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A C^0 INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD AND AN EQUILIBRATED *A POSTERIORI* ERROR ESTIMATOR FOR A NONLINEAR FOURTH ORDER ELLIPTIC BOUNDARY VALUE PROBLEM OF p-BIHARMONIC TYPE

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Abstract. We consider a C^0 Interior Penalty Discontinuous Galerkin (C0IPDG) approximation of a nonlinear fourth order elliptic boundary value problem of p-biharmonic type and an equilibrated *a posteriori* error estimator. The C0IPDG method can be derived from a discretization of the corresponding minimization problem involving a suitably defined reconstruction operator. The equilibrated *a posteriori* error estimator provides an upper bound for the discretization error in the broken $W^{2,p}$ norm in terms of the associated primal and dual energy functionals. It requires the construction of an equilibrated flux and an equilibrated moment tensor based on a three-field formulation of the C0IPDG approximation. The relationship with a residual-type *a posteriori* error estimator is studied as well. Numerical results illustrate the performance of the suggested approach.

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1. Introduction

The finite element solution of the biharmonic problem is well documented in the literature. In order to avoid computationally extremely expensive C^1-conforming elements, the early work concentrated on mixed methods (*cf.*, *e.g.*, [2, 3, 18, 36, 45] as well as the review article [32] and the references therein). An alternative to mixed methods are Discontinuous Galerkin (DG) approximations which have the advantage that they only require the numerical solution of one system of algebraic equations instead of two in case of mixed methods. In particular, Interior Penalty Discontinuous Galerkin (IPDG) and C^0 Interior Penalty Discontinuous Galerkin (C0IPDG) methods have been considered in [12, 29, 30, 33, 38, 39, 46]. For IPDG and C0IPDG approximations, adaptive mesh refinement has been realized based on residual-type *a posteriori* error estimators in [13, 19, 31] and on equilibrated *a posteriori* error estimators in [11]. As far as p-biharmonic problems and related nonlinear fourth order elliptic boundary value problems of p-biharmonic type are concerned, a lot of work has been devoted to analytical investigations [6, 7, 16, 21, 34, 37, 49, 50], but considerably less work has been done with regard to

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numerical solutions. Mixed methods have been developed in [35], DG methods in [40], and mesh-free methods in [41].

In this paper, we will study an adaptive C0IPDG method for a nonlinear fourth order elliptic boundary value problem of p-biharmonic type where the mesh refinement relies on an equilibrated *a posteriori* error estimator. The paper is organized as follows: After introducing some basic notations and preliminary results, in Section 2 we consider a nonlinear fourth order elliptic boundary value problem of p-biharmonic type with Navier boundary conditions which represents the optimality condition for the unconstrained minimization of a second order energy functional on the Sobolev space $W^{2,p}, 1 < p < \infty$. The following Section 3 is devoted to the C0IPDG method which can be derived as the optimality condition for a C0IPDG approximation of the related minimization problem involving a suitably defined recovery operator. We also present a three-field formulation of the C0IPDG method which will play a decisive role in the derivation of the equilibrated *a posteriori* error estimator. Section 4 deals with a computable upper bound for the global discretization error in the norm of the broken $W^{2,p}$ Sobolev space by means of the associated primal and dual energy functionals which can be derived by a general approach from [42]. In Section 5, we are concerned with the equilibrated *a posteriori* error estimator. This requires the construction of an equilibrated flux and an equilibrated moment tensor which can be done by means of Brezzi–Douglas–Marini finite elements with respect to the given triangulation of the computational domain. In the subsequent Section 6, we obtain the relationship with a residual-type *a posteriori* error estimator and finally, in Section 7 we present a documentation of numerical results which illustrate the performance of the suggested approach.

2. The nonlinear fourth order elliptic boundary value problem of p-biharmonic type and the associated primal and dual energy functionals

We use standard notation from Lebesgue and Sobolev space theory (*cf.*, *e.g.*, [47]). In particular, for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, with boundary $\Gamma = \partial\Omega$ and exterior unit normal $\mathbf{n}_\Gamma$ we refer to $L^p(\Omega;\mathbb{R}^d)$ and $L^p(\Omega;\mathbb{R}^{d\times d}), 1 < p < \infty$, as the Banach spaces of p-th power Lebesgue integrable functions and tensors on Ω with norms $\|\cdot\|_{L^p(\Omega;\mathbb{R}^d)}$ and $\|\cdot\|_{L^p(\Omega;\mathbb{R}^{d\times d})}$. In case $d = 1$ we will write $L^p(\Omega)$ instead of $L^p(\Omega;\mathbb{R})$. Matrix-valued functions in $L^p(\Omega;\mathbb{R}^{d\times d})$ will be denoted by $\underline{\underline{\mathbf{p}}} = (p_{ij})_{i,j=1}^d$ and for $\underline{\underline{\mathbf{p}}} \in L^p(\Omega;\mathbb{R}^{d\times d}), \underline{\underline{\mathbf{q}}} \in L^q(\Omega;\mathbb{R}^{d\times d}), 1/p + 1/q = 1$, we use the notation $\underline{\underline{\mathbf{p}}} : \underline{\underline{\mathbf{q}}}$ for $\underline{\underline{\mathbf{p}}} : \underline{\underline{\mathbf{q}}} := \sum_{i,j=1}^d p_{ij}q_{ij}$. Further, for $u \in W^{2,p}(\Omega)$, we refer to $D^2u := (\partial^2 u/\partial x_i \partial x_j)_{i,j=1}^2$ as the matrix of second partial derivatives.

We denote by $W^{s,p}(\Omega), s \in \mathbb{R}_+, 1 < p < \infty$, the Sobolev spaces with norms $\|\cdot\|_{W^{s,p}(\Omega)}$ and by $W_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,p}(\Omega)}$. Functions $u \in W^{2,p}(\Omega)$ have a trace $u|_\Gamma$ on the boundary $\Gamma = \partial\Omega$ with $u|_\Gamma \in W^{2-1/p,p}(\Gamma)$. For $u_D \in W^{2-1/p,p}(\Gamma)$ we set

$$W^{2,p}_{u_D,\Gamma}(\Omega) := \{v \in W^{2,p}(\Omega) \mid v|_\Gamma = u_D\}.$$

Further, we define $\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega)$ and $\underline{\underline{\mathbf{H}}}^{(p)}(\mathrm{div}^2,\Omega), 1 < p < \infty$, as the Banach spaces

$$\begin{aligned}\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega) &= \{\underline{\boldsymbol{\tau}} \in L^p(\Omega;\mathbb{R}^d) \mid \nabla\cdot\underline{\boldsymbol{\tau}} \in L^p(\Omega)\},\\ \underline{\underline{\mathbf{H}}}^{(p)}(\mathrm{div}^2,\Omega) &= \Big\{\underline{\underline{\boldsymbol{\tau}}} \in L^p(\Omega;\mathbb{R}^{d\times d}) \mid \nabla\cdot\underline{\underline{\boldsymbol{\tau}}} \in \underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega)\Big\}\end{aligned}$$

with the graph norms

$$\begin{aligned}\|\underline{\boldsymbol{\tau}}\|_{\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega)} &:= \Big(\|\underline{\boldsymbol{\tau}}\|^p_{L^p(\Omega;\mathbb{R}^d)} + \|\nabla\cdot\underline{\boldsymbol{\tau}}\|^p_{L^p(\Omega)}\Big)^{1/p},\\ \|\underline{\underline{\boldsymbol{\tau}}}\|_{\underline{\underline{\mathbf{H}}}^{(p)}(\mathrm{div}^2,\Omega)} &:= \Big(\|\underline{\underline{\boldsymbol{\tau}}}\|^p_{L^p(\Omega;\mathbb{R}^{d\times d})} + \|\nabla\cdot\underline{\underline{\boldsymbol{\tau}}}\|^p_{L^p(\Omega;\mathbb{R}^d)} + \|\nabla\cdot\nabla\cdot\underline{\underline{\boldsymbol{\tau}}}\|^p_{L^p(\Omega)}\Big)^{1/p}.\end{aligned}$$

For further properties of $\underline{\underline{\mathbf{H}}}^{(p)}(\mathrm{div},\Omega)$ we refer to [1]. We refer to $\underline{\underline{\mathbf{H}}}^{(p)}_{0,\Gamma}\big(\mathrm{div}^2,\Omega\big)$ as the subspace

$$\underline{\underline{\mathbf{H}}}^{(p)}_{0,\Gamma}\big(\mathrm{div}^2,\Omega\big) := \Big\{\underline{\underline{\boldsymbol{\tau}}} \in \underline{\underline{\mathbf{H}}}^{(p)}\big(\mathrm{div}^2,\Omega\big) \mid \mathbf{n}_\Gamma \cdot \nabla \cdot \underline{\underline{\boldsymbol{\tau}}} = 0 \text{ on } \Gamma\Big\}.$$

For later use we recall Young's inequality

$$\prod_{i=1}^{2} a_i \le \frac{\varepsilon}{p} a_1^p + \frac{\varepsilon^{-q/p}}{q} a_2^q \tag{2.1}$$

for $a_i > 0, 1 \le i \le 2$, and $1 < p, q < \infty, 1/p + 1/q = 1$, and any $\varepsilon > 0$, as well as the following inequality.

Let $w_i \in \mathbb{R}, 1 \le i \le 2$, and $0 \le r < \infty$. Then it holds (*cf.*, [44], page 136)

$$(|w_1| + |w_2|)^r \le 2^r (|w_1|^r + |w_2|^r), \quad r \ge 0. \tag{2.2}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\Gamma = \partial\Omega$ and exterior unit normal vector $\mathbf{n}_\Gamma$. Further, let $1 < p, q < \infty, 1/p + 1/q = 1$, and $f \in L^q(\Omega)$, $u_D \in W^{2-1/p,p}(\Gamma), u_N \in L^q(\Gamma)$. We consider the following nonlinear fourth order elliptic boundary value problem with Navier-type boundary conditions:

$$\nabla \cdot \nabla \cdot \Big(\big|D^2 u\big|^{p-2} D^2 u\Big) = f \qquad \text{in } \Omega, \tag{2.3a}$$

$$u = u_D \qquad \text{on } \Gamma, \tag{2.3b}$$

$$\Big(\big|D^2 u\big|^{p-2} D^2 u\Big)\, \mathbf{n}_\Gamma = u_N\, \mathbf{n}_\Gamma \qquad \text{on } \Gamma. \tag{2.3c}$$

The variational formulation of (2.3) requires the computation of $u \in W^{2,p}_{u_D,\Gamma}(\Omega)$ such that for all $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ it holds

$$\int_\Omega \big|D^2 u\big|^{p-2} D^2 u : D^2 v \,\mathrm{d}x = \ell(v), \tag{2.4a}$$

where the functional $\ell : V \to \mathbb{R}$ is given by

$$\ell(v) := \int_\Omega f v \,\mathrm{d}x + \int_\Gamma u_N\, \mathbf{n}_\Gamma \cdot \nabla v \,\mathrm{d}s. \tag{2.4b}$$

We note that (2.4) admits a solution and represents the necessary and sufficient optimality condition for the minimization problem

$$J_P(u) = \inf_{v \in W^{2,p}_{u_D,\Gamma}(\Omega)} J_P(v), \tag{2.5a}$$

where the objective functional J_P is given by

$$J_P(v) := \frac{1}{p}\int_\Omega |D^2 v|^p \,\mathrm{d}x - \int_\Omega f v \,\mathrm{d}x - \int_\Gamma u_N\, \mathbf{n}_\Gamma \cdot \nabla v \,\mathrm{d}s. \tag{2.5b}$$

The dual problem of (2.5) is given by (*cf.*, Chap. 4, Sect. 2.2 in [28])

$$J_D(\underline{\underline{\mathbf{p}}}) = \inf_{\underline{\underline{\mathbf{q}}} \in \underline{\underline{\mathbf{H}}}^{(q)}(\mathrm{div}^2;\Omega)} J_D(\underline{\underline{\mathbf{q}}}), \tag{2.6a}$$

subject to the equilibrium conditions

$$\nabla \cdot \nabla \cdot \underline{\underline{\mathbf{p}}} = f \text{ in } L^q(\Omega), \quad \underline{\underline{\mathbf{p}}}\, \mathbf{n}_\Gamma = u_N\, \mathbf{n}_\Gamma \text{ in } L^q\big(\Gamma;\mathbb{R}^2\big), \tag{2.6b}$$

where the objective functional J_D is given by

$$J_D(\underline{\underline{\mathbf{q}}}) := \frac{1}{q}\int_\Omega \big|\underline{\underline{\mathbf{q}}}\big|^q \,\mathrm{d}x + \int_\Gamma u_D\, \mathbf{n}_\Gamma \cdot \nabla \cdot \underline{\underline{\mathbf{q}}} \,\mathrm{d}s. \tag{2.6c}$$

3. C0IPDG approximation of the problem

Let $\mathcal{T}_h$ be a geometrically conforming, locally quasi-uniform, simplicial triangulation of the computational domain Ω. Given $D \subset \bar{\Omega}$, we denote by $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D)$ the set of vertices and edges of $\mathcal{T}_h$ in D, and we refer to $P_k(D)$, $k \in \mathbb{N}$, as the set of polynomials of degree $\leq k$ on D. Moreover, $h_K, K \in \mathcal{T}_h$, and h_E, $E \in \mathcal{E}_h$, stand for the diameter of K and the length of E, respectively. We define $h := \min\,\{h_K \mid K \in \mathcal{T}_h\}$. For two quantities $a, b \in \mathbb{R}$ we will write $a \lesssim b$, if there exists a constant $C > 0$, independent of h, such that $a \leq Cb$.

Due to the local quasi-uniformity of the triangulation there exists a constant $0 < c_R \leq C_R$ such that for all $K \in \mathcal{T}_h$ it holds

$$c_R h_K \leq h_E \leq C_R h_K, \quad E \in \mathcal{E}_h(\partial K). \tag{3.1}$$

We will use the following inverse inequality (*cf.*, *e.g.*, Thm. 3.2.6 in [17]): For $1 \leq p \leq \infty$ there exists a constant $C_{\text{inv}} > 0$, only depending on p, the polynomial degree k, and the local geometry of the triangulation, such that for $v_h \in P_k(K)$ and $E \in \mathcal{E}_h(\bar{\Omega})$ it holds

$$\|\nabla v_h\|_{L^p(K;\mathbb{R}^2)} \leq C_{\text{inv}} h_K^{-1} \|v_h\|_{L^p(K)}. \tag{3.2}$$

We will also use the following trace inequality (*cf.*, *e.g.*, [23]): For $1 \leq p \leq \infty$ there exists a constant $C_T > 0$, only depending on p, the polynomial degree k, and the local geometry of the triangulation, such that for $v_h \in P_k(K)$ and $K \in \mathcal{T}_h$ it holds

$$\|v_h\|_{L^p(\partial K)} \leq C_T h_K^{-1/p} \|v_h\|_{L^p(K)}. \tag{3.3}$$

For $E \in \mathcal{E}_h(\Omega)$, $E = K_+ \cap K_-$, $K_\pm \in \mathcal{T}_h(\Omega)$, and $v_h \in V_h$, we denote the average and jump of v_h across E by $\{v_h\}_E$ and $[v_h]_E$, *i.e.*,

$$\{v_h\}_E := \frac{1}{2}\big(v_h|_{E\cap K_+} + v_h|_{E\cap K_-}\big), \quad [v_h]_E := v_h|_{E\cap K_+} - v_h|_{E\cap K_-},$$

whereas for $E \in \mathcal{E}_h(\Gamma)$ we set

$$\{v_h\}_E := v_h|_E, \quad [v_h]_E := v_h|_E.$$

The averages $\{\nabla v_h\}_E, \{\underline{\boldsymbol{\tau}}_h\}_E$ and jumps $[\nabla v_h]_E, [\underline{\boldsymbol{\tau}}_h]_E$ of vector-valued functions ∇v_h and $\underline{\boldsymbol{\tau}}_h$ as well as the averages $\{D^2 v_h\}_E, \{\underline{\underline{\boldsymbol{\tau}}}_h\}_E$ and jumps $[D^2 v_h]_E, [\underline{\underline{\boldsymbol{\tau}}}_h]_E$ of matrix-valued functions $D^2 v_h$ and $\underline{\underline{\boldsymbol{\tau}}}_h$ are defined analogously. For $E \in \mathcal{E}_h(\bar{\Omega})$ it holds

$$\int_E [u_h v_h]_E \,\mathrm{d}s = \int_E \left(\{u_h\}_E\ [v_h]_E + [u_h]_E\ \{v_h\}_E\right) \mathrm{d}s. \tag{3.4}$$

We further denote by $\mathbf{n}_E, E \in \mathcal{E}_h(\Omega)$, with $E = K_+ \cap K_-$ the unit normal on E pointing from K_+ to K_- and by $\mathbf{n}_E, E \in \mathcal{E}_h(\Gamma)$, the exterior unit normal on E.

We define the broken $W^{2,p}$-space $W^{2,p}(\Omega;\mathcal{T}_h), 1 < p < \infty$, by

$$W^{2,p}(\Omega;\mathcal{T}_h) := \big\{v_h \in L^p(\Omega) \mid v_h|_K \in W^{2,p}(K), K \in \mathcal{T}_h\big\}, \tag{3.5}$$

equipped with the norm

$$\|v_h\|_{W^{2,p}(\Omega;\mathcal{T}_h)} := \left(\sum_{K\in\mathcal{T}_h} \|v_h\|^p_{W^{2,p}(K)}\right)^{1/p}, \tag{3.6}$$

and the broken spaces $\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega;\mathcal{T}_h)$ and $\underline{\underline{\mathbf{H}}}^{(p)}\big(\mathrm{div}^2,\Omega;\mathcal{T}_h\big)$ by

$$\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega;\mathcal{T}_h) := \Big\{\underline{\mathbf{q}}_h \in L^p(\Omega;\mathbb{R}^2) \mid \underline{\mathbf{q}}_h|_K \in \underline{\mathbf{H}}^{(p)}(\mathrm{div};K), K \in \mathcal{T}_h\Big\}, \tag{3.7a}$$

$$\underline{\underline{\mathbf{H}}}^{(p)}\big(\mathrm{div}^2,\Omega;\mathcal{T}_h\big) := \Big\{\underline{\underline{\mathbf{q}}}_h \in L^p(\Omega;\mathbb{R}^{2\times 2}) \mid \underline{\underline{\mathbf{q}}}_h|_K \in \underline{\underline{\mathbf{H}}}^{(p)}(\mathrm{div}^2;K), K \in \mathcal{T}_h\Big\}, \tag{3.7b}$$

equipped with the norms

$$\big\|\underline{\mathbf{q}}_h\big\|_{\underline{\mathbf{H}}^{(p)}(\mathrm{div},\Omega;\mathcal{T}_h)} := \left(\sum_{K\in\mathcal{T}_h} \big\|\underline{\mathbf{q}}_h\big\|^p_{\underline{\mathbf{H}}^{(p)}(\mathrm{div};K)}\right)^{1/p}, \tag{3.8a}$$

$$\big\|\underline{\underline{\mathbf{q}}}_h\big\|_{\underline{\underline{\mathbf{H}}}^{(p)}(\mathrm{div}^2,\Omega;\mathcal{T}_h)} := \left(\sum_{K\in\mathcal{T}_h} \big\|\underline{\underline{\mathbf{q}}}_h\big\|^p_{\underline{\underline{\mathbf{H}}}^{(p)}(\mathrm{div}^2;K)}\right)^{1/p}. \tag{3.8b}$$

For $v \in W^{2,p}(\Omega;\mathcal{T}_h)$ we redefine the primal energy functional (2.5b) according to

$$J_P(v) := \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K |D^2 v|^p\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K f v\,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Gamma)}\int_E u_N\ \mathbf{n}_\Gamma\cdot\nabla v\,\mathrm{d}s, \tag{3.9}$$

and note that it reduces to (2.5b) for $v \in W^{2,p}(\Omega)$.

We consider the finite element approximation with the DG spaces

$$V_h := \big\{v_h \in C(\bar\Omega) \mid v_h|_K \in P_k(K),\ K \in \mathcal{T}_h\big\}, \tag{3.10a}$$

$$\underline{\mathbf{V}}_h := \Big\{\underline{\mathbf{q}}_h : \bar\Omega \to \mathbb{R}^2 \mid \underline{\mathbf{q}}_h|_K \in P_{k-1}(K)^2,\ K \in \mathcal{T}_h\Big\}, \tag{3.10b}$$

$$\underline{\underline{\mathbf{V}}}_h := \Big\{\underline{\underline{\mathbf{q}}}_h : \bar\Omega \to \mathbb{R}^{2\times 2} \mid \underline{\underline{\mathbf{q}}}_h|_K \in P_k(K)^{2\times 2},\ K \in \mathcal{T}_h\Big\}. \tag{3.10c}$$

We note that for $k \ge 2$ we have $V_h \subset W^{2,p}(\Omega;\mathcal{T}_h)$. Moreover, for $\underline{\underline{\mathbf{q}}}_h \in \underline{\underline{\mathbf{V}}}_h$, we have $\nabla\cdot\underline{\underline{\mathbf{q}}}_h|_K \in P_{k-1}(K)^2$ and $\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{q}}}_h|_K \in P_{k-2}(K), K \in \mathcal{T}_h$.

For $u_h \in V_h$ we define the broken gradient $\nabla_h u_h$ and the broken Hessian $D_h^2 u_h$ by means of

$$\nabla_h u_h|_K := \nabla u_h|_K, \qquad K \in \mathcal{T}_h, \tag{3.11a}$$

$$D_h^2 u_h|_K := D^2 u_h|_K, \qquad K \in \mathcal{T}_h. \tag{3.11b}$$

Following [15,22], we further define recovery operators

$$\underline{\underline{\mathbf{R}}}_{h,i} : V_h \oplus W^{2,p}(\Omega) \to \underline{\underline{\mathbf{V}}}_h, 1 \le i \le 2,$$

according to

$$\int_\Omega \underline{\underline{\mathbf{R}}}_{h,1}(u) : \underline{\underline{\mathbf{q}}}_h\,\mathrm{d}x = \sum_{E\in\mathcal{E}_h(\Omega)}\int_E [\nabla u\otimes\mathbf{n}_E]_E : \{\underline{\underline{\mathbf{q}}}_h\}_E\,\mathrm{d}s\ -\sum_{E\in\mathcal{E}_h(\Gamma)} u\ \mathbf{n}_E\cdot\nabla\cdot\underline{\underline{\mathbf{q}}}_h\,\mathrm{d}s,\quad \underline{\underline{\mathbf{q}}}_h \in \underline{\underline{\mathbf{V}}}_h, \tag{3.12a}$$

$$\int_\Omega \underline{\underline{\mathbf{R}}}_{h,2}(u) : \underline{\underline{\mathbf{q}}}_h\,\mathrm{d}x = \sum_{E\in\mathcal{E}_h(\Gamma)}\int_E u\ \mathbf{n}_E\cdot\nabla\cdot\underline{\underline{\mathbf{q}}}_h\,\mathrm{d}s,\qquad \underline{\underline{\mathbf{q}}}_h \in \underline{\underline{\mathbf{V}}}_h, \tag{3.12b}$$

where $\nabla u\otimes\mathbf{n}_E$ stands for the tensor product of ∇u and $\mathbf{n}_E$.

Further, let u_D^* be chosen according to

$$u_D^* \in W^{2,p}(\Omega) \text{ such that } u_D^*|_\Gamma = u_D. \tag{3.13}$$

We define the broken DG Hessians $D^2_{\mathrm{DG},i} u_h, 1 \le i \le 2$, as follows:

$$D^2_{\mathrm{DG},1} u_h := D^2_h u_h - \underline{\underline{\mathbf{R}}}_{h,1}(u_h), \tag{3.14a}$$

$$D^2_{\mathrm{DG},2} u_h := D^2_{\mathrm{DG},1} u_h - \underline{\underline{\mathbf{R}}}_{h,2}(u_D^*). \tag{3.14b}$$

The following auxiliary result from [15] will enable us to estimate the L^p norm of $\underline{\underline{\mathbf{R}}}_{h,1}(u_h) + \underline{\underline{\mathbf{R}}}_{h,2}(u_D^*)$ for $u_h \in V_h$ (*cf.*, Lem. A2 in [15]).

Lemma 3.1. *For each $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$ there exists a constant $C_{\mathrm{IS}} > 0$, independent of h, such that it holds*

$$\inf_{\substack{u_h \in V_h \\ u_h \neq 0}} \sup_{\substack{\underline{\underline{\mathbf{q}}}_h \in \underline{\underline{\mathbf{V}}}_h \\ \underline{\underline{\mathbf{q}}}_h \neq \mathbf{0}}} \frac{\int_\Omega \Big(\underline{\underline{\mathbf{R}}}_{h,1}(u_h) + \underline{\underline{\mathbf{R}}}_{h,2}(u_D^*) \Big) : \underline{\underline{\mathbf{q}}}_h \, \mathrm{d}x}{\|u_h\|_{L^p(\Omega)} \Big\| \underline{\underline{\mathbf{q}}}_h \Big\|_{L^q(\Omega;\mathbb{R}^{2\times 2})}} \ge C_{\mathrm{IS}}. \tag{3.15}$$

Theorem 3.2. *Under the assumptions of Lemma 3.1 there exists a constant $C_{\mathrm{rec}} > 0$, independent of h, such that for $u_h \in V_h$ it holds*

$$\begin{aligned} \Big\| \underline{\underline{\mathbf{R}}}_{h,1}(u_h) + \underline{\underline{\mathbf{R}}}_{h,2}(u_D^*) \Big\|_{L^p(\Omega;\mathbb{R}^{2\times 2})} \le \ & C_{\mathrm{rec}} \Bigg(\Bigg(\sum_{E \in \mathcal{E}_h(\Omega)} h_E^{-p/q} \int_E |[\nabla u_h \otimes \mathbf{n}_E]_E|^p \, \mathrm{d}s \Bigg)^{1/p} \\ & + \Bigg(\sum_{E \in \mathcal{E}_h(\Gamma)} h_E^{-p(q+1)/q} \int_E |u_h - u_D|^p \, \mathrm{d}s \Bigg)^{1/p} \Bigg), \end{aligned} \tag{3.16a}$$

$$\Big\| \underline{\underline{\mathbf{R}}}_{h,2}(u_D^*) \Big\|_{L^p(\Omega;\mathbb{R}^{2\times 2})} \le C_{\mathrm{rec}} \Bigg(\sum_{E \in \mathcal{E}_h(\Gamma)} h_E^{-p(q+1)/q} \int_E |u_D|^p \, \mathrm{d}s \Bigg)^{1/p}. \tag{3.16b}$$

Proof. We have

$$\begin{aligned} \Big\| \underline{\underline{\mathbf{R}}}_{h,i}(u_h) \Big\|_{L^p(\Omega;\mathbb{R}^{2\times 2})} &= \sup_{\underline{\underline{\mathbf{q}}} \in L^q(\Omega;\mathbb{R}^{2\times 2})} \frac{\int_\Omega \underline{\underline{\mathbf{R}}}_{h,i}(u_h) : \underline{\underline{\mathbf{q}}} \, \mathrm{d}x}{\|\underline{\underline{\mathbf{q}}}\|_{L^q(\Omega;\mathbb{R}^{2\times 2})}} \\ &\ge \sup_{\underline{\underline{\mathbf{q}}}_h \in \underline{\underline{\mathbf{V}}}_h} \frac{\int_\Omega \underline{\underline{\mathbf{R}}}_{h,i}(u_h) : \underline{\underline{\mathbf{q}}}_h \, \mathrm{d}x}{\Big\|\underline{\underline{\mathbf{q}}}_h\Big\|_{L^q(\Omega;\mathbb{R}^{2\times 2})}}, \ 1 \le i \le 2. \end{aligned} \tag{3.17}$$

The inf-sup property (3.15) implies

$$\Big\| \underline{\underline{\mathbf{R}}}_{h,i}(u_h) \Big\|_{L^p(\Omega;\mathbb{R}^{2\times 2})} \le C_{\mathrm{IS}}^{-1} \sup_{\underline{\underline{\mathbf{q}}}_h \in \underline{\underline{\mathbf{V}}}_h} \frac{\int_\Omega \underline{\underline{\mathbf{R}}}_{h,i}(u_h) : \underline{\underline{\mathbf{q}}}_h \, \mathrm{d}x}{\Big\|\underline{\underline{\mathbf{q}}}_h\Big\|_{L^q(\Omega;\mathbb{R}^{2\times 2})}}. \tag{3.18}$$

Now, observing (3.12) and setting $E_1 := E_+, E_2 := E_-$ for $E \in \mathcal{E}_h(\Omega)$, we obtain

$$\begin{aligned} \int_\Omega \Big(\underline{\underline{\mathbf{R}}}_{h,1}(u_h) + \underline{\underline{\mathbf{R}}}_{h,2}(u_D^*) \Big) : \underline{\underline{\mathbf{q}}}_h \, \mathrm{d}x \le & \sum_{E \in \mathcal{E}_h(\Omega)} \int_E h_E^{-1/q} |[\nabla u_h \otimes \mathbf{n}_E]_E| \ h_E^{1/q} |\{\underline{\underline{\mathbf{q}}}_h\}_E| \, \mathrm{d}s \\ & + \sum_{E \in \mathcal{E}_h(\Gamma)} \int_E h_E^{-(1+1/q)} |u_h - u_D| \ h_E^{1+1/q} \Big| \mathbf{n}_E \cdot \Big\{ \nabla \cdot \underline{\underline{\mathbf{q}}}_h \Big\}_E \Big| \, \mathrm{d}s \end{aligned}$$

$$\leq \sum_{E\in\mathcal{E}_h(\Omega)} \left(\int_E h_E^{-p/q} |[\nabla u_h \otimes \mathbf{n}_E]_E|^p \,\mathrm{d}s\right)^{1/p} \left(\int_E h_E \Big|\underline{\underline{\mathbf{q}}}_h\Big|_{E_+} + \underline{\underline{\mathbf{q}}}_h|_{E_-}|^q \,\mathrm{d}s\right)^{1/q}$$
$$+ \sum_{E\in\mathcal{E}_h(\Gamma)} \left(\int_E h_E^{-p(q+1)/q} |u_h - u_D|^p \,\mathrm{d}s\right)^{1/p} \left(\int_E h_E^{q+1} \Big|\nabla\cdot \underline{\underline{\mathbf{q}}}_h\Big|^q \,\mathrm{d}s\right)^{1/q}. \tag{3.19}$$

Now, using (2.2), (3.1), the trace inequality (3.3) as well as the Cauchy–Schwarz inequality, the first term on the right-hand side of (3.19) can be bounded from above by

$$\begin{aligned}
&2\sum_{E\in\mathcal{E}_h(\Omega)} \left(\int_E h_E^{-p/q} |[\nabla u_h \otimes \mathbf{n}_E]_E|^p \,\mathrm{d}s\right)^{1/p} \left(\int_E h_E \left(\Big|\underline{\underline{\mathbf{q}}}_h\Big|_{E_+}|^q + \Big|\underline{\underline{\mathbf{q}}}_h\Big|_{E_-}|^q\right) \mathrm{d}s\right)^{1/q} \\
&\quad\leq 2C_R^{1/q} \left(\sum_{E\in\mathcal{E}_h(\Omega)} \int_E h_E^{-p/q} |[\nabla u_h \otimes \mathbf{n}_E]_E|^p \,\mathrm{d}s\right)^{1/p} \left(\sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K \,\Big|\underline{\underline{\mathbf{q}}}_h\Big|_{\partial K}|^q \,\mathrm{d}s\right)^{1/q} \\
&\quad\leq 2C_R^{1/q} C_T \left(\sum_{E\in\mathcal{E}_h(\Omega)} \int_E h_E^{-p/q} |[\nabla u_h \otimes \mathbf{n}_E]_E|^p \,\mathrm{d}s\right)^{1/p} \Big\|\underline{\underline{\mathbf{q}}}_h\Big\|_{L^q(\Omega;\mathbb{R}^{2\times 2})}.
\end{aligned} \tag{3.20}$$

Likewise, using additionally the inverse inequality (3.2), for the second term on the right-hand side of (3.19) we obtain the upper bound

$$\begin{aligned}
&C_R^{1+1/q} \left(\sum_{E\in\mathcal{E}_h(\Gamma)} \int_E h_E^{-p(q+1)/q} |u_h - u_D|^p \,\mathrm{d}s\right)^{1/p} \left(\sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^{1+q} \,\Big|\nabla\cdot\underline{\underline{\mathbf{q}}}_h\Big|_{\partial K}|^q \,\mathrm{d}s\right)^{1/q} \\
&\quad\leq C_R^{1+1/q} C_T \left(\sum_{E\in\mathcal{E}_h(\Gamma)} \int_E h_E^{-p(q+1)/q} |u_h - u_D|^p \,\mathrm{d}s\right)^{1/p} \left(\sum_{K\in\mathcal{T}_h} \int_K h_K^q \,\Big|\nabla\cdot\underline{\underline{\mathbf{q}}}_h\Big|_K|^q \,\mathrm{d}s\right)^{1/q} \\
&\quad\leq C_R^{1+1/q} C_{\mathrm{inv}} C_T \left(\sum_{E\in\mathcal{E}_h(\Gamma)} \int_E h_E^{-p(q+1)/q} |u_h - u_D|^p \,\mathrm{d}s\right)^{1/p} \left(\sum_{K\in\mathcal{T}_h} \int_K \Big|\underline{\underline{\mathbf{q}}}_h\Big|_K|^q \,\mathrm{d}s\right)^{1/q} \\
&\quad= C_R^{1+1/q} C_{\mathrm{inv}} C_T \left(\sum_{E\in\mathcal{E}_h(\Gamma)} \int_E h_E^{-p(q+1)/q} |u_h - u_D|^p \,\mathrm{d}s\right)^{1/p} \Big\|\underline{\underline{\mathbf{q}}}_h\Big\|_{L^q(\Omega;\mathbb{R}^{2\times 2})}.
\end{aligned} \tag{3.21}$$

Using (3.20) and (3.21) in (3.19) gives (3.16a). The assertion (3.16b) can be shown in a similar way. □

We denote by Π_k the orthogonal L^2 projection of $L^2(\Omega)$ onto V_h, which can be defined elementwise by

$$\begin{aligned}
\int_\Omega \Pi_k(v) v_h \,\mathrm{d}x &= \sum_{K\in\mathcal{T}_h} \int_K \Pi_{K,k}(v) v_h \,\mathrm{d}x, \quad v\in L^2(\Omega), \\
\int_K \Pi_{K,k}(v) p_k \,\mathrm{d}x &= \int_K v p_k \,\mathrm{d}x, \qquad\qquad p_k \in P_k(K), K\in\mathcal{T}_h.
\end{aligned} \tag{3.22}$$

We note that Π_k can be extended to $L^p(\Omega)$ for $p\in[1,2)$ and $p\in[2,\infty]$ (*cf.*, *e.g.*, [20]).

We further denote by $\underline{\underline{\mathbf{\Pi}}}_k$ and $\underline{\mathbf{\Pi}}_k$ the L^2 projections of $L^2(\Omega;\mathbb{R}^{2\times 2})$ onto $\underline{\underline{\mathbf{V}}}_h$ and of $L^2(\Omega;\mathbb{R}^2)$ onto $\underline{\mathbf{V}}_h$ which can also be defined elementwise similar to (3.22) involving $\underline{\underline{\mathbf{\Pi}}}_{K,k}$ and $\underline{\mathbf{\Pi}}_{K,k}$, $K \in \mathcal{T}_h$. The

L^2 projections of $L^2(\Gamma)$ onto $\{v_h \in L^2(\Gamma) \mid v_h|_E \in P_k(E), E \in \mathcal{E}_h(\Gamma)\}$ and of $L^2(\Gamma;\mathbb{R}^{2\times 2})$ onto $\Big\{\underline{\underline{\mathbf{q}}}_h \in L^2(\Gamma;\mathbb{R}^{2\times 2}) \mid \underline{\underline{\mathbf{q}}}_h|_E \in P_k(E)^{2\times 2}, E \in \mathcal{E}_h(\Gamma)\Big\}$, will be denoted by $\Pi_{\Gamma,k}$ and $\underline{\underline{\mathbf{\Pi}}}_{\Gamma,k}$, respectively.

We define the discrete data

$$f_h := \Pi_{k-2} f, \quad u_{h,D} := \Pi_{\Gamma,k-1} u_D, \quad u_{h,N} := \Pi_{\Gamma,k} u_N. \tag{3.23}$$

We consider the discrete minimization problem

$$J_{h,P}(u_h) = \inf_{v_h \in V_h} J_{h,P}(v_h), \tag{3.24a}$$

where the objective functional $J_{h,P}$ is given by

$$\begin{aligned} J_{h,P}(v_h) := {}& \frac{1}{p} \sum_{K\in\mathcal{T}_h} \int_K |D^2_{\mathrm{DG},2} v_h|^p \,\mathrm{d}x - \int_\Omega f_h v_h \,\mathrm{d}x - \int_\Gamma u_{h,N}\ \mathbf{n}_\Gamma \cdot \nabla v_h \,\mathrm{d}s \\ &+ \frac{\alpha_1}{p} \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q} \int_E |[\nabla v_h \otimes \mathbf{n}_E]_E|^p \,\mathrm{d}s + \frac{\alpha_2}{p} \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-p(q+1)/q} \int_E |[v_h - u_D]_E|^p \,\mathrm{d}s \end{aligned} \tag{3.24b}$$

and $\alpha_i > 0, 1 \le i \le 2$, are penalization parameters. The existence of a solution of (3.24) follows by standard arguments from the calculus of variations. The necessary and sufficient optimality condition gives rise to a discrete variational equation which represents the C0IPDG approximation of the nonlinear fourth order elliptic boundary value problem (2.3a)–(2.3c).

Find $u_h \in V_h$ such that for all $v_h \in V_h$ it holds

$$a_h^{\mathrm{DG}}(u_h, v_h) = \ell_h(v_h), \tag{3.25}$$

where, observing $\underline{\underline{\mathbf{\Pi}}}_k\big(D^2_{\mathrm{DG},1} v_h\big) = D^2_{\mathrm{DG},1} v_h$, the semilinear C0IPDG form $a_h^{\mathrm{DG}}(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ is given by

$$\begin{aligned} a_h^{\mathrm{DG}}(u_h, v_h) := {}& \sum_{K\in\mathcal{T}_h} \int_K \big|D^2_{\mathrm{DG},2} u_h\big|^{p-2} D^2_{\mathrm{DG},2} u_h : \underline{\underline{\mathbf{\Pi}}}_k\big(D^2_{\mathrm{DG},1} v_h\big) \,\mathrm{d}x \\ &+ \alpha_1 \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q} \int_E |[\nabla u_h \otimes \mathbf{n}_E]_E|^{p-2} [\nabla u_h \otimes \mathbf{n}_E]_E : [\nabla v_h \otimes \mathbf{n}_E]_E \,\mathrm{d}s \\ &+ \alpha_2 \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-p(q+1)/q} \int_E |u_h - u_D|^{p-2} (u_h - u_D)\ v_h \,\mathrm{d}s \\ = {}& \sum_{K\in\mathcal{T}_h} \int_K \underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2_{\mathrm{DG},2} u_h\big|^{p-2} D^2_{\mathrm{DG},2} u_h\Big) : D^2_{\mathrm{DG},1} v_h \,\mathrm{d}x \\ &+ \alpha_1 \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q} \int_E |[\nabla u_h \otimes \mathbf{n}_E]_E|^{p-2} [\nabla u_h \otimes \mathbf{n}_E]_E : [\nabla v_h \otimes \mathbf{n}_E]_E \,\mathrm{d}s \\ &+ \alpha_2 \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-p(q+1)/q} \int_E |u_h - u_D|^{p-2} (u_h - u_D)\ v_h \,\mathrm{d}s, \end{aligned} \tag{3.26}$$

and $\ell_h(\cdot) : V_h \to \mathbb{R}$ stands for the linear functional

$$\ell_h(v_h) := \sum_{K\in\mathcal{T}_h} \int_K f_h v_h \,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E u_{h,N}\ \mathbf{n}_E \cdot \nabla v_h \,\mathrm{d}s. \tag{3.27}$$

Lemma 3.3. *The C0IPDG approximation (3.25) is consistent with (2.3a)–(2.3c) in the sense that if $f = f_h$ and $u_N = u_{h,N}$ and u satisfies (2.3a)–(2.3c) pointwise almost everywhere, then for all $v_h \in \{v_h \in V_h \mid v_h|_E = 0,\ E \in \mathcal{E}_h(\Gamma)\}$ it holds*

$$a_h^{\mathrm{DG}}(u, v_h) = \ell_h(v_h). \tag{3.28}$$

Proof. Observing $[u]_E = [u_D^*]_E = 0, E \in \mathcal{E}_h(\Omega)$, and $(u - u_D^*)|_E = (u - u_D)|_E = 0, E \in \mathcal{E}_h(\Gamma)$, which implies $\underline{\mathbf{R}}_{h,1}(u) + \underline{\mathbf{R}}_{h,2}(u_D^*) = 0$, and observing further that $\underline{\underline{\mathbf{\Pi}}}_k(D^2 v_h) = D^2 v_h$ and hence

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \int_K \underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2u\big|^{p-2} D^2u\Big) : D^2 v_h \,\mathrm{d}x &= \sum_{K\in\mathcal{T}_h} \int_K \big|D^2u\big|^{p-2} D^2u : \underline{\underline{\mathbf{\Pi}}}_k\big(D^2 v_h\big) \,\mathrm{d}x \\ &= \sum_{K\in\mathcal{T}_h} \int_K \big|D^2u\big|^{p-2} D^2u : D^2 v_h \,\mathrm{d}x,\end{aligned}$$

it follows that

$$\begin{aligned} a_h^{\mathrm{DG}}(u, v_h) &= \sum_{K\in\mathcal{T}_h} \int_K \underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2u\big|^{p-2} D^2u\Big) : \big(D^2 v_h - \underline{\mathbf{R}}_{h,1}(v_h)\big) \,\mathrm{d}x \\ &= \sum_{K\in\mathcal{T}_h} \int_K \underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2u\big|^{p-2} D^2u\Big) : D^2 v_h \,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Omega)} \int_E \Big\{\underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2u\big|^{p-2} D^2u\Big)\Big\}_E : [\nabla v_h \otimes \mathbf{n}_E]_E \,\mathrm{d}s \\ &\quad + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E \mathbf{n}_E \cdot \Big(\nabla \cdot \underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2u\big|^{p-2} D^2u\Big)\Big)\, v_h \,\mathrm{d}s \\ &= \sum_{K\in\mathcal{T}_h} \int_K \big|D^2u\big|^{p-2} D^2u : D^2 v_h \,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Omega)} \int_E \Big\{\underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2u\big|^{p-2} D^2u\Big)\Big\}_E : [\nabla v_h \otimes \mathbf{n}_E]_E \,\mathrm{d}s \\ &\quad + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E \mathbf{n}_E \cdot \Big(\nabla \cdot \underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2u\big|^{p-2} D^2u\Big)\Big)\, v_h \,\mathrm{d}s. \end{aligned} \tag{3.29}$$

Applying Green's formula twice and taking $\underline{\underline{\mathbf{\Pi}}}_k((\nabla v_h \otimes \mathbf{n}_{\partial K})|_E) = (\nabla v_h \otimes \mathbf{n}_{\partial K})|_E$ and thus

$$\begin{aligned}\int_{\partial K} \big|D^2u\big|^{p-2} D^2u : (\nabla v_h \otimes \mathbf{n}_{\partial K})|_E \,\mathrm{d}s &= \int_{\partial K} \big|D^2u\big|^{p-2} D^2u : \underline{\underline{\mathbf{\Pi}}}_k((\nabla v_h \otimes \mathbf{n}_{\partial K})|_E) \,\mathrm{d}s \\ &= \int_{\partial K} \underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2u\big|^{p-2} D^2u\Big) : (\nabla v_h \otimes \mathbf{n}_{\partial K})|_E \,\mathrm{d}s\end{aligned}$$

as well as $\Pi_k(v_h) = v_h$ into account gives

$$\begin{aligned}\int_K \big|D^2u\big|^{p-2} D^2u : D^2 v_h \,\mathrm{d}x &= -\int_K \nabla \cdot \Big(\big|D^2u\big|^{p-2} D^2u\Big) \cdot \nabla v_h \,\mathrm{d}x + \int_{\partial K} \big|D^2u\big|^{p-2} D^2u : (\nabla v_h \otimes \mathbf{n}_{\partial K}) \,\mathrm{d}s \\ &= -\int_K \nabla \cdot \Big(\big|D^2u\big|^{p-2} D^2u\Big) \cdot \nabla v_h \,\mathrm{d}x + \int_{\partial K} \big|D^2u\big|^{p-2} D^2u : \underline{\underline{\mathbf{\Pi}}}_k(\nabla v_h \otimes \mathbf{n}_{\partial K}) \,\mathrm{d}s \\ &= -\int_K \nabla \cdot \Big(\big|D^2u\big|^{p-2} D^2u\Big) \cdot \nabla \Pi_k(v_h) \,\mathrm{d}x + \int_{\partial K} \big|D^2u\big|^{p-2} D^2u : \underline{\underline{\mathbf{\Pi}}}_k(\nabla v_h \otimes \mathbf{n}_{\partial K}) \,\mathrm{d}s \\ &= \int_K \nabla \cdot \nabla \cdot \Big(\big|D^2u\big|^{p-2} D^2u\Big) \Pi_k(v_h) \,\mathrm{d}x + \int_{\partial K} \underline{\underline{\mathbf{\Pi}}}_k\Big(\big|D^2u\big|^{p-2} D^2u\Big) : (\nabla v_h \otimes \mathbf{n}_{\partial K}) \,\mathrm{d}s \\ &\quad - \int_{\partial K} \mathbf{n}_{\partial K} \cdot \Big(\nabla \cdot \Big(\big|D^2u\big|^{p-2} D^2u\Big)\, \Pi_k(v_h)\Big) \,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \int_K \nabla\cdot\nabla\cdot\Big(\big|D^2u\big|^{p-2}D^2u\Big)\ v_h\,\mathrm{d}x + \int_{\partial K}\underline{\underline{\boldsymbol{\Pi}}}_k\Big(\big|D^2u\big|^{p-2}D^2u\Big) : (\nabla v_h\otimes \mathbf{n}_{\partial K})\,\mathrm{d}s\\
&\quad - \int_{\partial K}\Pi_k\Big(\mathbf{n}_{\partial K}\cdot\nabla\cdot\Big(\big|D^2u\big|^{p-2}D^2u\Big)\Big)\ v_h\,\mathrm{d}s.
\end{aligned}
$$

Summing over all $K\in\mathcal{T}_h$ and observing (3.4) as well as $[v_h]_E=0,\ E\in\mathcal{E}_h(\Omega)$ (due to the continuity of v_h across interior edges), and $v_h|_E=0,\ E\in\mathcal{E}_h(\Gamma)$, yields

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h}\int_K\big|D^2u\big|^{p-2}D^2u : D^2v_h\,\mathrm{d}x &= \sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\nabla\cdot\Big(\big|D^2u\big|^{p-2}D^2u\Big)v_h\,\mathrm{d}x\\
&\quad + \sum_{E\in\mathcal{E}_h(\bar\Omega)}\int_E\Big\{\underline{\underline{\boldsymbol{\Pi}}}_k\Big(\big|D^2u\big|^{p-2}D^2u\Big)\Big\}_E : [\nabla v_h\otimes\mathbf{n}_E]_E\,\mathrm{d}s\\
&\quad - \sum_{E\in\mathcal{E}_h(\Gamma)}\int_E\Pi_k\Big(\mathbf{n}_E\cdot\nabla\cdot\Big(\big|D^2u\big|^{p-2}D^2u\Big)\Big)v_h\,\mathrm{d}s\\
&= \sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\nabla\cdot\Big(\big|D^2u\big|^{p-2}D^2u\Big)v_h\,\mathrm{d}x\\
&\quad + \sum_{E\in\mathcal{E}_h(\Omega)}\int_E\Big\{\underline{\underline{\boldsymbol{\Pi}}}_k\Big(\big|D^2u\big|^{p-2}D^2u\Big)\Big\}_E : [\nabla v_h\otimes\mathbf{n}_E]_E\,\mathrm{d}s. \qquad (3.30)
\end{aligned}
$$

Using (3.30) in (3.29) and observing (2.3a)–(2.3c) results in

$$
\begin{aligned}
a_h^{\mathrm{DG}}(u,v_h) &= \sum_{K\in\mathcal{T}_h}\int_K\nabla\cdot\nabla\cdot\Big(\big|D^2u\big|^{p-2}D^2u\Big)\ v_h\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma)}\int_E\big|D^2u\big|^{p-2}D^2u\ \mathbf{n}_E\cdot\nabla v_h\,\mathrm{d}s\\
&= \int_\Omega f_h v_h\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma)}\int_E u_{h,N}\ \mathbf{n}_E\cdot\nabla v_h\,\mathrm{d}s,
\end{aligned}
$$

which is the assertion. □

Remark 3.4. We note that $u_h\notin W^{2,p}(\Omega)$, but a conforming finite element function $u_h^c\in V_h^c := V_h\cap W^{2,p}(\Omega)$ can be obtained from $u_h\in V_h$ by postprocessing. In particular, let V_h^c be the generalized version of the Hsieh-Clough-Tocher C ^{1} conforming finite element space as constructed in [27] and let $u_h^c = E_h(u_h)$ be the extension of u_h to V_h^c as constructed in [31]. By a generalization of a result from [31] to the case $p\neq 2$ there exists a constant $C_c>0$, depending only on the local geometry of the triangulation and on the penalty parameters $\alpha_i, 1\le i\le 2$, such that

$$
\|u_h-u_h^c\|^p_{W^{2,p}(\Omega;\mathcal{T}_h)} \le C_c\Bigg(\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q}\int_E|[\nabla u_h\otimes\mathbf{n}_E]_E|^p\,\mathrm{d}s + \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-p(q+1)/q}\int_E|u_h-u_{h,D}|^p\,\mathrm{d}s\Bigg). \qquad (3.31)
$$

Observing (3.14a) and (3.12a), for the first term on the right-hand side of (3.26) we find

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K\underline{\underline{\boldsymbol{\Pi}}}_k\Big(\big|D^2_{\mathrm{DG},2}u_h\big|^{p-2}D^2_{\mathrm{DG},2}u_h\Big) : D^2_{\mathrm{DG},1}v_h\,\mathrm{d}x\\
&= \sum_{K\in\mathcal{T}_h}\int_K\underline{\underline{\boldsymbol{\Pi}}}_k\Big(\big|D^2_{\mathrm{DG},2}u_h\big|^{p-2}D^2_{\mathrm{DG},2}u_h\Big) : D^2v_h\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K\underline{\underline{\boldsymbol{\Pi}}}_k\Big(\big|D^2_{\mathrm{DG},2}u_h\big|^{p-2}D^2_{\mathrm{DG},2}u_h\Big) : \underline{\underline{\mathbf{R}}}_{h,1}(v_h)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{T}_h}\int_K \left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h : \underline{\underline{\mathbf{\Pi}}}_k(D^2 v_h)\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{\Pi}}}_k\Big(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h\Big) : \underline{\underline{\mathbf{R}}}_{h,1}(v_h)\,\mathrm{d}x \\
&= \sum_{K\in\mathcal{T}_h}\int_K \left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h : D^2 v_h\,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Omega)}\int_E [\nabla v_h\otimes\mathbf{n}_E]_E : \Big\{\underline{\underline{\mathbf{\Pi}}}_k\Big(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h\Big)\Big\}_E\,\mathrm{d}s \\
&\quad + \sum_{E\in\mathcal{E}_h(\Gamma)} v_h\;\mathbf{n}_E\cdot\nabla\cdot\underline{\underline{\mathbf{\Pi}}}_k\Big(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h\Big)\,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}_h}\int_K \left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h : D^2 v_h\,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Omega)}\int_E [\nabla v_h\otimes\mathbf{n}_E]_E : \Big\{\underline{\underline{\mathbf{\Pi}}}_k\Big(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h\Big)\Big\}_E\,\mathrm{d}s \\
&\quad + \sum_{E\in\mathcal{E}_h(\Gamma)} v_h\;\mathbf{n}_E\cdot\nabla\cdot\underline{\underline{\mathbf{\Pi}}}_k\Big(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h\Big)\,\mathrm{d}s,
\end{aligned}
$$

and hence, we obtain

$$
\begin{aligned}
a_h^{\mathrm{DG}}(u_h,v_h) &= \sum_{K\in\mathcal{T}_h}\int_K \left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h : D^2 v_h\,\mathrm{d}x \\
&\quad - \sum_{E\in\mathcal{E}_h(\Omega)}\int_E [\nabla v_h\otimes\mathbf{n}_E]_E : \Big\{\underline{\underline{\mathbf{\Pi}}}_k\Big(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h\Big)\Big\}_E\,\mathrm{d}s \\
&\quad + \sum_{E\in\mathcal{E}_h(\Gamma)} v_h\mathbf{n}_E\cdot\nabla\cdot\underline{\underline{\mathbf{\Pi}}}_k\Big(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h\Big)\,\mathrm{d}s \\
&\quad + \alpha_1\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q}\int_E |[\nabla u_h\otimes\mathbf{n}_E]_E|^{p-2}[\nabla u_h\otimes\mathbf{n}_E]_E : [\nabla v_h\otimes\mathbf{n}_E]_E\,\mathrm{d}s \\
&\quad + \alpha_2\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-p(q+1)/q}\int_E |u_h-u_D|^{p-2}(u_h-u_D)\;v_h\,\mathrm{d}s. \qquad (3.32)
\end{aligned}
$$

We consider a three-field formulation of the C0IPDG approximation (3.25) which will be tantamount for the construction of an equilibrated flux in Section 5. To this end, we set

$$\underline{\underline{\mathbf{p}}}_h = \underline{\underline{\mathbf{\Pi}}}_k\Big(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h\Big), \qquad (3.33a)$$

$$\underline{\boldsymbol{\psi}}_h = \nabla_h\cdot\underline{\underline{\mathbf{p}}}_h, \qquad (3.33b)$$

$$\nabla_h\cdot\underline{\boldsymbol{\psi}}_h = f_h. \qquad (3.33c)$$

We consider (3.33a)–(3.33c) elementwise for each $K\in\mathcal{T}_h$, multiply (3.33a) by $\underline{\underline{\mathbf{q}}}_h\in\underline{\underline{\mathbf{V}}}_h$, equation (3.33b) by $\underline{\boldsymbol{\varphi}}_h\in\underline{\mathbf{V}}_h$, equation (3.33c) by $v_h\in V_h$, integrate and sum over all $K\in\mathcal{T}_h$, and apply Green's formula in case of (3.33b) and (3.33c). It follows that

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_h : \underline{\underline{\mathbf{q}}}_h\,\mathrm{d}x &= \sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{\Pi}}}_k\Big(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h\Big) : \underline{\underline{\mathbf{q}}}_h\,\mathrm{d}x \\
&= \sum_{K\in\mathcal{T}_h}\int_K \left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h : \underline{\underline{\mathbf{\Pi}}}_k\Big(\underline{\underline{\mathbf{q}}}_h\Big)\,\mathrm{d}x \\
&= \sum_{K\in\mathcal{T}_h}\int_K \left|D^2_{\mathrm{DG},2}u_h\right|^{p-2} D^2_{\mathrm{DG},2}u_h : \underline{\underline{\mathbf{q}}}_h\,\mathrm{d}x, \qquad (3.34a)
\end{aligned}
$$

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\boldsymbol{\psi}}_h\cdot\underline{\boldsymbol{\varphi}}_h\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\underline{\mathbf{p}}}_h\cdot\underline{\boldsymbol{\varphi}}_h\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_h : \nabla\underline{\boldsymbol{\varphi}}_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\underline{\underline{\mathbf{p}}}_h\;\mathbf{n}_{\partial K}\cdot\underline{\boldsymbol{\varphi}}_h\,\mathrm{d}s, \qquad (3.34b)$$

$$\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\boldsymbol{\psi}}_h\ v_h\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\boldsymbol{\psi}}_h\cdot\nabla v_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbf{n}_{\partial k}\cdot\underline{\boldsymbol{\psi}}_h\ v_h\,\mathrm{d}s = \sum_{K\in\mathcal{T}_h}\int_K f_h\ v_h\,\mathrm{d}x. \tag{3.34c}$$

We replace $\underline{\underline{\mathbf{p}}}_h|_{\partial K}$ in (3.34b) by $\underline{\hat{\mathbf{p}}}_{\partial K}$ and $\mathbf{n}_{\partial K}\cdot\underline{\boldsymbol{\psi}}_h$ in (3.34c) by $\mathbf{n}_{\partial K}\cdot\underline{\hat{\boldsymbol{\psi}}}_{\partial K}$, where $\underline{\hat{\mathbf{p}}}_{\partial K}$ and $\underline{\hat{\boldsymbol{\psi}}}_{\partial K}$ are numerical flux functions. We thus obtain the following system of discrete variational equations.

Find $\left(\underline{\underline{\mathbf{p}}}_h,\underline{\boldsymbol{\psi}}_h,u_h\right)\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h$ such that for all $\left(\underline{\underline{\mathbf{q}}}_h,\underline{\boldsymbol{\varphi}}_h,v_h\right)\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h$ it holds

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_h:\underline{\underline{\mathbf{q}}}_h\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h}\int_K \left|D^2_{\mathrm{DG},2}u_h\right|^{p-2}D^2_{\mathrm{DG},2}u_h:\underline{\underline{\mathbf{q}}}_h\,\mathrm{d}x, \tag{3.35a}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\boldsymbol{\psi}}_h\cdot\underline{\boldsymbol{\varphi}}_h\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_h:\nabla\underline{\boldsymbol{\varphi}}_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\underline{\hat{\mathbf{p}}}_{\partial K}\cdot\underline{\boldsymbol{\varphi}}_h\,\mathrm{d}s, \tag{3.35b}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\boldsymbol{\psi}}_h\ v_h\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\boldsymbol{\psi}}_h\cdot\nabla v_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbf{n}_{\partial K}\cdot\underline{\hat{\boldsymbol{\psi}}}_{\partial K}\ v_h\,\mathrm{d}s = \sum_{K\in\mathcal{T}_h}\int_K f_h\ v_h\,\mathrm{d}x. \tag{3.35c}$$

In particular, for the three-field formulation of the C0IPDG approximation (3.25) the numerical flux function $\underline{\hat{\mathbf{p}}}_{\partial K}$ and $\underline{\hat{\boldsymbol{\psi}}}_{\partial K}$ are chosen as follows:

$$\underline{\hat{\mathbf{p}}}_{\partial K}|_E := \begin{cases}\left(\left\{\underline{\underline{\boldsymbol{\Pi}}}_k(\underline{\underline{\mathbf{z}}}_h)\right\}_E - \alpha_1 h_E^{-p/q}\{\underline{\underline{\mathbf{w}}}_h\}_E\right)\mathbf{n}_E & E\in\mathcal{E}_h(\Omega)\\ u_{h,N}\ \mathbf{n}_E & E\in\mathcal{E}_h(\Gamma)\end{cases}, \tag{3.36a}$$

$$\underline{\hat{\boldsymbol{\psi}}}_{\partial K}|_E := \begin{cases}\mathbf{0} & E\in\mathcal{E}_h(\Omega)\\ \nabla\cdot\underline{\underline{\boldsymbol{\Pi}}}_k(\underline{\underline{\mathbf{z}}}_h) + \alpha_2\ h_E^{-p(q+1)/q}z_h\ \mathbf{n}_E & E\in\mathcal{E}_h(\Gamma)\end{cases}, \tag{3.36b}$$

where $\underline{\underline{\mathbf{z}}}_h := \left|D^2_{\mathrm{DG},2}u_h\right|^{p-2}D^2_{\mathrm{DG},2}u_h$, $\underline{\underline{\mathbf{w}}}_h := |\nabla u_h\otimes\mathbf{n}_E|^{p-2}(\nabla u_h\otimes\mathbf{n}_E)$, and $z_h := |u_h-u_D|^{p-2}(u_h-u_D)$.

Theorem 3.5. *The three-field formulation (3.35) with the numerical flux functions given by (3.36) is equivalent with (3.25). In particular, if $u_h\in V_h$ is the solution of (3.25), there exists a pair $(\underline{\underline{\mathbf{p}}}_h,\underline{\boldsymbol{\psi}}_h)\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h$ such that the triple $\left(\underline{\underline{\mathbf{p}}}_h,\underline{\boldsymbol{\psi}}_h,u_h\right)\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h$ satisfies (3.35). Conversely, if the triple $\left(\underline{\underline{\mathbf{p}}}_h,\underline{\boldsymbol{\psi}}_h,u_h\right)\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h$ satisfies (3.35), then $u_h\in V_h$ solves (3.25).*

Proof. Let $u_h\in V_h$ be a solution of (3.25). We then define $\underline{\underline{\mathbf{p}}}_h\in\underline{\underline{\mathbf{V}}}_h$ by means of (3.35a) and afterwards $\underline{\boldsymbol{\psi}}_h\in\underline{\mathbf{V}}_h$ according to (3.35b). We choose $\underline{\underline{\mathbf{q}}}_h = D^2v_h$ in (3.35a) and $\underline{\boldsymbol{\varphi}}_h=\nabla v_h$ in (3.35b) and insert the resulting expressions into (3.35c) observing (3.36). It follows that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\boldsymbol{\psi}}_h\ v_h\,\mathrm{d}x &= \sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_h:D^2v_h\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\underline{\hat{\mathbf{p}}}_{\partial K}\cdot\nabla v_h\,\mathrm{d}s + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbf{n}_{\partial K}\cdot\underline{\hat{\boldsymbol{\psi}}}_{\partial K}\ v_h\,\mathrm{d}s\\ &= \sum_{K\in\mathcal{T}_h}\int_K \left|D^2_{\mathrm{DG},2}u_h\right|^{p-2}D^2_{\mathrm{DG},2}u_h:D^2v_h\,\mathrm{d}x\\ &\quad - \sum_{E\in\mathcal{E}_h(\Omega)}\int_E \left\{\underline{\underline{\boldsymbol{\Pi}}}_k\left(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2}D^2_{\mathrm{DG},2}u_h\right)\right\}_E:[\nabla v_h\otimes\mathbf{n}_E]_E\,\mathrm{d}s\\ &\quad + \sum_{E\in\mathcal{E}_h(\Gamma)}\int_E \mathbf{n}_E\cdot\nabla\cdot\underline{\underline{\boldsymbol{\Pi}}}_k\left(\left|D^2_{\mathrm{DG},2}u_h\right|^{p-2}D^2_{\mathrm{DG},2}u_h\right)\ v_h\,\mathrm{d}s\\ &\quad + \alpha_1\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q}\int_E |[\nabla u_h\otimes\mathbf{n}_E]_E|^{p-2}[\nabla u_h\otimes\mathbf{n}_E]_E:[\nabla v_h\otimes\mathbf{n}_E]_E\,\mathrm{d}s\end{aligned}$$

$$+\,\alpha_2 \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-p\frac{q+1}{q}} \int_E |u_h-u_D|^{p-2}(u_h-u_D)\, v_h\,\mathrm{d}s - \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E u_{h,N}\, v_h\,\mathrm{d}s.$$

In view of (3.25) and (3.32) we deduce that the last equation in (3.35c) is satisfied.

Conversely, if the triple $\left(\underline{\underline{\mathbf{p}}}_h, \underline{\boldsymbol{\psi}}_h, u_h\right) \in \underline{\underline{\mathbf{V}}}_h \times \underline{\mathbf{V}}_h \times V_h$ satisfies (3.35), we choose $\underline{\underline{\mathbf{q}}}_h = D^2 v_h$ in (3.35a) and $\underline{\boldsymbol{\varphi}}_h = \nabla v_h$ in (3.35b) and insert (3.35a) and (3.35b) into (3.35c). Taking (3.36) into account this shows that $u_h \in V_h$ satisfies (3.25). □

4. An *a posteriori* error estimator for the global discretization error

Given reflexive Banach spaces V, Q with norms $\|\cdot\|_V, \|\cdot\|_Q$, convex and coercive objective functionals $C: V \to \mathbb{R}$, $D: Q \to \mathbb{R}$, and a bounded linear operator $\Lambda: V \to Q$, we consider the minimization problem

$$\inf_{u\in V} J(u) \tag{4.1}$$

for the objective functional

$$J(u) := C(u) + D(\Lambda u). \tag{4.2}$$

An abstract approach to the *a posteriori* error control for (4.1) has been provided in [42]. The *a posteriori* error control relies on the dual formulation of (4.1)

$$\sup_{q^*\in Q^*} J^*(q^*) \quad \text{or} \quad \inf_{q^*\in Q^*} (-J^*(q^*)), \tag{4.3}$$

in terms of the Fenchel conjugate J^* of J as given by

$$J^*(q^*) = -C^*(-\Lambda^* q^*) - D^*(q^*), \tag{4.4}$$

where C^* and D^* are the Fenchel conjugates of C and D and Λ^* stands for the adjoint of Λ.

Given some approximation $u_h \in V$ of the minimizer u of (4.1), the *a posteriori* error estimate Theorem 2.2 from [42] (*cf.*, also Sect. 3 in [5] and [43]) states that for any admissible function $q^* \in Q^*$ it holds

$$\Phi_\delta(\Lambda(u_h - u)) \le M_C(\Lambda^* q^*, u_h) + M_D(q^*, \Lambda u_h), \tag{4.5}$$

where $\Phi_\delta : Q \to \mathbb{R}_+$ is a continuous functional such that $\Phi_\delta(0) = 0$ and for all $q_i \in B(0,\delta) := \{q \in Q \mid ||q||_Q < \delta\},\ \delta > 0,\ 1 \le i \le 2$, it holds

$$D((q_1+q_2)/2) + \Phi_\delta(q_2 - q_1) \le (D(q_1) + D(q_2))/2$$

and

$$M_C(\Lambda^* q^*, u_h) := \frac{1}{2}(C(u_h) + C^*(\Lambda^* q^*) - < \Lambda^* q^*, u_h >_{V^*,V}),$$
$$M_D(q^*, \Lambda u_h) := \frac{1}{2}(D(\Lambda v) + D^*(-q^*) - < q^*, \Lambda u_h >_{Q^*,Q}).$$

Referring to D^2 as the Hessian, we apply the above result for $V = W^{2,p}(\Omega), Q := L^q(\Omega;\mathbb{R}^{2\times 2}), \Lambda = D^2$, and

$$C(u_h^c) := -\int_\Omega f u_h^c\,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E u_N \mathbf{n}_E \cdot \nabla u_h^c\,\mathrm{d}s, \tag{4.6a}$$

$$D(D^2 u_h^c) := \frac{1}{p} \sum_{K\in\mathcal{T}_h} \int_K |D^2 u_h^c|^p\,\mathrm{d}x + I_{K_1}(u_h^c), \tag{4.6b}$$

where I_{K_1} is the indicator function of the closed convex set

$$K_1 := \big\{v \in W^{2,p}(\Omega) \mid v = u_D \text{ on } \Gamma\big\}. \tag{4.6c}$$

We obtain:

$$C^*\Big(-\Lambda^* \underline{\underline{\mathbf{q}}}^*\Big) := I_{K_2}\Big(\underline{\underline{\mathbf{q}}}^*\Big), \qquad \underline{\underline{\mathbf{q}}}^* \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2;\Omega), \tag{4.7a}$$

$$D^*\Big(\underline{\underline{\mathbf{q}}}^*\Big) := \frac{1}{q}\int_\Omega |\underline{\underline{\mathbf{q}}}^*|^q \,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E u_D \;\mathbf{n}_E \cdot \nabla \cdot \underline{\underline{\mathbf{q}}}^* \,\mathrm{d}s, \quad \underline{\underline{\mathbf{q}}}^* \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2;\Omega), \tag{4.7b}$$

where I_{K_2} is the indicator function of the closed convex set

$$K_2 := \Big\{\underline{\underline{\mathbf{q}}}^* \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2;\Omega) \mid \nabla\cdot\nabla\cdot \underline{\underline{\mathbf{q}}}^* = f_h \text{ in } \Omega,\ \underline{\underline{\mathbf{q}}}^* \;\mathbf{n}_\Gamma = u_{h,N}\;\mathbf{n}_\Gamma \text{ on } \Gamma\Big\}. \tag{4.7c}$$

Similar to (3.8) in Example 2 (p-Laplace problem) of [42], the estimate (4.5) leads to

$$\|u-u_h\|_V^p \le C_{\mathrm{est}}(C(u_h) + C^*(-\Lambda^* q^*) + D(\Lambda u_h) + D^*(q^*)), \tag{4.8}$$

where $C_{\mathrm{est}} := 2^p p/2$.

We call $\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} \in \underline{\underline{\mathbf{V}}}_h$ an equilibrated moment tensor, if

$$\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} \in \underline{\underline{\mathbf{H}}}\big(\mathrm{div}^2;\Omega\big) \tag{4.9a}$$

and $\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}}$ satisfies the equilibrium conditions

$$\nabla\cdot\nabla\cdot \underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} = f_h \qquad \text{in } \Omega, \tag{4.9b}$$

$$\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}}\;\mathbf{n}_\Gamma = u_{h,N}\;\mathbf{n}_\Gamma \quad \text{on } \Gamma. \tag{4.9c}$$

Moreover, we choose $\underline{\underline{\mathbf{p}}}_c \in \Big\{\underline{\underline{\mathbf{q}}} \in \underline{\underline{\mathbf{H}}}_{0,\Gamma}^{(q)}\big(\mathrm{div}^2,\Omega\big) \mid \underline{\underline{\mathbf{q}}}\;\mathbf{n}_\Gamma \in L^q\big(\Gamma;\mathbb{R}^2\big)\Big\}$ such that

$$\nabla\cdot\nabla\cdot \underline{\underline{\mathbf{p}}}_c = f - f_h, \quad \underline{\underline{\mathbf{p}}}_c\;\mathbf{n}_\Gamma = (u_N - u_{h,N})\;\mathbf{n}_\Gamma \quad \text{on } \Gamma. \tag{4.10}$$

It follows that $\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} + \underline{\underline{\mathbf{p}}}_c \in K_2$, *i.e.*, $I_{K_2}\Big(\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} + \underline{\underline{\mathbf{p}}}_c\Big) = 0$, and hence, equation (4.8) reads as follows:

$$\|u-u_h^c\|_V^p \lesssim J_P(u_h^c) + I_{K_1}(u_h^c) + J_D\Big(\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} + \underline{\underline{\mathbf{p}}}_c\Big). \tag{4.11}$$

In view (2.6c) and $\mathbf{n}_\Gamma\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_c = 0$ on Γ we have

$$J_D\Big(\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} + \underline{\underline{\mathbf{p}}}_c\Big) = \frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K \Big|\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} + \underline{\underline{\mathbf{p}}}_c\Big|^q \,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma)}\int_E u_D\;\mathbf{n}_\Gamma\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}}\,\mathrm{d}s. \tag{4.12}$$

Using (2.2), we find

$$\frac{1}{q}\sum_{K\in\mathcal{T}_h}\int_K \Big|\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} + \underline{\underline{\mathbf{p}}}_c\Big|^q \,\mathrm{d}x \le \frac{1}{q}2^q\Bigg(\sum_{K\in\mathcal{T}_h}\int_K\Big|\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}}\Big|^q\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_K\Big|\underline{\underline{\mathbf{p}}}_c\Big|^q\,\mathrm{d}x\Bigg). \tag{4.13}$$

In order to estimate the second term on the right-hand side of (4.13) we use the Poincaré–Friedrichs inequalities

$$\Big\| v - |K|^{-1} \int_K v \, \mathrm{d}x \Big\|_{L^p(K)} \le C_{\mathrm{PF}}^{(1)} \, h_K \|\nabla v\|_{L^p(K)}, \quad v \in W^{1,p}(K), \; K \in \mathcal{T}_h, \tag{4.14a}$$

$$\Big\| v - |E|^{-1} \int_E v \, \mathrm{d}s \Big\|_{L^p(E)} \le C_{\mathrm{PF}}^{(2)} \, h_E \|\nabla v\|_{L^p(E)}, \quad v \in W^{1,p}(E), \; E \in \mathcal{E}_h(\Gamma), \tag{4.14b}$$

where $C_{\mathrm{PF}}^{(1)}$ depends on p and $|K|$, whereas $C_{\mathrm{PF}}^{(2)}$ depends on p and $|E|$. Both constants are uniformly bounded in p (*cf.*, *e.g.*, [26]).

Lemma 4.1. *Suppose that the following regularity assumption is satisfied: For* $\underline{\underline{\boldsymbol{\tau}}} \in \Big\{ \underline{\underline{\mathbf{q}}} \in \underline{\underline{\mathbf{H}}}_{0,\Gamma}^{(p)}(div, \Omega) \mid \underline{\underline{\boldsymbol{\tau}}} \, \mathbf{n}_\Gamma \in L^p\big(\Gamma; \mathbb{R}^2\big) \Big\}$ *and the weak solution* $z \in W^{2,p}(\Omega)$ *of the elliptic boundary value problem*

$$\begin{aligned} \nabla \cdot \nabla \cdot D^2 z &= \nabla \cdot \nabla \cdot \underline{\underline{\boldsymbol{\tau}}} && \text{in } \Omega, && (4.15a)\\ z &= 0 && \text{on } \Gamma, && (4.15b)\\ D^2 z \, \mathbf{n}_\Gamma &= \underline{\underline{\boldsymbol{\tau}}} \, \mathbf{n}_\Gamma && \text{on } \Gamma && (4.15c) \end{aligned}$$

there exists a constant $C_z^{(1)} > 0$ *such that*

$$D^2 z|_\Gamma \in L^p\big(\Gamma; \mathbb{R}^{2\times 2}\big), \quad \|D^2 z\|_{L^p(\Gamma;\mathbb{R}^{2\times 2})} \le C_z^{(1)}. \tag{4.16}$$

Moreover, there exists a constant $C_z^{(2)} > 0$ *such that*

$$\|\nabla z\|_{L^p(\Omega;\mathbb{R}^2)} \le C_z^{(2)}. \tag{4.17}$$

Then for $\underline{\underline{\mathbf{p}}}_c \in \Big\{ \underline{\underline{\mathbf{q}}} \in \underline{\underline{\mathbf{H}}}_{0,\Gamma}^{(q)}\big(div^2, \Omega\big) \mid \underline{\underline{\mathbf{q}}} \, \mathbf{n}_\Gamma \in L^q\big(\Gamma; \mathbb{R}^2\big) \Big\}$ *as given by* (4.10) *there exists a constant* $C_U > 0$, *depending on* $C_z^{(i)}, C_{\mathrm{PF}}^{(i)}, 1 \le i \le 2$, *and on* p, q *such that it holds*

$$\Big\| \underline{\underline{\mathbf{p}}}_c \Big\|_{L^q(\Omega;\mathbb{R}^{2\times 2})}^q \le C_U (\mathrm{osc}_{h,1} + \mathrm{osc}_{h,2}), \tag{4.18}$$

where $\mathrm{osc}_{h,1}$ *and* $\mathrm{osc}_{h,2}$ *refer to the data oscillations*

$$\mathrm{osc}_{h,1} := \sum_{K \in \mathcal{T}_h} \mathrm{osc}_{K,1}, \quad \mathrm{osc}_{K,1} := \begin{cases} h_K^q \int_K |f - f_h|^q \, \mathrm{d}x & k = 2 \\ h_K^{2q} \int_K |f - f_h|^q \, \mathrm{d}x & k \ge 3 \end{cases}, \tag{4.19a}$$

$$\mathrm{osc}_{h,2} := \sum_{K \in \mathcal{T}_h} \mathrm{osc}_{E,2}, \quad \mathrm{osc}_{E,2} := \sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} h_E^q \int_E |u_N - u_{h,N}|^q \, \mathrm{d}s. \tag{4.19b}$$

Proof. We have

$$\Big\| \underline{\underline{\mathbf{p}}}_c \Big\|_{L^q(\Omega;\mathbb{R}^{2\times 2})} = \sup \Big\{ \int_\Omega \underline{\underline{\mathbf{p}}}_c : \underline{\underline{\boldsymbol{\tau}}} \, \mathrm{d}x \mid \underline{\underline{\boldsymbol{\tau}}} \in \underline{\underline{\mathbf{H}}}_{0,\Gamma_D}^{(p)}\big(\mathrm{div}^2, \Omega\big), \; \big\| \underline{\underline{\boldsymbol{\tau}}} \big\|_{L^p(\Omega;\mathbb{R}^{2\times 2})} \le 1 \Big\}.$$

For $\underline{\underline{\boldsymbol{\tau}}} \in \Big\{ \underline{\underline{\mathbf{q}}} \in \underline{\underline{\mathbf{H}}}_{0,\Gamma}^{(p)}\big(\mathrm{div}^2, \Omega\big) \mid \underline{\underline{\mathbf{q}}} \, \mathbf{n}_\Gamma \in L^p\big(\Gamma; \mathbb{R}^2\big) \Big\}$ there exists $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\underline{\underline{\boldsymbol{\tau}}} = D^2 v$. In fact, v can be chosen as the weak solution of the boundary value problem (4.15). Hence, we have

$$\Big\| \underline{\underline{\mathbf{p}}}_c \Big\|_{L^q(\Omega,\mathbb{R}^{2\times 2})} \le \sup_{\|D^2 v\|_{L^p(\Omega;\mathbb{R}^{2\times 2})} \le 1} \int_\Omega \underline{\underline{\mathbf{p}}}_c : D^2 v \, \mathrm{d}x. \tag{4.20}$$

Applying Green's formula twice locally on each $K \in \mathcal{T}_h$ and observing (4.10), we get

$$\int_\Omega \underline{\underline{\mathbf{p}}}_c : D^2 z \,\mathrm{d}x = \sum_{K\in\mathcal{T}_h} \int_K \nabla\cdot\nabla\cdot \underline{\underline{\mathbf{p}}}_c \; z \,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E \underline{\underline{\mathbf{p}}}_c \;\mathbf{n}_\Gamma \cdot \nabla z \,\mathrm{d}s$$
$$= \sum_{K\in\mathcal{T}_h} \int_K (f - f_h)\; z\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E (u_N - u_{h,N})\; \mathbf{n}_\Gamma \cdot \nabla z \,\mathrm{d}s. \tag{4.21}$$

In order to estimate the first term on the right-hand side of (4.21) we first consider the case $k = 2$. In view of (3.23) we have

$$\sum_{K\in\mathcal{T}_h} \int_K (f - f_h)\; z\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h} \int_K (f - f_h)\; (z - p_0)\,\mathrm{d}x,$$

where $p_0 := |K|^{-1} \int_K z \,\mathrm{d}x$, and hence, an application of (4.14a) and (4.17) yields

$$\left| \sum_{K\in\mathcal{T}_h} \int_K (f - f_h)\; z\,\mathrm{d}x \right| \le \sum_{K\in\mathcal{T}_h} \left(\int_K |f - f_h|^q \,\mathrm{d}x \right)^{1/q} \left(\int_K |z - p_0|^p \,\mathrm{d}x \right)^{1/p}$$
$$\le C_{\mathrm{PF}}^{(1)} \left(\sum_{K\in\mathcal{T}_h} h_K^q \int_K |f - f_h|^q \,\mathrm{d}x \right)^{1/q} \left(\sum_{K\in\mathcal{T}_h} \int_K |\nabla z|^p \,\mathrm{d}x \right)^{1/p} \le C_z^{(2)} C_{\mathrm{PF}}^{(1)} \,\mathrm{osc}_{h,1}^{1/q}. \tag{4.22}$$

In case $k \ge 3$ we have

$$\sum_{K\in\mathcal{T}_h} \int_K (f - f_h)\; z\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h} \int_K (f - f_h)\; (z - p_1)\,\mathrm{d}x, \quad p_1 \in P_1(K).$$

We fix $p_1 \in P_1(K)$ by the interpolation conditions $\int_K p_1 \mathrm{d}x = |K|^{-1} \int_K z\,\mathrm{d}x$ and $\int_K \nabla p_1 \,\mathrm{d}x = |K|^{-1} \int_K \nabla z\,\mathrm{d}x$. An application of (4.14a) gives

$$\left| \sum_{K\in\mathcal{T}_h} \int_K (f - f_h)\; z\,\mathrm{d}x \right| \le \sum_{K\in\mathcal{T}_h} \left(\int_K |f - f_h|^q \,\mathrm{d}x \right)^{1/q} \left(\int_K |z - p_1|^p \,\mathrm{d}x \right)^{1/p}$$
$$\le C_{\mathrm{PF}}^{(1)} \sum_{K\in\mathcal{T}_h} \left(h_K^q \int_K |f - f_h|^q \,\mathrm{d}x \right)^{1/q} \left(\int_K |\nabla(z - p_1)|^p \,\mathrm{d}x \right)^{1/p}. \tag{4.23}$$

Setting $\nabla p_1 = (p_{11}, p_{12})^T$, another application of (4.14a) yields

$$\left\| \frac{\partial z}{\partial x_i} - p_{1i} \right\|_{L^p(K)} \le C_{\mathrm{PF}}^{(1)}\; h_K \left\| \nabla \frac{\partial z}{\partial x_i} \right\|_{L^p(K)}, \; 1 \le i \le 2.$$

Hence, using (2.2), we obtain

$$\left(\int_K |\nabla z - |K|^{-1} \int_K |\nabla(z - p_1)|^p \,\mathrm{d}x \right)^{1/p} \le \; 2\sqrt{2} \left(\left(\int_K \left| \frac{\partial z}{\partial x_1} - p_{11} \right|^p \mathrm{d}x \right)^{1/p} + \left(\int_K \left| \frac{\partial z}{\partial x_2} - p_{12} \right|^p \right) \mathrm{d}x \right)^{1/p}$$
$$\le 2\sqrt{2} C_{\mathrm{PF}}^{(1)} h_K \left(\int_K \left| D^2 z \right|^p \mathrm{d}x \right)^{1/p}. \tag{4.24}$$

Using (4.24) in (4.23) and observing $\|D^2 z\|_{L^p(\Omega;\mathbb{R}^{2\times 2})} \le 1$ it follows that

$$\begin{aligned}\Big|\sum_{K\in\mathcal{T}_h}\int_K (f-f_h)\, z\,\mathrm{d}x\Big| &\le 2\sqrt{2}\Big(C_{\mathrm{PF}}^{(1)}\Big)^2\Big(\sum_{K\in\mathcal{T}_h} h_K^{2q}\int_K |f-f_h|^q\,\mathrm{d}x\Big)^{1/q}\Big(\sum_{K\in\mathcal{T}_h}\int_K |D^2 z|^p\,\mathrm{d}x\Big)^{1/p}\\ &\le 2\sqrt{2}\Big(C_{\mathrm{PF}}^{(1)}\Big)^2\Big(\sum_{K\in\mathcal{T}_h} h_K^{2q}\int_K |f-f_h|^q\,\mathrm{d}x\Big)^{1/q}. \end{aligned}\tag{4.25}$$

Likewise, observing (3.23), (4.16), and choosing $\underline{\mathbf{p}}_2 := |E|^{-1}\int_E \nabla v\,\mathrm{d}s$, the Poincaré–Friedrichs inequality (4.14b) and similar arguments as before imply

$$\begin{aligned}\Big|\sum_{E\in\mathcal{E}_h(\Gamma)}\int_E (u_N-u_{h,N})\;\mathbf{n}_E\cdot\nabla v\,\mathrm{d}s\Big| &= \Big|\sum_{E\in\mathcal{E}_h(\Gamma)}\int_E (u_N-u_{h,N})\;\mathbf{n}_E\cdot(\nabla v-\underline{\mathbf{p}}_2)\,\mathrm{d}s\Big|\\ &\le 2\sqrt{2}C_z^{(1)}C_{\mathrm{PF}}^{(2)}\Big(\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^q\int_E |u_N-u_{h,N}|^q\,\mathrm{d}s\Big)^{1/q}. \end{aligned}\tag{4.26}$$

The assertion now follows from (4.22), (4.25), and (4.26). ∎

Remark 4.2. For $\underline{\underline{\tau}} \in \Big\{\underline{\underline{\mathbf{q}}} \in \underline{\underline{\mathbf{H}}}_{0,\Gamma}^{(p)}(\mathrm{div},\Omega)\mid \underline{\underline{\tau}}\;\mathbf{n}_\Gamma \in L^p\big(\Gamma;\mathbb{R}^2\big)\Big\}$ we have $\nabla\cdot\nabla\cdot\underline{\underline{\tau}} \in L^2(\Omega)$ and hence, we can expect (4.16) and (4.17) to hold true for convex domains by regularity results for the biharmonic equation with Navier boundary conditions.

Moreover, as far as $J_P(u_h^c)$ is concerned, we have

$$\begin{aligned} J_P(u_h^c) = J_P(u_h) &+ \frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K \big(|D^2u_h^c|^p - |D^2u_h|^p\big)\,\mathrm{d}x\\ &+ \sum_{K\in\mathcal{T}_h}\int_K f(u_h-u_h^c)\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma)}\int_E u_N\;\mathbf{n}_E\cdot(u_h-u_h^c)\,\mathrm{d}s. \end{aligned}\tag{4.27}$$

Lemma 4.3. *Let $u_h \in V_h$ be the solution of (3.25) and let $u_h^c \in V_h^c$ be its postprocessed finite element function. Then it holds*

$$\Big|J_P(u_h^c) - J_P(u_h)\Big| \lesssim \sum_{K\in\mathcal{T}_h} \kappa_K^{\mathrm{eq}},\tag{4.28}$$

where

$$\kappa_K^{\mathrm{eq}} := \|u_h-u_h^c\|_{W^{2,p}(K)}^p + \Big(\int_K |D^2u_h|^p\,\mathrm{d}x\Big)^{1/q} + \|f\|_{L^q(K)} + \sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma)}\|u_N\|_{L^q(E)}\|u_h-u_h^c\|_{W^{2,p}(K)}.\tag{4.29}$$

Proof. By Taylor expansion and using (2.2) as well as Hölder's inequality we find

$$\begin{aligned}&\Big|\frac{1}{p}\sum_{K\in\mathcal{T}_h}\int_K\Big(\big|D^2u_h^c\big|^p - \big|D^2u_h\big|^p\Big)\Big|\\ &\quad = \Big|\sum_{K\in\mathcal{T}_h}\int_K\int_0^1 \big|D^2u_h + \lambda D^2(u_h^c-u_h)\big|^{p-2}D^2u_h + \lambda D^2(u_h^c-u_h)\,\mathrm{d}\lambda : D^2(u_h^c-u_h)\,\mathrm{d}x\Big| \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{K\in\mathcal{T}_h} \int_K \int_0^1 \big|D^2u_h + \lambda D^2(u_h^c - u_h)\big|^{p-1}\big|D^2(u_h^c - u_h)\big| \,\mathrm{d}\lambda\,\mathrm{d}x \\
&\le 2^{p-1} \sum_{K\in\mathcal{T}_h} \int_K \int_0^1 |D^2u_h|^{p-1} + \lambda^{p-1}\big|D^2(u_h - u_h^c)\big|^{p-1}\big|D^2(u_h - u_h^c)\big| \,\mathrm{d}\lambda\,\mathrm{d}x \\
&\le 2^{p-1} \sum_{K\in\mathcal{T}_h} \left(\int_K |D^2u_h|^p\,\mathrm{d}x\right)^{1/q} \left(\int_K |D^2(u_h - u_h^c)|^p\,\mathrm{d}x\right)^{1/p} + \frac{1}{p}2^{p-1} \sum_{K\in\mathcal{T}_h} \int_K |D^2(u_h - u_h^c)|^p\,\mathrm{d}x. \quad (4.30)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\left| \sum_{K\in\mathcal{T}_h} \int_K f(u_h - u_h^c)\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E u_N\ \mathbf{n}_E\cdot\nabla(u_h - u_h^c)\,\mathrm{d}s \right| \le \sum_{K\in\mathcal{T}_h} \left(\int_K |f|^q\,\mathrm{d}x\right)^{1/q} \left(\int_K |u_h - u_h^c|^p\,\mathrm{d}x\right)^{1/p} \\
&\qquad + \sum_{E\in\mathcal{E}_h(\Gamma)} \left(\int_E |u_N|^q\,\mathrm{d}s\right)^{1/q} \left(\int_E |\nabla(u_h - u_h^c)|^p\,\mathrm{d}s\right)^{1/p}. \quad (4.31)
\end{aligned}
$$

The assertion now follows from (4.27) and (4.30), (4.31). □

For practical purposes, we further replace $I_{K_1}(u_h^c)$ by the penalty term

$$
\alpha_2 \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-p(q+1)/q} \int_E |u_h^c - u_D|^p\,\mathrm{d}s. \quad (4.32)
$$

In view of the construction of u_h^c we have $u_h^c|_E = u_{h,D}$ on $E \in \mathcal{E}_h(\Gamma)$ and hence, equation (4.32) gives rise to the data oscillation

$$
\alpha_2 \mathrm{osc}_{h,3} := \alpha_2 \sum_{K\in\mathcal{T}_h} \mathrm{osc}_{K,3}, \quad (4.33a)
$$

$$
\mathrm{osc}_{K,3} := \sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma)} h_E^{-p(q+1)/q} \int_E |u_D - u_{h,D}|^p\,\mathrm{d}s. \quad (4.33b)
$$

Using Lemmas 4.1 and 4.3 in (4.11) yields

$$
\|u - u_h\|^2_{W^{2,p}(\Omega;\mathcal{T}_h)} \lesssim \eta_{h,1}^{\mathrm{eq}} + \eta_{h,2}^{\mathrm{eq}}. \quad (4.34a)
$$

Here, $\eta_{h,1}^{\mathrm{eq}}$ and $\eta_{h,2}^{\mathrm{eq}}$ are given by

$$
\eta_{h,1}^{\mathrm{eq}} := \sum_{K\in\mathcal{T}_h} \eta_{K,1}^{\mathrm{eq}}, \quad \eta_{h,2}^{\mathrm{eq}} := \sum_{K\in\mathcal{T}_h} \eta_{K,2}^{\mathrm{eq}}, \quad (4.34b)
$$

where $\eta_{K,i}^{\mathrm{eq}}, 1 \le i \le 2$, read as follows:

$$
\begin{aligned}
\eta_{K,1}^{\mathrm{eq}} := &\frac{1}{p}\int_K |D^2u_h|^p\,\mathrm{d}x - \int_K f u_h\,\mathrm{d}x - \sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma)} \int_E u_N\ \mathbf{n}_E\cdot\nabla u_h\,\mathrm{d}s \\
&+ \frac{1}{q}\int_K \left|\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}}\right|^q\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma)} \int_E u_D\ \mathbf{n}_E\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}}\,\mathrm{d}s, \quad (4.34c)
\end{aligned}
$$

$$
\eta_{K,2}^{\mathrm{eq}} := \|u_h - u_h^c\|^2_{W^{2,p}(K)} + \kappa_K^{\mathrm{eq}} + \sum_{i=1}^{3} \mathrm{osc}_{K,i}. \quad (4.34d)
$$

The right-hand side in (4.34) is then a computable and localizable quantity for the *a posteriori* estimation of the global discretization error. It gives rise to the following equilibrated *a posteriori* error estimator

$$\eta_h^{\rm eq} := \eta_{h,1}^{\rm eq} + \eta_{h,2}^{\rm eq}, \quad \eta_{h,i}^{\rm eq} := \sum_{K\in\mathcal{T}_h(\Omega)} \eta_{K,i}^{\rm eq}, \; 1 \le i \le 2. \tag{4.35}$$

The construction of an equilibrated flux will be dealt with in the subsequent section.

5. Construction of an equilibrated flux

We construct an equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_h^{\rm eq} \in \underline{\underline{\mathbf{V}}}_h \cap \underline{\underline{\mathbf{H}}}(\operatorname{div}^2,\Omega)$ by an interpolation on each element. Thus it is a local procedure. In particular, denoting by $\mathbf{BDM}_k(K), k \in \mathbb{N}$, the Brezzi–Douglas–Marini finite element of order k (*cf.*, *e.g.*, [14]), we first construct an auxiliary vector field $\underline{\boldsymbol{\psi}}_h^{\rm eq} \in \underline{\mathbf{H}}(\operatorname{div},\Omega), \underline{\boldsymbol{\psi}}_h^{\rm eq}|_K \in \mathbf{BDM}_{k-1}(K), K \in \mathcal{T}_h(\Omega)$, satisfying

$$\nabla_h \cdot \underline{\boldsymbol{\psi}}_h^{\rm eq} = f_h, \tag{5.1}$$

and then an equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_h^{\rm eq} \in \underline{\underline{\mathbf{V}}}_h$ satisfying (4.9).

We construct the auxiliary flux function $\underline{\boldsymbol{\psi}}_h^{\rm eq}$ satisfying (4.9) following the procedure suggested in [9] (*cf.*, also [8]).

For a nodal point $x_i \in \mathcal{N}_h(\bar{\Omega}), 1 \le i \le n_h$, we associate a patch ω_i according to

$$\omega_i := \bigcup \{K \in \mathcal{T}_h \mid x_i \in \mathcal{T}_h(K)\}. \tag{5.2}$$

We assume that ω_i consists of N_i triangles $T_\ell, 1 \le \ell \le N_i$. We enumerate the interior edges $E_m, 1 \le m \le M_i$, counterclockwise (*cf.*, Fig. 1).

We construct $\underline{\boldsymbol{\psi}}_h^{\rm eq}$ such that

$$\underline{\boldsymbol{\psi}}_h^{\rm eq} = \sum_{i=1}^{n_h} \underline{\boldsymbol{\psi}}_h^{\omega_i}. \tag{5.3}$$

For the construction of the auxiliary vector field we recall the following result:

Lemma 5.1. *Any vector field* $\underline{\mathbf{q}} \in P_m(K)^2, m \in \mathbb{N}$, *is uniquely defined by the following degrees of freedom*

$$\int_E \mathbf{n}_E \cdot \underline{\mathbf{q}}\; p_m \,\mathrm{d}s, \qquad p_m \in P_m(E), \; E \in \mathcal{E}_h(\partial K), \tag{5.4a}$$

$$\int_K \underline{\mathbf{q}} \cdot \nabla p_{m-1} \,\mathrm{d}x, \qquad p_{m-1} \in P_{m-1}(K), \tag{5.4b}$$

$$\int_K \underline{\mathbf{q}} \cdot \mathbf{curl}(b_K p_{m-2}) \,\mathrm{d}x, \qquad p_{m-2} \in P_{m-2}(K). \tag{5.4c}$$

where b_K *in* (5.4c) *is the element bubble function on* K *given by* $b_K = \prod_{i=1}^3 \lambda_i^K$ *and* $\lambda_i^K, 1 \le i \le 3$, *are the barycentric coordinates of* K. *Moreover, there exists a positive constant* $C_E^{(1)}$, *depending only on* k *and the local geometry of the triangulation* $\mathcal{T}_h$, *such that*

$$\int_K |\underline{\mathbf{q}}|^q \,\mathrm{d}x \le C_E^{(1)} \Bigg(\sum_{E\in\mathcal{E}_h(\partial K)} h_E \int_E |\mathbf{n}_E \cdot \underline{\mathbf{q}}|^q \,\mathrm{d}s + h_K^q \int_K |\nabla \cdot \underline{\mathbf{q}}|^q \,\mathrm{d}x \\ + h_K^q \max\Big\{ \int_K |\underline{\mathbf{q}} \cdot \mathbf{curl}(b_K p_{m-2})|^q \,\mathrm{d}x \mid p_{m-2} \in P_{m-2}(K), \; \max_{x\in K} |p_{m-2}(x)| \le 1 \Big\} \Bigg). \tag{5.5}$$

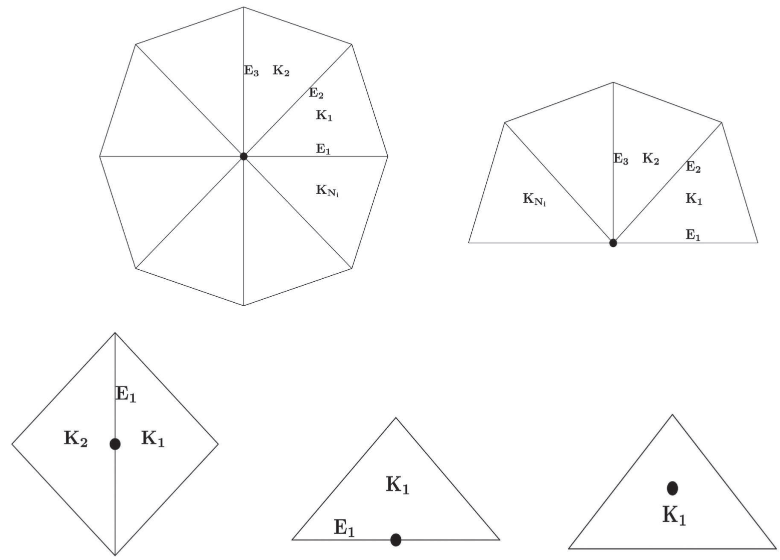

FIGURE 1. Patch ω_i associated with nodal point $x_i \in \mathcal{N}_h(\bar{\Omega})$ featuring N_i triangles $K_\ell, 1 \le \ell \le N_i$ ($x_i \in \mathcal{N}_h(\partial E \cap \Omega)$ (*top left*), $x_i \in \mathcal{N}_h(\partial E \cap \Gamma)$ (*top right*), $x_i \in \mathcal{N}_h(\mathrm{int} E \cap \Omega)$ (*bottom left*), $x_i \in \mathcal{N}_h(\mathrm{int} E \cap \Gamma)$ (*bottom middle*), and $x_i \in \mathcal{N}_h(\mathrm{int} K \cap \Omega)$ (*bottom right*)).

Proof. For the uniqueness result we refer to [14]. The estimate (5.5) can be derived by standard scaling arguments (*cf.*, Lem. 3.1 and Rem. 3.3 in [10] in case $q = 2$). □

For a patch ω_i, we construct $\underline{\boldsymbol{\psi}}_h^{\omega_i}$ such that

$$\begin{aligned} &\underline{\boldsymbol{\psi}}_h^{\omega_i}|_{K_\ell} \in \mathbf{BDM}_k(K_\ell), \\ &\nabla \cdot \underline{\boldsymbol{\psi}}_h^{\omega_i} = f_h^{\omega_i} \text{ in } \omega_i, \quad \mathbf{n}_E \cdot \underline{\boldsymbol{\psi}}_h^{\omega_i} = \mathbf{n}_E \cdot \hat{\underline{\boldsymbol{\psi}}}_{\partial K}^{\omega_i}|_E, \ E \in \mathcal{E}_h(\mathrm{int}\ \omega_i), \end{aligned} \tag{5.6}$$

where, denoting by $\varphi_h^{(x_i)} \in V_h^{(k-1)}$ the nodal basis function associated with x_i, $f_h^{\omega_i}$ and $\hat{\underline{\boldsymbol{\psi}}}_{\partial K}^{\omega_i}|_E$ are given by

$$f_h^{\omega_i} := \varphi_h^{(x_i)} f_h, \quad \hat{\underline{\boldsymbol{\psi}}}_{\partial K}^{\omega_i}|_E := \varphi_h^{(x_i)} \hat{\underline{\boldsymbol{\psi}}}_{\partial K}|_E, \ 1 \le \ell \le N_i. \tag{5.7}$$

Moreover, we define $\left(\nabla \cdot \underline{\underline{\mathbf{\Pi}}}_k\left(\underline{\underline{\mathbf{z}}}_h\right)\right)^{\omega_i}, 1 \le \ell \le N_i$, according to

$$\left(\nabla \cdot \underline{\underline{\mathbf{\Pi}}}_k\left(\underline{\underline{\mathbf{z}}}_h\right)\right)^{\omega_i} := \varphi_h^{(x_i)} \nabla \cdot \underline{\underline{\mathbf{\Pi}}}_k \underline{\underline{\mathbf{z}}}_h. \tag{5.8}$$

Case 1 ($x_i \in \mathcal{E}_h(\partial E \cap \Omega)$)**.** For $\ell = 1, 2, \cdots, N_i$ we compute $\underline{\boldsymbol{\psi}}_h^{\omega_i}|_{K_\ell} \in \mathbf{BDM}_{k-1}(K_\ell)$ according to

$$\int_{E_\ell} \mathbf{n}_{E_\ell \cap K_\ell} \cdot \underline{\boldsymbol{\psi}}_h^{\omega_i}|_{K_\ell} p_{k-1} \,\mathrm{d}s = \begin{cases} \int_{E_\ell} \mathbf{n}_{E_\ell \cap K_\ell} \cdot \hat{\underline{\boldsymbol{\psi}}}_{\partial K_\ell}^{\omega_i}|_{E_\ell} p_{k-1} \,\mathrm{d}s & \ell = 1 \\ \int_{E_\ell} \mathbf{n}_{E_\ell \cap K_\ell} \cdot \underline{\boldsymbol{\psi}}_h^{\omega_i}|_{K_{\ell-1}}|_{E_\ell} p_{k-1} \,\mathrm{d}s & \ell = 2, 3, \cdots, N_i \end{cases}, p_{k-1} \in P_{k-1}(E_\ell), \tag{5.9a}$$

$$\int_{E_{\ell+1}} \mathbf{n}_{E_{\ell+1} \cap K_\ell} \cdot \underline{\boldsymbol{\psi}}_h^{\omega_i}|_{K_\ell} p_{k-1} \,\mathrm{d}s = \begin{cases} \int_{E_\ell} \mathbf{n}_{E_{\ell+1} \cap K_\ell} \cdot \hat{\underline{\boldsymbol{\psi}}}_{\partial K_\ell}^{\omega_i}|_{E_\ell} p_{k-1} \,\mathrm{d}s & \ell = 1, 2, \cdots, N_{i-1} \\ \int_{E_{\ell+1}} \mathbf{n}_{E_{\ell+1} \cap K_\ell} \cdot \hat{\underline{\boldsymbol{\psi}}}_{\partial K_\ell}^{\omega_i}|_{E_1} \,\mathrm{d}s & \ell = N_i \end{cases}, p_{k-1} \in P_{k-1}(E_\ell), \tag{5.9b}$$

$$\mathbf{n}_E \cdot \underline{\boldsymbol{\psi}}_h^{\omega_i}|_{K_\ell} = 0,\ E \in \mathcal{E}_h(K_\ell \cap \partial\omega_i), \tag{5.9c}$$

$$\int_{K_\ell} \underline{\boldsymbol{\psi}}_h^{\omega_i}|_{K_\ell} \cdot \nabla p_{k-2}\,\mathrm{d}x = -\int_{K_\ell} f_h^{\omega_i} p_{k-2}\,\mathrm{d}x + \int_{\partial K_\ell} \mathbf{n}_{\partial K_\ell} \cdot \underline{\boldsymbol{\psi}}_h^{\omega_i}|_{K_\ell} p_{k-2}\,\mathrm{d}s,\ p_{k-2} \in P_{k-2}(K_\ell), \tag{5.9d}$$

$$\int_{K_\ell} \underline{\boldsymbol{\psi}}_h^{\omega_i}|_{K_\ell} \cdot \mathbf{curl}(b_{K_\ell} p_{k-3})\,\mathrm{d}x = \int_{K_\ell} (\nabla \cdot \underline{\underline{\boldsymbol{\Pi}}}_k(\underline{\underline{\mathbf{z}}}_h))^{\omega_i} \cdot \mathbf{curl}(b_{K_\ell} p_{k-3})\,\mathrm{d}x, p_{k-3} \in P_{k-3}(K_\ell). \tag{5.9e}$$

The cases Case 2 up to Case 5 can be dealt with accordingly.

For the construction of the equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}}$ we begin with the specification of the degrees of freedom for tensors $\underline{\underline{\mathbf{p}}} = (p_{ij})_{i,j=1}^d \in \underline{\underline{\mathbf{V}}}_h$. We note that

$$\dim\ P_k(K)^{2\times 2} = 2(k+1)(k+2). \tag{5.10}$$

Lemma 5.2. *Any $\underline{\underline{\mathbf{p}}} \in P_k(K)^{2\times 2}$ with $\underline{\mathbf{p}}^{(i)} = (p_{i1}, p_{i2})^T, 1 \le i \le 2$, is uniquely determined by the following degrees of freedom (DOF)*

$$\int_E \underline{\underline{\mathbf{p}}}\ \mathbf{n}_E \cdot \underline{\mathbf{p}}_k\,\mathrm{d}s, \qquad \underline{\mathbf{p}}_k \in P_k(E)^2,\ E \in \mathcal{E}_h(\partial K), \tag{5.11a}$$

$$\int_K \underline{\underline{\mathbf{p}}} : \nabla \underline{\mathbf{p}}_{k-1}\,\mathrm{d}x, \qquad \underline{\mathbf{p}}_{k-1} \in P_{k-1}(K)^2 \backslash P_0(K)^2, \tag{5.11b}$$

$$\int_K \underline{\mathbf{p}}^{(i)} \cdot \mathbf{curl}(b_K p_{k-2})\,\mathrm{d}x, \qquad p_{k-2} \in P_{k-2}(K),\ 1 \le i \le 2. \tag{5.11c}$$

The numbers of degrees of freedom (DOF) associated with (5.11a)–(5.11c) are as follows

$$\begin{aligned}\text{DOF (5.11a)} &= 6(k+1),\\ \text{DOF (5.11b)} &= k(k+1) - 2,\\ \text{DOF (5.11c)} &= (k-1)k\end{aligned}$$

and sum up to the right-hand side in (5.10).

Proof. The interpolation conditions for $\underline{\mathbf{p}}^{(1)}$ and $\underline{\mathbf{p}}^{(2)}$ are separated. The vector field $\underline{\mathbf{p}}^{(i)}$ (for $1 \le i \le 2$) is determined by the degrees of freedom

$$\begin{aligned}&\int_E \mathbf{n}_E \cdot \underline{\mathbf{p}}^{(i)} p_k\,\mathrm{d}s, && p_k \in P_k(E),\ E \in \mathcal{E}_h(\partial K),\\ &\int_K \underline{\mathbf{p}}^{(i)} \cdot \nabla p_{k-1}\,\mathrm{d}x, && p_{k-1} \in P_{k-1}(K) \backslash P_0(K),\\ &\int_K \underline{\mathbf{p}}^{(i)} \cdot \mathrm{curl}(b_K p_{k-2})\,\mathrm{d}x, && p_{k-2} \in P_{k-2}(K)\ .\end{aligned}$$

By applying Lemma 5.1 we conclude that there is a unique solution. ☐

Lemma 5.3. *Let $\underline{\underline{\mathbf{q}}} = \left(\underline{\underline{\mathbf{q}}}^{(1)}, \underline{\underline{\mathbf{q}}}^{(2)}\right) \in P_k(K)^{2\times 2}$. Then there exists a positive constant $C_E^{(2)}$, depending only on the polynomial degree k and the local geometry of the triangulation $\mathcal{T}_h$, such that*

$$\int_K \left|\underline{\underline{\mathbf{q}}}\right|^q \mathrm{d}x \le C_E^{(2)} \left(\sum_{E\in\mathcal{E}_h(\partial K)} h_E\ \int_E \left|\underline{\underline{\mathbf{q}}}\mathbf{n}_E\right|^q \mathrm{d}s + h_K^q\ \int_K \left|\nabla \cdot \underline{\underline{\mathbf{q}}}\right|^q \mathrm{d}x \right.$$
$$\left. + \ h_K^q \sum_{i=1}^2 \max\left\{ \int_K \left|\underline{\mathbf{q}}^{(i)} \cdot \mathbf{curl}(b_K p_{k-2})\right|^q \mathrm{d}x;\ p_{k-2} \in P_{k-2}, \max_{x\in K} |p_{k-2}(x)| \le 1 \right\} \right). \tag{5.12}$$

Proof. As in the proof of Lemma 5.1, the estimate (5.12) follows by standard scaling arguments. □

Now, for the construction of the equilibrated moment tensor we set

$$\underline{\mathbf{z}}_h^{(1)} := \left(\frac{\partial^2 u_h}{\partial x_1^2}, \frac{\partial^2 u_h}{\partial x_1 \partial x_2}\right)^T, \quad \underline{\mathbf{z}}_h^{(2)} := \left(\frac{\partial^2 u_h}{\partial x_1 \partial x_2}, \frac{\partial^2 u_h}{\partial x_2^2}\right)^T.$$

We construct $\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} = \left(p_{ij}^{h,\mathrm{eq}}\right)_{i,j=1}^2$, with $\underline{\mathbf{p}}_{h,\mathrm{eq}}^{(i)} = \left(p_{i1}^{h,\mathrm{eq}}, p_{i2}^{h,\mathrm{eq}}\right)^T, 1 \le i \le 2$, patchwise similar to the construction of $\underline{\boldsymbol{\psi}}_h^{\mathrm{eq}}$:

$$\underline{\underline{\mathbf{p}}}_h^{\mathrm{eq}} = \sum_{i=1}^{n_h} \underline{\underline{\mathbf{p}}}_h^{\omega_i}. \tag{5.13}$$

For a patch ω_i, we construct $\underline{\underline{\mathbf{p}}}_h^{\omega_i}$ such that

$$\begin{aligned} &\underline{\underline{\mathbf{p}}}_h^{\omega_i}|_{K_\ell} \in \mathbf{BDM}_k(K_\ell), \\ &\nabla \cdot \underline{\underline{\mathbf{p}}}_h^{\omega_i} = \underline{\boldsymbol{\psi}}_h^{\omega_i} \text{ in } \omega_i, \quad \underline{\underline{\mathbf{p}}}_h^{\omega_i} \mathbf{n}_E = \underline{\hat{\mathbf{p}}}_{\partial K}^{\omega_i}|_E,\ E \in \mathcal{E}_h(\mathrm{int}\ \omega_i),\ 1 \le \ell \le N_i, \end{aligned} \tag{5.14}$$

where, denoting by $\varphi_h^{(x_i)} \in V_h^{(k)}$ the nodal basis function associated with x_i, $\underline{\boldsymbol{\psi}}_h^{\omega_i}$ and $\underline{\hat{\mathbf{p}}}_{\partial K}^{\omega_i}|_E$ are given by

$$\underline{\boldsymbol{\psi}}_h^{\omega_i} := \varphi_h^{(x_i)} \underline{\boldsymbol{\psi}}_h^{\mathrm{eq}}, \quad \underline{\underline{\hat{\mathbf{p}}}}_{\partial K}^{\omega_i}|_E \mathbf{n}_E := \varphi_h^{(x_i)} \underline{\hat{\mathbf{p}}}_{\partial K}|_E, 1 \le i \le n_h. \tag{5.15}$$

Moreover, we define $\underline{\mathbf{z}}_h^{(\ell,\omega_i)}$ according to

$$\underline{\mathbf{z}}_h^{(\ell,\omega_i)} := \varphi_h^{(x_i)} \underline{\mathbf{z}}_h^{(\ell)},\ 1 \le \ell \le 2. \tag{5.16}$$

Case 1 ($x_i \in \mathcal{E}_h(\partial E \cap \Omega)$)**.** For $\ell = 1, 2, \cdots, N_i$ we compute $\underline{\underline{\mathbf{p}}}_h^{\omega_i}|_{K_\ell}$ with $\underline{\mathbf{p}}_h^{(m,\omega_i)}|_{K_\ell} \in \mathbf{BDM}_k(K_\ell), 1 \le m \le 2$, according to

$$\int_{E_\ell} \underline{\underline{\mathbf{p}}}_h^{\omega_i}|_{K_\ell} \mathbf{n}_{E_\ell \cap K_\ell} \cdot \underline{\mathbf{p}}_k \,\mathrm{d}s = \begin{cases} \int_{E_\ell} \underline{\hat{\boldsymbol{\psi}}}_{\partial K_\ell}^{\omega_i}|_{E_\ell} & \ell = 1 \\ \int_{E_\ell} \underline{\underline{\mathbf{p}}}_h^{\omega_i}|_{K_\ell} \mathbf{n}_{E_\ell \cap K_\ell} \cdot \underline{\mathbf{p}}_k \,\mathrm{d}s & \ell = 2, 3, \cdots, N_i \end{cases}, \underline{\mathbf{p}}_k \in P_k(E_\ell)^2, \tag{5.17a}$$

$$\int_{E_{\ell+1}} \underline{\underline{\mathbf{p}}}_h^{\omega_i}|_{K_\ell} \mathbf{n}_{E_{\ell+1} \cap K_\ell} \cdot \underline{\mathbf{p}}_k \,\mathrm{d}s = \begin{cases} \int_{E_{\ell+1}} \underline{\hat{\boldsymbol{\psi}}}_{\partial K_\ell}^{\omega_i}|_{E_{\ell+1}} \cdot \underline{\mathbf{p}}_k \,\mathrm{d}s & \ell = N_i \\ \int_{E_\ell} \underline{\underline{\hat{\mathbf{p}}}}_{\partial K}^{\omega_i} \mathbf{n}_{E_\ell} \cdot \underline{\mathbf{p}}_k \,\mathrm{d}s & \ell = 1, 2, \cdots, N_i, \end{cases}, \underline{\mathbf{p}}_k \in P_k(E_{\ell+1})^2, \tag{5.17b}$$

$$\underline{\underline{\mathbf{p}}}_h^{\omega_i} \mathbf{n}_E = \mathbf{0},\ E \in \mathcal{E}_h(K_\ell \cap \partial\omega_i), \tag{5.17c}$$

$$\begin{aligned} \int_{K_\ell} \underline{\underline{\mathbf{p}}}_h^{\omega_i}|_{K_\ell} : \nabla \underline{\mathbf{p}}_{k-1} \,\mathrm{d}x = &-\int_{K_\ell} \underline{\boldsymbol{\psi}}_h^{\omega_i} \cdot \underline{\mathbf{p}}_{k-1} \,\mathrm{d}x \\ &+ \int_{\partial K_\ell} \underline{\underline{\mathbf{p}}}_h^{\omega_i}|_{K_\ell} \mathbf{n}_{\partial K_\ell} \cdot \underline{\mathbf{p}}_{k-1} \,\mathrm{d}s,\ \underline{\mathbf{p}}_{k-1} \in P_{k-1}(K_\ell)^2, \end{aligned} \tag{5.17d}$$

$$\int_{K_\ell} \underline{\underline{\mathbf{p}}}_h^{(m,\omega_i)}|_{K_\ell} \cdot \mathbf{curl}(b_{K_\ell} p_{k-2}) \,\mathrm{d}x = \int_{K_\ell} \underline{\mathbf{z}}_h^{(m,\omega_i)} \cdot \mathbf{curl}(b_{K_\ell} p_{k-2}) \,\mathrm{d}x, 1 \le m \le 2,\ p_{k-2} \in P_{k-2}(K_\ell). \tag{5.17e}$$

Again, the cases Case 2 up to Case 5 can be treated similarly.

6. Relationship with a residual type *a posteriori* error estimator

A residual-type *a posteriori* error estimator for the IPDG approximation of the biharmonic problem with homogeneous Dirichlet boundary conditions has been derived and analyzed in [31]. Its generalization to arbitrary $1 < p < \infty$ reads as follows:

$$\eta_h^{\mathrm{res}} = \sum_{i=1}^{6} \eta_{h,i}^{\mathrm{res}} + \sum_{i=1}^{5} \widetilde{\eta}_{h,i}^{\mathrm{res}} + \sum_{i=4}^{5} \hat{\eta}_{h,i}^{\mathrm{res}}. \tag{6.1a}$$

Here, the element residual $\eta_{h,1}^{\mathrm{res}}$ and the edge residuals $\eta_{h,i}^{\mathrm{res}}, 2 \le i \le 6$, are given by

$$\eta_{h,1}^{\mathrm{res}} := \sum_{K\in\mathcal{T}_h} h_K^{2q} \int_K \left| f - \nabla\cdot\nabla\cdot \underline{\underline{\mathbf{\Pi}}}_k \left(\left| D^2_{\mathrm{DG},2} u_h \right|^{p-2} D^2_{\mathrm{DG},2} u_h \right) \right|^q \mathrm{d}x, \tag{6.1b}$$

$$\eta_{h,2}^{\mathrm{res}} := \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{q+1} \int_E \left| \mathbf{n}_E \cdot \left[\nabla\cdot \underline{\underline{\mathbf{\Pi}}}_k \left(\left| D^2_{\mathrm{DG},2} u_h \right|^{p-2} D^2_{\mathrm{DG},2} u_h \right) \right]_E \right|^q \mathrm{d}x, \tag{6.1c}$$

$$\eta_{h,3}^{\mathrm{res}} := \sum_{E\in\mathcal{E}_h(\Omega)} h_E \int_E \left| \left[\underline{\underline{\mathbf{\Pi}}}_k \left(\left| D^2_{\mathrm{DG},2} u_h \right|^{p-2} D^2_{\mathrm{DG},2} u_h \right) \right]_E \mathbf{n}_E \right|^q \mathrm{d}s, \tag{6.1d}$$

$$\eta_{h,4}^{\mathrm{res}} := \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-p/q} \int_E |[\nabla u_h \otimes \mathbf{n}_E]_E|^p \,\mathrm{d}s, \tag{6.1e}$$

$$\eta_{h,5}^{\mathrm{res}} := \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-p(q+1)/q} \int_E |u_h - u_D|^p \,\mathrm{d}s, \tag{6.1f}$$

$$\eta_{h,6}^{\mathrm{res}} := \sum_{E\in\mathcal{E}_h(\Gamma)} h_E \int_E \left| u_N\ \mathbf{n}_E - \underline{\underline{\mathbf{\Pi}}}_k \left(\left| D^2_{\mathrm{DG},2} u_h \right|^{p-2} D^2_{\mathrm{DG},2} u_h \right)\ \mathbf{n}_E \right|^q \mathrm{d}s. \tag{6.1g}$$

The residuals $\widetilde{\eta}_{h,i}^{\mathrm{res}}, 1 \le i \le 5$, and $\hat{\eta}_{h,i}^{\mathrm{res}}, 4 \le i \le 5$, read as follows:

$$\widetilde{\eta}_{h,i}^{\mathrm{res}} := \left(\eta_{h,i}^{\mathrm{res}}\right)^{1/q} \left| D^2_{\mathrm{DG},2} u_h \right|_{\mathrm{DG},\Omega}, \qquad 1 \le i \le 5, \tag{6.1h}$$

$$\hat{\eta}_{h,i}^{\mathrm{res}} := \left(\eta_{h,i}^{\mathrm{res}}\right)^{1/p} \left| D^2 u_h \right|_{\mathrm{DG},\Omega}, \qquad 4 \le i \le 5, \tag{6.1i}$$

where $\left| D^2_{\mathrm{DG},2} u_h \right|_{\mathrm{DG},\Omega}$ and $\left| D^2 u_h \right|_{\mathrm{DG},\Omega}$ are given by

$$\left| D^2_{\mathrm{DG},2} u_h \right|_{\mathrm{DG},\Omega} := \left(\sum_{K\in\mathcal{T}_h} \int_K \left| \underline{\underline{\mathbf{\Pi}}}_k \left| D^2_{\mathrm{DG},2} u_h \right|^{p-2} D^2_{\mathrm{DG},2} u_h \right|^q \mathrm{d}x \right)^{1/p}, \tag{6.1j}$$

$$|D^2 u_h|_{\mathrm{DG},\Omega} := \left(\sum_{K\in\mathcal{T}_h} \int_K |D^2 u_h||^p \,\mathrm{d}x \right)^{1/q}. \tag{6.1k}$$

We further define data oscillations $\widetilde{\mathrm{osc}}_{h,i}, 1 \le i \le 2$, according to

$$\widetilde{\mathrm{osc}}_{h,1} := \begin{cases} (\mathrm{osc}_{h,1})^{1/q}\ |\nabla u_h|_{\mathrm{DG},\Omega} & k = 2 \\ (\mathrm{osc}_{h,1})^{1/q}\ |D^2 u_h|_{\mathrm{DG},\Omega} & k \ge 3 \end{cases}, \tag{6.2a}$$

$$\widetilde{\mathrm{osc}}_{h,2} := (\mathrm{osc}_{h,2})^{1/q}\ |D^2 u_h|_{\mathrm{DG},\Gamma}, \tag{6.2b}$$

where $|\nabla u_h|_{\mathrm{DG},\Omega}$ and $\left|D^2 u_h\right|_{\mathrm{DG},\Gamma}$ are given by

$$|\nabla u_h|_{\mathrm{DG},\Omega} := \left(\sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^p\,\mathrm{d}x\right)^{1/p}, \tag{6.2c}$$

$$\left|D^2 u_h\right|_{\mathrm{DG},\Gamma} := \left(\sum_{E\in\mathcal{E}_h(\Gamma)}\int_E |D^2 u_h|^p\,\mathrm{d}s\right)^{1/p}. \tag{6.2d}$$

The following result establishes the relationship between the equilibrated and the residual *a posteriori* error estimator.

Theorem 6.1. *Let $u_h \in V_h$ be the C0IPDG approximation as given by (3.25) and let $\eta_h^{\mathrm{eq}}, \eta_{h,i}^{\mathrm{res}}, 1 \le i \le 6, \widetilde{\eta}_{h,i}^{\mathrm{res}}, 1 \le i \le 5, \hat{\eta}_{h,i}^{\mathrm{res}}, 4 \le i \le 5,$ and $\mathrm{osc}_{h,i}, 1 \le i \le 3, \widetilde{\mathrm{osc}}_{h,i}, 1 \le i \le 2,$ be the equilibrated and the residual a posteriori error estimators as well as the data oscillations as given by (4.35), (6.1), (4.19), (4.29), and (6.2). Then there exists a constant $C_{\mathrm{res}} > 0$, depending on $c_R, C_{\mathrm{rec}}, \alpha_i$, $C_E^{(i)}, C_{\mathrm{PF}}^{(i)}, 1 \le i \le 2$, and on p, q such that*

$$\eta_{h,1}^{\mathrm{eq}} \le C_{\mathrm{res}}\left(\sum_{i=1}^{6}\eta_{h,i}^{\mathrm{res}} + \sum_{i=4}^{5}\widetilde{\eta}_{h,i}^{\mathrm{res}} + \mathrm{osc}_{h,1} + \mathrm{osc}_{h,3} + \widetilde{\mathrm{osc}}_{h,1} + \widetilde{\mathrm{osc}}_{h,2}\right). \tag{6.3}$$

Moreover, if we use (3.31) in (4.34b), then $\eta_{h,2}^{\mathrm{eq}}$ can be estimated from above in terms of the residuals $\eta_{h,4}^{\mathrm{res}}$, $\eta_{h,5}^{\mathrm{res}}$, and the data oscillations $\mathrm{osc}_{h,i}, 1 \le i \le 3$.

The proof of (6.3) is fairly standard and will be omitted

Remark 6.2. Using techniques from [48], the local efficiency of the residual-type estimator can be established similarly as in case of the p-Laplacian (*cf.*, *e.g.*, [22]).

7. Numerical results

We have implemented the C0IPDG approximation (3.25) with the penalty parameters $\alpha_i, 1 \le i \le 2$, chosen as $\alpha_1 = 12.0\ k^2$ and $\alpha_2 = 2.5\ k^6$. Further, we have implemented the adaptive algorithm based on the equilibrated error estimator η_h^{eq} by Dörfler marking [25], *i.e.*, given a bulk parameter $\Theta \in (0,1)$, we have selected a set $\mathcal{M}_h \subset \mathcal{T}_h$ according to

$$\Theta \sum_{K\in\mathcal{T}_h}\eta_K^{\mathrm{eq}} \le \sum_{K\in\mathcal{M}_h}\eta_K^{\mathrm{eq}}$$

and we have refined elements $K \in \mathcal{M}_h$ by newest vertex bisection. In case of the residual-based error estimator η_h^{res} we have implemented the adaptive refinement likewise.

As numerical examples, we have chosen Ω as the L-shaped domain $\Omega := (-1,+1)^2 \setminus ([0,1) \times (-1,0])$. We have considered the cases $p = 1.5$ and $p = 3.0$ with the solution given by

$$u(r,\varphi) = r^{1+\gamma}\,\sin\left(\frac{2}{3}\varphi\right),$$
$$\gamma = \sigma(p) - \sqrt{\sigma^2(p) - 4/3}, \quad \sigma(p) = \frac{7p-6}{6(p-1)}\cdot$$

The right-hand side f in (2.3a), the Dirichlet data u_D in (2.3b), and the Neumann data u_N in (2.3c) have been chosen accordingly. The choice of γ is motivated by the singular behavior at the origin for the p-Laplacian [24]. We note that the solution belongs to $W^{2+\gamma-\varepsilon,p}(\Omega)$ for any $\varepsilon > 0$. In case $p = 2$ this corresponds to Example 2 in [31]. A similar regularity applies to the solution z of the boundary value problem

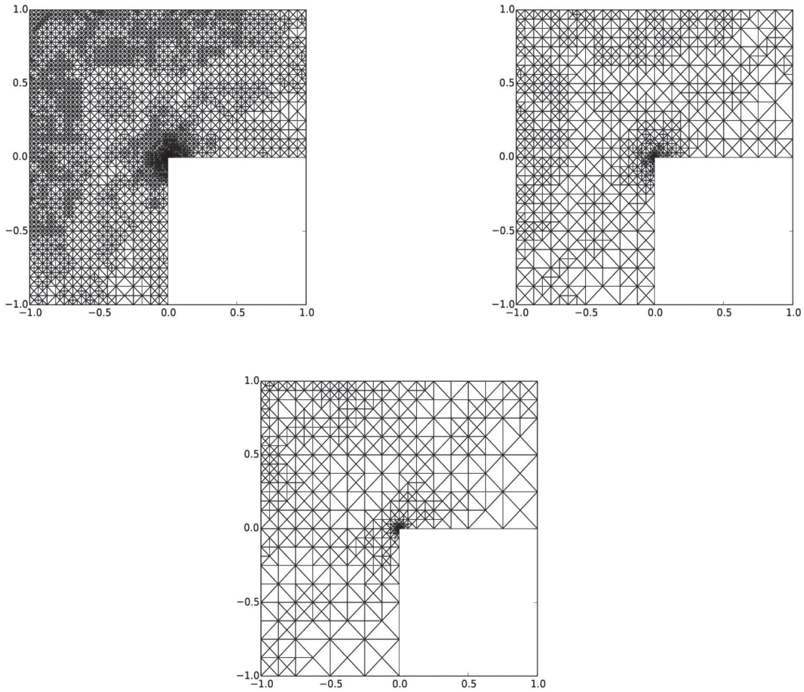

FIGURE 2. $p = 1.5$, $\Theta = 0.5$: adaptively generated meshes (equilibrated error estimator) for polynomial degree $k = 2$ (*top left*), $k = 3$ (*top right*), and $k = 4$ (*bottom*).

(4.15). However, we can not expect $D^2 z|_\Gamma \in L^p\big(\Gamma; \mathbb{R}^{2\times 2}\big)$ so that $\mathrm{osc}_{E,2}$ in (4.19b) has to be replaced by $\mathrm{osc}_{E,2} := \sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma)} h_E^\kappa \int_E |u_N - u_{h,N}|^\kappa \,\mathrm{d}s$ for some $\kappa \neq q$ (provided $u_N \in L^\kappa(\Gamma)$). Actually, in the numerical example we have $D^2 z|_\Gamma \in L^{2-1/p+\gamma}\big(\Gamma; \mathbb{R}^{2\times 2}\big)$. This yields $\kappa = ((2+\gamma)p - 1)/((1+(\gamma)p - 1)$ which is the conjugate of $2 - 1/p + \gamma$.

We have performed computations for $p = 1.5$ and $p = 3.0$ and the polynomial degrees $k = 2, k = 3$, and $k = 4$. The numerical solution of the nonlinear C0IPDG approximation (3.25) has been done by Newton's method with a relative tolerance of $\mathrm{tol} = 10^{-3}$ as termination criterion for the Newton iterates with respect to the Euclidean norm. The expected convergence rate for the discretization error in the broken $W^{2,p}$ norm is 0.5.
Figures 2 and 3 show the adaptively generated meshes in case $p = 1.5$ (Fig. 2) and $p = 3.0$ (Fig. 3) and bulk parameter $\Theta = 0.5$ for polynomial degree $k = 2$ (top left), $k = 3$ (top right), and $k = 4$ (bottom), where the adaptive mesh refinements were based on the equilibrated error estimator. As expected, we observe a pronounced refinement around the reentrant corner at the origin and substantially less refinement off the singularity for the higher polynomial degrees $k \geq 3$. The meshes obtained by the residual-based error estimator look similarly and are therefore omitted.

For $p = 1.5$ and $\Theta = 0.5$, Figure 4 displays the discretization error in the broken $W^{2,p}$ norm, the equilibrated error estimator η_h^{eq}, and the residual-based error estimator η_h^{res} as a function of the total number of degrees of freedom (DOFs) on a logarithmic scale. The result for the polynomial degree $k = 2$ is depicted top left, those for the polynomial degrees $k = 3$ and $k = 4$ top right and bottom. In all cases we observe the optimal convergence of 0.5. For $k = 3$ and $k = 4$ the decay is faster than 0.5 in the pre-asymptotic regime, but approaches 0.5 asymptotically. The equilibrated error estimator is smaller than the residual-based error estimator by approximately 3/4 of an order of magnitude.

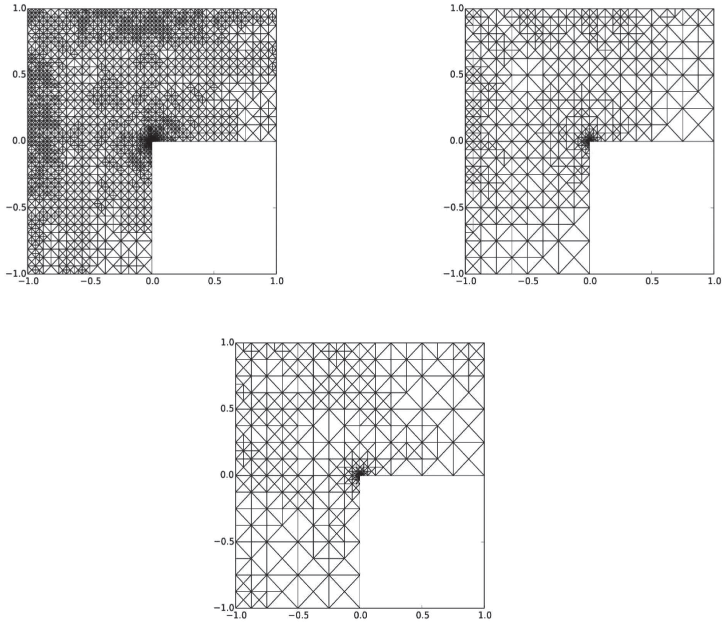

FIGURE 3. $p = 3.0$, $\Theta = 0.5$: adaptively generated meshes (equilibrated error estimator) for polynomial degree $k = 2$ (*top left*), $k = 3$ (*top right*), and $k = 4$ (*bottom*).

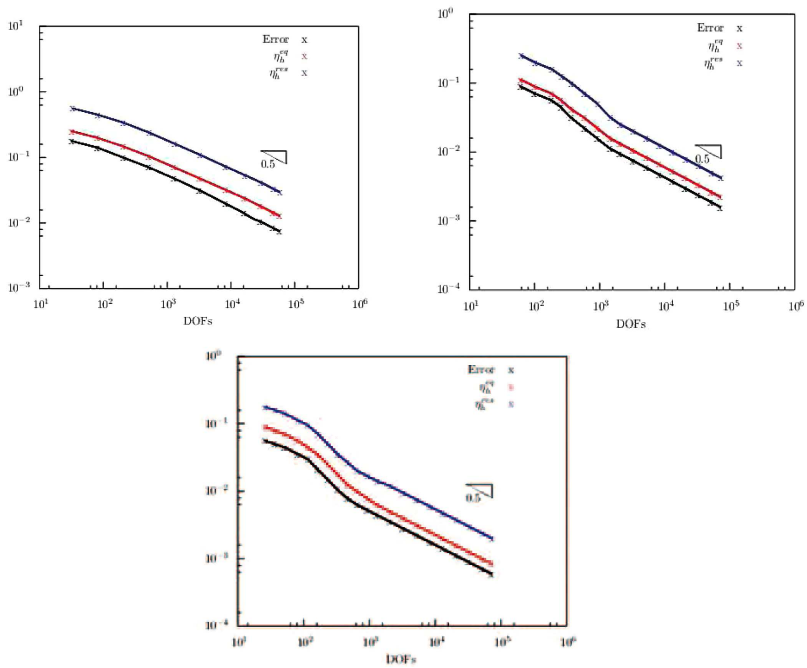

FIGURE 4. $p = 1.5$, $\Theta = 0.5$: the error in the broken $W^{2,p}$ norm (black), the equilibrated error estimator η_h^{eq} (red), and the residual-based error estimator η_h^{res} (blue) for polynomial degree $k = 2$ (*top left*), $k = 3$ (*top right*), and $k = 4$ (*bottom*).

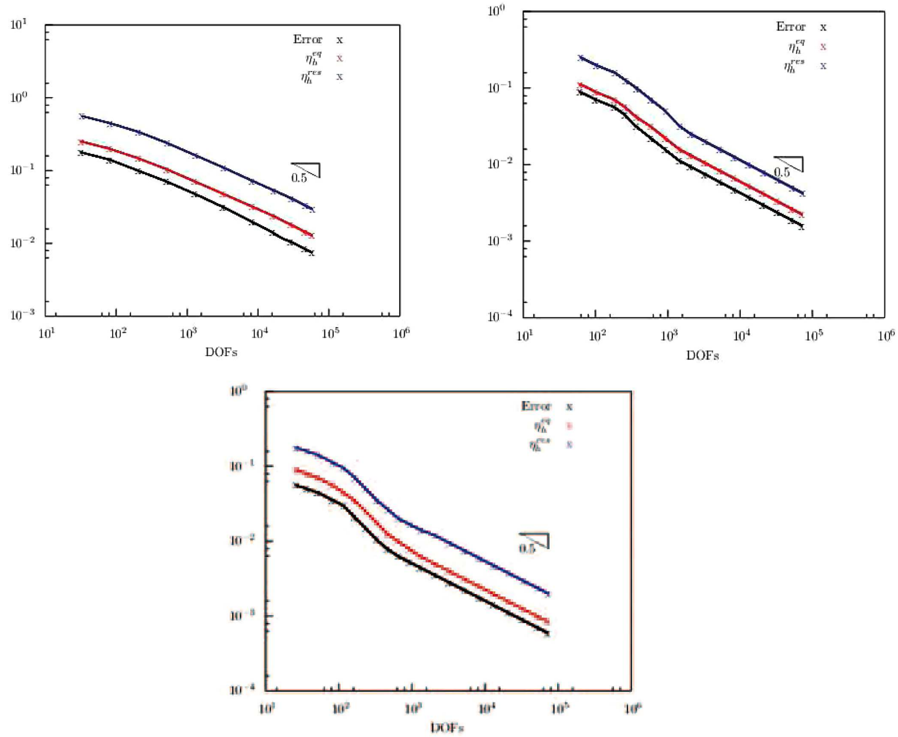

FIGURE 5. $p = 3.0$, $\Theta = 0.5$: the error in the broken $W^{2,p}$ norm (black), the equilibrated error estimator η_h^{eq} (red), and the residual-based error estimator η_h^{res} (blue) for polynomial degree $k = 2$ (*top left*), $k = 3$ (*top right*), and $k = 4$ (*bottom*).

Figure 5 shows the corresponding results for $p = 3.0$. We see a similar behavior as in case $p = 1.5$, but the error is slightly smaller due to the higher regularity of the solution.

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