Realization of Perfect Reconstruction Non-uniform Filter Banks via a Tree Structure

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Abstract. *Obviously, a tree structure filter bank can be realized via a non-uniform filter bank, and perfect reconstruction is achieved if and only if each branch of the tree structure can provide perfect reconstruction. In this paper, the converse of this problem is studied. We show that a perfect reconstruction non-uniform filter bank with decimation ratio {2,4,4} can be realized via a tree structure and each branch of the tree structure achieves perfect reconstruction.*

1. Introduction

It is well known that the tree structure filter bank shown in figure 1b can be realized via a non-uniform filter bank shown in figure 1a, and perfect reconstruction can be achieved if and only if each branch of the tree structure can provide perfect reconstruction [1-4]. However, is the converse true? That is, given any perfect reconstruction non-uniform filter shown in figure 1a, can it be realized via a tree structure shown in figure 1b? In general, a perfect reconstruction non-uniform filter bank cannot be realized by a tree structure [11]. This paper works on this problem.

There are some advantages of realizing a non-uniform filter bank via a tree structure, such as reducing the filter length in the filters [5], and improving the computation complexity and implementation speed [5]. In section 2, we show how a perfect reconstruction non-uniform filter bank can be converted to a tree structure filter bank. Some illustrative examples are demonstrated in section 3. Finally, a conclusion is given in section 4.

2. Realization of Non-uniform Filter Bank Via a Tree Structure

Theorem 1

A non-uniform filter bank with decimation ratio {*2,4,4*} achieves perfect reconstruction if and only if it can be realized via a tree structure and each branch of the tree structure achieves perfect reconstruction.

Proof:

Since the if part was well known [1-4], we only prove the only if part.

A non-uniform filter bank shown in figure 1a achieves perfect reconstruction if and only if $\exists c \in C$ and $\exists m \in Z$ such that:

$$\left(\begin{bmatrix} \dfrac{1}{2} \cdot H_0(z) & \dfrac{1}{4} \cdot H_1(z) & \dfrac{1}{4} \cdot H_2(z) \\ 0 & \dfrac{1}{4} \cdot H_1(z \cdot W) & \dfrac{1}{4} \cdot H_2(z \cdot W) \\ \dfrac{1}{2} \cdot H_0(z \cdot W^2) & \dfrac{1}{4} \cdot H_1(z \cdot W^2) & \dfrac{1}{4} \cdot H_2(z \cdot W^2) \\ 0 & \dfrac{1}{4} \cdot H_1(z \cdot W^3) & \dfrac{1}{4} \cdot H_2(z \cdot W^3) \end{bmatrix}\right) \cdot \begin{pmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{pmatrix} = \begin{pmatrix} c \cdot z^m \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

This directly implies that:

$$\det\left(\begin{bmatrix} H_1(z \cdot W) & H_2(z \cdot W) \\ H_1(z \cdot W^3) & H_2(z \cdot W^3) \end{bmatrix}\right) = 0. \tag{2}$$

Since

$$\det\left(\begin{bmatrix} H_1(z \cdot W) & H_2(z \cdot W) \\ H_1(z \cdot W^3) & H_2(z \cdot W^3) \end{bmatrix}\right) = 0 \Rightarrow \det\left(\begin{bmatrix} H_1(z) & H_2(z) \\ H_1(z \cdot W^2) & H_2(z \cdot W^2) \end{bmatrix}\right) = 0, \tag{3}$$

hence

$$\left(\begin{bmatrix} \dfrac{1}{2} \cdot H_0(z) & \dfrac{1}{4} \cdot H_1(z) & \dfrac{1}{4} \cdot H_2(z) \\ 0 & \dfrac{1}{4} \cdot H_1(z \cdot W) & \dfrac{1}{4} \cdot H_2(z \cdot W) \\ \dfrac{1}{2} \cdot H_0(z \cdot W^2) & \dfrac{1}{4} \cdot H_1(z \cdot W^2) & \dfrac{1}{4} \cdot H_2(z \cdot W^2) \\ 0 & \dfrac{1}{4} \cdot H_1(z \cdot W^3) & \dfrac{1}{4} \cdot H_2(z \cdot W^3) \end{bmatrix}\right) \cdot \begin{pmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{pmatrix} = \begin{pmatrix} c \cdot z^m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \det\left(\begin{bmatrix} H_1(z) & H_2(z) \\ H_1(z \cdot W) & H_2(z \cdot W) \end{bmatrix}\right) \neq 0$$

and

$$\det\left(\begin{bmatrix} H_0(z) & H_2(z) \\ H_0(z \cdot W^2) & H_2(z \cdot W^2) \end{bmatrix}\right) \neq 0. \tag{4}$$

The converse is also true. That is if:

$$\det\left(\begin{bmatrix} H_1(z) & H_2(z) \\ H_1(z \cdot W) & H_2(z \cdot W) \end{bmatrix}\right) \neq 0,$$

$$\det\left(\begin{bmatrix} H_0(z) & H_2(z) \\ H_0(z \cdot W^2) & H_2(z \cdot W^2) \end{bmatrix}\right) \neq 0$$

and

$$\det\left(\begin{bmatrix} H_1(z \cdot W) & H_2(z \cdot W) \\ H_1(z \cdot W^3) & H_2(z \cdot W^3) \end{bmatrix}\right) = 0,$$

then there exist $G_0(z)$, $G_1(z)$ and $G_2(z)$ such that:

$$\left(\begin{bmatrix} \dfrac{1}{2} \cdot H_0(z) & \dfrac{1}{4} \cdot H_1(z) & \dfrac{1}{4} \cdot H_2(z) \\ 0 & \dfrac{1}{4} \cdot H_1(z \cdot W) & \dfrac{1}{4} \cdot H_2(z \cdot W) \\ \dfrac{1}{2} \cdot H_0(z \cdot W^2) & \dfrac{1}{4} \cdot H_1(z \cdot W^2) & \dfrac{1}{4} \cdot H_2(z \cdot W^2) \\ 0 & \dfrac{1}{4} \cdot H_1(z \cdot W^3) & \dfrac{1}{4} \cdot H_2(z \cdot W^3) \end{bmatrix}\right) \cdot \begin{pmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{pmatrix} = \begin{pmatrix} c \cdot z^m \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5}$$

Let

$$H_i(z) = \sum_{l=0}^{3} z^{-l} \cdot E_{i,l}(z^4),$$

for $i=0,1,2$, then from equation (2), there exist $R(z)$, $R'(z)$ and $R''(z)$ such that:

$$E_{1,1}(z^4)=R(z^4) \cdot E_{1,0}(z^4) \text{ and } E_{2,1}(z^4)=R(z^4) \cdot E_{2,0}(z^4) \text{ and} \tag{6}$$

$$E_{1,3}(z^4)=R'(z^4) \cdot E_{1,2}(z^4) \text{ and } E_{2,3}(z^4)=R'(z^4) \cdot E_{2,2}(z^4) \text{ and} \tag{7}$$

$$\{R(z^4)=R'(z^4) \text{ or } \{E_{1,0}(z^4)=R''(z^4) \cdot E_{1,2}(z^4) \text{ and } E_{2,0}(z^4)=R''(z^4) \cdot E_{2,2}(z^4)\}\}. \tag{8}$$

But $E_{1,0}(z^4)=R''(z^4) \cdot E_{1,2}(z^4)$ and $E_{2,0}(z^4)=R''(z^4) \cdot E_{2,2}(z^4)$ contradict equation (4). Hence, we have $R(z^4)=R'(z^4)$. $R(z^4)=R'(z^4)$, which implies:

$$\frac{H_2(z)}{H_1(z)} = \frac{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4)}{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4)}. \tag{9}$$

Hence, there exist $F'_1(z)$, $F_0(z)$, and $F_1(z)$ such that $H_1(z)=F'_1(z)\cdot F_0(z^2)$ and $H_2(z)=F'_1(z)\cdot F_1(z^2)$, respectively. And the non-uniform filter bank shown in figure 1a can be realized via a tree structure shown in figure 1b.

From equation (4), we have:

$$E_{1,2}(z^4)\cdot E_{2,0}(z^4)-E_{1,0}(z^4)\cdot E_{2,2}(z^4)\neq 0 \text{ and} \tag{10}$$

$$\{E_{0,1}(z^4)\cdot E_{2,0}(z^4)-E_{0,0}(z^4)\cdot E_{2,1}(z^4)\neq 0 \text{ or} \tag{11}$$

$$E_{0,3}(z^4)\cdot E_{2,2}(z^4)-E_{0,2}(z^4)\cdot E_{2,3}(z^4)\neq 0 \text{ or} \tag{12}$$

$$E_{0,1}(z^4)\cdot E_{2,2}(z^4)+E_{0,3}(z^4)\cdot E_{2,0}(z^4)-E_{0,0}(z^4)\cdot E_{2,3}(z^4)-E_{0,2}(z^4)\cdot E_{2,1}(z^4)\neq 0\}. \tag{13}$$

Let $F_1(z^2)$ be the numerator of $H_2(z)/H_1(z)$, and $F_0(z^2)$ be the denominator of $H_2(z)/H_1(z)$, respectively. We have:

$$\det\left(\begin{bmatrix} F_0(z) & F_1(z) \\ F_0(-z) & F_1(-z) \end{bmatrix}\right)=2\cdot z^{-1}\cdot\left(E_{1,2}(z^2)\cdot E_{2,0}(z^2)-E_{1,0}(z^2)\cdot E_{2,2}(z^2)\right)\neq 0, \tag{14}$$

and

$$\det\left(\begin{bmatrix} H_0(z) & F'_1(z) \\ H_0(-z) & F'_1(-z) \end{bmatrix}\right)=\frac{2\cdot z^{-1}\cdot\left(E_{0,1}(z^4)\cdot E_{2,0}(z^4)-E_{0,0}(z^4)\cdot E_{2,1}(z^4)\right)+2\cdot z^{-5}\cdot\left(E_{0,3}(z^4)\cdot E_{2,2}(z^4)-E_{0,2}(z^4)\cdot E_{2,3}(z^4)\right)}{E_{2,0}(z^4)+z^{-2}\cdot E_{2,2}(z^4)} \tag{15}$$

$$+\frac{2\cdot z^{-3}\cdot\left(E_{0,1}(z^4)\cdot E_{2,2}(z^4)+E_{0,3}(z^4)\cdot E_{2,0}(z^4)-E_{0,0}(z^4)\cdot E_{2,3}(z^4)-E_{0,2}(z^4)\cdot E_{2,1}(z^4)\right)}{E_{2,0}(z^4)+z^{-2}\cdot E_{2,2}(z^4)}$$

$$\neq 0.$$

Hence, each branch of the tree structure achieves perfect reconstruction. ∎

3.   Illustrative Examples

3.1   Non-tree Structure Filter Bank

Consider an example of $H_0(z)=1+z^{-1}$, $H_1(z)=1-z^{-1}$, and $H_2(z)=z^{-2}$, respectively. Since $H_1(z)/H_2(z)=z^2\cdot(1-z^{-1})$, there does not exist $F'_1(z)$, $F_0(z)$, and $F_1(z)$ such that $H_1(z)=F'_1(z)\cdot F_0(z^2)$ and $H_2(z)=F'_1(z)\cdot F_1(z^2)$, respectively. Hence, this non-uniform filter bank cannot be realized via a tree structure. By theorem 1, this non-uniform filter bank does not achieve perfect reconstruction.

It is worth to note that by converting the non-uniform filter bank to a uniform filter bank shown in figure 2 [6-10], perfect reconstruction can be achieved. However, $G'_{-1}(z)\neq z^{-2}\cdot G'_0(z)$, this implies that the corresponding synthesis filter $G_0(z)$ shown in figure 1a is time varying.

3.2 Tree Structure Filter Bank

Consider another example with $H_0(z)=2 \cdot (1+z^{-1}+z^{-2}+z^{-3})$, $H_1(z)=4 \cdot (2+6 \cdot z^{-1}+4 \cdot z^{-2}+12 \cdot z^{-3})$, and $H_2(z)=4 \cdot (5+15 \cdot z^{-1}+7 \cdot z^{-2}+21 \cdot z^{-3})$, respectively. Since $H_1(z)/H_2(z)=(2+4 \cdot z^{-2})/(5+7 \cdot z^{-2})$, there exists $F'_1(z)$, $F_0(z)$, and $F_1(z)$ such that $H_1(z)=F'_1(z) \cdot F_0(z^2)$ and $H_2(z)=F'_1(z) \cdot F_1(z^2)$, respectively. Hence, this non-uniform filter bank can be realized via a tree structure. It can be checked easily that each branch in the tree structure achieves perfect reconstruction. Hence, this non-uniform filter bank achieves perfect reconstruction.

4. Conclusion

In this paper, we show that a non-uniform filter bank with decimation ratio $\{2,4,4\}$ achieves perfect reconstruction if and only if it can be realized via a tree structure and each branch of the tree structure achieves perfect reconstruction. The advantage of realizing a non-uniform filter bank via a tree structure is to reduce the computation complexity and provide a fast implementation for a non-uniform filter bank [5].

Acknowledgement

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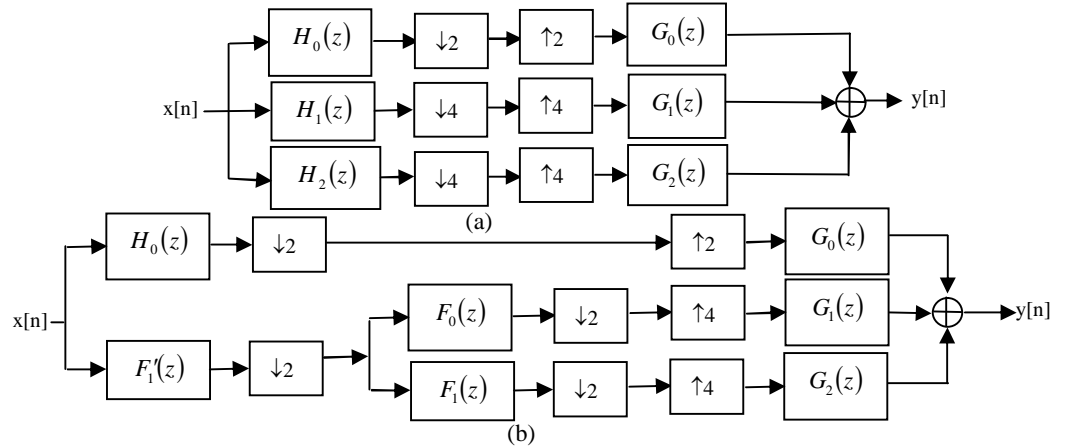
Fig. 1. (a) Non-uniform filter bank (b) Tree structure filter bank
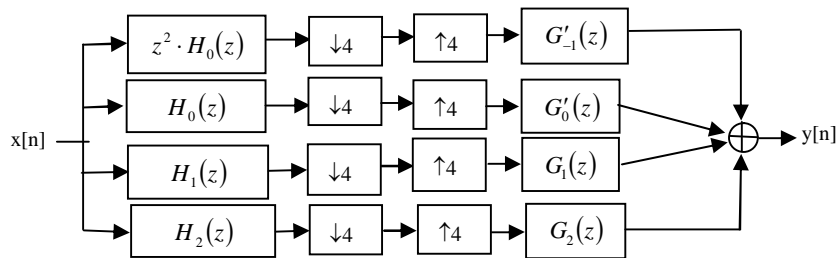
Fig. 2. Realization of non-uniform filter bank via a uniform filter bank