x, μ, v), we get

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{s\in[0,T]} |\gamma^\varepsilon(s)|^2\right] = 0.$$

Since the derivatives of f, σ, α and γ are bounded with respect to (x, μ, v), we have

$$\mathbb{E}\left[\sup_{s\in[0,t]} |\eta^\varepsilon(s)|^2\right] \leq C(t)\left\{\mathbb{E}\int_0^t |\eta^\varepsilon(s)|^2 ds + \mathbb{E}\left[\sup_{s\in[0,t]} |\gamma^\varepsilon(s)|^2\right]\right\}.$$

According to Gronwall's lemma, we obtain $\forall t \in [0, T]$

$$\mathbb{E}\left[\sup_{s\in[0,t]} |\eta^\varepsilon(s)|^2\right] \leq C(t)\left\{\mathbb{E}\left[\sup_{s\in[0,t]} |\gamma^\varepsilon(s)|^2\right] \exp\left\{\int_0^t C(s)\, ds\right\}\right\}.$$

Finally, putting $t = T$ and letting ε go to zero, the proof of Lemma 4.2 is complete. □

Now, we present the following lemma which play an important role in computing the variational inequality for the cost functional (4.7) subject to (4.2) and (4.5).

Lemma 4.3

Let assumption (H1) hold. Then, we have

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T} \mathbb{E}\left|\frac{\rho^\varepsilon(t) - \rho(t)}{\varepsilon} - \rho_1(t)\right|^2 = 0. \tag{4.16}$$

Proof. According to the definition of $\rho(\cdot)$ and $\rho_1(\cdot)$, we obtain

$$\begin{aligned}
\rho(t) + \varepsilon\rho_1(t) &= 1 + \int_0^t \rho(s)h(s)dY(s) \\
&\quad + \varepsilon\int_0^t [\rho_1(s)\, h(s) + \rho(s)h_x(s)\, \phi(s) + \rho(s)h_v(s)\, v(s)]\, dY(s) \\
&= 1 + \varepsilon\int_0^t \rho_1(s)h(s)dY(s) + \int_0^t \rho(s)\, h(s, x(s) + \varepsilon\phi(s), u(s) + \varepsilon v(s))dY(s) \\
&\quad - \varepsilon\int_0^t \rho(s)\,[A^\varepsilon(s)]\, dY(s),
\end{aligned}$$

where we take

$$\begin{aligned}
A^\varepsilon(s) &= \int_0^1 [h_x(s, x(s) + \lambda\varepsilon\phi(s), u(s) + \lambda\varepsilon v(s)) - h_x(s)]\, d\lambda\phi(s) \\
&\quad + \int_0^1 [h_v(s, x(s) + \lambda\varepsilon\phi(s), u(s) + \lambda\varepsilon v(s)) - h_v(s)]\, d\lambda v(s).
\end{aligned}$$

And, we have

$$
\begin{aligned}
&\rho^{\varepsilon}(t)-\rho(t)-\varepsilon\rho_1(t)\\
&=\int_0^t \rho^{\varepsilon}(s)\,h^{\varepsilon}(t)\,dY(s)-\varepsilon\int_0^t \rho_1(s)h(s)dY(s)\\
&\quad-\int_0^t \rho(s)h\left(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s)\right)dY(s)+\varepsilon\int_0^t \rho(s)\left[A^{\varepsilon}(s)\right]dY(s)\\
&=\int_0^t \left(\rho^{\varepsilon}(s)-\rho(s)-\varepsilon\rho_1(s)\right)h^{\varepsilon}(s)\,dY(s)\\
&\quad+\int_0^t \left(\rho(s)+\varepsilon\rho_1(s)\right)\left[h^{\varepsilon}(s)-h\left(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s)\right)\right]dY(s)\\
&\quad+\varepsilon\int_0^t \rho_1(s)\,h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))dY(s)\\
&\quad-\varepsilon\int_0^t \rho_1(s)\,h(s)dY(s)+\varepsilon\int_0^t \rho(s)\left[A^{\varepsilon}(s)\right]dY(s)\\
&=\int_0^t \left(\rho^{\varepsilon}(s)-\rho(s)-\varepsilon\rho_1(s)\right)h^{\varepsilon}(s)\,dY(s)\\
&\quad+\int_0^t (\rho(s)+\varepsilon\rho_1(s))\left[B_1^{\varepsilon}(s)\right]dY(s)+\varepsilon\int_0^t \rho_1(s)\left[B_2^{\varepsilon}(s)\right]dY(s)\\
&\quad+\varepsilon\int_0^t \rho(s)\left[A^{\varepsilon}(s)\right]dY(s),
\end{aligned}
$$

with

$$
\begin{aligned}
B_1^{\varepsilon}(s)&=h^{\varepsilon}(s)-h\left(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s)\right),\\
B_2^{\varepsilon}(s)&=h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))-h(s).
\end{aligned}
$$

We notice that

$$
B_1^{\varepsilon}(s)=\int_0^1 \left[h_x(s,x(s)+\varepsilon\phi(s)+\lambda(x^{\varepsilon}(s)-x(s)-\varepsilon\phi(s)),v^{\varepsilon}(s))\right]d\lambda(x^{\varepsilon}(s)-x(s)-\varepsilon\phi(s)).
$$

By Lemma 4.2 , we know that

$$
\mathbb{E}\int_0^t \left|(\rho(s)+\varepsilon\rho_1(s))B_1^{\varepsilon}(s)\right|^2 ds\leq C_{\varepsilon}\varepsilon^2, \tag{4.17}
$$

here C_{ε} denotes some nonnegative constant such that $C_{\varepsilon}\to 0$ as $\varepsilon\to 0$.

Furthermore, it is easy to see that

$$
\sup_{0\leq t\leq T}\mathbb{E}\left[\varepsilon\int_0^t \rho(s)A^{\varepsilon}(s)dY(s)\right]^2\leq C_{\varepsilon}\varepsilon^2, \tag{4.18}
$$

and

$$
\sup_{0\leq t\leq T}\mathbb{E}\left[\varepsilon\int_0^t \rho_1(s)B_2^{\varepsilon}(s)dY(s)\right]^2\leq C_{\varepsilon}\varepsilon^2. \tag{4.19}
$$

From (4.17), (4.18) and (4.19), we obtain

$$\begin{aligned}
&\mathbb{E}\left|(\rho^{\varepsilon}(t)-\rho(t))-\varepsilon\rho_1(t)\right|^2\\
&\leq C\left[\int_0^t \mathbb{E}\left|(\rho^{\varepsilon}(s)-\rho(s))-\varepsilon\rho_1(s)\right|^2+\mathbb{E}\int_0^t\left|(\rho(s)+\varepsilon\rho_1(s))B_1^{\varepsilon}(s)\right|^2 ds\right.\\
&\quad\left.+\sup_{0\leq s\leq t}\mathbb{E}\left(\varepsilon\int_0^t \rho(s)A^{\varepsilon}(s)dY(s)\right)^2+\sup_{0\leq s\leq t}\mathbb{E}\left(\varepsilon\int_0^t \rho_1(s)B_2^{\varepsilon}(s)dY(s)\right)^2\right]\\
&\leq C\int_0^t \mathbb{E}\left|\rho^{\varepsilon}(s)-\rho(s)-\varepsilon\rho_1(s)\right|^2 ds+C_{\varepsilon}\varepsilon^2.
\end{aligned}$$

Finally, by applying Gronwall's inequality, the proof of Lemma 4.3 is complete. □

Lemma 4.4

assumption (H1) hold. Then, we have

$$\begin{aligned}
0&\leq \mathbb{E}\int_0^T\left[\rho_1(t)\,l(t)+\rho(t)\,l_x(t)\phi(t)+\rho(t)\widehat{\mathbb{E}}\left[\partial_\mu l(t)\right]\phi(t)+\rho(t)l_v(t)v(t)\right]dt\\
&\quad+\mathbb{E}\left[\rho_1(T)\,\psi(x(T),\mathbb{P}_{x(T)})\right]+\mathbb{E}\left[\rho(T)\,\psi_x(x(T),\mathbb{P}_{x(T)})\phi(T)\right]\\
&\quad+\mathbb{E}\left[\rho(T)\,\widehat{\mathbb{E}}\left[\partial_\mu\psi(x(T),\mathbb{P}_{x(T)};\widehat{x}(T))\right]\phi(T)\right].
\end{aligned}\tag{4.20}$$

Proof. Applying the Taylor expansion, Lemmas 4.2 and 4.3 , we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\left[\rho^{\varepsilon}(T)\,\psi(x^{\varepsilon}(T),\mathbb{P}_{x^{\varepsilon}(T)})-\rho(T)\,\psi(x(T),\mathbb{P}_{x(T)})\right]\\
&=\mathbb{E}\left[\rho_1(T)\psi(x(T),\mathbb{P}_{x(T)})+\rho(T)\psi_x(x(T),\mathbb{P}_{x(T)})\phi(T)\right]\\
&+\mathbb{E}\left[\rho(T)\,\widehat{\mathbb{E}}\left[\partial_\mu\psi(x(T),\mathbb{P}_{x(T)};\widehat{x}(T))\right]\phi(T)\right],
\end{aligned}$$

and we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\int_0^T\left[\rho^{\varepsilon}(t)l^{\varepsilon}(t)-\rho(t)l(t)\right]dt\\
&=\mathbb{E}\int_0^T\left[\rho_1(t)l(t)+\rho(t)l_x(t)\phi(t)+\rho(t)\widehat{\mathbb{E}}\left[\partial_\mu l(t)\right]\widehat{\phi}(t)+\rho(t)l_v(t)v(t)\right]dt.
\end{aligned}$$

Then, by the fact that $\varepsilon^{-1}\left[J\left(v^{\varepsilon}(t)\right)-J\left(u(t)\right)\right]\geq 0$, we draw the desired conclusion. □

We are note that

$$\begin{cases}
d\widetilde{\rho}(t)=\left\{h_x(t)\phi(t)+h_v(t)v(t)\right\}d\widetilde{W}(t),\\
\widetilde{\rho}(0)=0,
\end{cases}\tag{4.21}$$

where $\widetilde{\rho}(t)=\rho^{-1}(t)\rho_1(t)$.

By using Itô's formula to $\Phi(t)\phi(t)$, $y(t)\widetilde{\rho}(t)$ and taking expectation respectively, where $\phi(0)=0$ and $\widetilde{\rho}(0)=0$, we get

$$\begin{aligned}
\mathbb{E}^u\left[\Phi(T)\phi(T)\right] &= \mathbb{E}^u\int_0^T \Phi(t)\,d\phi(t) + \mathbb{E}^u\int_0^T \phi(t)\,d\Phi(t) \\
&+ \mathbb{E}^u\int_0^T Q(t)\left[\sigma_x(t)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu\sigma(t)\widehat{\phi}(t)\right] + \sigma_v(t)v(t\right] dt \\
&+ \mathbb{E}^u\int_0^T \overline{Q}(t)\left[\alpha_x(t)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu\alpha(t)\widehat{\phi}(t)\right] + \alpha_v(t)v(t)\right] dt \\
&+ \mathbb{E}^u\int_0^T\int_\Theta R(t,\theta)\left[g_x(t,\theta)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu g(t,\theta)\,\widehat{\phi}(t)\right] + g_v(t,\theta)v(t)\right] m(d\theta)\,dt \\
&= \mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3+\mathbb{I}_4. \qquad (4.22)
\end{aligned}$$

We have

$$\begin{aligned}
\mathbb{I}_1 &= \mathbb{E}^u\int_0^T \Phi(t)\,d\phi(t) \\
&= \mathbb{E}^u\int_0^T \Phi(t)\left[f_x(t)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu f(t)\widehat{\phi}(t)\right] + f_v(t)v(t)\right] dt \\
&= \mathbb{E}^u\int_0^T \Phi(t)\,f_x(t)\phi(t)dt + \mathbb{E}^u\int_0^T \Phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu f(t)\widehat{\phi}(t)\right] dt \\
&+ \mathbb{E}^u\int_0^T \Phi(t)\,f_v(t)v(t)dt.
\end{aligned}$$

We proceed to estimate $\mathbb{I}_2$, According to equation (4.12), we have

$$\begin{aligned}
\mathbb{I}_2 &= \mathbb{E}^u\int_0^T \phi(t)\,d\Phi(t) \\
&= -\mathbb{E}^u\int_0^T \phi(t)\Big[f_x(t)\,\Phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu\widehat{f}(t)\,\widehat{\Phi}(t)\right] + \sigma_x(t)\,Q(t) \\
&+ \widehat{\mathbb{E}}\left[\partial_\mu\widehat{\sigma}(t)\,\widehat{Q}(t)\right] + \alpha_x(t)\,\overline{Q}(t) + \widehat{\mathbb{E}}\left[\partial_\mu\widehat{\alpha}(t)\,\widehat{\overline{Q}}(t)\right] + l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu\widehat{l}(t)\right] \\
&+ \int_\Theta\left[g_x(t,\theta)\,R(t,\theta) + \widehat{\mathbb{E}}\left[\partial_\mu\widehat{g}(t,\theta)\,\widehat{R}(t,\theta)\right]\right] m(d\theta) + h_x(t)\,K(t)\Big]\,dt.
\end{aligned}$$

By to make a simple computation, we have

$$\begin{aligned}
\mathbb{I}_2 = & -\mathbb{E}^u \int_0^T \phi(t) f_x(t) \Phi(t) dt - \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu \widehat{f}(t) \widehat{\Phi}(t) \right] dt \\
& - \mathbb{E}^u \int_0^T \phi(t) \sigma_x(t) Q(t) dt - \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu \widehat{\sigma}(t) \widehat{Q}(t) \right] dt \\
& - \mathbb{E}^u \int_0^T \phi(t) \alpha_x(t) \overline{Q}(t) dt - \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu \widehat{\alpha}(t) \widehat{\overline{Q}}(t) \right] dt \\
& - \mathbb{E}^u \int_0^T \phi(t) l_x(t) dt - \mathbb{E}^u \int_0^T \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu \widehat{l}(t) \right] dt \\
& - \mathbb{E}^u \int_0^T \int_\Theta \phi(t) g_x(t,\theta) R(t,\theta) m(d\theta) dt \\
& - \mathbb{E}^u \int_0^T \int_\Theta \phi(t) \widehat{\mathbb{E}} \left[\partial_\mu \widehat{g}(t,\theta) \widehat{R}(t,\theta) \right] m(d\theta) dt \\
& - \mathbb{E}^u \int_0^T \phi(t) h_x(t) K(t) dt.
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
\mathbb{I}_3 = & \mathbb{E}^u \int_0^T Q(t) \left[\sigma_x(t)\phi(t) + \widehat{\mathbb{E}} \left[\partial_\mu \sigma(t) \widehat{\phi}(t) \right] + \sigma_v(t) v(t) \right] dt \\
& + \mathbb{E}^u \int_0^T \overline{Q}(t) \left[\alpha_x(t)\phi(t) + \widehat{\mathbb{E}} \left[\partial_\mu \alpha(t) \widehat{\phi}(t) \right] + \alpha_v(t) v(t) \right] dt,
\end{aligned}$$

and

$$\mathbb{I}_4 = \mathbb{E}^u \int_0^T \int_\Theta R(t,\theta) \left[g_x(t,\theta)\phi(t) + \widehat{\mathbb{E}} \left[\partial_\mu g(t,\theta) \widehat{\phi}(t) \right] + g_v(t,\theta) v(t) \right] m(d\theta) dt.$$

Then, we are using Itô's formula to $y(t)\widetilde{\rho}(t)$ and taking expectation, we get

$$\begin{aligned}
\mathbb{E}^u \left[y(T) \widetilde{\rho}(T) \right] & = \mathbb{E}^u \int_0^T y(t) d\widetilde{\rho}(t) + \mathbb{E}^u \int_0^T \widetilde{\rho}(t) dy(t) \\
& + \mathbb{E}^u \int_0^T K(t) \left\{ h_x(t)\phi(t) + h_v(t) v(t) \right\} dt \\
& = \mathbb{J}_1 + \mathbb{J}_2 + \mathbb{J}_3,
\end{aligned} \tag{4.23}$$

where $\mathbb{J}_1 = \mathbb{E}^u \int_0^T y(t) d\widetilde{\rho}(t)$ is a martingale with zero expectation. Moreover, by to make a simple computation, we get

$$\mathbb{J}_2 = \mathbb{E}^u \int_0^T \widetilde{\rho}(t) dy(t) = -\mathbb{E}^u \int_0^T \widetilde{\rho}(t) l(t) dt,$$

and

$$\mathbb{J}_3 = \mathbb{E}^u \int_0^T K(t) \left[h_x(t)\phi(t) + h_v(t) v(t) \right] dt.$$

Now, by using Fubini's theorem, we obtain

$$\mathbb{E}^{u}\int_{0}^{T}\Phi\left(t\right)\widehat{\mathbb{E}}\left[\partial_{\mu}\widehat{f}\left(t\right)\widehat{\phi}\left(t\right)\right]dt=\mathbb{E}^{u}\int_{0}^{T}\phi\left(t\right)\widehat{\mathbb{E}}\left[\partial_{\mu}f(t)\widehat{\Phi}(t)\right]dt, \tag{4.24}$$

$$\mathbb{E}^{u}\int_{0}^{T}Q\left(t\right)\widehat{\mathbb{E}}\left[\partial_{\mu}\widehat{\sigma}\left(t\right)\widehat{\phi}\left(t\right)\right]dt=\mathbb{E}^{u}\int_{0}^{T}\phi\left(t\right)\widehat{\mathbb{E}}\left[\partial_{\mu}\sigma(t)\widehat{Q}(t)\right]dt, \tag{4.25}$$

$$\mathbb{E}^{u}\int_{0}^{T}\overline{Q}\left(t\right)\widehat{\mathbb{E}}\left[\partial_{\mu}\widehat{\alpha}\left(t\right)\widehat{\phi}\left(t\right)\right]dt=\mathbb{E}^{u}\int_{0}^{T}\phi\left(t\right)\widehat{\mathbb{E}}\left[\partial_{\mu}\alpha(t)\widehat{\overline{Q}}(t)\right]dt, \tag{4.26}$$

and we get

$$\mathbb{E}^{u}\int_{0}^{T}\int_{\Theta}R\left(t,\theta\right)\widehat{\mathbb{E}}\left[\partial_{\mu}\widehat{g}\left(t,\theta\right)\widehat{\phi}\left(t\right)\right]m\left(d\theta\right)dt=\mathbb{E}^{u}\int_{0}^{T}\int_{\Theta}\phi\left(t\right)\widehat{\mathbb{E}}\left[\partial_{\mu}g(t,\theta)\widehat{R}\left(t,\theta\right)\right]m\left(d\theta\right)dt. \tag{4.27}$$

Finally, substituting (4.22), (4.23), (4.24), (4.25), (4.26) and (4.27) into (4.20), this completes the proof of Theorem 4.1 . □

4.4 Partially observed McKean-Vlasov linear quadratic control problem with jumps

As an application, we would study partially observed optimal control problem for Mckean-Vlasov linear quadratic control problem with jump diffusion, where the stochastic system is described by a set of linear McKean-Vlasov stochastic differential equations and the cost is described by a quadratic function.

By using our stochastic maximum principle established in Sect. 3 and classical filtering theory, we obtain an explicit expression of the optimal control represented in feedback form involving both controlled state process $x\left(t\right)$ as well as its law represented by $\mathbb{E}\left[x\left(t\right)\right]$ via the solutions of ordinary differential equations (ODEs).

Let us consider the following partially observed control system

$$\begin{cases} dx^{v}\left(t\right)= & f\left(t,x^{v}\left(t\right),\mathbb{P}_{x^{v}(t)},v\left(t\right)\right)dt+\sigma\left(t,x^{v}\left(t\right),\mathbb{P}_{x^{v}(t)},v\left(t\right)\right)dW\left(t\right) \\ & +\alpha\left(t,x^{v}\left(t\right),\mathbb{P}_{x^{v}(t)},v\left(t\right)\right)d\widetilde{W}\left(t\right) \\ & +\int_{\Theta}g\left(t,x^{v}\left(t_{-}\right),\mathbb{P}_{x^{v}(t_{-})},v\left(t\right),\theta\right)\widetilde{N}\left(d\theta,dt\right) \\ x^{v}\left(0\right)= & x_{0}, \end{cases} \tag{4.28}$$

where the coefficients introduce as following

$$\begin{aligned} f\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) &= A(t)\, x(t) + B(t)\, \mathbb{E}[x(t)] + C(t)\, v(t), \\ \sigma\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) &= D(t), \\ \alpha\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) &= 0, \\ g\left(t, x^v(t_-), \mathbb{P}_{x^v(t-)}, v(t), \theta\right) &= F(t), \\ h(t, x^v(t), v(t)) &= G(t), \end{aligned}$$

and an observation

$$\begin{cases} dY(t) &= G(t)\, dt + d\widetilde{W}(t), \\ Y(0) &= 0, \end{cases} \tag{4.29}$$

and we take the quadratic cost functional as

$$J(v(\cdot)) = \mathbb{E}^u\left[\int_0^T L(t)\, v^2(t)\, dt + M_T x^2(T)\right]. \tag{4.30}$$

Here, the coefficients $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $F(\cdot)$, $G(\cdot)$ and $L(\cdot)$ are bounded continuous functions and $M_T \geq 0$. For any $v \in \mathcal{U}_{ad}([0,T])$, equations (4.28) and (4.29) have a unique solutions respectively.

Our objective is to find an explicitly optimal control to minimize the cost functional $J(v(\cdot))$ over $v(\cdot) \in \mathcal{U}_{ad}([0,T])$, subject to (4.28) and (4.29).

Now, we begin to seek the explicit expression of the optimal control by two steps.

First step. Find optimal control.

We begin by write down the Hamiltonian function H :

$$\begin{aligned} H(t, x, v, \Phi, Q, \overline{Q}, R(\cdot)) &= [A(t)\, x(t) + B(t)\, \mathbb{E}[x(t)] + C(t)\, v(t)]\, \Phi(t) + D(t)\, Q(t) \\ &\quad + G(t)\, K(t) + L(t)\, v^2(t) + \int_\Theta F(t)\, R(t, \theta)\, m(d\theta), \end{aligned} \tag{4.31}$$

where $x(\cdot)$ is the optimal trajectory, solution of equation (4.28) corresponding to the optimal control $u(\cdot)$.

From Theorem 4.1 and from (4.31), the optimal control $u(\cdot)$ satisfies the following expression:

$$u(t) = -\frac{1}{2} L^{-1}(t)\, C(t)\, \mathbb{E}\left[\Phi(t) \mid \mathcal{F}_t^Y\right], \tag{4.32}$$

where we have $\left(\Phi(\cdot), Q(\cdot), \overline{Q}(\cdot), R(\cdot,\cdot)\right)$ is the solution of the following BSDE

$$\begin{cases} -d\Phi(t) = \left[A(t)\Phi(t) + B(t)\mathbb{E}[\Phi(t)]\right]dt \\ \qquad\qquad -Q(t)\,dW(t) - \overline{Q}(t)d\widetilde{W}(t) - \int_\Theta R(t,\theta)\,d\widetilde{N}(d\theta,dt), \\ \Phi(T) = 2M_T x(T). \end{cases} \tag{4.33}$$

Second step. Give the explicit expression of the optimal control in (4.32).

According to Liptser & Shiryayev, 1979 ; Xiong 2008 , we can deduce the following group of filtering equations

$$\begin{cases} d\widehat{x}(t) = \left[A(t)\widehat{x}(t) + B(t)\mathbb{E}[\widehat{x}(t)] - \frac{1}{2}L^{-1}(t)C^2(t)\widehat{\Phi}(t)\right]dt \\ -d\widehat{\Phi}(t) = \left[A(t)\widehat{\Phi}(t) + B(t)\mathbb{E}\left[\widehat{\Phi}(t)\right]\right]dt - \widehat{\overline{Q}}(t)\,d\widetilde{W}(t), \\ \widehat{x}(0) = x_0,\ \widehat{\Phi}(T) = 2M_T\widehat{x}(T),\ \widehat{\overline{Q}}(t) = 0, \end{cases} \tag{4.34}$$

with $\widehat{\xi}(t) = \mathbb{E}^u\left[\xi(t) \mid \mathcal{F}_t^Y\right]$ is the filtering estimate of the state $\xi(t)$ depending on the observable filtration $\mathcal{F}_t^Y$, $\xi = x, \Phi, \overline{Q}$.

Now, to solve the above equation (4.34), we conjecture a process $\widehat{\Phi}(\cdot)$ of the form

$$\widehat{\Phi}(t) = \varphi(t)\widehat{x}(t) + \psi(t)\mathbb{E}[\widehat{x}(t)], \tag{4.35}$$

where $\varphi(\cdot), \psi(\cdot)$ are deterministic differential functions.

We derive (4.35) and comparing it with (4.34), we obtain

$$\begin{aligned} &-\left\{A(t)\left(\varphi(t)\widehat{x}(t) + \psi(t)\mathbb{E}[\widehat{x}(t)]\right) + B(t)\mathbb{E}\left[\varphi(t)\widehat{x}(t) + \psi(t)\mathbb{E}[\widehat{x}(t)]\right]\right\} \\ &= \dot{\varphi}(t)\widehat{x}(t) + \dot{\psi}(t)\mathbb{E}[\widehat{x}(t)] \\ &+ \varphi(t)\left\{A(t)\widehat{x}(t) + B(t)\mathbb{E}[\widehat{x}(t)] - \frac{1}{2}L^{-1}(t)C^2(t)\left(\varphi(t)\widehat{x}(t) + \psi(t)\mathbb{E}[\widehat{x}(t)]\right)\right\} \\ &+ \psi(t)\left\{(A(t) + B(t))\mathbb{E}[\widehat{x}(t)] - \frac{1}{2}L^{-1}(t)C^2(t)\mathbb{E}\left[\varphi(t)\widehat{x}(t) + \psi(t)\mathbb{E}[\widehat{x}(t)]\right]\right\}. \end{aligned} \tag{4.36}$$

By comparing the coefficients of $\widehat{x}(t)$ and $\mathbb{E}[\widehat{x}(t)]$ in (4.36), we obtain the following ODEs:

$$\begin{cases} \dot{\varphi}(t) + 2A(t)\varphi(t) - \frac{1}{2}L^{-1}(t)C^2(t)\varphi^2(t) = 0, \\ \varphi(T) = 2M_T, \end{cases} \tag{4.37}$$

and also

$$\begin{cases} \dot{\psi}(t) + 2(A(t) + B(t))\psi(t) + 2B(t)\varphi(t) \\ \quad - L^{-1}(t) C^2(t) \varphi(t) \psi(t) - \dfrac{1}{2} L^{-1}(t) C^2(t) \psi^2(t) = 0, \\ \psi(T) = 0. \end{cases} \tag{4.38}$$

Note that equations (4.37) and (4.38) are Bernoulli differential equation and Riccati differential equation respectively. To solve (4.37) and (4.38), we can utilize a method that is similar to method in Lakhdari, Miloudi & Hafayed, 2020 . Then, the optimal control $u(\cdot) \in \mathcal{U}_{ad}([0,T])$ for the problem (4.30) is given in the feedback form

$$u(t, \widehat{x}(t)) = -\frac{1}{2} L^{-1}(t) C(t) [\varphi(t) \widehat{x}(t) + \psi(t) \mathbb{E}[\widehat{x}(t)]],$$

where $\varphi(\cdot), \psi(\cdot)$ determined by (4.37) and (4.38) respectively.

Conclusion

In this thesis, we have developed a necessary conditions for partially observed stochastic optimal control problem, where the controlled state process in the first part of this study is governed by general McKean-Vlasov differential equations. We use Girsanov's theorem as well as standard variational technique to transform our partially observed optimal control problem to completely observable problem. Note that the results obtained here are based on the derivatives with respect to the probability law. As an illustration, we study partially observed linear-quadratic control problem where the control domain is assumed to be convex. For the second part, we consider a controlled state process governed by general McKean-Vlasov differential equations with jumps. By transforming the partial observation problem to a related problem with full information, a stochastic maximum principle for optimal control has been also established via the derivative with respect to probability measure. A partially observed linear-quadratic control problem with jumps has been solved explicitly to illustrate our theoretical results. The main feature of these results is to explicitly solve some mathematical finance problems such as conditional mean-variance portfolio selection problem in incomplete market.

Many interesting problems remain open. For example, study stochastic maximum principle for these control problems for a non convex control domain. And also as one possible problem is to establish some optimality conditions for partially observed stochastic optimal control for systems descriped by forward-backward stochastic differential equations of general McKean-Vlasov type with jumps with some applications.

Appendix

This Proposition, Theorem and lemma has been used a lot in this work.

Proposition A1.

Let $\mathcal{G}$ be the predictable $\sigma-$field on $\Omega \times [0,T]$, and f be a $\mathcal{G} \times \mathcal{B}(\Theta)-$measurable function such that

$$\mathbb{E} \int_0^T \int_\Theta |f(r,\theta)|^2 m(d\theta)dr < \infty,$$

then for all $p \geq 2$ there exists a positive constant $C = C(T, p, m(\Theta))$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \int_\Theta f(r,\theta) N(d\theta,dr)\right|^p\right] < C\mathbb{E}\left[\int_0^T \int_\Theta |f(r,\theta)|^p m(d\theta)dr\right].$$

Proof. See (Bouchard & Elie 2008, *Appendix*).

Theorem (Burkholder-Davis-Gundy inequality)

Let $(X_t)_{t\geq 0}$be a continuous local martingale defined on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ satisfying the usual conditions. Let $p > 0$. So there are two constants c_p and C_p, $0 < c_p < C_p < +\infty$ such that

$$1) \quad c_p \|X_\infty^*\|_p \leq \left\|[X,X]_\infty^{1/2}\right\|_p \qquad \text{et} \qquad 2) \quad \left\|[X,X]_\infty^{1/2}\right\|_p \leq C_p \|X_\infty^*\|_p$$

weher $X_t^* = \sup\{|X_s| \ / \ 0 \leq s \leq t\}$ and $[X^n, X^n]_k = \sum_{l=1}^{k}\left(X_l^n - X_{l-1}^n\right)^2$.

In the event that the martingale is not continuous, inequalities 1) and 2) remains valid only if $p \geq 1$. For more, see [60].

Proof. See for $p \in (1,\infty)$ Burkholder [9]. For $p \in (0,1]$ Burkholder and Gundy [10], and for the case $p = 1$ of (BDG) see Davis [16].

lemma (Gronwall's lemma)

Let $X(t)$ and $f(t)$ be nonnegative continuous functions on $0 \leq t \leq T$, for which the inequality

$$X(t) \leq c + \int_0^t f(s) X(s) ds, \qquad t \in [0,T]$$

holds, where $c \geq 0$ is a constant. Then

$$X(t) \leq c \exp\left(\int_0^t f(s) ds\right), \qquad t \in [0,T]$$

see [61].

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