Solutions and Approximations of Some Lévy-driven Stochastic (Partial) Differential Equations

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Zusammenfassung

In dieser Arbeit betrachten wir die Lösungen von stochastischen partiellen Differentialgleichungen (SPDE), deren Rauschen von einem Lévy-Prozess stammt. Dabei verstehen wir diese Lösungen im Kontext von Marcus-Integralen.
Das kanonische Marcus-Integral ist im Rahmen des Studiums gewöhnlicher stochastischer Differentialgleichungen (SDE) definiert. Wir wiederholen einige fundamentale Resultate zur Existenz von Flüssen von Lösungen der Marcus SDE and der Konvergenz von Wong-Zakai-Approximationen. Des Weiteren beweisen wir eine generalisierte Itô-Formel für die Lösungen der Marcus SDE und nutzen diese, um Formeln für den inversen Fluss herzuleiten.
Unser weiteres Ziel ist es, die Definition des Marcus-Integrals auf den Fall partieller stochastischer Differentialgleichungen auszuweiten und Lösungen für die enstprechenden Gleichungen zu finden. Unser Hauptfokus liegt dabei auf mehrdimensionalen Transportgleichungen erster Ordnung, die durch einen Lévy-Prozess gestört werden. Mit Hilfe der Methode der Charakteristiken weisen wir die Existenz und Eindeutigkeit von Lösungen dieser Gleichungen nach.
Für Transportgleichungen zweiter Ordnung beweisen wir die Existenz und Eindeutigkeit milder Lösungen für den Fall, dass das Rauschen durch einen reinen Sprungprozess gegeben ist. Dabei definieren wir Lösungen ebenfalls im Sinne des Marcus-Integrals.
Zuletzt untersuchen wir eine eindimensionale Gleichung zweiter Ordnung auf der Halbachse, deren Lévy-Rauschen auf dem Rand liegt. Wir betrachten sowohl Dirichlet als auch Neumann Randbedingungen and bestimmen die geschlossene Formel für eine milde Lösung. Des Weiteren definieren wir für die Gleichung Wong-Zakai-Approximationen, welche in der M_1-Topologie des Skorokhod-Raums gegen die stochastische Lösung konvergieren.

Abstract

In this work we look at solutions to stochastic partial differential equations (SPDEs) with noise induced by a Lévy process in the context of Marcus integrals.
The canonical Marcus integral is known from the study of SDEs with Lévy noise. We recapture the fundamental results on the existence of solution flows to the Marcus SDE and the convergence of Wong–Zakai approximations. We also prove a generalized Itô formula for said solutions and use this result to establish equations for the inverse flow.
We are then looking at extensions of Marcus integrals to the case of SPDEs and find solutions for these equations. Our focus mainly lies on multi-dimensional first-order transport equations driven by Lévy noise. Existence and uniqueness results for the Marcus SPDE are established using a method of characteristics.
For second-order equations we prove the existence and uniqueness of mild solutions for equations driven by pure jump Lévy processes, also in terms of Marcus SPDEs.
Finally, we study a one-dimensional second-order advection-diffusion equation on the half-line, with Lévy noise at the boundary. Both Dirichlet and Neumann boundary conditions are considered, and the closed form formulae for mild solutions are determined. We also define Wong–Zakai type approximations of the solution by classical solutions and show convergence in the setting of the M_1-topology in the Skorokhod space.

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Chapter 1

Introduction

Differential equations have been used for a long time to model various effects that occur in nature. In this work we are particularly interested in the first and second order partial differential equations. In the deterministic setting, one example is the first order PDE given by

$$\begin{cases} \partial_t u(t,x) = \nabla^T u(t,x) f(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

which can be used to describe the transport of a contaminant in an incompressible fluid, see e.g. [VdP07]. Here, $u(t,x)$ is the concentration of the contaminant at time instant t at the position $x \in \mathbb{R}^d$, and $-f(t,x)$ is the flow instant velocity at the position x at time t.

If, in addition, the contaminant transport is influenced by diffusion this yields the second order equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \nabla^T u(t,x) f(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x). \end{cases}$$

There are several reasons to consider transport equations perturbed by random noise. From a purely mathematical viewpoint the noise can have a regularizing effect; e.g. [Fla11, Chapter 4] details the effect of stochastic noise on the existence and regularity of the transport equation driven by Brownian motion. In certain scientific applications on the other hand, it is reasonable to consider natural phenomena which are influenced by random occurences. For example, in [WZ05] contaminant transport with random timing and magnitude of the source of contamination is considered and modeled using deterministic PDE with noise on the boundary.

Looking specifically at contaminant transport in fluids, the stochastic noise can also be used to model the turbulence of particles in a flow, see [FL20] and [MT07], this way simplifying physical models that are difficult to analyze in the deterministic setting. This naturally leads us to consider transport equations with multiplicative noise, an obvious example being the first order equation driven by Brownian motion, namely

$$\begin{aligned} \partial_t u(t,x) &= \nabla^T u(t,x) f(t,x) + \nabla^T u(t,x) F(t,x) \circ \dot{W}(t), \\ u(0,x) &= u_0(x). \end{aligned} \tag{1.2}$$

Here W is the Brownian motion, and $\circ \dot{W}$ denotes the noise term in the Stratonovich sense.

There is a great variety of works dedicated to studying the existence and properties of solutions to (1.2) and similar equations. We want to highlight two approaches here: In [Kun84] and [Kun97, Chapter 6], solutions to (1.2) are found using a method of stochastic characteristics, where the characteristic SDE is also given in the Stratonovich sense. This approach works analogously to the method of characteristics in the deterministic setting, see e.g. [Per07, Chapter 6], and makes use of a generalized Itô formula.

The second approach to mention here is the interpretation of (1.2) as the limit of a sequence of random PDEs. Approximations of irregular trajectories of random processes by smooth paths are well known in the literature under the name of Wong–Zakai approximations [WZ65a, WZ65b]. In particular for dynamical systems driven by Brownian motion there is a number of results in both finite and infinite dimensional settings which state that the approximations converge in the uniform topology to the solution of the Stratonovich equation, see, e.g. [BF95, TZ06, Twa91].

Up to this point we have only mentioned results regarding Brownian motion, some of which can be extended more generally to continuous semimartingales. However, in some applications it is useful to allow the stochastic noise to have jumps. Coming back to the example of contaminant transport in fluids we find that in addition to the influence of Brownian motion it is sometimes desirable to model extreme flow events that cause instantaneous changes in contaminant concentrations. In [OT10] and [TH19] for example, Poisson processes are used to describe these changes. Generalizing this idea, Lévy processes provide an attainable class of stochastic processes covering both jumps and continuous noise in a unifying approach.

In this thesis in particular we look at transport equations driven by Lévy pro-

cesses. The literature already provides some results on solutions to Lévy-driven partial differential equations. A good introduction to the general theory can be found in [PZ07], and further results on solutions to the transport equation within the frame of white noise theory are given in [Pro04a].
However, looking at Lévy-driven SPDE in this thesis, the goal is to transfer the aforementioned approaches of stochastic characteristics and Wong–Zakai approximations. For this purpose it is necessary to find a framework of stochastic calculus that works for Lévy processes equivalently to how the Stratonovich integral works for continuous semimartingales. This leads us to study Lévy-driven differential equations in the Marcus sense which can be written as

$$\begin{aligned} \partial_t u(t,x) &= \nabla^T u(t,x) f(t,x) + \nabla^T u(t,x) F(t,x) \circ \dot{W}(t) + \nabla^T u(t,x)\varphi(t,x) \diamond \dot{Z}(t), \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.3}$$

where W is the Brownian motion part of the Lévy process and Z is the purely discontinuous part. The noise term in the Marcus sense is denoted by $\diamond\dot{Z}(t)$. The precise meaning of this expression will be given in Chapter 4.

The canonical (Marcus) SDE was introduced in [Mar78] and several results on the properties of Marcus integrals were obtained in the last decades. Existence and uniqueness of solutions to the Marcus SDE are proven in [KPP95] in the classical setting, along with an Itô formula for Marcus integrals. In [FK99a] and [FK99b] the Marcus SDE as well as its derivatives and the inverse flow are studied in a more general setting. Results on the convergence of approximations can be found in [Mar81] as well as [Kun95] and [KKP19].
In this thesis we extend these results by a generalized Itô formula for Marcus SDEs and some results on the inverse flow. Furthermore, we define Marcus integrals for Lévy-driven transport equations and show that in the Marcus framework the method of stochastic characteristics can in fact be used to obtain solutions to these equations.

The plan of this thesis is as follows: In Chapter 2 we establish the necessary notation and give a quick overview on stochastic integration w.r.t. Lévy processes. Chapter 3 gives the reader an understanding of the Marcus SDE. We recall results on the existence and properties of solutions and study the weak convergence of Wong–Zakai approximations. In this chapter, a new generalized Itô formula for solutions to Marcus SDE is proven. This result is then used to show how the inverse flow of solutions to the Marcus SDE solves a specific Marcus equation and to get an Itô formula specifically for the inverse flow.

This representation of the inverse will be needed in Chapter 4, where a first-order Marcus SPDE is studied. Namely, we prove the existence and uniqueness of solutions to the Marcus transport equation driven by Lévy noise with bounded jumps.

Chapter 5 deals with Marcus SPDE with purely discontinuous noise and the existence of mild solutions to the transport equation with a second-order diffusion term which generates a C_0-semigroup is shown.

In the last chapter we consider a slightly different problem. Looking at a second-order advection-diffusion equation, we change the way the noise is introduced. Instead of looking at a multiplicative noise, the Lévy noise is inserted through a boundary condition. We then prove the existence of mild solutions and show convergence of Wong–Zakai approximations for this equation.

Chapter 2

Preliminaries

2.1 Semimartingales and Stochastic Integration

In this chapter we give a brief summary of stochastic integration and the properties of stochastic integrals. We will not dwell much on the intricacies of stochastic integration, since most of the properties we are using in this work are results that can be found in the literature. For proofs and further reading we refer the reader for example to [App04] and [Pro04b].

We start by introducing some basic notations used throughout this thesis. In the following let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where the filtration $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ satisfies the usual conditions, i.e. it is right continuous and complete (see [Pro04b, Chapter I.5] for more details).

An adapted stochastic process $M = (M_t)_{t\geq 0}$ is called a martingale, if $\mathbb{E}|M_t| < \infty$ and $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ a.s. for all $0 \leq s \leq t$. It is called square integrable if $\mathbb{E}|M_t|^2 < \infty$, $t \geq 0$.

By Doob-Meyer's decomposition theorem, for every square integrable martingale M there is a unique integrable predictable increasing process $A = (A_t)_{t\geq 0}, A_0 = 0$, such that $M^2 - A$ is again a martingale. We denote $\langle M \rangle := A$. This definition can be extended to locally square integrable martingales with the help of the localization technique.

For two square integrable martingales M and N we define the angle bracket $\langle M, N \rangle$ as

$$\langle M, N \rangle_t = \frac{1}{4}(\langle M + N \rangle_t - \langle M - N \rangle_t),\ t \geq 0,$$

which obviously implies $\langle M, M \rangle_t = \langle M \rangle_t$. This allows us to define the Hilbert space of predictable, integrable processes $f\colon \mathbb{R} \to \mathbb{R}^d$ w.r.t. the square integrable martingale $M = (M_t)_{t\in[0,T]}$ by

$$L^2(\langle M\rangle) = \Big\{ f\colon f \text{ is predictable and } \mathbb{E}\Big[\int_0^T |f(s)|^2 \,\mathrm{d}\langle M\rangle_s\Big] < \infty \Big\}$$

with the norm

$$\|f\|_{L^2(\langle M\rangle)} = \mathbb{E}\Big[\int_0^T |f(s)|^2 \,\mathrm{d}\langle M\rangle_s\Big]^{\frac{1}{2}}.$$

For $f \in L^2(\langle M\rangle)$, we denote by $\int_0^t f(s)\,\mathrm{d}M_s$ the Itô integral w.r.t. M.
In the multidimensional case, i.e. $f\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}^{d\times m}$ and M takes values in $\mathbb{R}^m$,

$$\int_0^t f(s)\,\mathrm{d}M_s = \sum_{i=1}^d \int_0^t f_i(s)\,\mathrm{d}M_s^i,$$

where f_i is the i-th column vector of the $(d \times m)$-matrix f.

These integrals are known to always have a càdlàg modification, that means a modification that is right continuous and has left limits. In the following we will always mean this modification when writing the integral. Note also that if f is not predictable but is adapted and càdlàg, then we can still consider the integral of $f(s-) = \lim_{r\to s, r<s} f(r)$, since this process is then predictable.
The well-known Itô isometry gives us the following properties of the stochastic integral for $f \in L^2(\langle M\rangle)$:

$$\Big\langle \int_0^\cdot f(s)\,\mathrm{d}M_s \Big\rangle_t = \int_0^t |f(s)|^2 \,\mathrm{d}\langle M\rangle_s,$$

and

$$\mathbb{E}|\int_0^t f(s)\,\mathrm{d}M_s|^2 = \mathbb{E}\int_0^t |f(s)|^2 \,\mathrm{d}\langle M\rangle_s.$$

For example, for integrals w.r.t. a standard Wiener process W this gives us

$$\Big\langle \int_0^\cdot f(s)\,\mathrm{d}W_s \Big\rangle_t = \int_0^t |f(s)|^2 \,\mathrm{d}s.$$

We now turn our attention to semimartingales:

Definition 2.1. *Let $X = (X_t)_{t\geq 0}$ be an adapted stochastic process with càdlàg paths. We call X a semimartingale if there is a decomposition*

$$X_t = X_0 + M_t + A_t,$$

into $\mathbb{F}$-adapted processes M and A, such that M is a local martingale and A has *locally finite variation, with* $A_0 = 0$ *and* $M_0 = 0$.

We write X_{t-} for the left hand side limit of X and $\Delta X_t = X_t - X_{t-}$ for the jump at time t. To define integration w.r.t. semimartingales, first note that since A has locally finite variation we can define the integral w.r.t. A as the usual Stieltjes integral. For a predictable process f, which is integrable w.r.t. M and A, we define the Itô integral as

$$\int_0^t f(s)\,\mathrm{d}X_s := \int_0^t f(s)\,\mathrm{d}M_s + \int_0^t f(s)\mathrm{d}A_s.$$

Before we state some important results on the integration for semimartingales we first take a look at the quadratic variation. For more details we refer the reader to [JS03].
Let X and Y be two semimartingales. Then their quadratic co-variation $[X,Y]_t$ can be defined as

$$[X,Y]_t = X_tY_t - X_0Y_0 - \int_0^t X_{s-}\,\mathrm{d}Y_s - \int_0^t Y_{s-}\,\mathrm{d}X_s. \tag{2.1}$$

The quadratic variation of X is defined as $[X]_t = [X,X]_t$. Note that for a continuous martingale M we have $[M]_t = \langle M\rangle_t$.

Lemma 2.2 ([JS03], Theorem 1.4.52). *Let X and Y be semimartingales. We denote the continuous martingale part of X and Y by M^c and N^c respectively. Then*

$$[X,Y]_t = [X,Y]_t^c + \sum_{0\le s\le t} \Delta X_s \Delta Y_s \;=\; \langle M^c, N^c\rangle_t + \sum_{0\le s\le t} \Delta X_s \Delta Y_s.$$

We use the quadratic variation to define the Stratonovich integral. For the purposes of this thesis it is enough to restrict the definition to the case where the integrator is a continuous semimartingale, e.g. Brownian motion. Specifically, let Y be a semimartingale and X be a continuous semimartingale. We define the Stratonovich integral of Y w.r.t. X as

$$\int_0^t Y_s \circ \mathrm{d}X_s = \int_0^t Y_{s-}\,\mathrm{d}X_s + \frac{1}{2}[Y,X]_s = \int_0^t Y_{s-}\,\mathrm{d}X_s + \frac{1}{2}\langle Y^c, X\rangle_s.$$

For a semimartingale X we now get Itô's formula (cf. [Pro04b, Chapter II.7]):

Theorem 2.3 (Itô formula). *Let $F : \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable.*

Then F(X) is again a semimartingale and

$$F(X_t) = F(X_0) + \sum_i \int_0^t \frac{\partial F}{\partial x_i}(X_{s-})\,\mathrm{d}X_s^i + \frac{1}{2}\sum_{i,j}\int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_{s-})\,\mathrm{d}[X^i, X^j]_s^c$$
$$+ \sum_{0<s\le t}\left(F(X_s) - F(X_{s-}) - \sum_i \frac{\partial F}{\partial x_i}(X_{s-})\Delta X_s^i\right).$$

The last result we mention here is the Stochastic Fubini Theorem. For our purposes it suffices to use a version of this theorem for bounded functions. For the proof and more details see [Pro04b, Chapter IV.6].

Theorem 2.4. *Let X be a semimartingale, $f(s,x,\omega)$ a measurable bounded function and μ a finite measure on $\mathbb{R}^d$. Then for every Borel set B,*

$$\int_0^t \int_B f(s,x,\omega)\mu(\mathrm{d}x)\,\mathrm{d}X_s = \int_B \int_0^t f(s,x,\omega)\,\mathrm{d}X_s\mu(\mathrm{d}x) \quad a.s.$$

2.2 Lévy Processes

The focus of this work is to study solutions to differential equations w.r.t. stochastic processes with jumps, instead of assuming the driving semimartingale to be continuous. The reason to study Lévy processes in particular comes from the fact that this is a very broad class of possibly discontinuous semimartingales. In this section we take a brief look at the useful properties of Lévy processes. We start with the general definition of an $\mathbb{R}^d$-valued Lévy process:

Definition 2.5. *We call an $\mathbb{R}^d$-valued stochastic process $L = (L_t)_{t\ge 0}$ a Lévy process if $L_0 = 0$ a.s., L has independent and stationary increments, and L is stochastically continuous, that means for every $t \ge 0$, L is continuous in t with probability 1.*

The Poisson random measure

The most famous examples of Lévy processes are the Brownian motion and compound Poisson processes. We will see later that if we decompose a Lévy process into the continuous and purely discontinuous part $L = L^c + L^d$, the continuous part is given by the sum of a drift μ and a Brownian motion W. For the discontinuous part we give a few more details now.

First note that the sum of the jumps of a Lévy process is not necessarily absolutely convergent, meaning that there are Lévy processes L, s.t.

$$\sum_{0\le s\le t} |\Delta L_s| = \infty.$$

This is due to the fact that the Lévy process can have an infinite number of small jumps. To deal with this difficulty, we split the jumps into "large" and "small" jumps. We start with the large jumps:
For $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$ and A bounded away from 0, we define

$$N(A,t) = \sharp\{0 \le s \le t \colon \Delta L_s \in A\} = \sum_{0\le s\le t} \mathbb{1}_A(\Delta L(s)).$$

First note that $N(t,\cdot)$ is a counting measure on $\mathcal{B}(\mathbb{R}^d - \{0\})$ (cf. [App04, p. 100]). A needs to be bounded away from zero because of the reasons mentioned above. For $A \in \mathcal{B}(\mathbb{R}^d)$ with $0 \in A$, we could get $N(A,t) = \infty$.
N is called the Poisson random measure (PRM) of L. For more on random measures see [App04, Section 2.3.1].
For $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$ we define the Lévy measure $\nu(\mathrm{d}z)$ of L, where

$$\nu(A) = \mathbb{E}(N(A,1)) = \int N(A,1)\,\mathrm{d}\mathbb{P},$$

and the compensated Poisson random measure as

$$\tilde{N}(A,t) = N(A,t) - t\nu(A), \quad t \ge 0.$$

Now, for a Borel function $f\colon \mathbb{R}^d \to \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$ the integral w.r.t. the PRM is defined as

$$\int_A f(z)\,N(\mathrm{d}z,t) = \sum_{z\in A} f(z)\,N(\{z\},t) = \sum_{0\le s\le t} f(\Delta L_s)\,\mathbb{1}_A(\Delta L_s),$$

see [App04, p. 106]. With this definition it is possible to write the large jumps of the Lévy process as an integral:

$$\int_{|z|>1} z\,N(\mathrm{d}z,t) = \sum_{0\le s\le t} \Delta L_s\,\mathbb{1}\{|\Delta L_s| > 1\}.$$

Due to the accumulation of infinite numbers of small jumps, this definition cannot simply be extended to $|z| < 1$. Instead we subtract the mean value and thus define

$$\int_{|z|\le 1} f(z)\,\tilde{N}(\mathrm{d}z,t) = \lim_{\varepsilon\to 0,\varepsilon>0}\left[\int_{\varepsilon<|z|\le 1} f(z)\,N(\mathrm{d}z,t) - t\int_{\varepsilon<|z|\le 1} f(z)\nu(\mathrm{d}z)\right],$$

where the limit is taken in probability. To see that the limit exists one can show that the sequence on the right hand side is a Cauchy sequence in L^2.

We now have the necessary notation to state the result of the Lévy-Itô-decomposition. For proof of the theorem and more details we refer the reader to [App04, Theorem 2.4.16].

Theorem 2.6. *For every $\mathbb{R}^d$-valued Lévy process L there exists a drift $\mu \in \mathbb{R}^d$, an $\mathbb{R}^d$-valued Brownian motion W with covariance matrix Σ and an independent Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ with Lévy measure ν, s.t.*

$$\int_{\mathbb{R}^d} (|z|^2 \wedge 1)\,\nu(\mathrm{d}z) < \infty,$$

and for each $t \ge 0$

$$L_t = \mu t + W_t + \int_{|z|\le 1} z\,\tilde{N}(\mathrm{d}z,t) + \int_{|z|>1} z\,N(\mathrm{d}z,t).$$

This decomposition is unique up to the setting of the threshold of large jumps.

For the sake of completeness we also give the Lévy-Khintchin formula for the characteristic function of $\Phi_t(\alpha) = \mathbb{E}[\mathrm{e}^{\mathrm{i}(\alpha,L_t)}]$, with $\alpha \in \mathbb{R}^d$ and $t \ge 0$:

$$\Phi_t(\alpha) = \exp\left[t\left(\mathrm{i}(\mu,\alpha) - \frac{1}{2}(\alpha,\Sigma\alpha) + \int_{\mathbb{R}^d-\{0\}} (\mathrm{e}^{\mathrm{i}(\alpha,z)} - 1 - \mathrm{i}(\alpha,z)\,\mathbb{1}_{|z|\le 1}\,\nu(\mathrm{d}z)\right)\right],$$

where Σ is the covariance matrix of the Brownian motion W.
We see from this formula that the Lévy process can be determined by the so-called generating triplet (Σ,ν,μ) – which is the covariance matrix Σ of the Brownian motion part, the Lévy measure ν and the drift term μ. Depending on the jumps, there are two noteworthy special cases of Lévy processes:
First, consider the case where L has only bounded jumps, i.e. there is some $A > 1$ s.t. for any $t > 0$, $|\Delta L_t| < A$ a.s.. The Poisson process is an obvious example for this case. Then the integral above can be reduced to

$$L_t = \tilde{\mu}t + W_t + \int_{|z|\le A} z\,\tilde{N}(\mathrm{d}z,\mathrm{d}s),$$

by defining $\tilde{\mu} = \mu + \int_{1<|z|\le A} z\;\nu(\mathrm{d}z)$.

On the other hand, if the sum of the jumps of L is absolutely convergent, i.e. $\int_{|z|\leq 1} |z|\, \nu(\mathrm{d}z) < \infty$, then the compensated Poisson random measure is not needed and the Lévy process is given by

$$L_t = \bar{\mu} t + W_t + \int_{|z|>0} z\, N(\mathrm{d}z, \mathrm{d}s),$$

with $\bar{\mu} = \mu - \int_{|z|\leq 1} z\, \nu(\mathrm{d}z)$.

Quadratic variation and Itô's formula for Lévy-type integrals

Using the Lévy-Itô decomposition we now turn to integration w.r.t. Lévy processes. It will be convenient to work in the framework of the following Lévy-type stochastic integral (cf. [App04, p. 251]).

$$\begin{aligned} X_t = X_0 + \int_0^t g(s)\,\mathrm{d}s + \int_0^t G(s)\,\mathrm{d}W_s \\ + \int_0^t \int_{|z|\leq 1} \varphi(s,z)\,\tilde{N}(\mathrm{d}z,\mathrm{d}s) + \int_0^t \int_{|z|>1} \psi(s,z)\,N(\mathrm{d}z,\mathrm{d}s), \end{aligned} \tag{2.2}$$

with $g\colon \mathbb{R}_+ \to \mathbb{R}^d$, $G\colon \mathbb{R}_+ \to \mathbb{R}^{d\times m}$ and $\varphi, \psi\colon \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^{d\times m}$ being predictable processes, such that

$$\mathbb{E}\int_0^t |g(s)|\,\mathrm{d}s < \infty, \quad \mathbb{E}\int_0^t |G(s)|^2\,\mathrm{d}s < \infty \text{ and } \mathbb{E}\int_0^t \int_{|z|\leq 1} |\varphi(s)|^2 \nu(\mathrm{d}z)\,\mathrm{d}s < \infty.$$

The integral w.r.t. the compensated Poisson random measure is a discontinuous martingale. We further see (cf. [Kun04, Lemma 2.4]) that

$$\left\langle \int_0^\cdot \int_{|z|\leq 1} \varphi(s,z)\,\tilde{N}(\mathrm{d}z,\mathrm{d}s) \right\rangle_t = \int_0^t \int_{|z|\leq 1} |\varphi(s,z)|^2\,\nu(\mathrm{d}z)\,\mathrm{d}s,$$

and

$$\mathbb{E}\left|\int_0^t \int_{|z|\leq 1} \varphi(s,z)\,\tilde{N}(\mathrm{d}z,\mathrm{d}s)\right|^2 = \mathbb{E}\left[\int_0^t \int_{|z|\leq 1} |\varphi(s,z)|^2\,\nu(\mathrm{d}z)\,\mathrm{d}s\right].$$

The process X given in equation (2.2) is a semimartingale, which means that Itô's formula can be applied to X. Note for this that we can neatly distinguish the parts of X: The integrals w.r.t. the Lebesgue measure and the PRM N are processes of finite variation. On the other hand, the integral w.r.t. the Brownian motion is a continuous martingale and the integral w.r.t. $\tilde{N}$ a purely discontinuous martingale. This means, applying Itô's formula to X gives us the following theorem.

Theorem 2.7. *Let $F\colon \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable. Then*

$$
\begin{aligned}
F(X_t)-&F(X_0) \\
&= \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}(X_{s-}) g_i(s)\,\mathrm{d}s + \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}(X_{s-}) G_i(s)\,\mathrm{d}W_s \\
&+ \sum_{i,j=1}^{d} \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(Y_{s-}) G_i(s) G_j(s)\,\mathrm{d}\langle W, W\rangle_s \\
&+ \int_0^t \int_{|z|\le 1} \left[F(X_{s-}+\varphi(s,z)) - F(X_{s-})\right]\,\tilde{N}(\mathrm{d}z,\mathrm{d}s) \\
&+ \int_0^t \int_{|z|>1} \left[F(X_{s-}+\psi(s,z)) - F(X_{s-})\right]\,N(\mathrm{d}z,\mathrm{d}s) \\
&+ \int_0^t \int_{|z|\le 1} F(X_{s-}+\varphi(s,z)) - F(X_{s-}) - \sum_{i=1}^{d} \varphi^i(s,z) \frac{\partial F}{\partial x_i}(X_{s-})\ \nu(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}
$$

2.3 Integration w.r.t. the Compensated Poisson Random Measure in a Banach Space

In Chapters 5 and 6 the integrands of the stochastic integrals take values in a Hilbert space. The construction of stochastic integrals in Hilbert spaces w.r.t. martingales is standard, see for example [PZ07, Chapter 8.2]. However, for Chapter 5 a particular version of the stochastic integral w.r.t. a compensated Poisson random measure $\tilde{N}$ is used, that works for integrands which are not necessarily predictable. In this section, we present a definition for integration which was introduced in [BH09] and works for progressively measurable integrands in spaces of martingale type p for $p \in (1, 2]$. We start with some basic definitions.

Definition 2.8. *A stochastic process $(X_t)_{t\ge 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a Banach space E is called progressively measurable w.r.t. $\mathbb{F}$, if the map $X\colon [0,t]\times\Omega \to E$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.*

Definition 2.9. *Let $p \in [1, 2]$ be fixed and E be a Banach space. We say that E is of martingale type p, if there exists a constant $L_p(E) > 0$, s.t. for every E-valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds:*

$$
\sup_n \mathbb{E}|M_n|^p \le L_p(E) \sum_{n=0}^{N} \mathbb{E}|M_n - M_{n-1}|^p,
$$

where we set $M_{-1} = 0$.

Let now N be a PRM on a measurable space $(S, \mathcal{S})$, with $\tilde{N}$ being the compensated PRM and ν the (non-negative) intensity measure. Furthermore, we denote by $\mathcal{M}^p_{step}(0, \infty; L_p(S, \nu; E))$ the space of progressively measurable step processes $\xi \colon \mathbb{R}_+ \to L_p(S, \nu; E)$ with

$$\mathbb{E} \int_0^\infty |\xi(t)|^p \, \mathrm{d}t < \infty.$$

Definition 2.10. *Let E be a real separable Banach space of martingale type p, with $p \in (1, 2]$. Let ξ be in $\mathcal{M}^p_{step}(0, \infty; L_p(S, \nu; E))$ and have the representation*

$$\xi(t, x) = \sum_{j=1}^n \xi(t_j, x) \mathbb{I}_{(t_{j-1}, t_j]}(t)$$

for some $0 = t_0 < t_1 < \cdots < t_n < \infty$. We define the integral of ξ w.r.t. the compensated Poisson random measure $\tilde{N}$ as

$$\tilde{I}(\xi) = \sum_{j=1}^n \int_S \xi(t_j, x) \, \tilde{N}(\mathrm{d}x, (t_{j-1}, t_j]).$$

It is shown in [BH09, Appendix C] that there is a unique bounded linear operator

$$I : \mathcal{M}^p(0, \infty; L_p(S, \nu; E)) \to L_p(\Omega, \mathcal{F}, E),$$

that extends the operator $\tilde{I}$ from the dense set of step functions to all functions in the space $\mathcal{M}^p(0, \infty; L_p(S, \nu; E))$, which is the space of progressively measurable processes ξ with

$$\mathbb{E} \int_0^\infty \|\xi(t)\|^p_{L_p(S, \nu; E)} \, \mathrm{d}t < \infty.$$

Furthermore the following important inequality is proven, see [BH09, Theorem C.1]:

Lemma 2.11. *For every $\xi \in \mathcal{M}^p(0, \infty; L_p(S, \nu; E))$*

$$\mathbb{E} \left| \int_0^t \int_S \xi(r, z) \, \tilde{N}(\mathrm{d}z, \mathrm{d}r) \right|^p_E \leq C_p \mathbb{E} \int_0^t \int_S |\xi(r, z)|^p_E \, \nu(\mathrm{d}z) \, \mathrm{d}r, \quad t \geq 0,$$

where C_p is independent of ξ.

The Banach space E will mostly be a fractional Sobolev space H^θ in the following, and S will be $\mathbb{R}^d$. When we write the integrals w.r.t. to the compensated PRM in these spaces, we will always mean the definition above.

2.4 Analytical Preliminaries

Function spaces and basic notation

We first give some basic notations regarding function spaces and differentiability: Let $d, m, n \geq 1$. By C^n we denote n times continuously differentiable functions. For a mapping $F\colon \mathbb{R}^d \mapsto \mathbb{R}^m$, DF is the Jacobian (gradient) matrix, namely,

$$DF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_d} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_d} \end{pmatrix}, \tag{2.3}$$

and in particular, for $F\colon \mathbb{R}^d \mapsto \mathbb{R}$, $DF = (\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_d}) = \nabla^T F$.
We write ∂^α for the partial derivative with multiindex α and ∂_x for the partial derivative w.r.t. the coordinate x.
We denote by C_b^n the space of n times continuously differentiable bounded functions with bounded derivatives and write $\|f\|$ for the supremum norm of $f \in C_b^n$. C_c^n denotes the space of continuously differentiable functions with compact support.

In Chapters 5 and 6 we will be mostly working in fractional Sobolev spaces. We give some definitions, results and remarks on notation here. For further information we refer the reader to [HT08] and also the appendix of this work.
Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing functions and let $\mathcal{S}'(\mathbb{R}^d)$ be its dual space, also known as the space of tempered distributions. Let $L^2(\mathbb{R}^d)$ be the Hilbert space of equivalence classes of square-integrable functions $f\colon \mathbb{R}^d \to \mathbb{C}$ with scalar product $\langle f, g\rangle_2 = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,\mathrm{d}x$ and the associated norm $\|f\|_2^2 := \int_{\mathbb{R}^d} |f(x)|^2\,\mathrm{d}x$. On $\mathcal{S}(\mathbb{R}^d)$ or $\mathcal{S}'(\mathbb{R}^d)$ respectively we define the Fourier transform $\mathcal{F}$ such that for $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$(\mathcal{F}\varphi)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\xi}\varphi(x)\,\mathrm{d}x,$$

and for $T \in \mathcal{S}'(\mathbb{R}^d)$, $\mathcal{F}T$ is the functional on $\mathcal{S}'(\mathbb{R}^d)$, such that $\mathcal{F}T(\varphi) = T(\mathcal{F}\varphi)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$.
For $\theta \in \mathbb{R}$, $H^\theta(\mathbb{R}^d)$ denotes the fractional Sobolev space, namely a separable Hilbert space

$$H^\theta(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d)\colon (1+\xi^2)^{\theta/2}(\mathcal{F}f)(\xi) \in L^2(\mathbb{R}^d)\}$$

with the norm
$$\|f\|_{\theta,2} := \left\|(1+\xi^2)^{\theta/2}(\mathcal{F}f)(\xi)\right\|_2,$$
see e.g. [HT08]. In Chapter 6 we consider the one-dimensional case and work in the restriction of $H^\theta(\mathbb{R})$ to $\mathbb{R}_+$, denoted by $H^\theta(\mathbb{R}_+)$. We equip this space with the norm
$$\|g\|_{H^\theta(\mathbb{R}_+)} := \inf_{\widetilde{g}|_{\mathbb{R}_+}=g} \|\widetilde{g}\|_{\theta,2}.$$
To define the associated scalar product in $H^\theta(\mathbb{R}_+)$, note that for every $f \in H^\theta(\mathbb{R}_+)$ there is a unique extension $\operatorname{ext} f$ to $\mathbb{R}$, such that $\|f\|_{H^\theta(\mathbb{R}_+)} = \|\operatorname{ext} f\|_{\theta,2}$ and such that the relation

$$\langle f, g\rangle_{H^\theta(\mathbb{R}_+)} := \langle \operatorname{ext} f, \operatorname{ext} g\rangle_{\theta,2}$$

defines a scalar product on $H^\theta(\mathbb{R}_+)$, see Lemma A.1. Completeness and separability of $H^\theta(\mathbb{R}_+)$ then follow from the completeness and separability of $H^\theta(\mathbb{R})$.

At some points we will assume for $H_2^\theta(\mathbb{R}^d)$ that $\theta > \frac{d}{2}$. This is due to the Sobolev embedding Theorem (see for example [HT08, Theorem 3.32]), which states that for $\theta > \frac{d}{2} + n$, the space $H_2^\theta(\mathbb{R}^d)$ can be embedded into $C^n(\mathbb{R}^d)$. This specifically means, that for $\theta > \frac{d}{2}$ we find a continuous function in every equivalence class in $H_2^\theta(\mathbb{R})$. Thus, for $f \in H_2^\theta(\mathbb{R}^d)$ and fixed $x \in \mathbb{R}^d$, we can evaluate f in the point x, defining $f(x)$ as the value of the continuous representative from the equivalence class at the point x.

C_0-semigroups of contractions

We give a very short introduction to the theory of C_0-semigroups. It is based on [Paz83]. Throughout this section let E and F be Banach spaces. We say an operator $A\colon E \to F$ is bounded, if there is $c \in \mathbb{R}$, such that

$$\|Ax\|_F \le c\|x\|_E \text{ for all } x \in X.$$

The space $L(E,F)$ of all bounded, linear operators $A\colon E \longrightarrow F$ becomes a Banach space, when equipped with the norm

$$\|A\|_{L(E,F)} = \sup\{\|Ax\|_F : x \in E \text{ with } \|x\|_E \le 1\}.$$

We write Id for the identity operator on E and $L(E) := L(E,E)$.

Definition 2.12. *Let $(S(t))_{t\ge 0}$, $S(t)\colon E \to E$, be a family of bounded linear* *operators. We call $S(t)$ a strongly continuous semigroup or C_0-semigroup, if the following conditions are satisfied:*

(i) $S(0) = \mathrm{Id}$,

(ii) $S(t+s) = S(t)S(s)$ *for all* $t, s \geq 0$,

(iii) $\lim_{t \searrow 0} S(t)x = x$ *for all* $x \in E$.

If in addition $\|S(t)\|_{L(E)} \leq 1$ for all $t, s \geq 0$, we call S(t) contractive or C_0-semigroup of contractions.

Definition 2.13. *Let $S(t)$ be a C_0-semigroup and*

$$D(A) := \{x \in E \colon \lim_{t \searrow 0} \frac{S(t)x - x}{t} \text{ exists}\}.$$

We call the linear operator A defined on $D(A)$ by

$$Ax = \lim_{t \searrow 0} \frac{S(t)x - x}{t}$$

infinitesimal generator of the semigroup $S(t)$, and $D(A)$ the domain of A.

In the following we state some properties of C_0-semigroups which will be needed later. For proofs and further reading we refer the reader to [Paz83].

Lemma 2.14. *Let $S(t)$, $t \geq 0$, be a C_0-semigroup and A its infinitesimal generator. Then*

(i) for all $x \in D(A)$ we have $S(t)x \in D(A)$ and

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax,$$

(ii) and for all $x \in E$ we have $\int_0^t S(s)xds \in D(A)$ and

$$A\left(\int_0^t S(s)xds\right) = S(t)x - x.$$

Definition 2.15. *An operator $A \in L(E)$ is called closed, if for every sequence (x_n) in $D(A)$ with $Ax_n \xrightarrow[X]{} y \in X$ there is $x \in D(A)$ with $x_n \xrightarrow[E]{} x$ and $Ax = y$.*

Definition 2.16. *Let A be a linear operator in E. Then the resolvent set $\rho(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which $(\lambda \,\mathrm{Id} - A)^{-1}$ is a bounded linear operator in E.*

The resolvent set is needed for the well-known Hille-Yosida Theorem for C_0-semigroups. The following version of the theorem is taken from [Paz83, Theorem 3.1], and gives sufficient conditions for A to be the generator of a semigroup, which is not only strongly continuous but also contractive.

Theorem 2.17. *A linear (possibly unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions in E if and only if*

(i) A is closed and $D(A)$ is dense in E,

(ii) $(0,\infty) \subset \rho(A)$ and for all $\lambda > 0$

$$\left\|(\lambda \operatorname{Id} - A)^{-1}\right\|_{L(E,D(A))} \leq \frac{1}{\lambda}.$$

2.5 Deterministic Differential Equations

Looking at some simple examples of deterministic differential equations, namely non-autonomous ODEs and simple transport equations, is a crucial step to solving stochastic differential equations with Lévy noise in the sense of the Marcus integration. On the one hand, solving deterministic differential equations is an essential part of the construction of the Marcus integral. On the other hand the proofs in this section also exemplify the use of characteristics to solve the transport equation, which works similarly in the deterministic setting as for the stochastic PDE in Chapter 4.

Ordinary differential equations

We start by reiterating some very basic results about existence and uniqueness of ODEs and continue with some important qualitative results about the solutions. The results given in this section are based on the first part of [Tes12] and [Kön04, Section 4.6].

For our purposes it is enough to look at global solutions in $\mathbb{R}^d$. For some $T > 0$, let $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. We consider solutions to the equation

$$\dot{v}(t) = f(t, v(t)), \quad v(0) = x, \tag{2.4}$$

where $x \in \mathbb{R}^d$ and $\dot{v}$ denotes the derivative of v w.r.t. t.
It is well-known that there is a unique global solution for (2.4) if we assume a

global Lipschitz condition for f, namely

$$\sup_{x\neq y\in\mathbb{R}^d} \frac{|f(t,x)-f(t,y)|}{|x-y|} < \infty, \quad t\in[0,T],$$

see for example [Tes12, Corollary 2.6].

Looking at the dependence of the solution on the initial condition we see that if f is in C^1, the solution defines a flow of diffeomorphisms $x\mapsto v(t;x)$, and the gradient matrix $Dv(u,x)$ w.r.t. x solves the equation

$$w(t,x) = \mathrm{Id} + \int_0^t Df(u,v(u,x))w(u,x)\ \mathrm{d}u, \tag{2.5}$$

where Id is the identity matrix of size $d\times d$.

When used for the construction of Marcus integrals, the deterministic differential equations get a parameter $z\in\mathbb{R}^m$ from the jumps of the Lévy process. Specifically, in Chapter 3 we consider the solution to

$$\dot{v}(s,x,z) = f(s,v,z), \quad v(0,x,z) = x.$$

Assume for $s\geq 0$, $z\in\mathbb{R}^m$ and $k\geq 2$, that $f(s,\cdot,z)\in C^k(\mathbb{R}^d,\mathbb{R}^d)$. Then the derivatives of v w.r.t. z exist up to order k.
One special case is when $f(s,x,z) = \varphi(x)z$, where $\varphi\in C^2(\mathbb{R}^d,\mathbb{R}^{d\times m})$. From [Tes12, Theorem 2.12.] we can gather that in this case the derivatives up to order 2 solve the following differential equations for $1\leq j\leq m$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\partial_{z_j}v(t,z)\right) &= \varphi_j(v) + D(\varphi(x)z)|_{x=v(t,z)}\partial_{z_j}v(t,z)\\ \partial_{z_j}v(0,z) &= 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\partial_{z_jz_i}v(t,z)\right) = \Big(&D(\varphi(x)z)\partial_{z_jz_i}v(t,z) + D(D(\varphi(x)z)\partial_{z_i}v(t,z))\partial_{z_j}v(t,z)\\ &+D(\varphi_j(x))\partial_{z_i}v(t,z) + D(\varphi_i(x))\partial_{z_j}v(t,z)\Big)\Big|_{x=v(t,z)}\\ \partial_{z_jz_i}v(0,z) = 0,&\end{aligned}$$

where φ_j is the j-th column vector of φ.

The transport equation

The first-order transport equation is given by

$$\begin{aligned} \dot{g}(t,x) &= \nabla^T g(t,x)\alpha(x), \\ g(0,x) &= g_0(x), \end{aligned} \tag{2.6}$$

where $\alpha\colon \mathbb{R}^d \to \mathbb{R}^d$ is a C^1 function of linear growth and $g_0\colon \mathbb{R}^d \to \mathbb{R}$ is some smooth initial condition.
Let $v(t,x)$ be the solution flow generated by equation (2.4) with $f(x) = -\alpha(x)$. The solution to the transport equation can be constructed using the inverse flow of $v(t,x)$. Consider the non-autonomous ODE

$$y(t,x) = x + \int_0^t (Dv(u,y(u,x))^{-1}\alpha(v(u,y(u))) \,\mathrm{d}u.$$

We easily see that the solution $y(t,x)$ to this equation gives us the inverse flow of $v(t,x)$: Obviously, $v(0,y(0,x)) \equiv x$. For $t > 0$ we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}v(t,y(t,x)) &= \dot{v}(t,y(t,x)) + Dv(t,y(t,x))\dot{y}(t,x) \\ &= -\alpha(v(t,y(t,x))) + Dv(t,y(t,x))(Dv(t,y(t,x)))^{-1}\alpha(v(t,y(t,x))) \\ &= 0. \end{aligned}$$

We can now prove the existence of a solution to (2.6).

Lemma 2.18. *We define*

$$g(t,x) = g_0(y(t,x)). \tag{2.7}$$

Then g solves the equation (2.6)*.*

Proof. For the initial condition we immediately see that $g_0(v(0,x)) = g_0(x)$. The chain rule then implies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}g_0(y(t,x)) =& \nabla^T g_0(y(t,x)) \cdot \dot{y}(u,x) \\ =& \nabla^T g_0(y(t,x)) \cdot (Dv(u,y(u,x))^{-1}\alpha(v(u,y(u))). \end{aligned}$$

Since y is the inverse flow of v we also have

$$\begin{aligned} \alpha(v(u,y(u))) &= \alpha(x), \\ (Dv(u,y(u,x))^{-1} &= Dy(u,x), \end{aligned}$$

and therefore

$$\begin{aligned}\nabla^T\Big(g_0(y(t,x))\Big)\alpha(x) &= \nabla^T g_0(y(t,x))Dy(t,x)\alpha(x)\\ &= \nabla^T g_0(y(t,x))\Big(Dv(t,y(t,x))\Big)^{-1}\alpha(x).\end{aligned}$$

□

Eventually, we consider the slightly more complex first order equation of the following form:

$$\begin{aligned}&\dot g(t,x) = \nabla^T g(t,x)\alpha(x) + g(t,x)\beta(x) + \sigma(x), \quad t\in[0,T]\\ &g(0,x) = g_0(x),\end{aligned}$$

where α, β and σ are again C^1 functions of linear growth. This is the deterministic analogon of the stochastic equation we study in Chapter 4. To solve it we use the method of characteristics which gives us the following result:

Lemma 2.19. *Let v be the solution to* (2.4) *with $f(t,x) = -\alpha(x)$, and y the inverse flow of v. Then the solution g satisfies*

$$g(t,x) = \mathrm{e}^{\int_0^t \beta(y(s,x))\,\mathrm{d}s}\Big[g_0(y(t,x)) + \int_0^t \mathrm{e}^{\int_0^s \beta(y(r,x))\,\mathrm{d}r}\sigma(y(s,x))\,\mathrm{d}s\Big].$$

Proof. We look at the enlarged $(d+2)$-dimensional system of differential equations

$$\dot X_t = \begin{pmatrix}\dot v(t,x)\\ \dot X_t^{d+1}\\ \dot X_t^{d+1}\end{pmatrix} = \begin{pmatrix}-\alpha(v(t))\\ -X_t^{d+1}\beta(v(t))\\ -X_t^{d+1}\sigma(v(t))\end{pmatrix}$$

The inverse flow is X_{-t}. Consider the functions

$$\Psi(x, x_{d+1}, x_{d+2}) := x_{d+1}g_0(x) + x_{d+2}$$

and

$$g(t,x,x_{d+1},x_{d+2}) := \Psi(X_{-t}(x,x_{d+1},x_{d+2})) = X_{-t}^{d+1}g_0(v(-t,x)) + X_{-t}^{d+2}$$

We write $\bar{x}$ for the vector $(x, x_{d+1}, x_{d+2})^T$. Then

$$\begin{aligned}
&g(t, x, x_{d+1}, x_{d+2}) \\
&= \Psi(x, x_{d+1}, x_{d+2}) + \int_0^t \nabla_{\bar{x}}^T \Psi(X_{-s}(\bar{x})) \dot{X}_{-s}(\bar{x}) \,\mathrm{d}s \\
&= \Psi(x, x_{d+1}, x_{d+2}) + \int_0^t \nabla_{\bar{x}}^T \Psi(X_{-s}(\bar{x})) \Big(DX_s(X_{-s}(\bar{x})) \Big)^{-1} \begin{pmatrix} \alpha(x) \\ x^{d+1}\beta(x) \\ x^{d+1}\sigma(x) \end{pmatrix} \mathrm{d}s \\
&= \Psi(x, x_{d+1}, x_{d+2}) + \int_0^t \nabla_{\bar{x}}^T \Psi(X_{-s}(\bar{x})) DX_{-s}(\bar{x}) \begin{pmatrix} \alpha(x) \\ x^{d+1}\beta(x) \\ x^{d+1}\sigma(x) \end{pmatrix} \mathrm{d}s \\
&= \Psi(x, x_{d+1}, x_{d+2}) + \int_0^t \nabla_{\bar{x}}^T \Big(\Psi(X_{-s}(\bar{x})) \Big) \begin{pmatrix} \alpha(x) \\ x^{d+1}\beta(x) \\ x^{d+1}\sigma(x) \end{pmatrix} \mathrm{d}s
\end{aligned}$$

Thus setting $x_{d+1} = 1$, $x_{d+2} = 0$ we get

$$\begin{aligned}
\dot{g}(t, x) = \dot{g}(t, x, 1, 0) &= \nabla_{\bar{x}}^T \Big(\Psi(X_{-s}(x, 1, 0)) \Big) \begin{pmatrix} \alpha(x) \\ \beta(x) \\ \sigma(x) \end{pmatrix} \\
&= \nabla_x^T g(t, x)\alpha(x) \Big|_{x_{d+1}=1, x_{d+2}=0} + \partial_{x_{d+1}} g(t, x, x_{d+1}, x_{d+1})\beta(x) \Big|_{x_{d+1}=1, x_{d+2}=0} \\
&\qquad + \partial_{x_{d+2}} g(t, x, x_{d+1}, x_{d+1})\sigma(x) \Big|_{x_{d+1}=1, x_{d+2}=0}
\end{aligned}$$

Here

$$\begin{aligned}
X_t^{d+1}(x, x_{d+1}) &= x_{d+1} \mathrm{e}^{-\int_0^t \beta(v(s,x))} \,\mathrm{d}s, \\
X_t^{d+2}(x, x_{d+1}, x_{d+2}) &= x_{d+2} - x_{d+1} \int_0^t \mathrm{e}^{-\int_0^s \beta(v(r,x)) \,\mathrm{d}r} \sigma(v(s, x)) \,\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\partial_{x_{d+1}} X_t^{d+1}(x, x_{d+1}) &= \mathrm{e}^{-\int_0^t \beta(v(s,x))} \,\mathrm{d}s = X_t^{d+1}(x, 1), \\
\partial_{x_{d+1}} X_t^{d+2}(x, x_{d+1}, x_{d+2}) &= -\int_0^t \mathrm{e}^{-\int_0^s \beta(v(r,x)) \,\mathrm{d}r} \sigma(v(s, x)) \,\mathrm{d}s = X_t^{d+1}(x, 1), \\
\partial_{x_{d+2}} X_t^{d+1}(x, x_{d+1}, x_{d+2}) &= 0, \text{ and} \\
\partial_{x_{d+2}} X_t^{d+2}(x, x_{d+1}, x_{d+2}) &= 1.
\end{aligned}$$

Hence

$$\dot{g}(t, x) = \nabla_x^T g(t, x)\alpha(x) + g(t, x)\beta(x) + \sigma(x).$$

$\square$

Chapter 3

The Marcus SDE

Looking at stochastic differential equations, there are different ways to interpret an equation. The difference between the Itô and the Stratonovich SDE is well-known and there are famous results on the convergence of Wong–Zakai approximations to the solution of a Stratonovich type equation in the case of the Brownian motion. In the context of SDEs driven by Lévy processes similar results for the convergence of Wong–Zakai approximations as for the Brownian motion can be obtained using the canonical (Marcus) integral.
After an introduction to the Marcus SDE in the first section, this chapter is focused on Itô-type formulae for solutions of Marcus SDE, the inverse flows and Wong–Zakai approximations. The results of Sections 3.2 and 3.3 were accepted to be published in [HPar].

3.1 Existence and Properties of the Solution to the Marcus SDE

Let W be an $\mathbb{R}^m$-valued Brownian motion and Z an m-dimensional pure jump Lévy process given by

$$Z_t = \int_0^t \int_{|z|\leq 1} z\,\tilde{N}(\mathrm{d}z, \mathrm{d}t) + \int_0^t \int_{|z|>1} z\,N(\mathrm{d}z, \mathrm{d}t),$$

where N is the Poisson random measure with intensity measure $\nu(\mathrm{d}z)$ and $\tilde{N}$ is the compensated PRM. For $d \geq 1$, let

$$\begin{aligned} f(x, r, \omega)&\colon \mathbb{R}^d \times \mathbb{R}_+ \times \Omega \to \mathbb{R}^d, \\ F(x, r, \omega)&\colon \mathbb{R}^d \times \mathbb{R}_+ \times \Omega \to \mathbb{R}^{d\times m}, \\ \varphi(x, r, z, \omega)&\colon \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^m \times \Omega \to \mathbb{R}^d \end{aligned}$$

be predictable processes with parameters x and (x, z) respectively. In what follows, we will often omit the dependence on $\omega \in \Omega$.

Consider a semimartingale Φ with parameter x given by

$$\begin{aligned} \Phi(x, t) = &\int_0^t f(x, r)\,\mathrm{d}r + \int_0^t F(x, r)\,\mathrm{d}W_r \\ &+ \int_0^t \int_{|z|\leq 1} \varphi(x, r, z)\,\tilde{N}(\mathrm{d}z, \mathrm{d}r) + \int_0^t \int_{|z|>1} \varphi(x, r, z)\,N(\mathrm{d}z, \mathrm{d}r). \end{aligned} \tag{3.1}$$

We define the canonical Marcus SDE with the generator Φ, which we formally write as

$$X_t(x) = x + \int_0^t \Phi(X_r(x), \diamond\,\mathrm{d}r). \tag{3.2}$$

Before giving a detailed formula for what this equation means, it is important to understand how to treat the jumps of the Lévy process in this equation. Whenever the driving Lévy process Z makes a jump, the solution flow should make a jump at the same time but not simply with the same height as the Lévy process. Instead, the solution of the Marcus SDE shall fly with infinite speed along the integral curve of the vector field $\varphi(\cdot, t, \Delta Z_t)$.

To this end, for each $r \geq 0$, $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ and $\omega \in \Omega$, consider an ODE (the Marcus ODE) for the function $v = v(u) = v(u; x, r, z)$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}u} v(u) = \varphi(v(u), r, z), \quad u \in [0, 1], \\ v(0) = x. \end{cases} \tag{3.3}$$

Assuming for now that φ is regular enough, there is a global solution v for $u \in \mathbb{R}$ which we denote by

$$\mathbf{e}^{u\varphi(\cdot;r,z)}(x) := v(u; x, r, z), \quad u \in \mathbb{R}. \tag{3.4}$$

In particular we define the exponential mapping

$$\mathbf{e}^{\varphi(\cdot;r,z)}(x) := v(1; x, r, z). \tag{3.5}$$

The Marcus SDE (3.2) is then defined as the Itô SDE

$$
\begin{aligned}
X_t(x) = x &+ \int_0^t f(X_{r-}(x), r)\,\mathrm{d}r + \int_0^t F(X_{r-}(x), r) \circ \mathrm{d}W_r \\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(x)) - X_{r-}(x)\Big)\,\tilde{N}(\mathrm{d}z, \mathrm{d}r) \\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(x)) - X_{r-}(x) - \varphi(X_{r-}(x), r, z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r \\
&+ \int_0^t \int_{|z|> 1} \Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}) - X_{r-}\Big)\,N(\mathrm{d}z, \mathrm{d}r).
\end{aligned}
$$

The existence and uniqueness of solutions to these equations have been studied for example by [KPP95] and [FK99a]. From the general theory of SDEs we naturally expect the coefficients to be Lipschitz continuous. Due to the nature of the Stratonovich integral, which includes terms of the kind $DF \cdot F$, it is necessary to impose slightly more regularity than in the case of Itô SDEs. The following Theorem 3.2 uses slightly stricter assumptions on the regularity of the coefficients than would be needed for just the existence of solutions, giving us in turn more regularity for the solution. Namely, we derive the following conditions from Condition A^* and Condition B^* of [FK99a]:

Assumption 3.1. *There is $\delta \in (0,1)$, s.t.*

***H**$_f$:*

$$
\begin{aligned}
&f(\cdot, r) \in C^{2+\delta}(\mathbb{R}^d, \mathbb{R}^d), \\
&\sup_x \frac{|f(x,r)|}{1+|x|} \le K, \\
&\|\partial^\alpha f^i(\cdot, r)\| \le K, \quad 1 \le i \le d, \quad |\alpha| = 1, 2, \\
&\|\partial^\alpha f^i(x, r) - \partial^\alpha f^i(y, r)\| \le L\|x-y\|^\delta, \quad 1 \le i \le d, \quad |\alpha| = 2.
\end{aligned}
$$

***H**$_F$:*

$$
\begin{aligned}
&F(\cdot, r) \in C^{3+\delta}(\mathbb{R}^d, \mathbb{R}^{d\times m}), \\
&\sup_x \frac{|F^i_j(x,r)|}{1+|x|} \le K, \quad 1 \le i \le d,\ 1 \le j \le m, \\
&\|\partial^\alpha F^i_j(\cdot, r)\| \le K, \quad 1 \le i \le d,\ 1 \le j \le m, \quad |\alpha| = 1, 2, \\
&\|\partial^\alpha F^i_j(\cdot, r) F^k_l(\cdot, r)\| \le K, \quad 1 \le i,k \le d,\ 1 \le j,l \le m, \quad |\alpha| = 2, 3, \\
&\|\partial^\alpha F^i_j(x, r) - \partial^\alpha F^i_j(y, r)\| \le L\|x-y\|^\delta, \quad 1 \le i \le d,\ 1 \le j \le m, \quad |\alpha| = 3.
\end{aligned}
$$

***H**$_\varphi$:*

$$\varphi(\cdot, r, z) \in C^{2+\delta}(\mathbb{R}^d, \mathbb{R}^{d\times m}),$$

and there are non-negative functions $K_0(z)$, $K_1(z)$ *and* $L_1(z)$ *and* $L_2(z)$, *such that*

$$\int_{|z|\le 1} \Big(K_0(z)^2 + K_1(z)^2 + L_0(z) + L_1(z)^2\Big)\, \nu(\mathrm{d}z) < \infty$$

and

$$\begin{aligned}
&\sup_x \frac{|\varphi^i_j(\cdot, r, z)|}{1+|x|} \le K_0(z), \quad 1 \le i \le d,\ 1 \le j \le m,\\
&\|\partial^\alpha \varphi^i_j(\cdot, r, z)\| \le K_1(z), \quad 1 \le i, j \le d, \quad 1 \le |\alpha| \le 2,\\
&\|\partial^\alpha \varphi^i_j(x, r, z)\varphi^k_l(x, r, z) - \partial^\alpha \varphi^i_j(y, r, z)\varphi^k_l(y, r, z)\| \le L_0(z)\|x-y\|,\\
&\hspace{16em} 1 \le i, k \le d,\ 1 \le j, l \le m, \quad |\alpha| = 1,\\
&\|\partial^\alpha \varphi^i_j(x, r, z) - \partial^\alpha \varphi^i_j(y, r, z)\| \le L_1(z)\|x-y\|^\delta,\\
&\hspace{16em} 1 \le i \le d,\ 1 \le j \le m, \quad |\alpha| = 2.
\end{aligned}$$

We then get the following theorem from Corollaries 3.2 and 4.2 of [FK99a].

Theorem 3.2. *Under the Assumption 3.1, the Marcus SDE* (3.2) *has a unique solution. The solution has a càdlàg modification* $(X_t)_{t\ge 0}$*, such that the map* $X_t : \mathbb{R}^d \to \mathbb{R}^d$ *is an onto* C^2*-diffeomorphism for any* $t \ge 0$ *almost surely.*

Idea of the proof. The idea of the proof is to show that equation (3.2) can be written as an Itô SDE, which satisfies the conditions of [CN90, Theorem IV.I]. □

Marcus SDE of separating type

In Section 3.4, convergence of Wong–Zakai approximations will be studied for solutions to the classical Marcus SDE, i.e. the case where the coefficients f, F and φ only depend on x. This is also called Marcus SDE of separating type. Formally, we use the following definition:
Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$, $F, \varphi\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be deterministic functions. We call the equation

$$\begin{aligned}
X_t(x) &= x + \int_0^t f(X_{r-}(x))\,\mathrm{d}r + \int_0^t F(X_{r-}(x)) \circ \mathrm{d}W_r\\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\varphi(\cdot)z}(X_{r-}(x)) - X_{r-}(x)\Big)\,\tilde{N}(\mathrm{d}z, \mathrm{d}r)\\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\varphi(\cdot)z}(X_{r-}(x)) - X_{r-}(x) - \varphi(X_{r-}(x), z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r\\
&+ \int_0^t \int_{|z|>1} \Big(\mathbf{e}^{\varphi(\cdot)z}(X_{r-}) - X_{r-}\Big)\,N(\mathrm{d}z, \mathrm{d}r).
\end{aligned} \tag{3.6}$$

Marcus SDE of separating type.
In this case equation (3.6) can be rewritten as

$$X_t(x) = x + \int_0^t f(X_r)\,\mathrm{d}r + \int_0^t F(X_r) \circ \mathrm{d}W_r + \int_0^t \varphi(X_{r-})\,\mathrm{d}Z_r \\ + \sum_{0 \le r \le t} \left(\mathrm{e}^{\varphi(\cdot)\Delta Z_r}(X_{r-}) - X_{r-} - \varphi(X_{r-})\Delta Z_r \right),$$

which coincides with the notation used in [KPP95].

The proof of convergence of approximations in Section 3.4 uses slightly more information on the derivatives of the solution. The following Lemma is based on [KKP19, Theorem 2.2].

Lemma 3.3. *Under Assumption 3.1, let X be the solution to* (3.6)*. Then for any g in C_b^2 there is $C > 0$, s.t. for every $x \in \mathbb{R}^d$, every t in $[0, T]$ and any multiindex α with $1 \le |\alpha| \le 2$*

$$|\partial^\alpha \mathbb{E}_x g(X_t)| \le C.$$

Proof. The proof is given in [KKP19, Section 7]. The only difference here are the weaker assumptions on the coefficients, which only give us the result for $|\alpha| \le 2$, instead of $|\alpha| \le 4$ in [KKP19]. □

3.2 Generalized Itô Formula for Marcus SDEs

Let $X_t(x)$ be a semimartingale with spatial parameter and Y another semimartingale. In this section, the aim is to give a formula for the expression $X_t(Y_t)$.
In the case of continuous semimartingales, a generalized Itô formula, also called the Itô–Wentzell formula, can be found for example in [Kun97, Section 3.3] and [CN90, Section III.3], but these results do not work in the case where both X and Y have jumps.
The main result of this section is the generalized Itô formula for the case where X and Y are solutions to Marcus SDEs driven by the same Poisson random measure. We start by recalling the conventional Itô formula for the Marcus SDE, which works similar to [KPP95, Proposition 4.2].

Theorem 3.4 (Itô's formula for solutions of canonical SDEs)**.** *Let X be the solution of the SDE* (3.2) *and let $\Theta \in C^2(\mathbb{R}^d, \mathbb{R})$. Then*

$$\Theta(X_t) = \Theta(x) + \int_0^t D\Theta\Phi(X_r, \diamond\,\mathrm{d}r), \tag{3.7}$$

where the canonical integral in the r.h.s. of (3.7) *is equal to*

$$\begin{aligned}
&\int_0^t D\Theta(X_{r-}) f(X_{r-}, r)\,\mathrm{d}r + \int_0^t D\Theta(X_{r-}) F(X_{r-}, r) \circ \mathrm{d}W_r \\
&\quad + \int_0^t \int_{|z|\le 1} \Big(\Theta(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-})) - \Theta(X_{r-})\Big) \tilde{N}(\mathrm{d}z, \mathrm{d}r) \\
&\quad + \int_0^t \int_{|z|\le 1} \Big(\Theta(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-})) - \Theta(X_{r-}) - \varphi(X_{r-}, r, z)\Big) \nu(\mathrm{d}z)\,\mathrm{d}r \\
&\quad + \int_0^t \int_{|z|> 1} \Big(\Theta(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-})) - \Theta(X_{r-})\Big) N(\mathrm{d}z, \mathrm{d}r).
\end{aligned}$$

Remark 3.5. *Note that the process* $\Theta(X)$ *has the jumps*

$$\begin{aligned}
\Theta(X_r) - \Theta(X_{r-}) &= \Theta(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-})) - \Theta(X_{r-}) \\
&= \int_0^1 D\Theta(\mathbf{e}^{u\varphi(\cdot,r,z)}(X_{r-}))\varphi(\mathbf{e}^{u\varphi(\cdot,r,z)}(X_{r-}), r, z)\,\mathrm{d}u,
\end{aligned}$$

which justifies the formal writing (3.7).

We now consider a second semimartingale Y which solves the SDE with generator Ψ given by

$$\begin{aligned}
\Psi(x,t) = &\int_0^t g(x,r)\,\mathrm{d}r + \int_0^t G(x,r)\,\mathrm{d}W_r \\
&+ \int_0^t \int_{|z|\le 1} \psi(x,r,z)\,\tilde{N}(\mathrm{d}z, \mathrm{d}r) + \int_0^t \int_{|z|>1} \psi(x,r,z)\,N(\mathrm{d}z, \mathrm{d}r).
\end{aligned}$$

The generalized Itô formula for $X_t(Y_t)$ is given in following Theorem:

Theorem 3.6 (generalized Itô formula for canonical SDEs)**.** *Consider solutions of canonical SDEs with generators* Φ *and* Ψ *such that the functions* f, F, φ *and* g, G, ψ *satisfy Assumptions 3.1 respectively,*

$$X_t(x) = x + \int_0^t \Phi(X_r(x), \diamond\,\mathrm{d}r), \tag{3.8}$$

$$Y_t = y + \int_0^t \Psi(Y_r, \diamond\,\mathrm{d}r). \tag{3.9}$$

Then $X_t(Y_t)$ *satisfies the following formula*

$$X_t(Y_t) = y + \int_0^t \Phi(X_r(Y_r), \diamond\,\mathrm{d}r) + \int_0^t DX\Psi(Y_r, \diamond\,\mathrm{d}r), \tag{3.10}$$

where the latter integrals are understood as

$$
\begin{aligned}
&\int_0^t \Phi(X_r(Y_r), \diamond\,\mathrm{d}r) \\
&\quad = \int_0^t f(X_{r-}(Y_{r-}), r)\,\mathrm{d}r + \int_0^t F(X_{r-}(Y_{r-}), r)\,\mathrm{d}W_r \\
&\quad + \frac{1}{2}\sum_{j=1}^m \int_0^t DF_j(X_{r-}(Y_{r-}), r)F_j(X_{r-}(Y_{r-}), r)\,\mathrm{d}r \\
&\quad + \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-})) - X_{r-}(Y_{r-})\Big)\tilde{N}(\mathrm{d}z, \mathrm{d}r) \\
&\quad + \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-})) - X_{r-}(Y_{r-}) - \varphi(X_{r-}(Y_{r-}), r, z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r \\
&\quad + \int_0^t \int_{|z|>1} \Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-})) - X_{r-}(Y_{r-})\Big)N(\mathrm{d}z, \mathrm{d}r)
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
&\int_0^t DX\Psi(Y_r, \diamond\,\mathrm{d}r) \\
&= \int_0^t DX_{r-}(Y_{r-})g(Y_{r-}, r)\,\mathrm{d}r + \sum_{j=1}^m \frac{1}{2}\int_0^t DX_{r-}(Y_{r-})DG_j(Y_{r-}, r)G_j(Y_{r-}, r)\,\mathrm{d}r \\
&\quad + \frac{1}{2}\sum_{j,k=1}^m \int_0^t (D(DX_{r-})_j)_k(Y_{r-})G_j(Y_{r-}, r)G_k(Y_{r-}, r)\,\mathrm{d}r \\
&\quad + \frac{1}{2}\sum_{j=1}^m \int_0^t DX_{r-}(Y_{r-})DF_j(X_{r-}(Y_{r-}), r)G_j(Y_{r-}, r)\,\mathrm{d}r \\
&\quad + \int_0^t DX_{r-}(Y_{r-})G(Y_{r-}, r)\,\mathrm{d}W_r \\
&\quad + \int_0^t \int_{|z|\le 1} \Big[\mathbf{e}^{\varphi(\cdot,r,z)}\Big(X_{r-}\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-})\Big) - \mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-}))\Big]\,\tilde{N}(\mathrm{d}z, \mathrm{d}r) \\
&\quad + \int_0^t \int_{|z|\le 1} \Big[\mathbf{e}^{\varphi(\cdot,r,z)}\Big(X_{r-}\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-})\Big)\Big) - \mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-})) \\
&\qquad\qquad - DX_{r-}(Y_{r-})\psi(Y_{r-}, r, z)\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}r \\
&\quad + \int_0^t \int_{|z|>1} \Big[\mathbf{e}^{\varphi(\cdot,r,z)}\Big(X_{r-}\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-})\Big) - \mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-}))\Big]\,N(\mathrm{d}z, \mathrm{d}r).
\end{aligned}
\tag{3.12}
$$

Before moving on to the proof of Theorem 3.6, we give some basic estimates for the exponential map $\mathbf{e}^{\varphi(\cdot,r,z)}$.

Lemma 3.7. *Assume that $\varphi(\cdot, r, z) \in C^{2+\delta}(\mathbb{R}^d, \mathbb{R}^d)$ satisfies H_φ in Assumption 3.1. Then $\mathbf{e}^{\varphi(\cdot,r,z)}$ is twice differentiable w.r.t. x and we get the following estimates:*

$$
\begin{aligned}
&(1) \quad |\mathbf{e}^{\varphi(\cdot,r,z)}(x) - x| \le (1 + |x|)K_0(z)\mathrm{e}^{K_0(z)},\\
&(2) \quad |\nabla \mathbf{e}^{\varphi(\cdot,r,z)}(x) - \mathrm{Id}\,| \le K_1(z)\mathrm{e}^{K_1(z)},\\
&(3) \quad |\nabla^2 \mathbf{e}^{\varphi(\cdot,r,z)}(x)| \le (K_1(z))^3\mathrm{e}^{3K_1(z)},\\
&(4) \quad |\mathbf{e}^{\varphi(\cdot,r,z)}(x) - x - \varphi(x, r, z)| \le K_0(z)K_1(z)\mathrm{e}^{K_0(z)}(1 + |x|).
\end{aligned}
$$

Proof. For the first equation we see that

$$
\begin{aligned}
|\mathbf{e}^{\varphi(\cdot,r,z)}(x) - x| &\le \int_0^1 \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{e}^{u\varphi(\cdot,r,z)}(x)\,\mathrm{d}u \le \int_0^1 |\varphi(\mathbf{e}^{u\varphi(\cdot,r,z)}(x), r, z)|\,\mathrm{d}u\\
&\le K_0(z) + K_0(z)\int_0^1 |x|\,\mathrm{d}u + K_0(z)\int_0^1 |\mathbf{e}^{u\varphi(\cdot,r,z)}(x) - x|\,\mathrm{d}u.
\end{aligned}
$$

Then it follows from Gronwall's inequality that

$$
|\mathbf{e}^{\varphi(\cdot,r,z)}(x) - x| \le (1 + |x|)K_0(z)\mathrm{e}^{K_0(z)}.
$$

The derivatives can also be written as solutions to the following differential equations, see for example [Tes12, Theorem 2.10.]:

$$
\begin{aligned}
w(t) &= \mathrm{Id} + \int_0^t \nabla\varphi(\mathbf{e}^{u\varphi(\cdot,r,z)}, r, z)w(u)\,\mathrm{d}u,\\
y(t) &= \int_0^t \nabla^2\varphi(\mathbf{e}^{u\varphi(\cdot,r,z)}, r, z)(w(u))^2 + \nabla\varphi(\mathbf{e}^{u\varphi(\cdot,r,z)}, r, z)y(u)\,\mathrm{d}u,
\end{aligned}
$$

where $\nabla\mathbf{e}^{u\varphi(\cdot,r,z)} = w(u)$ and $\nabla^2\mathbf{e}^{u\varphi(\cdot,r,z)} = y(u)$. Using Gronwall's inequality again gives us the estimates.

To get the last estimate we omit the dependence on r and z for a moment and write in one dimension in favour of better readability. We then define

$$
k(x, u) = \mathbf{e}^{\varphi(\cdot)u}(x) - x - \varphi(x)u,
$$

and see that for some $\xi \in (0, 1)$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}\left(k(x, u) + \varphi(x)u\right) &= \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{e}^{\varphi(\cdot)u}(x) = \varphi(\mathbf{e}^{\varphi(\cdot)u}(x))\\
&= \varphi(k(x, u) + x + \varphi(x)u)\\
&= \varphi(x) + \varphi'(\xi)(k(x, u) + \varphi(x)u).
\end{aligned}
$$

We further see that

$$\frac{\mathrm{d}}{\mathrm{d}u} k(x,u) = \varphi'(\xi)(k(x,u) + \varphi(x)u),$$

and thus

$$\begin{aligned} |k(x,u)| &\leq |\varphi'(\xi)| \int_0^u |k(x,s) + \varphi(x)s| \,\mathrm{d}s \\ &\leq K_1(z) \int_0^u |\mathbf{e}^{\varphi(\cdot)s}(x) - x| \,\mathrm{d}s \\ &\leq K_1(z)(1+|x|)K_0(z)\mathrm{e}^{K_0(z)}. \end{aligned}$$

□

Proof of Theorem 3.6. For the proof of the generalized Itô formula we apply the method of [CN90, Theorem III.3.3]. To simplify the notation, we assume that X and Y, as well as W and Z are one-dimensional processes, i.e. $d = m = 1$. We also assume that $\nu([-1,1]^c) = 0$. Adding large jumps is straightforward.

We write the SDEs for X and Y in the Itô form:

$$\begin{aligned} X_t(x) = x &+ \int_0^t f(X_{r-}(x), r)\,\mathrm{d}r + \int_0^t F(X_{r-}(x), r)\,\mathrm{d}W_r \\ &+ \frac{1}{2}\int_0^t F'(X_{r-}(x), r)F(X_{r-}(x), r)\,\mathrm{d}r \\ &+ \int_0^t \int_{|z|\leq 1} \Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(x)) - X_{r-}(x)\Big)\,\tilde{N}(\mathrm{d}z, \mathrm{d}r) \\ &+ \int_0^t \int_{|z|\leq 1} \Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(x)) - X_{r-}(x) - \varphi(X_{r-}(x), r, z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r \\ Y_t = y &+ \int_0^t g(Y_{r-}, r)\,\mathrm{d}r + \int_0^t G(Y_{r-}, r)\,\mathrm{d}W_r + \frac{1}{2}\int_0^t G'(Y_{r-}, r)G(Y_{r-}, r)\,\mathrm{d}r \\ &+ \int_0^t \int_{|z|\leq 1} \Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - Y_{r-}\Big)\,\tilde{N}(\mathrm{d}z, \mathrm{d}r) \\ &+ \int_0^t \int_{|z|\leq 1} \Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - Y_{r-} - \psi(Y_{r-}, r, z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r. \end{aligned} \tag{3.13}$$

We first perform localizations of the semimartingales $X(x)$ and Y, so that we can assume them to be bounded for the rest of the proof.
We start with Y: Lemma 3.7 implies that

$$|\mathbf{e}^{\psi(\cdot,r,z)}(y) - y| \leq K_0^{\psi}(z)\mathrm{e}^{K_0^{\psi}(z)}(1+|y|) \leq K^{\psi}(1+|y|), \quad |z| \leq 1, y \in \mathbb{R}.$$

Let the initial value $y \in \mathbb{R}$ be fixed. For each $n \geq 1$, let $\tau_n = \inf\{t \geq 0 \colon |Y_t| > n\}$.

Then the stopped process is bounded by construction and (3.2) since

$$|Y_t^{\tau_n}| \le n + |\Delta Y_{\tau_n}| \le n + K^\psi(1+n) =: C(n).$$

In particular,

$$|\mathbf{e}^{\varphi(\cdot,r,z)}(x) - x| \le K^\varphi(1+|x|), \quad |z| \le 1, x \in \mathbb{R}. \tag{3.14}$$

For X we denote

$$A_t^k = \sup_{|y|\le k+1} \Big(|X_t(y)| + \Big|\nabla X_t(y)\Big| + \Big|\nabla^2 X_t(y)\Big| \Big)$$

and let $\sigma_{m,k} = \inf\{t \ge 0 \colon A_t^k > m\}$. By Lemma 3.7 the jumps of $X(x)$ and its derivatives can be estimated by

$$\begin{aligned}
|X_t(x) - X_{t-}^{\sigma_{m,k}}(x)| &= |\mathbf{e}^{\varphi(\cdot,t,z)}(X_{t-}(x)) - X_{t-}(x)| \le (1+|X_{t-}(x)|)K_0^\varphi(z)\mathrm{e}^{K_0^\varphi(z)},\\
|\nabla X_t(x) - \nabla X_{t-}(x)| &= \Big|\nabla \mathbf{e}^{\varphi(\cdot,t,z)}(X_{t-}(x)) - 1\Big| \cdot |\nabla X_{t-}(x)|\\
&\le K_1^\varphi(z)\mathrm{e}^{K_1^\varphi(z)}|\nabla X_{t-}(x)|,
\end{aligned}$$

$$\begin{aligned}
|\nabla^2 X_t(x) - \nabla^2 X_{t-}(x)| &= \Big|\nabla^2 \mathbf{e}^{\varphi(\cdot,t,z)}(X_{t-}(x))\Big(\nabla X_{t-}(x)\Big)^2\Big|\\
&\quad + \Big|\nabla \mathbf{e}^{\varphi(\cdot,t,z)}(X_{t-}(x)) - 1\Big||\nabla^2 X_{t-}(x)|\\
&\le (K_1^\varphi(z))^3 \mathrm{e}^{3K_1^\varphi(z)}|\nabla X_{t-}(x)|^2 + K_1^\varphi(z)\mathrm{e}^{K_1^\varphi(z)}|\nabla^2 X_{t-}(x)|.
\end{aligned}$$

For the stopped process this means

$$\begin{aligned}
\sup_{|x|\le k+1} |X_t^{\sigma_{m,k}}(x)| &\le m + \sup_{|x|\le k+1} |\Delta X_{\sigma_{m,k}-}(x))| \le m + K_0^\varphi(z)\mathrm{e}^{K_0^\varphi(z)}(1+m)\\
&\le C_0(m),
\end{aligned}$$

and analogously

$$\sup_{|x|\le k} |\nabla X_t^{\sigma_{m,k}}(x)| \le C_1(m),$$

$$\sup_{|x|\le k} |\nabla^2 X_t^{\sigma_{m,k}}(x)| \le C_2(m).$$

Clearly, for each $k \ge 1$, $\lim_m \sigma_{m,k} = +\infty$ and $\lim_n \tau_n = +\infty$. Hence we can choose subsequences n_l, m_l, k_l, and define

$$T_l := \tau_{n_l} \wedge \sigma_{m_l,k_l} \to +\infty$$

such that the stopped processes Y^{T_l} and X^{T_l} satisfy for $t \in [0,T]$:

$$\begin{aligned}|Y_t^{T_l}| &\leq l,\\ \sup_{|x|\leq l+1}\Big(|X_t^{T_l}(x)| + \Big|\nabla X_t^{T_l}(x)\Big| + \Big|\nabla^2 X_t^{T_l}(x)\Big|\Big) &\leq C(l),\\ |\Delta Y_t^{T_l}| + \sup_{|x|\leq l+1}\Big(|\Delta X_t^{T_l}(x)| + |\Delta(\nabla X_t^{T_l}(x))| + |\Delta(\nabla^2 X_t^{T_l}(x))|\Big) &\leq D(l)\end{aligned}$$

for some sequences $C(l) \uparrow +\infty$ and $D(l) \uparrow +\infty$.
From now on we will work with the stopped semimartingales and can therefore assume the coefficients g, G, ψ to be uniformly bounded and f, F, φ to be bounded and thus uniformly continuous in the ball of the radius $l+1$.

Consider a sequence of mollifiers $h_n \in C_K^\infty(\mathbb{R},\mathbb{R})$ given by

$$h_n(x) := nh(nx),$$

where $h \in C_K^\infty(\mathbb{R}^d,\mathbb{R})$ is supported on a unit ball $|x| \leq 1$, $h(x) \geq 0$, and such that $\int_{\mathbb{R}} h(x)\,\mathrm{d}x = 1$. Then for each $x \in \mathbb{R}$, the classical Itô formula applied to the semimartingale Y yields

$$\begin{aligned}h_n(Y_t - x) = h_n(y-x) &+ \int_0^t h_n'(Y_{r-} - x) g(Y_{r-}, r)\,\mathrm{d}r\\ &+ \int_0^t h_n'(Y_{r-} - x) G(Y_{r-}, r)\,\mathrm{d}W_r\\ &+ \frac{1}{2}\int_0^t h_n'(Y_{r-} - x) G'(Y_{r-}, r) G(Y_{r-}, r)\,\mathrm{d}r\\ &+ \frac{1}{2}\int_0^t h_n''(Y_{r-} - x) G^2(Y_{r-}, r)\,\mathrm{d}r\\ &+ \int_0^t\int_{|z|\leq 1} h_n'(Y_{r-} - x)\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - Y_{r-} - \psi(Y_{r-}, r, z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r\\ &+ \int_0^t\int_{|z|\leq 1}\Big[h_n\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - x\Big) - h_n(Y_{r-} - x)\Big]\,\tilde{N}(\mathrm{d}z,\mathrm{d}r)\\ &+ \int_0^t\int_{|z|\leq 1}\Big[h_n\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - x\Big) - h_n(Y_{r-} - x)\\ &\qquad - h_n'(Y_{r-} - x)\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - Y_{r-}\Big)\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}r.\end{aligned} \tag{3.15}$$

Next we apply the Itô product formula (see equation (2.1)) to $X(x)h(Y-x)$ to get

$$
\begin{aligned}
&X_t(x)h_n(Y_t-x) = xh_n(y-x)\\
&\begin{aligned}
I_1 &= \\ I_2 &= \\ I_3 &= \\ \\ I_4 &=
\end{aligned}
\left\{\begin{aligned}
&+\int_0^t h_n(Y_{r-}-x)f(X_{r-}(x),r)\,\mathrm{d}r\\
&+\frac{1}{2}\int_0^t h_n(Y_{r-}-x)F'(X_{r-}(x),r)F(X_{r-}(x),r)\,\mathrm{d}r\\
&+\int_0^t\int_{|z|\le 1} h_n(Y_{r-}-x)\times\\
&\qquad\times\Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(x))-X_{r-}(x)-\varphi(X_{r-}(x),r,z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r\\
&+\int_0^t h_n(Y_{r-}-x)F(X_{r-}(x),r)\,\mathrm{d}W_r
\end{aligned}\right.\\
&\begin{aligned}
J_1 &= \\ J_2 &= \\ J_3 &= \\ J_4 &= \\ \\ J_5 &= \\ \\ J_6 &= \\ J_7 &= \\ J_8 &= \\ \\
\end{aligned}
\left\{\begin{aligned}
&+\int_0^t X_{r-}(x)h_n'(Y_r-x)g(Y_{r-},r)\,\mathrm{d}r\\
&+\frac{1}{2}\int_0^t X_{r-}(x)h_n'(Y_r-x)G'(Y_{r-},r)G(Y_{r-},r)\,\mathrm{d}r\\
&+\frac{1}{2}\int_0^t X_{r-}(x)h_n''(Y_r-x)G^2(Y_{r-},r)\,\mathrm{d}r\\
&+\int_0^t\int_{|z|\le 1} X_{r-}(x)\Big[h_n\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-})-x\Big)-h_n(Y_{r-}-x)\\
&\qquad -h_n'(Y_{r-}-x)\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-})-Y_{r-}\Big)\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}r\\
&+\int_0^t\int_{|z|\le 1} X_{r-}(x)h_n'(Y_{r-}-x)\times\\
&\qquad\times\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-})-Y_{r-}-\psi(Y_{r-},r,z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r\\
&+\int_0^t X_{r-}(x)h_n'(Y_r-x)G(Y_{r-},r)\,\mathrm{d}W_r\\
&+\int_0^t F(X_{r-}(x),r)h_n'(Y_r-x)G(Y_{r-},r)\,\mathrm{d}r\\
&+\int_0^t\int_{|z|\le 1}\Big[\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(x))-X_{r-}(x)\Big]\times\\
&\qquad\times\Big[h_n\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-})-x\Big)-h_n(Y_{r-}-x)\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}r
\end{aligned}\right.\\
&K_1 = \left\{\begin{aligned}
&+\int_0^t\int_{|z|\le 1}\Big[\mathrm{e}^{\varphi(\cdot,r,z)}(X_{r-}(x))h_n\Big(\mathrm{e}^{\psi(\cdot,r,z)}(Y_{r-})-x\Big)\\
&\qquad -X_{r-}(x)h_n(Y_{r-}-x)\Big]\,\tilde{N}(\mathrm{d}z,\mathrm{d}r).
\end{aligned}\right.
\end{aligned}
\tag{3.16}
$$

We decompose the term K_1 further into the sum

$$\begin{cases} I_5 = & \displaystyle\int_0^t \int_{|z|\leq 1} \Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(x)) - X_{r-}(x)\Big) h_n(Y_{r-} - x)\, \tilde{N}(\mathrm{d}z, \mathrm{d}r) \\ J_9 = & \displaystyle + \int_0^t \int_{|z|\leq 1} \mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(x)) \times \\ & \qquad \times \Big[h_n\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - x\Big) - h_n(Y_{r-} - x)\Big]\, \tilde{N}(\mathrm{d}z, \mathrm{d}r). \end{cases}$$

All the (stochastic) integrals in (3.16) exist due to the integrability assumptions on the functions φ and ψ and the estimates for the exponential mappings from Lemma 3.7.

In the next part of the proof we go through all the terms from above one by one. The plan is to integrate w.r.t. x and pass to the limit. For all terms the strategy will be roughly the same: We first use the (stochastic) Fubini Theorem to change the order of integration. Then, using the properties of the mollifiers (see for example [Eva02, Appendix C.4, Theorem 6]) and Lebesgue's theorem on dominated convergence, we get the limit for $n \to \infty$. We note here that all integrals contain the term $h_n(Y_{r-} - x)$ or one of its derivatives. Since Y is bounded by some constant l, these terms disappear for $|x| \geq l$. This means that in the following we can assume f, F and φ to be also uniformly bounded to simplify the notation.

We distinguish between the Lebesgue integrals w.r.t. $\mathrm{d}r$, the Itô integrals w.r.t. $\mathrm{d}W$ and the compensated Poisson random measure $\tilde{N}$, and the terms containing the derivatives h_n' and h_n''. We start with the terms coming from the integral $\int_0^t h_n(Y_{r-} - x)\, \mathrm{d}X_r$ which will converge to the first integral in (3.10).

Initial and end points. It follows directly from the properties of the mollifiers and the continuity of $x \mapsto X_t(x)$ that

$$\lim_{n\to\infty} \int_{\mathbb{R}} X_t(x) h_n(Y_t - x)\, \mathrm{d}x = X_t(Y_t)$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}} x h_n(y - x)\, \mathrm{d}x = y.$$

Terms I_1 and I_2. We start with the Lebesgue integrals coming from the drift part and the noise-induced drift appearing in the Stratonovich integrals w.r.t. W.

For each $\omega \in \Omega$,

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} |h_n(Y_{r-} - x) f(X_{r-}(x), r)| \,\mathrm{d}x \,\mathrm{d}r &\leq \|f\| \int_0^t \int_{\mathbb{R}} h_n(Y_{r-} - x) \,\mathrm{d}x \,\mathrm{d}r \\ &= t \cdot \|f\| < \infty\end{aligned}$$

and the Fubini theorem yields

$$\begin{aligned}\int_{\mathbb{R}} \int_0^t h_n(Y_{r-} - x) f(X_{r-}(x), r) \,\mathrm{d}r \,\mathrm{d}x &= \int_0^t \int_{\mathbb{R}^d} h_n(Y_{r-} - x) f(X_{r-}(x), r) \,\mathrm{d}x \,\mathrm{d}r \\ &= \int_0^t \int_{\|x\| \leq 1/n} h_n(x) f(X_{r-}(Y_{r-} - x), r) \,\mathrm{d}x \,\mathrm{d}r.\end{aligned}$$

For each $r \in [0, T]$, the function $y \mapsto f(X_{r-}(y), r)$ is continuous and by [Eva02, Appendix C.4, Theorem 6(iii)]

$$\lim_{n \to \infty} \Big| \int_{|x| \leq 1/n} h_n(x) f(X_{r-}(y - x), r) \,\mathrm{d}x - f(X_{r-}(y), r) \Big| = 0.$$

Since for $r \in [0, T]$

$$\Big| \int_{|x| \leq 1/n} h_n(x) f(X_{r-}(Y_{r-} - x), r) \,\mathrm{d}x \Big| \leq \|f\|$$

the Lebesgue's dominated convergence theorem implies that

$$\int_0^t \int_{\mathbb{R}} h_n(Y_{r-} - x) f(X_{r-}(x), r) \,\mathrm{d}x \,\mathrm{d}r \to \int_0^t f(X_{r-}(Y_{r-}), r) \,\mathrm{d}r.$$

Analogously we get the convergence of the term I_2.

Term I_3. Consider the function

$$H^{I_3}(x, r) = \int_{|z| \leq 1} \left(\mathrm{e}^{\varphi(\cdot, r, z)}(x) - x - \varphi(x, r, z) \right) \nu(\mathrm{d}z).$$

It follows from Lemma 3.7 that

$$|\mathrm{e}^{\varphi(\cdot, r, z)}(x) - x - \varphi(x, r, z)| \leq K_0^{\varphi}(z) K_1^{\varphi}(z) \mathrm{e}^{K_0^{\varphi}(z)} (1 + |x|),$$

so $x \mapsto H^{I_3}(x, r)$ is well-defined and continuous.

Recalling that X and Y are assumed to be bounded, the argument of the previous step applies and

$$\begin{aligned}\int_0^t h_n(Y_{r-}-x)\int_{|z|\le 1}\Big(\mathrm{e}^{\varphi(\cdot,r,z)}(X_{r-}(x))-X_{r-}(x)-\varphi(X_{r-}(x),r,z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r\\ \to\int_0^t\int_{|z|\le 1}\Big(\mathrm{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-}))-X_{r-}(Y_{r-})-\varphi(X_{r-}(Y_{r-}),r,z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r.\end{aligned}$$

Term I_4. By Fubini's theorem for stochastic integrals (see Theorem 2.4), for each $|y|\le l$

$$\int_{\mathbb{R}}\int_0^t h_n(y-x)F(X_{r-}(x),r)\,\mathrm{d}W_r\,\mathrm{d}x=\int_0^t\int_{\mathbb{R}} h_n(y-x)F(X_{r-}(x),r)\,\mathrm{d}x\,\mathrm{d}W_r.$$

By the properties of mollifiers for each $r\in[0,T]$ and $|y|\le l$

$$\Big|\int_{\mathbb{R}} h_n(y-x)F(X_{r-}(x),r)\,\mathrm{d}x-F(X_{r-}(y),r)\Big|\le 2C_F$$

and by the Itô isometry and again Lebesgue's dominated convergence

$$\begin{aligned}&\mathbb{E}\Big|\int_0^t\int_{\mathbb{R}} h_n(Y_{r-}-x)F(X_{r-}(x),r)\,\mathrm{d}x\,\mathrm{d}W_r-\int_0^t F(X_{r-}(Y_{r-}),r)\,\mathrm{d}W_r\Big|^2\\ &\quad=\int_0^t\mathbb{E}\Big|\int_{\mathbb{R}} h_n(Y_{r-}-x)F(X_{r-}(x),r)\,\mathrm{d}x-F(X_{r-}(Y_{r-}),r)\Big|^2\,\mathrm{d}r\\ &\quad\to 0.\end{aligned}$$

Term I_5. The jump term I_5 is estimated analogously with the help of the Itô isometry for stochastic integrals w.r.t. a compensated PRM.

Terms J_1, J_2 and J_7. Consider the Lebesgue integrals and again apply Fubini's theorem for each ω. To deal with h_n', we integrate by parts, using that X is a C^1-diffeomorphism. Convergence follows from applying Lebesgue's dominated convergence:

$$\begin{aligned}\int_{\mathbb{R}}\int_0^t X_r(x)h_n'(Y_r-x)g(Y_r,r)\,\mathrm{d}r\,\mathrm{d}x&=\int_0^t\Big[\int_{\mathbb{R}} X_r(x)h_n'(Y_r-x)\,\mathrm{d}x\Big]\,g(Y_r,r)\,\mathrm{d}r\\ &=\int_0^t\Big[\int_{\mathbb{R}} X_r'(x)h_n(Y_r-x)\,\mathrm{d}x\Big]g(Y_r,r)\,\mathrm{d}r\\ &\to\int_0^t X_r'(Y_r)g(Y_r,r)\,\mathrm{d}r.\end{aligned}$$

The terms J_2 and J_7 are treated analogously.

Term J_3. This step works similarly as for the term J_1. Using that X is a

C^2-diffeomorphism to integrate by parts twice we get

$$\begin{aligned}\int_{\mathbb{R}}\int_0^t & X_{r-}(x)h_n''(Y_{r-}-x)G^2(Y_{r-},r)\,\mathrm{d}r\,\mathrm{d}x \\ &= \int_0^t \Big[\int_{\mathbb{R}} X_{r-}(x)h_n''(Y_{r-}-x)\,\mathrm{d}x\Big]\,G^2(Y_{r-},r)\,\mathrm{d}r \\ &= \int_0^t \Big[\int_{\mathbb{R}} X_{r-}''(x)h_n(Y_{r-}-x)\,\mathrm{d}x\Big]\,G^2(Y_{r-},r)\,\mathrm{d}r \\ &\to \int_0^t X_{r-}''(Y_{r-})G^2(Y_{r-},r)\,\mathrm{d}r.\end{aligned}$$

Terms J_4, J_5, J_8. First note that the sum of the terms $J_4+J_5+J_8$ is well defined: For each $n \geq 1$, the function h_n is Lipschitz continuous, and so the integrability of the term from J_8 follows from Lemma 3.7 and the integrability assumptions on K_0^{φ} and K_0^{ψ}.
For J_5 integrability follows from the same arguments as for the term in I_3.
To show the integrability for J_4 we use Taylor's theorem to estimate

$$\begin{aligned}&\Big|h_n\Big(\mathrm{e}^{\psi(\cdot,r,z)}(y)-x\Big)-h_n(y-x)-h_n'(y-x)\Big(\mathrm{e}^{\psi(\cdot,r,z)}(y)-y\Big)\Big| \\ &\leq \int_0^1 \Big|h_n''\Big(y-x+\theta(\mathrm{e}^{\psi(\cdot,r,z)}(y)-y)\Big)\Big|(1-\theta)\,\mathrm{d}\theta\cdot\Big(\mathrm{e}^{\psi(\cdot,r,z)}(y)-y\Big)^2 \\ &\leq \int_0^1 \Big|h_n''\Big(y-x+\theta(\mathrm{e}^{\psi(\cdot,r,z)}(y)-y)\Big)\Big|(1-\theta)\,\mathrm{d}\theta\cdot(K_0^{\psi}(z))^2\mathrm{e}^{2K_0^{\psi}(z)}(1+|y|)^2.\end{aligned}$$

Since h_n'' and Y_{r-} are bounded for every $n \geq 1$, the integrability assumption on $K_0^{\psi}(z))^2$ suffices to get integrability of the whole term. Hence the sum $J_4+J_5+J_8$ is simplified to

$$\begin{aligned}J_4+J_5+J_8 = \int_0^t\int_{|z|\leq 1} \Big[&\mathrm{e}^{\varphi(\cdot,r,z)}(X_{r-}(x))\Big(h_n\Big(\mathrm{e}^{\psi(\cdot,r,z)}(Y_{r-})-x\Big)-h_n(Y_{r-}-x)\Big) \\ &- X_{r-}(x)\psi(Y_{r-},r,z)h_n'(Y_{r-}-x)\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}r\end{aligned}$$

and by Fubini's theorem, the integration by parts and the dominated convergence theorem we get

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}}(J_4+J_5+J_8)\,\mathrm{d}x \\ &= \int_0^t\int_{|z|\leq 1}\Big[\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}((\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-})))-\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-})) \\ &\qquad\qquad -X_{r-}'(Y_{r-})\psi(Y_{r-},r,z)\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}r.\end{aligned}$$

Term J_6**.** The term J_6 is treated with the help of integration by parts analogously to the term I_4.

Term J_9**.** The term J_9 is treated analogously to the term I_5. □

The formula from Theorem 3.6 is interesting in and of itself, but it will also be an important tool in the next section, which is dealing with the inverse flow of solutions to the Marcus SDE. In this context, another result will be needed, which is a slightly different take on the generalized Itô formula for Marcus SDEs. Here, we consider a semimartingale Φ instead of the solution X. We get the following result:

Theorem 3.8. *Let Φ be a one-dimensional semimartingale given by* (3.1) *with a d-dimensional parameter x and let Y be a solution of the d-dimensional canonical SDE* (3.9)*. Then*

$$\Phi(Y_t,t) = \int_0^t \Phi(Y_{r-},\mathrm{d}r) + \int_0^t \nabla^T\Phi(Y_r,\diamond\,\mathrm{d}r), \tag{3.17}$$

where

$$\begin{aligned}\int_0^t \Phi(Y_{r-},\mathrm{d}r) &= \int_0^t f(Y_{r-},r)\,\mathrm{d}r + \int_0^t F(Y_{r-},r)\,\mathrm{d}W_r \\ &\quad + \int_0^t\int_{|z|\le 1}\varphi(Y_{r-},r,z)\,\tilde N(\mathrm{d}z,\mathrm{d}r) + \int_0^t\int_{|z|>1}\varphi(Y_{r-},r,z)\,N(\mathrm{d}z,\mathrm{d}r)\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&\int_0^t \nabla^T\Phi(Y_r,\diamond\,\mathrm{d}r)\\ &= \int_0^t \nabla^T\Phi(Y_{r-},r)g(Y_{r-},r)\,\mathrm{d}r + \int_0^t \nabla^T\Phi(Y_{r-},r)G(Y_{r-},r)\circ\mathrm{d}W_r\\ &+ \int_0^t\int_{|z|\le 1}\int_0^1 \Big(\nabla^T\Phi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r-) + \nabla^T\varphi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r,z))\Big)\times\\ &\qquad\qquad\times\psi(\mathrm{e}^{\psi(\cdot,r,z)}(Y_{r-}),r,z)\,\mathrm{d}u\,\tilde N(\mathrm{d}z,\mathrm{d}r)\\ &+ \int_0^t\int_{|z|\le 1}\int_0^1 \Big[\Big(\nabla^T\Phi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r-) + \nabla^T\varphi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r,z))\Big)\times\\ &\qquad\qquad\times\psi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r,z)\Big]\,\mathrm{d}u - \nabla^T\Phi(Y_{r-},r-)\psi(Y_{r-},r,z)\,\nu(\mathrm{d}z)\,\mathrm{d}r\\ &+ \int_0^t\int_{|z|>1}\int_0^1 \Big(\nabla^T\Phi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r-) + \nabla^T\varphi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r,z))\Big)\times\\ &\qquad\qquad\times\psi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r,z)\,\mathrm{d}u\,N(\mathrm{d}z,\mathrm{d}r).\end{aligned} \tag{3.19}$$

Proof. The proof of this theorem is analogous to the proof of Theorem 3.6. After localization, and application of a mollifier h_n to Y we obtain again the formula (3.15). The product formula for $\Phi(x,t)h_n(Y_t - x)$ takes a slightly different form, namely

$$\begin{aligned}
\Phi(x,t)h_n(Y_t - x) &= \Phi(x,0)h_n(y-x) \\
&+ \int_0^t h_n(Y_{r-} - x) f(X_{r-}(x), r)\,\mathrm{d}r \\
&+ \int_0^t h_n(Y_{r-} - x) F(X_{r-}(x), r)\,\mathrm{d}W_r \\
&+ \int_0^t \Phi(x,r-)h_n'(Y_{r-} - x) g(Y_{r-}, r)\,\mathrm{d}r \\
&+ \frac{1}{2}\int_0^t \Phi(x,r-)h_n'(Y_{r-} - x) G'(Y_{r-}, r) G(Y_{r-}, r)\,\mathrm{d}r \\
&+ \frac{1}{2}\int_0^t \Phi(x,r-)h_n''(Y_{r-} - x) G^2(Y_{r-}, r)\,\mathrm{d}r \\
&+ \int_0^t \int_{|z|\le 1} \Phi(x,r-)\Big[h_n\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - x\Big) - h_n(Y_{r-} - x) \\
&\qquad\qquad -h_n'(Y_{r-} - x)\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - Y_{r-}\Big)\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}r \\
&+ \int_0^t \int_{|z|\le 1} \Phi(x,r-)h_n'(Y_{r-} - x)\times \\
&\qquad\qquad \times\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - Y_{r-} - \psi(Y_{r-}, r, z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r \\
&+ \int_0^t \Phi(x,r-)h_n'(Y_r - x) G(Y_{r-}, r)\,\mathrm{d}W_r \\
&+ \frac{1}{2}\int_0^t F(X_{r-}(x), r) h_n'(Y_r - x) G(Y_{r-}, r)\,\mathrm{d}r \\
&+ \int_0^t \int_{|z|\le 1} \varphi(x,r,z)\Big(h_n\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - x\Big) - h_n(Y_{r-} - x)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r \\
&+ \int_0^t \int_{|z|\le 1} \Big[\Big(\Phi(x,r-) + \varphi(x,r,z)\Big)h_n\Big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}) - x\Big) \\
&\qquad\qquad -\Phi(x,r-)h_n(Y_{r-} - x)\Big]\,\tilde{N}(\mathrm{d}z, \mathrm{d}r).
\end{aligned} \tag{3.20}$$

Note that the initial condition disappears since $\Phi(x,0) = 0$. Integrating w.r.t. x and passing to the limit with $n \to \infty$ as in the proof of Theorem 3.6 we then get the formula

$$
\begin{aligned}
\Phi(Y_t,t) &= \int_0^t f(Y_{r-},r)\,\mathrm{d}r + \int_0^t F(Y_{r-},r)\,\mathrm{d}W_r \\
&+ \int_0^t \nabla^T\Phi(Y_{r-},r-)g(Y_{r-},r)\,\mathrm{d}r + \frac{1}{2}\int_0^t \nabla^T\Phi(Y_{r-},r-)G^2(Y_{r-},r)\,\mathrm{d}r \\
&+ \frac{1}{2}\int_0^t \nabla^T\nabla\Phi(Y_{r-},r-)G'(Y_{r-},r)G(Y_{r-},r)\,\mathrm{d}r \\
&+ \int_0^t\int_{|z|\le 1} \Big(\Phi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r-)) - \Phi(Y_{r-},r-) \\
&\qquad\qquad -\nabla^T\Phi(Y_{r-},r-)\psi(Y_{r-},r,z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r \\
&+ \int_0^t \nabla^T\Phi(Y_{r-},r-)G(Y_{r-},r)\,\mathrm{d}W_r + \frac{1}{2}\int_0^t \nabla^T F(Y_{r-},r)G(Y_{r-},r)\,\mathrm{d}r \\
&+ \int_0^t\int_{|z|\le 1} \Big(\varphi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r,z) - \varphi(Y_{r-},r,z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r \\
&+ \int_0^t\int_{|z|\le 1} \Big[\Phi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r-) + \varphi(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-}),r,z) \\
&\qquad\qquad -\Phi(Y_{r-},r-)\Big]\,\tilde{N}(\mathrm{d}z,\mathrm{d}r).
\end{aligned}
$$

which can be transformed to (3.17), (3.18), (3.19). □

3.3 Equations for the Inverse Flows

By Theorem 3.2, the solution $x \mapsto X_t(x)$, $t \ge 0$, maps $\mathbb{R}^d$ onto itself diffeomorphically, and there exists a modification such that $X_{s,t} := X_t \circ X_s^{-1}$ defines the stochastic flow of diffeomorphisms. Denote by $DX_t(x)$ its Jacobian matrix and let $(DX_t(x))^{-1}$ be its matrix inverse.

Consider the inverse flow $X_{t,0} := X_{0,t}^{-1}$, $t \ge 0$. We show that the inverse flow satisfies the following formula.

Theorem 3.9. *The inverse flow $t \mapsto X_{t,0}$, $t \ge 0$ satisfies the canonical SDE*

$$X_{t,0}(x) = x - \int_0^t \Big(DX_{0,r}(X_{r,0}(x))\Big)^{-1}\Phi(x,\diamond\,\mathrm{d}r) \tag{3.21}$$

which is understood in the following sense:

$$
\begin{aligned}
X_{t,0}(x) = x &- \int_0^t \Big(DX_{0,r-}(X_{r-,0}(x))\Big)^{-1} f(x,r)\,\mathrm{d}r \\
&- \int_0^t \Big(DX_{0,r-}(X_{r-,0}(x))\Big)^{-1} F(x,r)\circ \mathrm{d}W_r \\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\psi(\cdot,r,z)}(X_{r-,0}(x)) - X_{r-,0}(x)\Big)\,\tilde N(\mathrm{d}z,\mathrm{d}r) \\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\psi(\cdot,r,z)}(X_{r-,0}(x)) - X_{r-,0}(x) \\
&\qquad\qquad - (DX_{0,r-}(X_{r-,0}(x)))^{-1}\varphi(x,r,z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r \\
&+ \int_0^t \int_{|z|> 1} \Big(\mathbf{e}^{\psi(\cdot,r,z)}(X_{r-,0}(x)) - X_{r-,0}(x)\Big)\,N(\mathrm{d}z,\mathrm{d}r).
\end{aligned}
\tag{3.22}
$$

where $\mathbf{e}^{\psi(\cdot,r,z)}$ is the exponential mapping defined with the help of the solution $w = w(u;x) = w(u;x,r,z)$ of the ODE

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}u} w(u;y) &= -\Big(D\mathbf{e}^{u\varphi(\cdot,r,z)}(X_{r-}(\cdot))\Big)^{-1}\varphi(\mathbf{e}^{u\varphi(\cdot,r,z)}(X_{r-}(\cdot)),r,z)\circ w(u;y), \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u\in[0,1], \\
w(0;y) &= y,
\end{cases}
\tag{3.23}
$$

i.e. $\mathbf{e}^{\psi(\cdot,r,z)}(y) := \mathbf{e}^{\psi(r,z)}(y) := w(1;y)$.

Proof. For brevity we assume that $\nu(|z|>1) = 0$ and denote $X_{0,t} = X_t$, and $DX_{0,t} = DX_t$. $DX_t(x)$ is a right stochastic exponent, see [Pro04b, Section V.9] and [FK99a, Section 4] for more detail. It is well defined and is invertible.

Define the following drift, diffusion and jump coefficients:

$$
\begin{aligned}
g(y,r) &= -\Big(DX_{r-}(y)\Big)^{-1} f(X_{r-}(y),r), \\
G(y,r) &= -\Big(DX_{r-}(y)\Big)^{-1} F(X_{r-}(y),r), \\
\psi(y,r,z,u) &= -\Big(D\mathbf{e}^{u\varphi(\cdot,r,z)}(X_{r-}(y))\Big)^{-1}\varphi(\mathbf{e}^{u\varphi(\cdot,r,z)}(X_{r-}(y)),r,z).
\end{aligned}
$$

In particular,

$$
\psi(y,r,z,0) = -\Big(DX_{r-}(y)\Big)^{-1}\varphi(X_{r-}(y),r,z).
$$

The functions g, G and ψ are predictable and with the help of localization we can assume that they satisfy Assumptions 3.1. Then equation (3.23) has a unique global solution.

Consider the supplementary SDE

$$
\begin{aligned}
Y_t = x &+ \int_0^t g(Y_{r-}, r)\,\mathrm{d}r + \int_0^t G(Y_{r-}, r) \circ \mathrm{d}W_r \\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\psi(r,z)}(Y_{r-}) - Y_{r-}\Big)\,\tilde N(\mathrm{d}z, \mathrm{d}r) \\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\psi(r,z)}(Y_{r-}) - Y_{r-} - \Big(DX_{r-}(Y_{r-})\Big)^{-1} \varphi(X_{r-}(Y_{r-}), r, z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r
\end{aligned}
$$

where $\mathbf{e}^{\psi(r,z)}$ is defined in (3.23).

We show that for each $T > 0$ and for any localized solution we have $X_t(Y_t) \equiv x$ on $[0, T]$. Let us again consider the one-dimensional case. We apply the generalized Itô formula and show that all the integral terms vanish. Indeed, for the drift term we get

$$
\begin{aligned}
&f(X_{r-}(Y_{r-}), r, z) + DX_{r-}Y_{r-}g(Y_{r-}, r, z) \\
&\qquad = f(X_{r-}(Y_{r-}), r, z) - DX_{r-}Y_{r-}\Big(DX_{r-}(Y_{r-})\Big)^{-1} f(X_{r-}(Y_{r-}), r) \equiv 0.
\end{aligned}
$$

The other Lebesgue and Itô stochastic integrals w.r.t. W vanish analogously. To treat the jump terms we consider the function $h(u; x) := \mathbf{e}^{u\varphi(\cdot,r,z)}(X_{r-}(x))$ where the mapping $(u, x) \mapsto \mathbf{e}^{u\varphi}(x)$ has been defined in (3.3), (3.4), (3.5), so that $h(0; x) := X_{r-}(x)$ and $h(1; x) := X_r(x)$. Then taking into account (3.23) we obtain that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u} h(u; w(u; y)) &= \frac{\partial}{\partial u} h(u; w(u; y)) + \frac{\partial}{\partial x} h(u; w(u; y)) \frac{\mathrm{d}}{\mathrm{d}u} w(u; y) \\
&= \varphi(h(u; w(u; y)); r, z) \\
&\quad - \frac{\partial}{\partial x} h(u; w(u; y)) \Big(\frac{\partial}{\partial x} h(u; w(u; y))\Big)^{-1} \cdot \varphi(h(u; w(u; y)); r, z) \\
&\equiv 0.
\end{aligned}
\tag{3.24}
$$

In other words, we have

$$
\begin{aligned}
\mathbf{e}^{\varphi(r,z)}(X_{r-}(\mathbf{e}^{\psi(r,z)}(Y_{r-})) - X_{r-}(Y_{r-}) &= \Big(h(1; w(1; y)) - h(0; w(0, y))\Big)\Big|_{y=Y_{r-}} \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}u} h(u; w(u; y))\,\mathrm{d}u\Big|_{y=Y_{r-}} \\
&\equiv 0.
\end{aligned}
$$

Furthermore, putting together the compensated terms in the generalized Itô formula we get

$$
\begin{aligned}
&\Big(\mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-})) - X_{r-}(Y_{r-}) - \varphi(X_{r-}(Y_{r-}), r, z)\Big)\\
&+ \Big(\mathbf{e}^{\varphi(\cdot,r,z)}\big(X_{r-}\big(\mathbf{e}^{\psi(\cdot,r,z)}(Y_{r-})\big)\big) - \mathbf{e}^{\varphi(\cdot,r,z)}(X_{r-}(Y_{r-})) - DX_{r-}(Y_{r-})\psi(Y_{r-}, r, z)\Big)\\
&\equiv 0.
\end{aligned}
$$

Hence $Y_t(y) = X_t^{-1}(y)$ for each localized solution Y. Since X exists on $[0,T]$, passing to the limit in the localization sequence we get that the Y is the inverse flow and satisfies the SDE (3.22). □

Theorem 3.10 (Itô's formula for the inverse flow w.r.t. the first variable)**.** *Let $\Theta \in C^2(\mathbb{R}^d, \mathbb{R})$. Then the inverse flow $t \mapsto X_{t,0}$ satisfies the canonical SDE*

$$
\begin{aligned}
\Theta(X_{t,0}(x)) &= \Theta(x) - \int_0^t D\Theta(X_{r,0}(x))DX_{r,0}(x)\Phi(x, \diamond\,\mathrm{d}r)\\
&= \Theta(x) - \int_0^t D\Big(\Theta \circ X_{r,0}(x)\Big)\Phi(x, \diamond\,\mathrm{d}r).
\end{aligned}
\tag{3.25}
$$

which is understood as follows:

$$
\begin{aligned}
\Theta(X_{t,0}(x)) = \Theta(x) &- \int_0^t D\Theta(X_{r-,0}(x))DX_{r-,0}(x)f(x,r)\,\mathrm{d}r\\
&- \int_0^t D\Theta(X_{r-,0}(x))DX_{r-,0}(x)F(x,r)\circ \mathrm{d}W_r\\
&+ \int_0^t\int_{|z|\le 1}\Big(\Theta(\mathbf{e}^{-\varphi(\cdot,r,z)}(X_{r-,0}(x))) - \Theta(X_{r-,s}(x))\Big)\tilde{N}(\mathrm{d}z,\mathrm{d}r)\\
&+ \int_0^t\int_{|z|\le 1}\Big(\Theta(\mathbf{e}^{-\varphi(\cdot,r,z)}(X_{r-,0}(x))) - \Theta(X_{r-,0}(x))\\
&\qquad\qquad + D\Theta(X_{r-,0}(x))DX_{r-,0}(x)\varphi(x,r,z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r\\
&+ \int_0^t\int_{|z|>1}\Big(\Theta(\mathbf{e}^{-\varphi(\cdot,r,z)}(X_{r-,0}(x))) - \Theta(X_{r-,0}(x))\Big)N(\mathrm{d}z,\mathrm{d}r).
\end{aligned}
\tag{3.26}
$$

Proof. The proof goes along the lines of the proof of [Kun97, Theorem 4.4.5]. For brevity we denote the forward flow by $X_t := X_{0,t}$ and the inverse flow by $Y_t := X_{t,0}$, $t \in [0,T]$. We have shown that the inverse flow Y satisfies the SDE (3.21).

First note that since $X_t(Y_t(x)) \equiv x$, the gradient matrices $DX_t(x)$ and $DY_t(x)$ satisfy the relation

$$DX_t(Y_t(x)) \cdot DY_t(x) = \mathrm{Id}$$

or equivalently,

$$\Big(DX_t(Y_t(x))\Big)^{-1} = DY_t(x).$$

Second, taking into account (3.24) we get that

$$h(u; w(u; x)) \equiv x, \quad u \in [0, 1],$$

and hence

$$Dh(u; w(u; x))Dw(u; x) = \mathrm{Id} \tag{3.27}$$

or equivalently,

$$\Big(Dh(u; w(u; x))\Big)^{-1} = Dw(u; x), \quad u \in [0, 1].$$

Thus the equation (3.23) for w takes the form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}u} w(u; x) & = -Dw(u; x)\varphi(x, r, z), \quad u \in [0, 1], \\ w(0; x) & = x. \end{cases}$$

This is the first order transport equation, its solution is given by $\mathbf{e}^{-u\varphi(\cdot,r,z)}(x)$ and hence

$$\mathbf{e}^{\psi(\cdot,r,z)}(\cdot) = \mathbf{e}^{-\varphi(\cdot,r,z)}(\cdot). \tag{3.28}$$

Applying the Itô formula to the equation (3.22) and taking into account (3.27) and (3.28) yields

$$\begin{aligned} \Theta(Y_t(x)) = x &- \int_0^t \nabla^T \Theta(Y_{r-}(x)) DY_{r-}(x) f(x, r)\, \mathrm{d}r \\ &- \int_0^t \nabla^T \Theta(Y_{r-}(x)) DY_{r-}(x) F(x, r) \circ \mathrm{d}W_r \\ &+ \int_0^t \int_{|z|\le 1} \Big(\Theta(\mathbf{e}^{-\varphi(\cdot,r,z)}(Y_{r-})) - \Theta(Y_{r-})\Big)\, \tilde{N}(\mathrm{d}z, \mathrm{d}r) \\ &+ \int_0^t \int_{|z|\le 1} \Big(\Theta(\mathbf{e}^{-\varphi(\cdot,r,z)}(Y_{r-})) - \Theta(Y_{r-}) \\ &\qquad\qquad + \nabla^T \Theta(Y_{r-}(x)) DY_{r-}(x) \varphi(x, r, z)\Big)\, \nu(\mathrm{d}z)\, \mathrm{d}r, \end{aligned}$$

or equivalently

$$
\begin{aligned}
&\Theta(Y_t(x)) = x - \int_0^t \nabla^T \Big(\Theta \circ Y_{r-}(x)\Big) f(x,r)\,\mathrm{d}r \\
&- \int_0^t \nabla^T \Big(\Theta \circ Y_{r-}(x)\Big) F(x,r) \circ \mathrm{d}W_r \\
&+ \int_0^t \int_{|z|\le 1} \Big(\Theta(\mathbf{e}^{-\varphi(\cdot,r,z)}(Y_{r-})) - \Theta(Y_{r-})\Big)\, \tilde{N}(\mathrm{d}z,\mathrm{d}r) \\
&+ \int_0^t \int_{|z|\le 1} \Big(\Theta(\mathbf{e}^{-\varphi(\cdot,r,z)}(Y_{r-})) - \Theta(Y_{r-}) + \nabla^T \Big(\Theta \circ Y_{r-}(x)\Big)\varphi(x,r,z)\Big)\, \nu(\mathrm{d}z)\,\mathrm{d}r,
\end{aligned}
$$

The latter formula can be formally written in the canonical form (3.25). □

3.4 Wong–Zakai Approximations

One way to motivate the analysis of the Marcus SDE can be found looking at Wong–Zakai approximations. These approximations are mostly known from the results on stochastic (partial) differential equations w.r.t. the Brownian motion, see for example the original paper by Wong and Zakai [WZ65a] for SDEs and [BF95] for SPDEs.

In the case of Lévy processes however, the jump part needs to be considered differently, leading to the Marcus SDE. Results on the convergence of Wong–Zakai approximations to the solution of the Marcus SDE can for example be found in [KPP95] and [Kun95].

Before we get into the specific results let us specify the kind of Wong–Zakai approximations we consider: Let W be again an $\mathbb{R}^m$-valued Brownian motion and Z an m-dimensional pure jump Lévy process with N, $\tilde{N}$ and ν being the PRM, compensated PRM and the Lévy measure, respectively. Denote by X the solution to the the Marcus SDE of separating type

$$
\begin{aligned}
X_t(x) = x &+ \int_0^t f(X_{r-}(x))\,\mathrm{d}r + \int_0^t F(X_{r-}(x)) \circ \mathrm{d}W_r \\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\varphi(\cdot)z}(X_{r-}(x)) - X_{r-}(x)\Big)\, \tilde{N}(\mathrm{d}z,\mathrm{d}r) \\
&+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\varphi(\cdot)z}(X_{r-}(x)) - X_{r-}(x) - \varphi(X_{r-}(x),z)\Big)\, \nu(\mathrm{d}z)\,\mathrm{d}r \\
&+ \int_0^t \int_{|z|>1} \Big(\mathbf{e}^{\varphi(\cdot)z}(X_{r-}) - X_{r-}\Big)\, N(\mathrm{d}z,\mathrm{d}r),
\end{aligned}
\tag{3.29}
$$

which exists under the conditions of Theorem 3.2.

For $h > 0$ we define the following piecewise linear approximations of the Brownian motion W and the jump part Z:

$$W_t^h = W_{kh} + \frac{t - kh}{h}\left(W_{(k+1)h} - W_{kh}\right), \quad t \in [kh, (k+1)h],$$
$$Z_t^h = Z_{kh} + \frac{t - kh}{h}\left(Z_{(k+1)h} - Z_{kh}\right), \quad t \in [kh, (k+1)h], \quad k \geq 0.$$

The Wong–Zakai approximations X^h are now defined piecewise as solution to the random ODEs with initial value

$$X_0^h = x,$$

and for $t \in [kh, (k+1)h]$

$$X_t^h = X_{kh} + \int_{kh}^t f(X_s^h)\,\mathrm{d}s + \int_{kh}^t F(X_s^h)\frac{\Delta_{kh}W}{h}\,\mathrm{d}s + \int_{kh}^t \varphi(X_s^h)\frac{\Delta_{kh}Z}{h}\,\mathrm{d}s,$$

where $\Delta_{kh}W := W_{(k+1)h} - W_{kh}$ and $\Delta_{kh}Z := Z_{(k+1)h} - Z_{kh}$.

Weak convergence

Weak convergence for this kind of equations has already been studied in the literature. In [Kun95, Theorem 3 and Theorem 4] and [Mar78] weak convergence is shown and [KKP19] gives results on the rate of convergence.
The following Theorem 3.11 follows from [Kun95, Theorem 4]. But since in this paper the author looks at a more general case of equations on Lie manifolds and here we are only interested in weak convergence for the specific case of the classical Marcus SDE, we will prove the theorem directly.
We will restrict ourselves to the case where f, F and φ are bounded. This is not strictly necessary, as shown for example in [KKP19]. However, it is not implausible to assume boundedness, since this renders any further assumptions on the regularity of the derivatives of f, F and φ unnecessary, apart from those needed to ensure existence of solutions. In a broader context, it is often useful to assume boundedness to guarantee the solution to be a Feller process, see [Kol11, Theorem 4.6.1].

Theorem 3.11. *Let $f = f(x)$, $F = F(x)$ and $\varphi \cdot z = \varphi(x)z$ satisfy assumption 3.1 and furthermore be bounded. Then for fixed x, any $t \in [0, T]$ and any function*

$g \in C_b^2(\mathbb{R}^d, \mathbb{R})$,

$$\lim_{h\to 0,\, h>0} \mathbb{E}g(X_t^h) = \mathbb{E}g(X_t).$$

Proof. We will prove the Theorem for the one-dimensional case only. This is solely done in the interest of readability; the arguments remain the same in the multi-dimensional case.
We first show that we can assume the jumps of the Lévy process Z to be bounded: Fix the function $g \in C_b^2$ and assume it is bounded by $K > 0$. The probability that $|\Delta Z_t| > A$ for a constant $A > 0$ is given by $\exp(-T\int_{|z|>A} \nu(\mathrm{d}z))$. For any $\varepsilon > 0$, we can choose A large enough, s.t.

$$\exp\Big(-T\int_{|z|>A} \nu(\mathrm{d}z)\Big) \leq \frac{\varepsilon}{2K}.$$

Define the bounded Lévy process

$$Z_t^A := \int_{|z|\leq 1} z\,\tilde{N}(\mathrm{d}z, t) + \int_{1<|z|\leq A} z\,N(\mathrm{d}z, t),$$

and see that

$$\mathbb{P}(Z_t = Z_t^A, t \in [0,t]) = 1 - \exp(-T\int_{|z|>A} \nu(\mathrm{d}z)) \geq 1 - \frac{\varepsilon}{2K}.$$

Now, write $\tilde{X}$ for the solution to (3.29) w.r.t. the Lévy process with bounded jumps and $\tilde{X}^h$ for the approximation of $\tilde{X}$. We see that

$$\begin{aligned}
\left|\mathbb{E}\left[g(X_{nh}^h) - g(X_{nh})\right]\right| &\leq \left|\mathbb{E}\left[g(X_{nh}^h) - g(X_{nh})\middle| Z_t = Z_t^A, t \in [0,T]\right]\right| \\
&\quad + \left|\mathbb{E}\left[g(X_{nh}^h) - g(X_{nh})\middle| |\Delta Z_t| > A \text{ for some } t \in [0,T]\right]\right| \\
&\leq \left|\mathbb{E}\left[g(\tilde{X}_{nh}^h) - g(\tilde{X}_{nh})\right]\right| + \varepsilon.
\end{aligned}$$

From now on we write X^h and X again but assume the jumps to be bounded. Under this assumption the jump part can be written as

$$Z_t = t\mu_A + \int_{|z|\leq A} z\,\tilde{N}(\mathrm{d}z, t), \tag{3.30}$$

and the solution $X_t(x)$ as

$$\begin{aligned}X_t(x) &= x + \int_0^t f(X_s)\,\mathrm{d}s + \frac{1}{2}\int_0^t FF'(X_s)\,\mathrm{d}s + \mu_A \int_0^t \varphi(X_{s-})\,\mathrm{d}s \\ &\quad + \int_0^t F(X_s)\,\mathrm{d}W_s + \int_0^t \int_{|z|\le A} \left(\mathbf{e}^{\varphi(\cdot)z}(X_{s-}) - X_{s-}\right) \tilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &\quad + \int_0^t \int_{|z|\le A} \left(\mathbf{e}^{\varphi(\cdot)z}(X_{s-}) - X_{s-} - \varphi(X_{s-})z\right)\nu(\mathrm{d}z)\,\mathrm{d}s,\end{aligned}$$

where

$$\mu_A = \int_{1\le|z|\le A} z\ \nu(\mathrm{d}z).$$

Now assume that h is fixed and $n \ge 1$. We will first give an estimate for the node point hn. For $0 \le k \le n$, consider the processes

$$Y_t^{k,h}(x) := \begin{cases} X_t^h(x), & t \in [0,kh], \\ X_{t-kh}(X_{kh}^h(x)), & t \in [kh,nh], \end{cases}$$

which implies

$$\begin{aligned} Y_t^{0,h} &\equiv X_t, \\ Y_t^{n,h} &\equiv X_t^h, \quad t \in [0,nh]. \end{aligned}$$

Heuristically, the process $Y^{k,h}$ can be described as following the approximation for k steps and from then on we solve the SDE for another $n-k$ steps. Comparing the approximations "next to each other" we get the following:

$$\begin{aligned} |\mathbb{E}_x g(X_{nh}^h) - g(X_{nh})| &= \Big| \sum_{k=0}^{n-1} \mathbb{E}_x\big(g(Y_{nh}^{k,h}) - g(Y_{nh}^{k+1,h})\big)\Big| \\ &= \Big| \sum_{k=0}^{n-1} \mathbb{E}_x \mathbb{E}\Big[g(Y_{nh}^{k,h}) - g(Y_{nh}^{k+1,h})\Big|\mathscr{F}_{kh}\Big]\Big| \\ &= \Big| \sum_{k=0}^{n-1} \mathbb{E}_x \mathbb{E}_{X_{(k-1)h}^h}\Big[\mathbb{E}_{X_h^h} g(X_{(n-k)h}) - \mathbb{E}_{X_h} g(X_{(n-k)h})\Big]\Big|. \end{aligned} \tag{3.31}$$

Note that we have used the Markov property for $(X_{kh}^h)_{k\ge0}$ here and that because of this,

$$\mathbb{E}_{X_{kh}^h} g(X_{(n-k)h}) = \mathbb{E}_{X_{(k-1)h}^h} \mathbb{E}_{X_h^h} g(X_{(n-k)h}).$$

We define the functions

$$g_k(x) = \mathbb{E}_x g(X_{(n-k)h}), \quad 1 \le k \le n-1.$$

Note that for $k \geq 1$, g_k is in C_b^2: The boundedness easily follows from the fact that g is bounded as well. The existence and boundedness of the derivatives follows from Lemma 3.3.

We write the last expression in the sum as

$$\mathbb{E}_x \mathbb{E}_{X^h_{(k-1)h}} \mathbb{E}_{X^h_h} g(X_{(n-k)h}) - \mathbb{E}_{X_h} g(X_{(n-k)h}) = \mathbb{E}_x \mathbb{E}_{X^h_{(k-1)h}} (g_k(X^h_h) - g_k(X_h)).$$

This means that it is enough to show that we can give the one-step estimate for $|\mathbb{E}_x g(X^h_h(x)) - \mathbb{E}_x g(X_h(x))|$. For this, we show in the next step that

$$\Big|\mathbb{E}_x g_k(X^h_h(x)) - \mathbb{E}_x g_k(X_h(x))\Big| \leq C n^{-\frac{3}{2}},$$

where C does not depend on k. The representation in (3.31) then gives us

$$|\mathbb{E}_x g(X_{nh}) - \mathbb{E}_x g(X^h_{nh})| \leq \sum_{k=0}^{n-1} C n^{-\frac{3}{2}} \leq C n^{-\frac{1}{2}}, \tag{3.32}$$

which tends to zero for $n \to \infty$.

We estimate the expected value of $g_k(X_{nh})$ using the generator of the Markov process. By [Kol11, Theorem 4.6.1] we get that $X_t(x)$ is a Markov process with the generator A given by

$$\begin{aligned} A g_k(x) =& g_k'(x)(f(x) + \mu_A \varphi(x)) + \frac{1}{2}(g_k'(x)F(x)F'(x) + g_k''(x)F^2(x)) \\ &+ \int_{|z|\leq A} \big[g_k(\mathbf{e}^{\varphi(\cdot)z}(x)) - g_k(x) - z g_k'(x)\varphi(x)\big]\nu(\mathrm{d}z). \end{aligned}$$

By Lemma 3.3 we have $\|g_k'\| < C$ and $\|g_k''\| < C$, where C is some constant independent of k. Using the Lipschitz continuity of f, F, $F'F$, φ and $\mathbf{e}^{\varphi(\cdot)z}$ for $|z| < A$, we see that

$$|A g_k(x) - A g_k(y)| \leq C|x - y|,$$

where again, C does not depend on k.
Furthermore, for any $g \in C_b^2(\mathbb{R}^d)$,

$$\mathbb{E}_x g(X_h(x)) = g(x) + \mathbb{E}_x \int_0^h A g(X_s(x))\, \mathrm{d}s.$$

We then get

$$\begin{aligned}\Big|\mathbb{E}_x g_k(X_h) - g_k(x) - \int_0^h Ag_k(x)\,\mathrm{d}s\Big| &= \Big|\int_0^h \mathbb{E}_x Ag_k(X_s) - Ag_k(x)\,\mathrm{d}s\Big| \\ &\le C\int_0^h \left(\mathbb{E}_x|X_s(x)-x|^2\right)^{\frac{1}{2}}\,\mathrm{d}s\end{aligned} \tag{3.33}$$

Furthermore, for some constant $C > 0$ we get

$$\sup_x \mathbb{E}|X_t(x) - x|^2 \le Ct, \quad t \in [0, T]. \tag{3.34}$$

To show this, we write

$$X_t(x) - x = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \int_0^t f(X_s) + \frac{1}{2}FF'(X_s) + \mu_A\ \varphi(X_{s-})\,\mathrm{d}s, \\ I_2 &= \int_0^t F(X_s)\,\mathrm{d}W_s, \\ I_3 &= \int_0^t \int_{|z|\le A} \left(\mathbf{e}^{\varphi(\cdot)z}(X_{s-}) - X_{s-}\right)\tilde{N}(\mathrm{d}s,\mathrm{d}z), \\ I_4 &= \int_0^t \int_{|z|\le A} \left(\mathbf{e}^{\varphi(\cdot)z}(X_{s-}) - X_{s-} - \varphi(X_{s-})z\right)\nu(\mathrm{d}z)\,\mathrm{d}s. \end{aligned}$$

First note that

$$\begin{aligned}\mathbb{E}|I_1|^2 &\le \left(\|f\| + \|FF'\| + \mu_A\ \|\varphi\|\right)^2 t^2 \\ &\le \left(\|f\| + \|FF'\| + \mu_A\ \|\varphi\|\right)^2 T\cdot t \le Ct,\end{aligned}$$

and

$$\mathbb{E}|I_2|^2 \le \left(\mathbb{E}\Big|\int_0^t F(X_s)\,\mathrm{d}W_s\Big|^2\right) \le \left(\mathbb{E}\int_0^t |F(X_s)|^2\,\mathrm{d}s\right) \le \|F\|t.$$

For the jump part we note that like in the proof of Lemma 3.7 we can show that

$$\begin{aligned}\sup_x |\mathbf{e}^{\varphi(\cdot)z}(x) - x| &\le \sup_x \int_0^1 |\varphi(\mathbf{e}^{u\varphi(\cdot)z}(x))z|\,\mathrm{d}u \\ &\le \|\varphi\|\cdot z,\end{aligned}$$

and

$$\begin{aligned}\sup_x |\mathbf{e}^{\varphi(\cdot)z}(x) - x - \varphi(x)z| &\le K_1(z)\sup_x \int_0^1 |\mathbf{e}^{u\varphi(\cdot)z}(x) - x|\,\mathrm{d}u \\ &\le K_1(z)z\|\varphi\|,\end{aligned}$$

where $K_1(z)$ is the constant from H_φ in Assumption 3.1. We then get the following estimates:

$$\mathbb{E}|I_3|^2 \leq \Big(\int_{|z|\leq A} |\varphi^z(x) - x|^2 \nu(\mathrm{d}z) \Big) t \leq Ct,$$

and

$$\mathbb{E}|I_4| \leq \|\varphi\| \int_0^t \int_{|z|\leq A} |K_1(z)z| \; \nu(\mathrm{d}z)\, \mathrm{d}s \leq Ct.$$

And so for all $x \in \mathbb{R}$,

$$\mathbb{E}|X_t(x) - x|^2 \leq Ct.$$

Combining this with (3.33) we get

$$\begin{aligned} \Big| \mathbb{E}_x g_k(X_h) - g_k(x) - \int_0^h A g_k(x)\, \mathrm{d}s \Big| &\leq C \int_0^h \sqrt{s}\; \mathrm{d}s \\ &\leq C h^{\frac{3}{2}}, \end{aligned}$$

which finishes this step of the proof.

In the next step, we do the same for the approximations. To estimate the term $\mathbb{E}_x g(X_h^h(x))$ slightly more work is needed than in the case of the real solution. Consider the ODE

$$\begin{aligned} &\dot{\psi}(t) = f(\psi(t))h + F(\psi(t))w + \varphi(\psi(t))z, \\ &\psi(0) = x, \quad t \in [0,1] \end{aligned}$$

and denote

$$\Psi(h, w, z; x) = \psi(1; x; h, w, z).$$

Consider the Itô process

$$Y_t(x) = \Psi(t, W_t, Z_t; x), \quad t \geq 0.$$

Then

$$X_h^h(x) = Y_h(x).$$

Applying the Itô formula to the function $g(\Psi)$ and the three-dimensional Lévy process (t, W_t, Z_t), we get

$$
\begin{aligned}
g(Y_h) &= g(\Psi(0,0,0)) + \int_0^t g'(\Psi(s,W_s,Z_s))\Psi_h(s,W_s,Z_s)\,\mathrm{d}s \\
&+\mu_A \int_0^h g'(\Psi(s,W_s,Z_s))\Psi_z(s,W_s,Z_s)\,\mathrm{d}s \\
&+\int_0^h g'(\Psi(s,W_s,Z_s))\Psi_w(s,W_s,Z_s)\,\mathrm{d}W_s \\
&+\frac{1}{2}\int_0^h g'(\Psi(s,W_s,Z_s))\Psi_{ww}(s,W_s,Z_s) + g''(\Psi(s,W_s,Z_s))\Psi_w^2(s,W_s,Z_s)\,\mathrm{d}s \\
&+\int_0^h \int_{|z|\le A} \Big(g(\Psi(s,W_s,Z_{s-}+z)) - g(\Psi(s,W_s,Z_{s-}))\,\tilde{N}(\mathrm{d}s,\mathrm{d}z) \\
&+\int_0^h \int_{|z|\le A} \Big(g(\Psi(s,W_s,Z_{s-}+z)) - g(\Psi(s,W_s,Z_{s-})) \\
&- zg'(\Psi(s,W_s,Z_s))\Psi_z(s-,W_{s-},Z_{s-})\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}s,
\end{aligned}
$$

where the term with μ_A comes from the shape of the Lévy process Z, see (3.30). Taking the expected value of $g(Y_h)$, we can eliminate the martingale parts, i.e. the integrals w.r.t. W_s and $\tilde{N}$.

The derivatives of Ψ satisfy certain differential equations, see Section 2.5, which allows us to give explicit formulae for them. We only give the formula for Ψ_w here, the rest follows from the symmetry of the ODE.

$$
\Psi_w(h,w,z;x) = \int_0^1 F(\psi(s;x))\mathrm{e}^{\int_s^1 f'(\psi(r))h + F'(\psi(r))w + \varphi'(\psi(r))z\,\mathrm{d}r}\,\mathrm{d}s.
$$

We see from this that

$$
\begin{aligned}
\Psi(0,0,0;x) &= x, \\
\Psi_h(0,0,0;x) &= f(x), \\
\Psi_w(0,0,0;x) &= F(x), \\
\Psi_z(0,0,0;x) &= \varphi(x), \\
\Psi(0,0,z;x) &= \mathbf{e}^{\varphi(\cdot)z}(x).
\end{aligned}
$$

Furthermore, dealing with the second derivative analogously, we also get

$$
\Psi_{ww}(0,0,0;x) = F(x)F'(x).
$$

In other words, writing $Ag(x)$ in terms of Ψ, we get:

$$\begin{aligned}\int_0^h Ag(x)\,\mathrm{d}s &= \int_0^h g'(\Psi(0,0,0))\Psi_h(0,0,0)\,\mathrm{d}s \\ &+\mu_A \int_0^h g'(\Psi(0,0,0)))\Psi_z(0,0,0)\,\mathrm{d}s \\ &+\frac{1}{2}\int_0^h g'(\Psi(0,0,0))\Psi_{ww}(0,0,0) + g''(\Psi(0,0,0))\Psi_w^2(0,0,0))\,\mathrm{d}s \\ &+\int_0^h\int_{|z|\le A}\Big(g(\Psi(0,0,z)) - g(\Psi(0,0,0)) \\ &\qquad -zg'(\Psi(0,0,0))\Psi_z(0,0,0)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}s.\end{aligned}$$

Keeping in mind that g' and g are bounded and checking that Ψ, as well as the derivatives w.r.t. h, w and z are Lipschitz continuous, we now see that

$$\begin{aligned}|\mathbb{E}_x g(X_h^h) - g(x) - \int_0^h Ag(x)\,\mathrm{d}s| &\le |\mathbb{E}_x C\int_0^h (s+W_s+Z_s)\,\mathrm{d}s| \\ &\le C\int_0^h s + \sqrt{\mathbb{E}_x(W_s^2)} + \sqrt{\mathbb{E}_x(Z_s^2)}\,\mathrm{d}s \\ &\le C\int_0^h s^{\frac{1}{2}}\,\mathrm{d}s \\ &\le Ch^{\frac{3}{2}}.\end{aligned}$$

Combined with (3.33) this gives us

$$\begin{aligned}&|\mathbb{E}_x g(X_h^h) - \mathbb{E}_x g(X_h)| \\ &\le \left|\mathbb{E}_x g(X_h^h) - f(x) - \int_0^h Ag(x)\,\mathrm{d}s\right| + \left|\mathbb{E}_x g(X_h) - f(x) - \int_0^h Ag(x)\,\mathrm{d}s\right| \\ &\le Ch^{\frac{3}{2}}.\end{aligned}$$

Now, for the last step let $t \in [0,T]$ be fixed. Assume we have $h > 0$ and $k \ge 1$, s.t. $t \in [kh,(k+1)h]$. We can write

$$\begin{aligned}&|\mathbb{E}_x g(X_t^h) - \mathbb{E}_x g(X_t)| \\ &\le |\mathbb{E}_x g(X_t^h) - \mathbb{E}_x g(X_{kh}^h)| + |\mathbb{E}_x g(X_{kh}^h) - \mathbb{E}_x g(X_{kh})| + |\mathbb{E}_x g(X_{kh}) - \mathbb{E}_x g(X_t)|.\end{aligned}$$

Since g is Lipschitz continuous, we can estimate the first term with

$$\begin{aligned}
|\mathbb{E}_x g(X_t^h) - \mathbb{E}_x g(X_{kh}^h)| &\leq C\mathbb{E}_x |X_t^h - X_{kh}^h| \\
&\leq C\mathbb{E}_x \Big| \int_{kh}^t f(X_s^h) + F(X_s^h)\frac{\Delta_{kh} W}{h} + \varphi(X_s^h)\frac{\Delta_{kh} Z}{h} \, \mathrm{d}s \Big| \\
&\leq C \sup\{\|f\|, \|F\|, \|\varphi\|\} \int_{kh}^t 1 + \frac{\mathbb{E}_x|W_h|}{h} + \frac{\mathbb{E}_x|Z_h|}{h} \, \mathrm{d}s \\
&\leq C \int_{kh}^t \frac{1}{\sqrt{h}} \, \mathrm{d}s \\
&\leq C\sqrt{h}.
\end{aligned}$$

The estimates for the second and third term follow from (3.32) and (3.34), using the Markov property of X. The final result then gives us that for any $\varepsilon > 0$ we can find $h_0 > 0$, such that for every $h < h_0$ there is $k \geq 1$, so that

$$\begin{aligned}
&|\mathbb{E}_x g(X_t^h) - \mathbb{E}_x g(X_t)| \\
&\leq |\mathbb{E}_x g(X_t^h) - \mathbb{E}_x g(X_{kh}^h)| + |\mathbb{E}_x g(X_{kh}^h) - \mathbb{E}_x g(X_{kh})| + |\mathbb{E}_x g(X_{kh}) - \mathbb{E}_x g(X_t)| \\
&\leq 3 \cdot C \cdot \sqrt{h} \\
&< \varepsilon.
\end{aligned}$$

□

Chapter 4

First Order Linear Marcus SPDEs

Setting and main result

We consider an m-dimensional Brownian motion W and an m-dimensional pure jump Lévy process Z,

$$Z_t = \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}z, \mathrm{d}s) + \int_0^t \int_{|z|>1} zN(\mathrm{d}z, \mathrm{d}s).$$

Let

$$\begin{aligned} a\colon \mathbb{R}^d &\to \mathbb{R}^d, & b, c\colon \mathbb{R}^d &\to \mathbb{R}, \\ A, \alpha\colon \mathbb{R}^d &\to \mathbb{R}^{d\times m}, & B, C, \beta, \sigma\colon \mathbb{R}^d &\to \mathbb{R}^m, \end{aligned}$$

be measurable functions.

For $u \in C^1(\mathbb{R}^d, \mathbb{R})$, consider first order operators

$$\begin{aligned} \mathcal{P}u(x) &= \nabla^T u(x)a(x) + u(x)b(x) + c(x), \\ \mathcal{R}u(x) &= \nabla^T u(x)A(x) + u(x)B(x) + C(x), \\ \mathcal{Q}u(x) &= \nabla^T u(x)\alpha(x) + u(x)\beta(x) + \sigma(x), \end{aligned}$$

and the first order linear equation written in the compact differential form as

$$\begin{aligned} \mathrm{d}u(t, x) &= \mathcal{P}u(t, x)\,\mathrm{d}t + \mathcal{R}u(t, x) \circ \mathrm{d}W_t + \mathcal{Q}u(t, x) \diamond \mathrm{d}Z_t, \\ u(0, x) &= u_0(x), \quad x \in \mathbb{R}^d,\ t \in [0, T], \end{aligned} \tag{4.1}$$

with some initial condition $u_0 : \mathbb{R}^d \to \mathbb{R}$.

More precisely, this equation is understood as the integral equation as follows:

$$\begin{aligned} u(t,x) = u_0(x) &+ \int_0^t \Big[\nabla^T u(s,x)a(x) + u(s,x)b(x) + c(x)\Big]\,\mathrm{d}s \\ &+ \int_0^t \Big[\nabla^T u_x(s,x)A(x) + u(s,x)B(x) + C(x)\Big] \circ \mathrm{d}W_s \\ &+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\mathcal{Q}z}u(s-,x) - u(s-,x)\Big)\tilde{N}(\mathrm{d}z,\mathrm{d}s) \\ &+ \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\mathcal{Q}z}u(s-,x) - u(s-,x) - \mathcal{Q}u(s-,x)z\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}s \\ &+ \int_0^t \int_{|z|>1} \Big(\mathbf{e}^{\mathcal{Q}z}u(s-,x) - u(s-,x)\Big)N(\mathrm{d}z,\mathrm{d}s), \end{aligned} \tag{4.2}$$

where for each $z \in \mathbb{R}^m$ the mapping

$$u(\cdot) \mapsto \mathbf{e}^{\mathcal{Q}z}u(\cdot)$$

is defined with the help of the solution of the first order, linear, time autonomous partial differential equation

$$\begin{aligned} \partial_r g(r,x) &= \nabla^T g(r,x)\alpha(x)z + g(r,x)\beta(x)z + \sigma(x)z, \quad r \in [0,1], \\ g(0,x) &= u(x), \end{aligned} \tag{4.3}$$

and

$$\mathbf{e}^{\mathcal{Q}z}u(x) := g(1;x,z).$$

Definition 4.1. *We say that u is a solution of the equation (4.1) if $t \mapsto u(t,\cdot,\cdot)$ is a càdlàg adapted process, $x \mapsto u(\cdot,x,\cdot)$ is a C^1-function, and the equation (4.1) is satisfied almost surely.*

The goal is to solve (4.1) with the help of the method of stochastic characteristics. The results of this chapter have been accepted to be published in [HPar].
The main result is as follows. Assume that the functions satisfy the following conditions.

Assumption 4.2.

$$\begin{gathered} a \in C_b^3(\mathbb{R}^d,\mathbb{R}^d) \quad \textit{and} \quad b,c \in C_b^3(\mathbb{R}^d,\mathbb{R}) \\ A,\alpha \in C_b^4(\mathbb{R}^d,\mathbb{R}^{d\times m}) \quad \textit{and} \quad B,c,\beta,\sigma \in C_b^4(\mathbb{R}^d,\mathbb{R}^m). \end{gathered} \tag{4.4}$$

Consider a $(d+2)$-dimensional system of Marcus SDEs (characteristics equations)

$$\varphi_{0,t}(x) = x - \int_0^t a(\varphi_{0,r}(x))\,\mathrm{d}r - \int_0^t A(\varphi_{0,r}(x)) \circ \mathrm{d}W_r - \int_0^t \alpha(\varphi_{0,r}(x)) \diamond \mathrm{d}Z_r, \tag{4.5}$$

$$\begin{aligned}\xi_{0,t}(x,\xi_0) = \xi_0 - \int_0^t \xi_{0,r}(x,\xi_0) b(\varphi_{0,r}(x))\,\mathrm{d}r - \int_0^t \xi_{0,r}(x,\xi_0) B_j(\varphi_{0,r}(x)) \circ \mathrm{d}W_r \\ - \int_0^t \xi_{0,r}(x,\xi_0)\beta_j(\varphi_{0,r}(x)) \diamond \mathrm{d}Z_r,\end{aligned} \tag{4.6}$$

$$\begin{aligned}\zeta_{0,t}(x,\xi_0,\zeta_0) = \zeta_0 - \int_0^t \xi_{0,r}(x,\xi_0) c(\varphi_{0,r}(x))\,\mathrm{d}r - \int_0^t \xi_{0,r}(x,\xi_0) C_j(\varphi_{0,r}(x)) \circ \mathrm{d}W_r \\ - \int_0^t \xi_{0,r}(x,\xi_0)\sigma(\varphi_{0,r}(x)) \diamond \mathrm{d}Z_r.\end{aligned} \tag{4.7}$$

We will see that under Assumptions 4.2, there exists a unique strong solution to the SDEs (4.5), (4.6) and (4.7). Furthermore the associated solution flow is a C^2-flow of diffeomorphisms of $\mathbb{R}^{d+2}$.
Let $(\varphi_{t,0}, \xi_{t,0}, \zeta_{t,0})$ be the inverse flow of $(\varphi_{0,t}, \xi_{0,t}, \zeta_{0,t})$.

Theorem 4.3. *Let Assumptions 4.2 be true and let $u_0 \in C^2(\mathbb{R}^d, \mathbb{R})$. Then the function*

$$u(t,x) := \xi_{t,0}(x,1) u_0(\varphi_{t,0}(x)) + \zeta_{t,0}(x,1,0) \tag{4.8}$$

is the unique solution of (4.1).

The intuition behind the formula (4.8) is described in the sequel: In the deterministic case, $A = \alpha = 0$, $B = C = \beta = \sigma = 0$, formula (4.8) is reduced to the formula from Lemma 2.19. In the continuous case, $\alpha = 0$, $B = \beta = 0$, the formula (4.8) was derived by Kunita, see [Kun97]. The important feature of the equation (4.1) in the continuous case is that the stochastic integrals have to be considered in the Stratonovich sense. Informally this can be justified by the following consideration. It is well-known that the Stratonovich stochastic integral can be approximated pathwise by Wong–Zakai approximations. Each of these approximations can be treated pathwise as a deterministic first order PDE that has a solution (4.8), and hence the limit should have the same form. In the case of jump noise, the role of the Stratonovich stochastic differential equations is played by the Marcus (canonical) stochastic differential equations. As we have seen in Chapter 3, these equations can also be seen as a limit of continuous Wong–

Zakai approximations and enjoy the Newton–Leibniz change of variables formula of conventional calculus. Hence it is intuitively clear that the equation (4.1) has to be considered as a Marcus equation.

Example 4.4 (transport equation). *Consider the transport equation*

$$\begin{aligned} u(t,x) = u_0(x) &+ \int_0^t \nabla^T u(r,x)a(x)\,\mathrm{d}r \\ &+ \int_0^t \nabla^T u(r,x)A(x)\circ \mathrm{d}W_r + \int_0^t \nabla^T u(r,x)\alpha(x)\diamond \mathrm{d}Z_r. \end{aligned} \tag{4.9}$$

In this case, the characteristics equation is a d-dimensional Marcus SDE

$$\varphi_{0,t}(x) = x - \int_0^t a(\varphi_{0,r}(x))\,\mathrm{d}r - \int_0^t A(\varphi_{0,r}(x))\circ \mathrm{d}W_r - \int_0^t \alpha(\varphi_{0,r}(x))\diamond \mathrm{d}Z_r.$$

Then the solution has the form

$$u(t,x) = u_0(\varphi_{t,0}(x)).$$

Example 4.5 (explicit one-dimensional solution). *In dimension $m = d = 1$ if $a(x) = A(x) = \alpha(x)$, the equation* (4.9) *can be solved explicitly with the help of the Itô formula for Marcus SDEs. Indeed, let Z be a general (not necessarily pure jump) one-dimensional Lévy process, and consider the equation*

$$u(t,x) = u_0(x) + \int_0^t \partial_x u(r,x)\alpha(x)\diamond \mathrm{d}Z_r. \tag{4.10}$$

Assume that $\alpha(x) > 0$ and denote

$$H(x) = \int_0^x \frac{\mathrm{d}y}{\alpha(y)}, \qquad x \in \mathbb{R}.$$

Then the characteristics equation

$$\varphi_{0,t}(x) = x - \int_0^t \alpha(\varphi_{0,r}(x))\diamond \mathrm{d}Z_r$$

has the solution

$$\varphi_{0,t}(x) = H^{-1}(H(x) - Z_t)$$

and the inverse flow $\varphi_{t,0}$ can be found by a straightforward calculation as

$$\varphi_{t,0}(x) = H^{-1}(H(x) + Z_t).$$

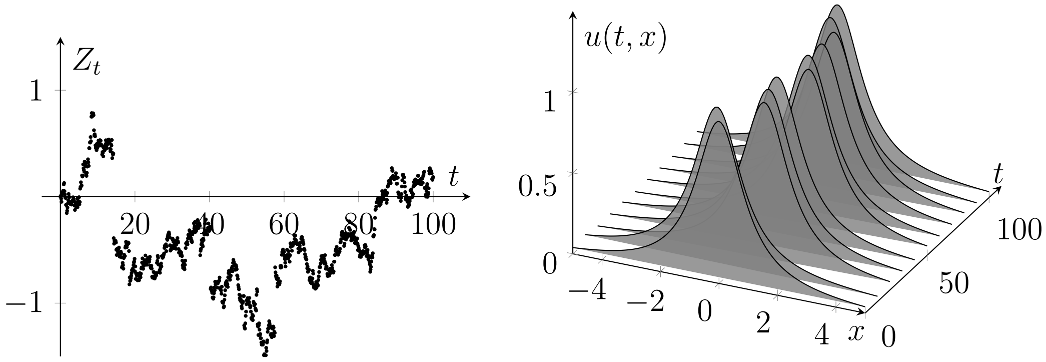

Figure 4.1: A sample path of a symmetric α-stable Lévy process Z with $\mathbb{E}e^{i\lambda Z_1} = e^{-0.1|\lambda|^\alpha}$ for $\alpha = 1.75$ (left); the solution $u(t,x)$ with the initial condition $u_0(x) = 1/(1+x^2)$ sampled at $t = 0, 10, 20, \dots, 100$ (right).

Let us show that

$$u(t,x) := u_0(H^{-1}(H(x) + Z_t)) \tag{4.11}$$

satisfies (4.10)*. Indeed,* $u(0,x) = u_0(x)$*, and*

$$\partial_x u(t,x) = \partial_x u_0(\varphi_{t,0}(x))\partial_x \varphi_{t,0}(x) = \partial_x u_0(\varphi_{t,0}(x))\frac{\alpha(H(x)+Z_t)}{\alpha(x)}.$$

On the other hand, the Itô formula for Marcus SDEs (see Theorem 3.4) yields

$$\begin{aligned} u(t,x) = u_0(H^{-1}(H(x)+Z_t)) &= u_0(x) + \int_0^t \frac{\partial}{\partial Z} u_0(H^{-1}(H(x)+Z_r)) \diamond \mathrm{d}Z_r \\ &= u_0(x) + \int_0^t \partial_x u_0(\varphi_{0,r}(x))\alpha(H(x)+Z_r) \diamond \mathrm{d}Z_r \\ &= u_0(x) + \int_0^t \partial_x u(r,x)\alpha(x) \diamond \mathrm{d}Z_r. \end{aligned}$$

Example 4.6. *In this example we apply formula* (4.11) *to the first order equation*

$$\partial_t u(t,x) = \partial_x u(t,x)\sqrt{x^2+1} \diamond \mathrm{d}Z_t, \quad u(0,x) = u_0(x). \tag{4.12}$$

In this case, for $x \in \mathbb{R}$

$$H(x) = \operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2+1})$$

and

$$H^{-1}(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Hence, (4.12) *has the explicit solution*

$$u(t,x) = u_0\Big(\sinh(\operatorname{arcsinh}(x) + Z_t)\Big), \quad t \geq 0,\ x \in \mathbb{R}.$$

Sample paths of solutions u driven by a symmetric α-stable Lévy process are presented in Fig. 4.1.

Proof of Theorem 4.3

Existence

For simplicity assume that $\nu(|z| > 0) = 0$. Consider the linear equation (4.1). To solve it, we consider the $(d+2)$-dimensional system of characteristics Marcus SDEs (4.5), (4.6), (4.7). Note that φ is a d-dimensional process whereas ξ and ζ are one-dimensional.

Denote $X = (\varphi^1, \ldots, \varphi^d, \xi, \zeta) = (X^1, \ldots, X^d, X^{d+1}, X^{d+2}) \in \mathbb{R}^{d+2}$, and

$$f(X) = \begin{pmatrix} a^1(X^1,\ldots,X^d) \\ \vdots \\ a^d(X^1,\ldots,X^d) \\ X^{d+1} b(X^1,\ldots,X^d) \\ X^{d+1} c(X^1,\ldots,X^d) \end{pmatrix},$$

$$F(X) = \begin{pmatrix} A_1^1(X^1,\ldots,X^d) & \cdots & A_1^m(X^1,\ldots,X^d) \\ \vdots & & \\ A_d^1(X^1,\ldots,X^d) & \cdots & A_d^m(X^1,\ldots,X^d) \\ X^{d+1}B_1(X^1,\ldots,X^d) & \cdots & X^{d+1}B_m(X^1,\ldots,X^d) \\ X^{d+1}C_1(X^1,\ldots,X^d) & \cdots & X^{d+1}C_m(X^1,\ldots,X^d) \end{pmatrix},$$

$$\Sigma(X) = \begin{pmatrix} \alpha_1^1(X^1,\ldots,X^d) & \cdots & \alpha_1^m(X^1,\ldots,X^d) \\ \vdots & & \\ \alpha_d^1(X^1,\ldots,X^d) & \cdots & \alpha_d^m(X^1,\ldots,X^d) \\ X^{d+1}\beta_1(X^1,\ldots,X^d) & \cdots & X^{d+1}\beta_m(X^1,\ldots,X^d) \\ X^{d+1}\sigma_1(X^1,\ldots,X^d) & \cdots & X^{d+1}\sigma_m(X^1,\ldots,X^d) \end{pmatrix}. \tag{4.13}$$

In the matrix form, the system (4.5), (4.6) and (4.7) reads as a canonical equation of the type (3.2)

$$X_{0,t} = X_0 - \int_0^t \Phi(X_{0,r}, \diamond\, \mathrm{d}r),$$

with $\varphi(X,r,z) = \Sigma(X)z$ (here we allow an abuse of notation). There is a unique solution which is a C^2-flow. To see this, first note that a, A and α satisfy Assumptions 3.1 and thus $\varphi_{0,t}$ is a C^2-flow according to Theorem 3.2. On the other hand, the other two equations can be solved explicitly, $\zeta_{0,t}$ being linear and $\xi_{0,t}$ given by the exponential

$$\begin{aligned}&\xi_{0,t}(x,\xi_0)\\ &=\xi_0\exp\Big(-\int_0^t b(\varphi_{0,r}(x))\,\mathrm{d}r-\int_0^t B(\varphi_{0,r}(x))\circ\mathrm{d}W_r-\int_0^t\beta(\varphi_{0,r}(x))\diamond\mathrm{d}Z_r\Big).\end{aligned}\tag{4.14}$$

Denote $Y_t = X_{t,0} = X_{0,t}^{-1} = (\varphi^1_{t,0},\dots,\varphi^d_{t,0},\xi_{t,0},\zeta_{t,0})$ the inverse flow.

Consider a function

$$\Theta(x,\xi_0,\zeta_0)=\xi_0u_0(x)+\zeta_0$$

and for $x\in\mathbb{R}^d,\ \xi_0,\zeta_0\in\mathbb{R}$ define a process

$$u(t;x,\xi_0,\zeta_0)=\Theta(Y_t(x,\xi_0,\zeta_0))=\xi_{t,0}(x,\xi_0)u_0(\varphi_{t,0}(x))+\zeta_{t,0}(x,\xi_0,\zeta_0).$$

Then by Theorem 3.10 we get that

$$\begin{aligned}u(t,x,&\xi_0,\zeta_0)=\Theta(Y_t)=\Theta(x,\xi_0,\zeta_0)-\int_0^t\nabla^T\Big(\Theta\circ Y_r(x,\xi_0,\zeta_0)\Big)\Phi(x,\xi_0,\zeta_0,\diamond\,\mathrm{d}r)\\ &=u_0(0,x,\xi_0,\zeta_0)+\int_0^t\nabla_x^Tu(r-,x,\xi_0,\zeta_0)a(x)\,\mathrm{d}r\\ &\quad+\xi_0\int_0^t\partial_{\xi_0}u(r-,x,\xi_0,\zeta_0)b(x)\,\mathrm{d}r+\xi_0\int_0^t\partial_{\zeta_0}u(r-,x,\xi_0,\zeta_0)c(x)\,\mathrm{d}r\\ &\quad+\int_0^t\nabla_x^Tu(r-,x,\xi_0,\zeta_0)A(x)\circ\mathrm{d}W_r+\xi_0\int_0^t\partial_{\xi_0}u(r-,x,\xi_0,\zeta_0)B(x)\circ\mathrm{d}W_r\\ &\quad+\xi_0\int_0^t\partial_{\zeta_0}u(r-,x,\xi_0,\zeta_0)C(x)\circ\mathrm{d}W_r\\ &\quad+\int_0^t\int_{|z|\le1}\Big[\Theta(\mathbf{e}^{-\Sigma(\cdot)z}(Y_{r-}(x,\xi_0,\zeta_0)))-u(r-,x,\xi_0,\zeta_0)\Big]\tilde{N}(\mathrm{d}z,\mathrm{d}r)\\ &\quad+\int_0^t\int_{|z|\le1}\Big[\Theta(\mathbf{e}^{-\Sigma(\cdot)z}(Y_{r-}(x,\xi_0,\zeta_0)))-u(r-,x,\xi_0,\zeta_0)\\ &\qquad\qquad+\nabla_x^Tu(r-,x,\xi_0,\zeta_0)\alpha(x)z+\xi_0\partial_{\xi_0}u(r-,x,\xi_0,\zeta_0)\beta(x)z\\ &\qquad\qquad\qquad+\xi_0\partial_{\zeta_0}u(r-,x,\xi_0,\zeta_0)\sigma(x)z\Big]\,\nu(\mathrm{d}z)\,\mathrm{d}r.\end{aligned}$$

Let us study the derivatives $\partial_{\xi_0}u$ and $\partial_{\zeta_0}u$. We already know that the process ξ is found explicitly as the exponential (4.14). We then see that the derivative is equal to

$$
\begin{aligned}
&\partial_{\xi_0}\xi_{0,t}(x,\xi_0)\\
&\quad=\exp\Big(-\int_0^t b(\varphi_{0,r}(x))\,\mathrm{d}r-\int_0^t B(\varphi_{0,r}(x))\circ\mathrm{d}W_r-\int_0^t \beta(\varphi_{0,r}(x))\diamond\mathrm{d}Z_r\Big)
\end{aligned}
$$

and it satisfies (4.6) with the initial values $(x,\xi_0)=(x,1)$. By the formula of the derivative of the inverse function we get

$$
\begin{aligned}
\partial_{\xi_0}\xi_{t,0}(x,\xi_0)&=\Big(\partial_{\xi_0}\xi_{0,t}(\varphi_{t,0}(x),\xi_{t,0}(x,\xi_0))\Big)^{-1}\\
&=\Big(\xi_{0,t}(\varphi_{t,0}(x),1)\Big)^{-1}.
\end{aligned}\tag{4.15}
$$

On the other hand, we see that

$$
\xi_{0,t}\Big(\varphi_{t,0}(x),\big(\xi_{0,t}(\varphi_{t,0}(x),1)\big)^{-1}\Big)\equiv 1,
$$

which means that

$$
\Big(\xi_{0,t}(\varphi_{t,0}(x),1)\Big)^{-1}=\xi_{t,0}(x,1).
$$

Analogously,

$$
\begin{aligned}
&\partial_{\xi_0}\zeta_{t,0}(x,\xi_0,\zeta_0)=\zeta_{t,0}(x,1,0),\\
&\partial_{\zeta_0}\zeta_{t,0}(x,\xi_0,\zeta_0)=1.
\end{aligned}\tag{4.16}
$$

Thus taking into account (4.15) and (4.16) we can write

$$
\begin{aligned}
\partial_{\xi_0}u(r,x,\xi_0,\zeta_0)&=\partial_{\xi_0}\Big(\Theta(\varphi_{t,0}(x),\xi_{t,0}(x,\xi_0),\zeta_{t,0}(x,\xi_0,\zeta_0))\Big)\\
&=u_0(\varphi_{t,0}(x))\partial_{\xi_0}\xi_{t,0}(x,\xi_0)+\partial_{\xi_0}\zeta_{t,0}(x,\xi_0,\zeta_0)\\
&=u_0(\varphi_{t,0}(x))\xi_{t,0}(x,1)+\zeta_{t,0}(x,1,0)\\
&=u(r,x,1,0)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_{\zeta_0}u(r,x,\xi_0,\zeta_0)&=\partial_{\zeta_0}\Big(\Theta(\varphi_{t,0}(x),\xi_{t,0}(x,\xi_0),\zeta_{t,0}(x,\xi_0,\zeta_0))\Big)\\
&=\partial_{\zeta_0}\zeta_{t,0}(x,\xi_0,\zeta_0)\\
&=1.
\end{aligned}
$$

Inspecting the structure of the matrix function Σ in (4.13) we get that the mapping $\mathbf{e}^{\Sigma(\cdot)z}\colon \mathbb{R}^d \to \mathbb{R}^{d+2}$ has the following form:

$$\mathbf{e}^{\Sigma(\cdot)z}\begin{pmatrix} x \\ \xi_0 \\ \zeta_0 \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{-\alpha(\cdot)z}(x) \\ \xi_0 \exp\Big(-\int_0^1 \beta(\mathbf{e}^{-r\alpha(\cdot)z}(x))z\,\mathrm{d}r\Big) \\ \zeta_0 - \xi_0 \int_0^1 \exp\Big(-\int_0^s \beta(\mathbf{e}^{-r\alpha(\cdot)z}(x))z\,\mathrm{d}r\Big)\sigma(\mathbf{e}^{-r\alpha(\cdot)z}(x))\,\mathrm{d}s \end{pmatrix}.$$

Recalling from Lemma 2.19 that

$$\mathbf{e}^{\mathcal{Q}z}(\Theta(\cdot)) = \Theta(\mathbf{e}^{-\Sigma(\cdot)z}(\cdot)),$$

we get the equality

$$\begin{aligned}
&u(t,x,1,0) \\
&= u_0(x) + \int_0^t \nabla^T u(r-,x,1,0)a(x)\,\mathrm{d}r + \int_0^t u(r-,x,1,0)b(x)\,\mathrm{d}r + c(x)t \\
&\quad + \int_0^t \nabla^T u(r-,x,1,0)A(x)\circ \mathrm{d}W_r + \int_0^t u(r-,x,1,0)B(x)\circ\mathrm{d}W_r + C(x)W_t \\
&\quad + \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\mathcal{Q}z}(u(r-,x,1,0)) - u(r-,x,1,0)\Big)\,\tilde{N}(\mathrm{d}z,\mathrm{d}r) \\
&\quad + \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\mathcal{Q}z}(u(r-,x,1,0)) - u(r-,x,1,0) - \mathcal{Q}u(r-,x,1,0)z\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r.
\end{aligned}$$

This means that $u(t,x) = u(t,x,1,0)$ is the solution of (4.2).

Uniqueness

To show uniqueness, we use the same approach as in [Kun97, Theorem 6.1.2.]. Let us first assume that $b(x) = 0$, $B(x) = 0$, $\beta(x) = 0$, and $c(x) = 0$, $C(x) = 0$, $\sigma(x) = 0$. In this case, the solution defined by the characteristics has the form

$$u(t,x) = u_0(\varphi_{t,0}(x)). \tag{4.17}$$

Let v be another semimartingale solution of the form

$$\begin{aligned}
v(t,x) = u_0(x) &+ \int_0^t f(x,r)\,\mathrm{d}r + \int_0^t F(x,r)\,\mathrm{d}W_r \\
&+ \int_0^t \int_{|z|\le 1} \varphi(x,r,z)\,\tilde{N}(\mathrm{d}z,\mathrm{d}r)
\end{aligned}$$

with some f, F and φ.

Then Theorem 3.8 yields that

$$
\begin{aligned}
&v(t,\varphi_{0,t}(x))\\
&= u_0(x) + \int_0^t f(\varphi_{0,r-}(x),r)\,\mathrm{d}r + \int_0^t F(\varphi_{0,r-}(x),r)\,\mathrm{d}W_r\\
&\quad + \int_0^t \int_{|z|\le 1} \varphi(\varphi_{0,r-}(x),r,z)\,\tilde{N}(\mathrm{d}z,\mathrm{d}r) - \int_0^t \partial_x v(r-,\varphi_{0,r-}(x)) a(\varphi_{0,r-}(x))\,\mathrm{d}r\\
&\quad - \int_0^t \partial_x v(r-,\varphi_{0,r-}(x)) A(\varphi_{0,r-}(x)) \circ \mathrm{d}W_r\\
&\quad + \int_0^t \int_{|z|\le 1} \Big(v(r-,\mathbf{e}^{-\alpha}(\varphi_{0,r-})) + \varphi(\mathbf{e}^{-\alpha}(\varphi_{0,r-})),r,z) - v(r-,\varphi_{0,r-})\\
&\qquad\qquad - \varphi(r-,\varphi_{0,r-},r,z)\Big)\,\tilde{N}(\mathrm{d}z,\mathrm{d}r)\\
&\quad + \int_0^t \int_{|z|\le 1} \Big(v(r-,\mathbf{e}^{-\alpha}(\varphi_{0,r-})) + \varphi(\mathbf{e}^{-\alpha}(\varphi_{0,r-})),r,z) - v(r-,\varphi_{0,r-})\\
&\qquad\qquad + \partial_x v(r-,\varphi_{0,r-})\alpha(\varphi_{0,r-}) - \varphi(r-,\varphi_{0,r-},r,z)\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r,
\end{aligned}
$$

where we know that φ is given by

$$\varphi(x,r,z) = \mathbf{e}^{\mathcal{Q}z}(v(r-,x)) - v(r-,x).$$

Let us take a closer look at the jump terms. On the one hand we know that

$$v(r,\mathbf{e}^{\alpha(\cdot)z}(\varphi_{0,r-}(x))) = \mathbf{e}^{\mathcal{Q}z} v(r,\varphi_{0,r-}(x)).$$

On the other hand, inverting the sign of the jump size z is equivalent to the reversion of the fictitious time in the Marcus ODE for $\mathrm{e}^{\alpha(\cdot)z}$. Hence we obtain that

$$\mathbf{e}^{\mathcal{Q}z}(v(r,\mathbf{e}^{-\alpha(\cdot)z}(\varphi_{0,r-}(x)))) = v(r,\varphi_{0,r-}(x)).$$

We also see that

$$\partial_x v(r-,\varphi_{0,r-}(x))\alpha(\varphi_{0,r-}(x)) = \mathcal{Q}v(r-,\varphi_{0,r-}(x)),$$

and thus we get

$$
\begin{aligned}
&v(t,\varphi_{0,t}(x))\\
&= u_0(x) + \int_0^t \partial_t v(r-,\varphi_{0,r-}(x)) \diamond \mathrm{d}\varphi_{0,r}(x)\\
&\quad - \int_0^t \partial_x v(r-,\varphi_{0,r-}(x)) a(\varphi_{0,r-}(x))\,\mathrm{d}r\\
&\quad - \int_0^t \partial_x v(r-,\varphi_{0,r-}(x)) A(\varphi_{0,r-}(x)) \circ \mathrm{d}W_r\\
&\quad - \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\mathcal{Q}z}(v(r-,\varphi_{0,r-}(x)) - v(r-,\varphi_{0,r-}(x))\Big)\,\tilde{N}(\mathrm{d}z,\mathrm{d}r)\\
&\quad - \int_0^t \int_{|z|\le 1} \Big(\mathbf{e}^{\mathcal{Q}z}(v(r-,\varphi_{0,r-}(x))) - v(r-,\varphi_{0,r-}(x))\\
&\qquad\qquad\qquad\qquad\qquad\qquad - \mathcal{Q}v(r-,\varphi_{0,r-}(x))z\Big)\,\nu(\mathrm{d}z)\,\mathrm{d}r\\
&\quad = u_0(x).
\end{aligned}
$$

Hence, $v(t,x)$ coincides with the solution u given by (4.17).

In the presence of linear terms b, B and β, the process $u(t,\varphi_{t,0}(x))$ given by the characteristics solution has the form

$$
\begin{aligned}
u(t,\varphi_{t,0}(x)) &= u_0(\varphi_{t,0}(x)) \times\\
&\times \exp\Big(\int_0^t b(\varphi_{0,r}(x))\,\mathrm{d}r + \int_0^t B(\varphi_{0,r}(x)) \circ \mathrm{d}W_r + \int_0^t \beta(\varphi_{0,r}(x)) \diamond \mathrm{d}Z_r\Big)
\end{aligned}
$$

and the difference $d(t,x) := v(t,\varphi_{0,t}(x)) - u(t,\varphi_{t,0}(x))$ satisfies the linear equation

$$
\begin{aligned}
d(t,x) = \int_0^t d(r-,x) b(\varphi_{r-}(x))\,\mathrm{d}r &+ \int_0^t d(r-,x) B(\varphi_{r-}(x)) \circ \mathrm{d}W_r\\
&+ \int_0^t d(r-,x)\beta(\varphi_{r-}(x)) \diamond \mathrm{d}Z_r
\end{aligned}
$$

and thus $d \equiv 0$. The same relation holds for the difference of the non-homogeneous equations, which proves the uniqueness of the solution.

Chapter 5

Mild Solutions to Second Order Equations

Setting and definitions

A common technique to solve stochastic differential equations is to consider mild solutions. We now turn our attention to second order equations with Lévy noise and see how mild solutions work in this framework.
The second order equation is given by

$$\begin{cases} \mathrm{d}u(t,x) & = Au(t,x) + Bu(t,x) \diamond \mathrm{d}Z_t, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) & = u_0(x), \end{cases} \tag{5.1}$$

with A being a second order differential operator, e.g. the Laplacian, and B characterizing a first order differential operator.
We will restrict ourselves to the case where the Lévy process is a pure jump process with bounded jumps, given by

$$Z_t = \int_{|z|\leq\alpha} z\,\tilde{N}(\mathrm{d}s,\mathrm{d}x),$$

for some $\alpha > 0$.
At the end of the chapter some comments on the incorporation of the Brownian motion and the drift will be made.

In this chapter we will be working in the framework of Sobolev spaces. For the reasons explained in Section 2.4, we are looking for $H_2^\theta(\mathbb{R}^d)$-valued solutions with $\theta > \frac{d}{2}$. For convenience we just write $H^\theta = H_2^\theta(\mathbb{R}^d)$.
Again, because the Marcus integral is not defined in a general sense, it is necessary to specify what equation (5.1) means.

Assumption 5.1. *Let $B : \mathbb{R}^m \to L(H^\theta, H^{\theta-1})$ be a bounded linear operator of the following form: For $z \in \mathbb{R}^m$ and $u \in H^\theta$, $(B_z)u$ is defined as*

$$(B_z u)(x) = \langle b(x)z, \nabla u(x)\rangle,$$

where b is in $C_c^k(\mathbb{R}^d, \mathbb{R}^{d\times m})$, with $k := \min\{l \in \mathbb{N} : \theta < l\}$.

Under Assumption 5.1, B_z generates a group $(\mathbf{e}^{rB_z})_{r\in\mathbb{R}}$, s.t. $\mathbf{e}^{rB_z}v_0(\cdot) = v(r,\cdot)$, where v solves

$$\begin{cases} \dot{v}(r,x) &= B_z v(r,x), \quad (r,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ v(0,x) &= v_0(x). \end{cases} \tag{5.2}$$

Under the above restrictions on Z, we understand the solution to (5.1) as solution to the equation

$$\begin{aligned} \mathrm{d}u(t,x) =& Au(t,x) + \int_{|z|\le\alpha} (\mathbf{e}^{B_z} - \mathrm{Id})u(t,x)\tilde{N}(\mathrm{d}t,\mathrm{d}z) \\ &+ \int_{|z|\le\alpha} (\mathbf{e}^{B_z} - \mathrm{Id} - B_z)u(t,x)\nu(\mathrm{d}z)\,\mathrm{d}t. \end{aligned}$$

Note here that the integral w.r.t. the compensated PRM is understood in the sense of Section 2.3. This means that the integral is well defined if the solution process u is progressively measurable.

Assumption 5.2. *Let $A : H^\theta \to H^{\theta-2}$ be a linear operator that generates an analytic semigroup $(S(t))_{t\ge0}$ of contractions with $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A.*

Note that Assumption 5.2 especially implies that $(S(t))_{t\ge0}$ is a strongly continuous semigroup, see [Paz83, Chapter 2.5].

Definition 5.3. *We define a mild solution to (5.1) as a progressively measurable process $u\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{C}$, which is càdlàg in time and satisfies*

$$\begin{aligned} u(t,x) = S(t)u(0) &+ \int_0^t \int_{|z|\le\alpha} S(t-s)(\mathbf{e}^{B_z} - \mathrm{Id})u(s,x)\tilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &+ \int_0^t \int_{|z|\le\alpha} S(t-s)(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)u(s,x)\nu(\mathrm{d}z)\,\mathrm{d}s. \end{aligned}$$

Existence of the mild solution

To prove the existence of solutions to (5.1) one more technical assumption on the Lévy process is needed. It is known that while $\int_{|z|\leq\alpha} |z|\,\nu(\mathrm{d}z)$ may diverge, for the square of the jumps $\int_{|z|\leq\alpha} |z|^2\,\nu(\mathrm{d}z) < \infty$ holds true. The next assumption slightly strengthens this statement:

Assumption 5.4. *Let the Lévy measure ν of Z satisfy*

$$\int_{|z|\leq\alpha} |z|^p\,\nu(\mathrm{d}z) < \infty$$

for some $1 < p < 2$.

The main result of this chapter is the following existence theorem:

Theorem 5.5. *Let u_0 be in H^θ with $\theta > \frac{d}{2} + 2$. Under the Assumptions 5.1, 5.2 and 5.4 there is a unique mild solution $u : [0,T] \to H^\theta$ to (5.1).*

Proof. Consider the Banach space $\mathcal{M}^2 = \mathcal{M}^2(0,T;H^\theta)$ of progressively measurable, integrable, H^θ-valued processes with the family of equivalent norms $\{\|\cdot\|_\lambda\}_{\lambda>0}$,

$$\|u\|_\lambda = \left(\mathbb{E}\int_0^T \mathrm{e}^{-\lambda s}\|u(s)\|_{H^\theta}^2\,\mathrm{d}s\right)^{\frac{1}{2}}.$$

We take the operator

$$\begin{aligned} I(u)(x) =& S(t)u_0(x) + \int_0^t \int_{|z|\leq\alpha} S(t-s)(\mathbf{e}^{B_z} - \mathrm{Id})u(s,x)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &+ \int_0^t \int_{|z|\leq\alpha} S(t-s)(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)u(s,x)\,\nu(\mathrm{d}z)\,\mathrm{d}s, \end{aligned}$$

and show that $I(u) \in \mathcal{M}^2$ for $u \in \mathcal{M}^2$. It is then enough to show that I is a contraction on $\mathcal{M}^2$, to see that there is a unique fixed point which solves the SPDE in the mild sense.
One benefit of mild solutions is the smoothing property of the analytic semigroup, which follows from using [Paz83, Theorem 2.6.13] on $A^{\frac{r}{2}}$ for $0 < r \leq 2$. Namely, we get

$$\|S(t)\|_{L(H^{\theta-r},H^\theta)} \leq Ct^{-\frac{r}{2}}, \tag{5.3}$$

which will be used for both integral terms.

We start with the integral regarding the compensated Poisson random measure. Let us first take a look at the expression $(\mathrm{e}^{B_z} - \mathrm{Id})$. Since e^{B_z} solves the transport equation, for v in the domain of B_z we have

$$(\mathbf{e}^{B_z} - \mathrm{Id})v = \int_0^1 \mathbf{e}^{xB_z}(B_z)v \, \mathrm{d}s. \tag{5.4}$$

By using first Fubini's Theorem and then the Minkowski inequality for integrals (see (A.21)), we get:

$$\begin{aligned}
&\left\| \int_0^1 \mathbf{e}^{sB_z} B_z v \, \mathrm{d}s \right\|_{H^{\theta-1}} \\
&= \left(\int_{\mathbb{R}^d} \left| (1+\xi^2)^{\frac{\theta-1}{2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \mathrm{e}^{-i\xi x} \int_0^1 \mathbf{e}^{sB_z} B_z v(x) \, \mathrm{d}s \, \mathrm{d}x \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^d} \left| \int_0^1 (1+\xi^2)^{\frac{\theta-1}{2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \mathrm{e}^{-i\xi x} \mathbf{e}^{sB_z} B_z v(x) \, \mathrm{d}s \, \mathrm{d}x \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&\leq \int_0^1 \left(\int_{\mathbb{R}^d} \left| (1+\xi^2)^{\frac{\theta-1}{2}} (\mathcal{F}\mathbf{e}^{sBz} B_z v)(\xi) \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \mathrm{d}s \\
&= \int_0^1 \|\mathbf{e}^{sB_z} B_z v\|_{H^{\theta-1}} \, \mathrm{d}s.
\end{aligned}$$

Note here that using Fubini's Theorem in the first step is not trivial and this step actually uses some conditions that otherwise would not be needed for the theorem. Obviously $|\mathrm{e}^{-i\xi x}| = 1$, and it is enough to check that $|\mathbf{e}^{sB_z} B_z u(x)|$ is in fact integrable. Due to the Sobolev embedding we can consider this function to be continuous. On the other hand, Assumption 5.1 ensures that the function will disappear outside of a ball around the origin, hence we can assume integrability.

The operator B_z is bounded, specifically

$$\|B_z u\|_{H^{\theta-1}} \leq C|z| \|u\|_{H^\theta}.$$

Furthermore, for $|z| \leq \alpha$, $\|\mathbf{e}^{B_z}\|_{L(H^{\theta-1}, H^{\theta-1})}$ is bounded and thus

$$\|(\mathbf{e}^{B_z} - \mathrm{Id})\|_{L(H^\theta, H^{\theta-1})} = \sup_{u : \|u\|_H^\theta = 1} \|(\mathbf{e}^{B_z} - \mathrm{Id})u\|_{H^{\theta-1}} \leq C|z|.$$

Taking the norm of the integral w.r.t. $\tilde{N}$ and using Lemma 2.11 we get

$$
\begin{aligned}
&\left\|\int_0^t \int_{0<|z|\le\alpha} S(t-s)(\mathbf{e}^{B_z}-\mathrm{Id})u(s,\cdot)\,\tilde{N}(\mathrm{d}z,\mathrm{d}s)\right\|_\lambda \\
&= \left(\mathbb{E}\int_0^T \mathrm{e}^{-\lambda t}\left\|\int_0^t \int_{0<|z|\le\alpha} S(t-s)(\mathbf{e}^{B_z}-\mathrm{Id})u(s,\cdot)\,\tilde{N}(\mathrm{d}z,\mathrm{d}s)\right\|_{H^\theta}^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\le C\left(\mathbb{E}\int_0^T \mathrm{e}^{-\lambda t}\left(\int_0^t \int_{0<|z|\le\alpha} \|S(t-s)(\mathbf{e}^{B_z}-\mathrm{Id})u(s,\cdot)\|_{H^\theta}^p\,\nu(\mathrm{d}z)\,\mathrm{d}s\right)^{\frac{2}{p}} \mathrm{d}t\right)^{\frac{1}{2}} \\
&\le C\left(\mathbb{E}\int_0^T \mathrm{e}^{-\lambda t}\left(\int_0^t (t-s)^{-\frac{p}{2}}\|u(s,\cdot)\|_{H^\theta}^p \int_{0<|z|\le\alpha} |z|^p\,\nu(\mathrm{d}z)\,\mathrm{d}s\right)^{\frac{2}{p}} \mathrm{d}t\right)^{\frac{1}{2}} \\
&\le C\left(\mathbb{E}\int_0^T \mathrm{e}^{-\lambda t}\left(\int_0^t (t-s)^{-\frac{p}{2}}\|u(s,\cdot)\|_{H^\theta}^p\,\mathrm{d}s\right)^{\frac{2}{p}} \mathrm{d}t\right)^{\frac{1}{2}},
\end{aligned}
$$

where the second inequality follows from

$$
\begin{aligned}
&\|S(t-s)(\mathbf{e}^{B_z}-\mathrm{Id})u(s,\cdot)\|_{H^\theta}^p \\
&\le \|S(t-s)\|_{L(H^\theta,H^{\theta-1})}^p \cdot \|(\mathbf{e}^{B_z}-\mathrm{Id})\|_{L(H^{\theta-1},H^\theta)}^p \cdot \|u(s,\cdot)\|_{H^\theta}^p \\
&\le C(t-s)^{-\frac{p}{2}}\,|z|^p\,\|u(s,\cdot)\|_{H^\theta}^p.
\end{aligned}
$$

We now rewrite the integral and use Young's inequality for convolution integrals (see (A.20) for $r=1$ and $q=2/p$):

$$
\begin{aligned}
&\left(\mathbb{E}\int_0^T \mathrm{e}^{-\lambda t}\left(\int_0^t (t-s)^{-\frac{p}{2}}\|u(s,\cdot)\|_{H^\theta}^p\,\mathrm{d}s\right)^{\frac{2}{p}} \mathrm{d}t\right)^{\frac{1}{2}} \\
&= \left(\mathbb{E}\int_0^T \left(\int_0^t \mathrm{e}^{-\frac{p\lambda}{2}(t-s)}(t-s)^{-\frac{p}{2}}\mathrm{e}^{-\frac{p\lambda}{2}s}\|u(s,\cdot)\|_{H^\theta}^p\,\mathrm{d}s\right)^{\frac{2}{p}} \mathrm{d}t\right)^{\frac{1}{2}} \\
&\le \left(\mathbb{E}\left(\int_0^T \mathrm{e}^{-\frac{p\lambda}{2}(t)}t^{-\frac{p}{2}}\,\mathrm{d}t\right)^{\frac{2}{p}}\left(\int_0^T \mathrm{e}^{-\lambda(t)}\|u(s,\cdot)\|_{H^\theta}^2\,\mathrm{d}t\right)\right)^{\frac{1}{2}} \\
&= \left(\int_0^T \mathrm{e}^{-\frac{p\lambda}{2}(t)}t^{-\frac{p}{2}}\,\mathrm{d}t\right)^{\frac{1}{p}}\|u\|_\lambda.
\end{aligned}
$$

A simple substitution then gives us in conclusion

$$\left\| \int_0^t \int_{0<|z|\le 1} S(t-s)(\mathbf{e}^{B_z} - \operatorname{Id})u(s,\cdot)\tilde{N}(\mathrm{d}z,\mathrm{d}s) \right\|_\lambda \le C \left(\int_0^T \mathrm{e}^{-\frac{p\lambda}{2}(t)} t^{-\frac{p}{2}} \,\mathrm{d}t \right)^{\frac{1}{p}} \|u\|_\lambda$$
$$\le C \left(\frac{2}{p\lambda}\right)^{\frac{1}{p}-\frac{1}{2}} \left(\int_0^T \mathrm{e}^{-(t)} t^{-\frac{p}{2}} \,\mathrm{d}t \right)^{\frac{1}{p}} \|u\|_\lambda \le C \left(\frac{2}{p\lambda}\right)^{\frac{1}{p}-\frac{1}{2}} \|u\|_\lambda.$$

Note that the constant C depends on b, p and θ, but not on u or λ, which is the first step to show that I maps $\mathcal{M}^2$ into $\mathcal{M}^2$. On the other hand, $p < 2$ and so $\frac{1}{p} - \frac{1}{2} > 0$. This means, λ can be chosen big enough, so that $C\left(\frac{2}{p\lambda}\right)^{\frac{1}{p}-\frac{1}{2}} < 1$. Replacing u with $u - v$ for $u, v \in \mathcal{M}^p$, this also gives us the first step to showing that I is in fact a contraction.

We now turn to the integral w.r.t. the Lévy measure ν. Most of the steps from above work analogously for this part, but a few additional steps are needed beforehand. It is clear that

$$\|(\mathbf{e}^{B_z} - \operatorname{Id} - B_z)\|_{L(H^\theta, H^{\theta-1})} \le \|(\mathbf{e}^{B_z} - \operatorname{Id})\|_{L(H^\theta, H^{\theta-1})} + \|B_z\|_{L(H^\theta, H^{\theta-1})} \le C|z|, \tag{5.5}$$

but of course this is not good enough, since $\int_{|z|\le 1} |z| \,\nu(\mathrm{d}z)$ may diverge. It is necessary to estimate the term $(\mathrm{e}^{B_z} - \operatorname{Id} - B_z)$ in a way that gives us $|z|^p$ to work with, which will take a little more effort.
Treating the expression $(\mathrm{e}^{B_z} - \operatorname{Id} - B_z)$ similar to (5.4) before gives us

$$(\mathbf{e}^{B_z} - \operatorname{Id} - B_z)v = \int_0^1 \int_0^r \mathbf{e}^{sB_z} (B_z)^2 v \,\mathrm{d}s \,\mathrm{d}r.$$

Repeating the steps we did for $(\mathbf{e}^{B_z} - \operatorname{Id})$, we finally get

$$\|(\mathbf{e}^{B_z} - \operatorname{Id} - B_z)\|_{L(H^\theta, H^{\theta-2})} \le C|z|^2. \tag{5.6}$$

This looks sufficient at first glance, because $|z|^2$ is integrable. However, using this would mean that we could not benefit from the smoothing property of the semigroup, see (5.3), since

$$\int_0^t \|S(t-s)\|_{L(H^{\theta-2}, H^\theta)} \,\mathrm{d}s \le C \int_0^t t^{-1} \,\mathrm{d}t = \infty$$

is not useful.
The solution to this problem comes from some interpolation theory, see the appendix for details. Using (5.5) and (5.6) with $\vartheta = p - 1$, Theorem A.4 gives

us

$$\|(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)\|_{L(H^\theta, H^{\theta-p})} \le C|z|^p.$$

Thus the integral w.r.t. ν can also be estimated by a constant

$$\int_{0<|z|\le\alpha} \|(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)\|_{L(H^\theta, H^{\theta-p})}\, \nu(\mathrm{d}z) < \infty,$$

and using the semigroup as before we see that

$$\begin{aligned}
&\int_0^t \int_{0<|z|\le\alpha} \|S(t-s)(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)u(s,\cdot)\|_{H^\theta}\, \nu(\mathrm{d}z)\, \mathrm{d}s \\
&\le C \int_0^t \int_{0<|z|\le\alpha} \|S(t-s)\|_{L(H^{\theta-p}, H^\theta)} \|(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)u(s,\cdot)\|_{H^{\theta-p}}\, \nu(\mathrm{d}z)\, \mathrm{d}s \\
&\le C \int_0^t (t-s)^{-\frac{p}{2}} \|u(s,\cdot)\|_{H^\theta}\, \mathrm{d}s \int_{0<|z|\le\alpha} \|(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)\|_{L(H^\theta, H^{\theta-p})}\, \nu(\mathrm{d}z) \\
&\le C \int_0^t (t-s)^{-\frac{p}{2}} \|u(s,\cdot)\|_{H^\theta}\, \mathrm{d}s.
\end{aligned}$$

Using the same tactics as before we get

$$\begin{aligned}
&\left\| \int_0^t \int_{|z|\le\alpha} S(t-s)(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)u(s,\cdot)\, \nu(\mathrm{d}z)\, \mathrm{d}s \right\|_\lambda \\
&= \left(\mathbb{E} \int_0^T \mathrm{e}^{-\lambda t} \left\| \int_0^t \int_{|z|\le\alpha} S(t-s)(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)u(s,\cdot)\, \nu(\mathrm{d}z)\, \mathrm{d}s \right\|_{H^\theta}^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\le C \left(\mathbb{E} \int_0^T \mathrm{e}^{-\lambda t} \left(\int_0^t \int_{|z|\le\alpha} \|S(t-s)(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)u(s,\cdot)\|_{H^\theta}\, \nu(\mathrm{d}z)\, \mathrm{d}s \right)^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\le C \left(\mathbb{E} \int_0^T \mathrm{e}^{-\lambda t} \left(\int_0^t (t-s)^{-\frac{p}{2}} \|u(s,\cdot)\|_{H^\theta}\, \mathrm{d}s \right)^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&= C \left(\mathbb{E} \int_0^T \left(\int_0^t \mathrm{e}^{-\frac{\lambda}{2}(t-s)} (t-s)^{-\frac{p}{2}} \mathrm{e}^{-\frac{\lambda}{2}s} \|u(s,\cdot)\|_{H^\theta}\, \mathrm{d}s \right)^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\le C \int_0^T \mathrm{e}^{-\frac{\lambda}{2}s} s^{-\frac{p}{2}}\, \mathrm{d}s \; \left(\mathbb{E} \int_0^T \mathrm{e}^{-\lambda t} \|u(t,\cdot)\|_{H^\theta}^2\, \mathrm{d}t \right)^{\frac{1}{2}}.
\end{aligned}$$

Another substitution then gives us in conclusion

$$\begin{aligned}\left\| \int_0^t \int_{|z|\le\alpha} S(t-s)(\mathbf{e}^{B_z} - \mathrm{Id} - B_z)u(s,\cdot)\,\nu(\mathrm{d}z)\,\mathrm{d}s \right\|_\lambda \\ &\le C\left(\frac{2}{\lambda}\right)^{1-p/2} \int_0^\infty \mathrm{e}^{-t} t^{-\frac{p}{2}}\,\mathrm{d}t \cdot \|u\|_\lambda \\ &\le C\left(\frac{2}{\lambda}\right)^{1-p/2} \cdot \|u\|_\lambda.\end{aligned}$$

Again, we see that this integral also stays in $\mathcal{M}^2$ and that for λ big enough we get a contraction in $\|\cdot\|_\lambda$.
We have not mentioned the initial condition until now. However, by [Paz83, Theorem 1.2.2], it is immediately clear that $\|S(t)u_0(x)\|_\lambda < \infty$. This suffices to finish the proof. □

Example 5.6. *Let $d = m = 1$. We look at the equation*

$$\mathrm{d}u(t,x) = \Delta u(t,x)\mathrm{d}t + b(x)\nabla u(t,x) \diamond \mathrm{d}Z_t,$$

where Δ is the Laplacian and b is some $\mathbb{R}$-valued function. It is well-known, that Δ satisfies the Assumption (5.2). To ensure the existence of the solution according to the theorem we only need to choose b regular enough, for example take some $b \in H^2$ with compact support. Then, Theorem A.3 with $p = p_1 = p_2 = 2$, $\theta_1 = \theta - 1$, $\theta_2 = 1$, $f = \nabla u$, $g = b$ gives us

$$\begin{aligned}\|Bu\|_{H^{\theta-1}} &= \|b\cdot\nabla u\|_{H^{\theta-1}} \\ &\le C\|\nabla u\|_{H^{\theta-1}}\|b\|_{H^{\theta-1}} \\ &\le C\|\Delta^{-\frac{1}{2}}\nabla u\|_{H^\theta} \\ &\le C\|u\|_{H^\theta},\end{aligned}$$

where C depends on b and the last part comes from the Fourier multiplier property.

Mild solution for general Lévy processes

Obviously, the next step is to ask what happens when the drift and the Brownian motion part are added to the Lévy process. Consider the Lévy process $L_t = \mu t + W_t + Z_t$, where $\mu \in \mathbb{R}^m$, W is a Brownian motion and Z is the pure jump process from before.

Now consider the new equation:

$$\mathrm{d}u(t,x) = Au(t,x) + \mathcal{B}u(t,x)\mu \ \mathrm{d}t + \mathcal{B}u(t,x) \diamond \mathrm{d}Z_t + \mathcal{C}u(t,x)\,\mathrm{d}W, \tag{5.7}$$

where $\mathcal{B}$ is the same linear operator as in Theorem (5.5) and $\mathcal{C}$ is another linear operator.
We first see that the drift part can easily be incorporated using the same arguments as in the proof for the discontinuous part. However, this does not work as well for the Brownian motion.
The case when $\mathcal{C}$ is bounded in the norm of $L(H^\theta, H^{\theta-\gamma})$ with $\gamma < 1$ is easy to deal with: Obviously it follows from the smoothing property of A that

$$\int_0^t \|S(t-s)Cu(s,\cdot)\|_{H^\theta}^2 \,\mathrm{d}t \leq \|C\|_{L(H^\theta,H^{\theta-\gamma})}^2 \int_0^t t^{-\gamma}\|u(s,\cdot)\|_{H^\theta}\,\mathrm{d}t.$$

It is enough to apply the Itô isometry and Young's inequality in addition to the steps of the proof of the Theorem to incorporate the Brownian motion part.
However, in the case where $\mathcal{C}$ is a first order differential operator, e.g. $C = \nabla$, we would have $\gamma = 1$ and could not proceed in the same way as above. If we tried to use the smoothing property of the semigroup the result would not be integrable.

Chapter 6

Advection-Diffusion Equation with Noise on the Boundary

This chapter is motivated by a physically important model of a contaminant transport in a one-dimensional semi-infinite pipe with a constant flow velocity and diffusion. In the classical setting, the contaminant concentration $u = u(t,x)$ satisfies the advection-diffusion equation

$$\begin{cases} \partial_t u(t,x) = \partial_{xx} u(t,x) - \mu \partial_x u(t,x), \quad t > 0,\ x > 0, \\ u(0,x) = 0, \\ Bu(t,0) = g(t), \quad t \geq 0, \end{cases} \tag{6.1}$$

with zero initial concentration and continuously differentiable source process $g = g(t)$, which affects the concentration of the contaminant at the boundary point $x = 0$. We assume that the diffusion coefficient equals 1, the flow velocity $\mu \in \mathbb{R}$. The boundary conditions are treated in a unified way with the help of the boundary operator B, namely we set

$$B_D u(t,0) = \lim_{x \downarrow 0} u(t,x), \tag{6.2}$$

$$B_N u(t,0) = -\lim_{x \downarrow 0} \partial_x u(t,x), \tag{6.3}$$

for the Dirichlet and Neumann problems respectively. In the Dirichlet case, the source g prescribes the concentration of the contaminant at the boundary, in the Neumann setting it determines the transfer rate through the boundary. If the input function $g \in C_b^1(\mathbb{R}_+, \mathbb{R})$, the solution (6.1) is well known in the closed form, see e.g. [CJ86, Pol02].

In realistic models, the assumption that g is smooth and deterministic is too restrictive. For instance, in [WZ05, CS06, MT07] the authors allow g to be a random source of contamination in an open channel flow. In particular, the contaminant can be released at random time instants, so that g can consist of a random train of delta-spikes or be a Brownian noise.

In this chapter we tackle two problems. First, we will solve equation (6.1) with a general boundary Lévy noise $g = \dot{Z}$, including Brownian motion or Poisson process. We will determine the mild solution of (6.1) as process with values in the fractional Sobolev space $H^\theta(\mathbb{R}_+)$, find its explicit form as a convolution integral w.r.t. the driving Lévy process and determine its law in the large time limit. Second, we study the Wong–Zakai approximations of solutions, namely we consider absolutely continuous approximations of the driving process Z and study convergence of classical solutions to the mild solution of the original equation in the non-standard M_1-Skorokhod topology.

The results of this chapter have been published in [HP19]. More details on the theory of PDEs with boundary noise can be found e.g. in [DZ93, CS04, BDS09]. PDEs with Lévy noise on the boundary were considered in [PZ07], and more recently in [HR15] and [BGPR15]. In the deterministic case, controllability of the one-dimensional heat equation on the half line was studied in [MZ00], whereas [AB02b, AB02a, FG09, Mas10] considered the one-dimensional heat equation on the half line with white Gaussian noise on the boundary.

Eventually, we mention the works [KZ78, WZ05, MT07] for applications of the mathematical model (6.1) to hydrology, [PvG84] for a discussion on the proper choice of boundary conditions from the physical point of view, and [JF92] for a bibliography on transport of chemicals through soil.

6.1 Existence of Solutions

Let $Z = (Z(t))_{t\geq 0}$ be an $\mathbb{R}$-valued Lévy process with the characteristic function

$$\begin{aligned}
&\mathbf{E}e^{i\lambda Z(t)} = e^{t\Phi(\lambda)}, \quad \lambda \in \mathbb{R}, \\
&\Phi(\lambda) = -\frac{\sigma^2}{2}\lambda^2 + ia\lambda + \int_{\mathbb{R}} (e^{i\lambda z} - 1 - i\lambda z \mathbb{I}(|z| \leq 1))\, \mu(dz),
\end{aligned} \tag{6.4}$$

where the Gaussian variance, drift and the Lévy measure satisfy $\sigma^2 \geq 0$, $a \in \mathbb{R}$, and $\mu(\{0\}) = 0$, $\int (z^2 \wedge 1)\, \mu(dz) < \infty$.

We solve the advection-diffusion equation

$$\begin{cases} \partial_t u(t,x) = \partial_{xx} u(t,x) - \mu \partial_x u(t,x), \quad t > 0,\ x > 0, \\ u(0,x) = 0, \\ Bu(t,0) = \dot{Z}(t), \quad t \geq 0, \end{cases} \tag{6.5}$$

where B is given by a Dirichlet or Neumann boundary operator (6.2) or (6.3). Following [Bal81, FG09, PZ07], we consider (6.5) as an evolution equation in an appropriate Hilbert space $\mathcal{H}$ and derive an integral formula for its solution. The space $\mathcal{H}$ should satisfy two properties. First, the operator $A_B = \partial_{xx} - \mu\partial_x$ with the domain $\mathcal{D}(A_B) = \{u \in \mathcal{H} \colon A_B u \in \mathcal{H},\ Bu = 0\}$ should generate a C_0-semigroup $(S_B(t))_{t \geq 0}$ in $\mathcal{H}$. Second, $\mathcal{H}$ should be rich enough to guarantee that the solution of (6.5) is càdlàg, so that we can talk about convergence in the Skorokhod topology. It turns out that it is convenient to work in fractional Sobolev spaces $\mathcal{H} = H^\theta(\mathbb{R}_+)$, $\theta \in \mathbb{R}$ (see Section 2.4 for definitions).

Define the Dirichlet map operator $D_B \colon \mathbb{R} \mapsto C_b^2(\mathbb{R}_+, \mathbb{R})$ by the relation $D_B a = \varphi$, where φ is a unique bounded solution of the ordinary differential equation

$$\begin{cases} \varphi''(x) - \mu\varphi'(x) = (1+\mu)\varphi(x), \quad x > 0, \\ B\varphi(0) = a. \end{cases}$$

A straightforward calculation yields that

$$(D_B a)(x) = a\mathrm{e}^{-x}.$$

Assume for a moment that we are in the classical setting (6.1) and the input $g \equiv \dot{Z}$ is a smooth function, $g \in C^1(\mathbb{R})$. Consider the non-homogeneous equation

$$\begin{cases} \partial_t \tilde{u}(t,x) = A\tilde{u}(t,x)\,\mathrm{d}t + \Big((1+\mu)D_B g(t) - D_B \dot{g}(t)\Big)(x), \quad t > 0, x > 0, \\ \tilde{u}(0,x) = -(D_B(g(0))(x), \\ B\tilde{u}(t,0) = 0. \end{cases} \tag{6.6}$$

We claim that $u(t,x) = \tilde{u}(t,x) + \Big(D_B(g(t))\Big)(x)$. Indeed, the direct substitution yields

$$\begin{aligned}
\partial_t u(t,x) &= \partial_t \tilde{u}(t,x) + \Big(D_B(\dot{g}(t))\Big)(x) \\
&= \partial_{xx}\tilde{u}(t,x) - \mu\partial_x \tilde{u}(t,x) + \Big((1+\mu)D_B(g(t))\Big)(x) \\
&= \partial_{xx}u(t,x) - \mu\partial_x u(t,x),
\end{aligned}$$

and the initial and boundary conditions of (6.1) are also satisfied:

$$\begin{aligned}
u(0,x) &= \tilde{u}(0,x) + \Big(D_B(g(0))\Big)(x) = 0, \\
Bu(t,0) &= B\tilde{u}(t,0) + \Big(BD_B(g(t))\Big)(0) = g(t).
\end{aligned}$$

The solution to the problem (6.6) is found with the help of the convolution formula (Duhamel's principle). Let S_B be the C_0-semigroup of the operator $A_B = \partial_{xx} - \mu\partial_x$ on the domain $\mathcal{D}(A_B)$. Then $\tilde{u}$ is found explicitly as

$$\tilde{u}(t) = -S_B(t)D_B(g(0)) + \int_0^t S_B(t-s)\Big((1+\mu)D_B g(t) - D_B\dot{g}(t)\Big)\,\mathrm{d}s. \tag{6.7}$$

Using the C_0-continuity of S_B we note that

$$\frac{\mathrm{d}}{\mathrm{d}s}S_B(t-s) = -A_B S_B(t-s),$$

so that the integration by parts gives

$$\int_0^t S_B(t-s)D_B(\dot{g}(s))\,\mathrm{d}s = S_B(t-s)D_B(g(s))\Big|_0^t - \int_0^t A_B S_B(t-s)D_B(g(s))\,\mathrm{d}s.$$

Together with (6.7) this gives

$$u(t,x) = \int_0^t \Big((1+\mu)\,\mathrm{Id} - A_B\Big)S_B(t-s)D_B(g(s))(x)\,\mathrm{d}s. \tag{6.8}$$

The formula (6.8) allows us to work with the following definition.

Definition 6.1. *We call the process*

$$u(t,x) := \int_0^t \Big((1+\mu)\,\mathrm{Id} - A_B\Big)(S_B(t-s)D_B)(x)\,\mathrm{d}Z(s)$$

a mild solution of (6.5) *in the state space* $\mathcal{H}$.

The latter definition presupposes that the integral on the right hand side exists. The construction of an integral of Hilbert-valued deterministic integrand w.r.t. a Lévy process is standard, see e.g. [CM87, Rie15].

The semigroup S_B has a well-known explicit representation in terms of the Green function of the heat equation, see [CJ86, Pol02]:

$$S_B(t)f(x) = \int_0^\infty f(y)\Lambda_B(x,y,t)\,\mathrm{d}y, \quad f \in C_b(\mathbb{R}_+,\mathbb{R}),$$

where

$$\begin{aligned}
\Lambda_D(x,y,t) &= \frac{1}{2\sqrt{\pi t}}\mathrm{e}^{-\frac{\mu(y-x)}{2}-\frac{\mu^2 t}{4}}\Big(\mathrm{e}^{-\frac{(x-y)^2}{4t}} - \mathrm{e}^{-\frac{(x+y)^2}{4t}}\Big),\\
\Lambda_N(x,y,t) &= \frac{1}{2\sqrt{\pi t}}\mathrm{e}^{-\frac{\mu^2}{4}t+\frac{\mu(x-y)}{2}}\Big(\mathrm{e}^{-\frac{(x-y)^2}{4t}} + \mathrm{e}^{-\frac{(x+y)^2}{4t}}\Big) + \frac{\mu}{2}\mathrm{e}^{-y\mu}\operatorname{erfc}\Big(\frac{x+y-\mu t}{2\sqrt{t}}\Big),\\
\operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y.
\end{aligned}$$

Hence straightforward integration allows us to simplify

$$G_D(x,t) := \Big((1+\mu)\operatorname{Id} - A_D\Big)S_D(t)\mathrm{e}^{-x} = \frac{x}{2\sqrt{\pi}t^{3/2}}\,\mathrm{e}^{-\frac{(x-\mu t)^2}{4t}}, \tag{6.9}$$

$$G_N(x,t) := \Big((1+\mu)\operatorname{Id} - A_N\Big)S_N(t)\mathrm{e}^{-x} = \frac{1}{\sqrt{\pi t}}\,\mathrm{e}^{-\frac{(x-\mu t)^2}{4t}} + \frac{\mu}{2}\operatorname{erfc}\Big(\frac{x-\mu t}{2\sqrt{t}}\Big), \tag{6.10}$$

which yields the closed form solution for u.

To show the existence of a mild solution we first have to determine a suitable Hilbert space $\mathcal{H}$, so that the operator $A_B = \partial_{xx} - \mu\partial_x$ with the boundary condition B generates a C_0-semigroup. We will find that in this case fractional Sobolev spaces are a good choice. First note that if $Au \in H^\theta(\mathbb{R}_+)$ then $u \in H^{\theta+2}(\mathbb{R}_+)$ (see the following Lemma 6.2 and its proof). However, since Sobolev spaces are spaces of equivalence classes of functions, the meaning of the boundary conditions $u(0) = 0$ and $\partial_x u(0) = 0$ for $u \in H^{\theta+2}(\mathbb{R}_+)$ may not be obvious. For this reason we write $\mathcal{D}(\mathbb{R}_+)$ for the space of infinitely differentiable functions $f\colon \mathbb{R}_+ \to \mathbb{C}$ with compact support in $(0,\infty)$. We use this space to give meaning to the boundary condition of operator A, since for $u \in \mathcal{D}(\mathbb{R}_+)$ these conditions are well defined. So in what follows, the expressions $u(0) = 0$ and $\partial_x u(0) = 0$ will be understood in the sense of closures of $\mathcal{D}(\mathbb{R}_+)$ in $H^{\theta+2}(\mathbb{R}_+)$. This relies on the important fact that if $\theta < \frac{1}{2}$, then $\mathcal{D}(\mathbb{R}_+)$ is dense in $H^\theta(\mathbb{R}_+)$, see Lemma A.2 in the appendix for details.

Lemma 6.2. *(i) For $\theta < -\frac{3}{2}$, the operator $A = A_D$ with the domain $D(A) = \{u \in H^\theta(\mathbb{R}_+) \colon Au \in H^\theta(\mathbb{R}_+), u(0) = 0\}$ generates a C_0-semigroup in $H^\theta(\mathbb{R}_+)$.*

(ii) For $\theta < -\frac{1}{2}$, the operator $A = A_N$ with the domain $D(A) = \{u \in H^\theta(\mathbb{R}_+) \colon Au \in H^\theta(\mathbb{R}_+), \partial_x u(0) = 0\}$ generates a C_0-semigroup in $H^\theta(\mathbb{R}_+)$.

Proof. In the following we write $\mathcal{H}$ for $H^\theta(\mathbb{R}_+)$.
We use the Hille–Yosida Theorem for contractive C_0-semigroups (see Theorem 2.17). First we show, that $D(A) = H^{\theta+2}(\mathbb{R}_+)$ and therefore $D(A)$ is dense in $\mathcal{H}$. For the Laplace operator Δ_D on $\mathbb{R}_+$ with the Dirichlet boundary condition we have $D(\Delta_D) = H^{\theta+2}(\mathbb{R}_+)$ for $\theta < -\frac{3}{2}$. Now note, that

$$\|Au\|_{\mathcal{H}} < \infty \Rightarrow \|\Delta_D u\|_{\mathcal{H}} < \infty$$

and on the other hand

$$\|u\|_{H^{\theta+2}(\mathbb{R}_+)} < \infty \Rightarrow \|Au\|_{\mathcal{H}} < \infty$$

and so,

$$D(\Delta_D) \subseteq D(A) \subseteq H^{\theta+2}(\mathbb{R}_+) = D(\Delta_D).$$

Now, we need to take a look at the resolvent set $\rho(A)$ and show that $(0,\infty) \subseteq \rho(A)$ and for all $\lambda > 0$

$$\left\|(\lambda \operatorname{Id} - A)^{-1}\right\|_{L(H^\theta(\mathbb{R}_+), D(A))} \leq \frac{1}{\lambda}.$$

Let $f \in \mathcal{H}$. We define

$$h := \mathcal{F}^{-1}\Big(\frac{(\mathcal{F}\operatorname{ext}(f))(\xi)}{\lambda + \xi^2 + \mathrm{i}\mu\xi}\Big)$$

and $g := h|_{\mathbb{R}_+}$. Then, because of the properties of the Fourier transform,

$$g = (\lambda \operatorname{Id} - A)^{-1} f.$$

Furthermore

$$\begin{aligned}
\|g\|_{\mathcal{H}}^2 \leq \|h\|_{\theta,2}^2 = & \int_{-\infty}^{\infty} (1+\xi^2)^\theta \frac{|\mathcal{F}\operatorname{ext}(f)|^2}{(\lambda+\xi^2)^2 + (\mu\xi)^2} \,\mathrm{d}\xi \\
\leq & \frac{1}{\lambda^2} \int_{-\infty}^{\infty} (1+\xi^2)^\theta |\mathcal{F}\operatorname{ext}(f)|^2 \,\mathrm{d}\xi \\
= & \frac{1}{\lambda^2} \|\operatorname{ext}(f)\|_{\theta,2}^2 = \frac{1}{\lambda^2} \|f\|_{\mathcal{H}}^2.
\end{aligned}$$

Noting that

$$\frac{1}{\lambda^2}\int_{-\infty}^{\infty}(1+\xi^2)^\theta|\mathcal{F}\operatorname{ext}(f)|^2\,\mathrm{d}\xi<\infty$$
$$\Rightarrow\quad\int_{-\infty}^{\infty}(1+\xi^2)^{\theta+2}\frac{|\mathcal{F}\operatorname{ext}(f)|^2}{(\lambda+\xi^2)^2+(\mu\xi)^2}\,\mathrm{d}\xi<\infty,$$

we also get $g\in H^{\theta+2}(\mathbb{R}_+)=D(A)$ and thus

$$\left\|(\lambda\operatorname{Id}-A)^{-1}\right\|_{L(H^\theta(\mathbb{R}_+),D(A))}\le\sup_{\|f\|_{\mathcal{H}}\le 1}\frac{1}{\lambda}\|f\|_{\mathcal{H}}=\frac{1}{\lambda}.$$

□

From Lemma 6.2 we can now easily deduce the first result of this chapter which is formulated in the following Theorem and gives us the existence of the mild solution.

Theorem 6.3. (D) *The equation* (6.5) *with a Dirichlet boundary condition has a mild solution in* $H^\theta(\mathbb{R}_+)$ *for* $\theta<-\frac{3}{2}$ *which has the explicit form*

$$u(t,x)=\int_0^t\frac{x}{2\sqrt{\pi}(t-s)^{3/2}}\,\mathrm{e}^{-\frac{(x-\mu(t-s))^2}{4(t-s)}}\,\mathrm{d}Z(s).$$

(N) *The equation* (6.5) *with a Neumann boundary condition has a mild solution in* $H^\theta(\mathbb{R}_+)$ *for* $\theta<-\frac{1}{2}$ *which has the explicit form*

$$u(t,x)=\int_0^t\Big(\frac{1}{\sqrt{\pi(t-s)}}\,\mathrm{e}^{-\frac{(x-\mu(t-s))^2}{4(t-s)}}+\frac{\mu}{2}\operatorname{erfc}\Big(\frac{x-\mu(t-s)}{2\sqrt{t-s}}\Big)\Big)\,\mathrm{d}Z(s).$$

In all cases the mild solution is unique and the paths $t\mapsto u(t,\cdot)$ *are càdlàg in* $H^\theta(\mathbb{R}_+)$ *a.s. Moreover, for any* $x>0$, *the paths* $t\mapsto u(t,x)$ *are continuous in* $\mathbb{R}$.

Sample paths of solutions u driven by an α-stable Lévy subordinator and a symmetric α-stable Lévy process are presented in Fig. 6.1 and Fig. 6.2. Note that negative jumps of the noise may cause negative values of the solution. This explains why Lévy subordinators should be used to model contaminant concentrations.

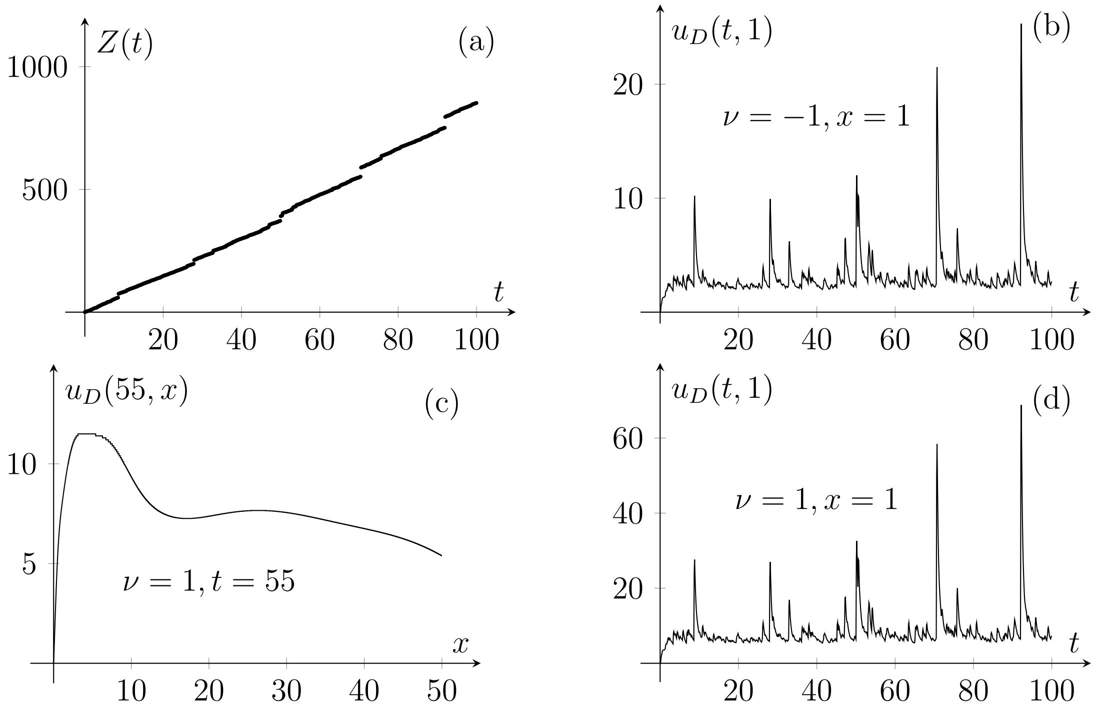

Figure 6.1: A sample path of an α-stable Lévy subordinator Z with $\mathbf{E}\mathrm{e}^{-\lambda Z_1} = \mathrm{e}^{-\lambda^\alpha}$ for $\alpha = 0.9$ (a); solutions $t \mapsto u_D(t,x)$ with Dirichlet boundary noise for $\nu = -1$, $x = 1$ (b) and $\nu = 1$, $x = 1$ (d); the concentration curve $x \mapsto u_D(t,x)$ for $\nu = 1$, $t = 55$ (c).

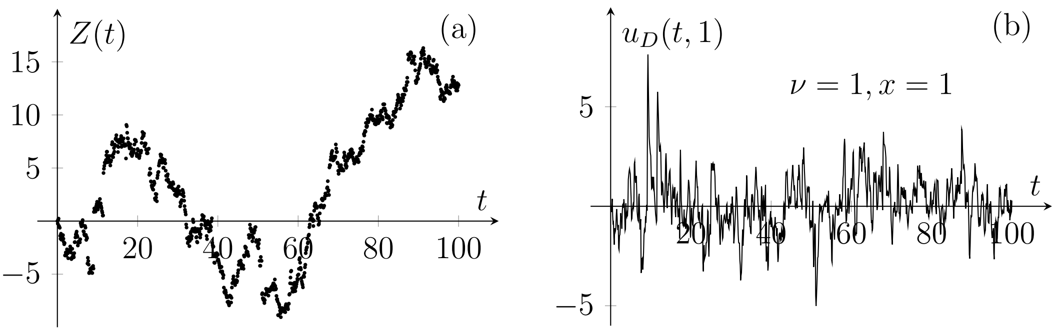

Figure 6.2: A sample path of a symmetric α-stable Lévy process Z with $\mathbf{E}\mathrm{e}^{-\mathrm{i}\lambda Z_1} = \mathrm{e}^{-|\lambda|^\alpha}$ for $\alpha = 1.75$ (a); the solution $t \mapsto u_D(t,x)$ with Dirichlet boundary noise for $\nu = 1$, $x = 1$.

6.2 Limiting Probability Distribution of the Contaminant Concentration

The explicit form of the solution allows us to calculate the stationary contaminant distribution in the large time limit.

To determine the limiting distribution of u in the stationary regime, we consider the equation (6.5) on the time interval $[-\tau, 0]$, $\tau > 0$, driven by a shifted Lévy process $Z_\tau = (Z(t-\tau))_{t\geq\tau}$. Let $u_\tau = u_\tau(t,x)$, $t \in [-\tau, 0]$ be its solution. For $x > 0$, we consider the limit in law

$$u(x) = \lim_{-\tau\to-\infty} u_\tau(0,x) = \lim_{-\tau\to-\infty} \int_{-\tau}^{0} G(-s,x)\,\mathrm{d}Z_\tau(s) \stackrel{d}{=} \int_0^\infty G(s,x)\,\mathrm{d}Z(s),$$

provided the integral on the r.h.s. exists. Recalling (6.4) we find the Fourier transform of $u(x)$ explicitly as

$$\mathbf{E}\mathrm{e}^{\mathrm{i}\lambda u(x)} = \exp\Big(\int_0^\infty \Phi\Big(G(s,x)\lambda\Big)\,\mathrm{d}s\Big), \quad \lambda \in \mathbb{R},$$

provided the integral in the exponent exists.

In physically meaningful models, the process Z does not take negative values, i.e. is a Lévy subordinator with the Laplace transform

$$\mathbf{E}\mathrm{e}^{-\lambda Z(t)} = \mathrm{e}^{t\Psi(\lambda)}, \quad \lambda \geq 0,$$
$$\Psi(\lambda) = -b\lambda + \int_0^\infty (\mathrm{e}^{-\lambda z} - 1)\,\mu(\mathrm{d}z),$$

with $b \geq 0$ and the jump measure satisfying $\mu(\{0\}) = 0$, $\int_0^\infty (z \wedge 1)\,\mu(\mathrm{d}z) < \infty$. In this case, $u(t,x) \geq 0$ a.s. and its Laplace transform is

$$\mathbf{E}\mathrm{e}^{-\lambda u(x)} = \exp\Big(\int_0^\infty \Psi\Big(G(s,x)\lambda\Big)\,\mathrm{d}s\Big), \quad \lambda \geq 0.$$

It is instructive to calculate the limiting law in the following particular case.

Let Z be an α-stable subordinator with $\Psi(\lambda) = -c\lambda^\alpha$, $c > 0$ being the scale parameter and $\alpha \in (0,1)$ the stability index. Then

$$\mathbf{E}\mathrm{e}^{-\lambda u(x)} = \exp\Big(-c\lambda^\alpha \int_0^\infty G(s,x)^\alpha\,\mathrm{d}s\Big), \quad \lambda \geq 0.$$

In other words, the limiting concentration $u(x)$ at the location $x > 0$ has a

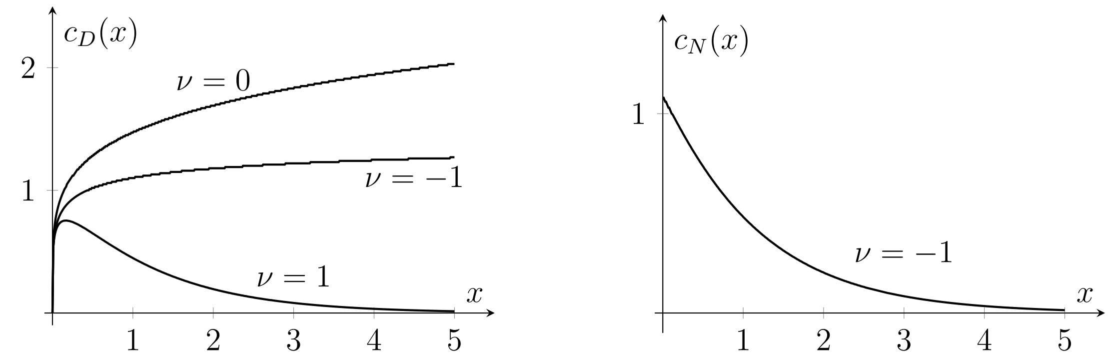

Figure 6.3: The scales $c(x)$ of the limiting distribution in the Dirichlet case for $\nu = \pm 1, 0$ (left), and the Neumann case for $\nu = -1$ (right); $\alpha = 0.9$, $c = 1$.

spectrally positive α-stable distribution with the scale

$$c(x) = c \int_0^\infty G(s,x)^\alpha \, \mathrm{d}s. \tag{6.11}$$

The straightforward integration allows to determine the limiting scale $c_D(x)$ in case of the Dirichlet boundary noise as

$$c_D(x) = \begin{cases} c \cdot \dfrac{2^{1-\alpha}}{\pi^{\alpha/2}} |\mu|^{\frac{3\alpha-2}{2}} \cdot \mathrm{e}^{-\frac{1}{2}x\alpha\mu} x^{1-\frac{\alpha}{2}} K_{\frac{3\alpha-2}{2}}\Big(\dfrac{\alpha x |\mu|}{2}\Big), \quad \mu \neq 0,\ \alpha \in (0,1), \\ c \cdot \Gamma\Big(\dfrac{3\alpha-2}{2}\Big) \dfrac{\alpha^{\frac{2-3\alpha}{2}}}{2^{2(1-\alpha)}\pi^{\alpha/2}} \cdot x^{2(1-\alpha)}, \quad \mu = 0,\ \alpha \in (2/3, 1), \\ +\infty, \quad \mu = 0,\ \alpha \in (0, 2/3]. \end{cases}$$

where K_μ is the modified Bessel function of the second kind. Taking into account the asymptotic expansion

$$K_\mu(x) = \sqrt{\frac{\pi}{2}} x^{-1/2} \mathrm{e}^{-x} (1 + \mathcal{O}(|x^{-1}|)), \quad |x| \to \infty,$$

we get that for large values of x and $\alpha \in (0,1)$

$$c_D(x) \approx \begin{cases} c \cdot \dfrac{2^{1-\alpha}\pi^{\frac{1-\alpha}{2}}}{\alpha^{1/2}} \mu^{\frac{3}{2}(\alpha-1)} \cdot \mathrm{e}^{-x\alpha\mu} x^{\frac{1-\alpha}{2}}, \quad \mu > 0, \\ c \cdot \dfrac{2^{1-\alpha}\pi^{\frac{1-\alpha}{2}}}{\alpha^{1/2}} |\mu|^{\frac{3}{2}(\alpha-1)} \cdot x^{\frac{1-\alpha}{2}}, \quad \mu < 0, \end{cases}$$

see Fig. 6.3 (left).

In the Neumann case, it is clear that $c_N(x) = +\infty$ for $\mu \geq 0$. For $\mu < 0$, the result of numerical integration is presented in Fig. 6.3 (right).

It is interesting to note that the integral (6.11) diverges for $\mu = 0$ and $\alpha \in$

$(0, 2/3]$ in the Dirichlet case. The same critical value $\alpha = 2/3$ was discovered in [PS08] in the analysis of limiting distributions of Lévy driven transport dynamics.

6.3 Wong–Zakai Approximations

From the point of view of applications, the boundary noise $\dot{Z}$ in (6.5) is an idealization of a very fast continuous injection process taking place at the opening of the pipe. A natural question about the convergence of approximations to the solution u arises.

Commonly used examples of absolutely continuous approximations of a Lévy process Z are polygonal approximations,

$$Z^n(t) = Z\Big(\frac{k-1}{n}\Big) + n\Big(Z\Big(\frac{k}{n}\Big) - Z\Big(\frac{k-1}{n}\Big)\Big)\Big(t - \frac{k-1}{n}\Big),\ k \geq 1,\ n \geq 1, \tag{6.12}$$

red noise approximations

$$Z^n(t) = \int_0^t \Big(1 - \mathrm{e}^{-n(t-s)}\Big)\,\mathrm{d}Z(s), \quad n \geq 1, \tag{6.13}$$

or short memory averaging

$$Z^n(t) = n\int_{(t-n^{-1})\wedge 0}^t Z(s)\,\mathrm{d}s, \quad n \geq 1. \tag{6.14}$$

A common feature of these approximations is that they approximate a continuous process Z (i.e. a Brownian motion with drift) in the uniform topology. If Z has jumps, these jumps are approximated continuously and in a monotonous way. Such type of approximations can be very well described with the help of the so-called M_1-Skorokhod topology.

Let V be a separable Banach space with the norm $\|\cdot\|$. In this section, we will mainly deal with $V = \mathbb{R}$ for approximations of the Lévy process Z and $V = H^\theta(\mathbb{R}_+)$ for approximations of solutions of the equation (6.5). For a fixed time $T > 0$, the space of V-valued càdlàg functions is denoted by $D([0,T], V)$. Each $f \in D([0,T], V)$ may have at most countably many discontinuities.

For two elements $v_1, v_2 \in V$ we define a segment $[\![v_1, v_2]\!]$ as a straight line between v_1 and v_2:

$$[\![v_1, v_2]\!] := \{v \in V \colon v = \alpha v_1 + (1-\alpha)v_2 \text{ for } \alpha \in [0,1]\}.$$

In order to define the so-called (strong) M_1 metric on $D([0,T];V)$, we define for each $f \in D([0,T],V)$ the extended graph of f by

$$\Gamma(f) := \{(t,v) \in [0,T] \times V : v \in [\![f(t-), f(t)]\!]\},$$

where $f(0-) := f(0)$. A total order relation on $\Gamma(f)$ is given by

$$(t_1, v_1) \leq (t_2, v_2) \;\Leftrightarrow\; \begin{cases} t_1 < t_2 \quad \text{or} \\ t_1 = t_2 \text{ and } \|f_1(t_1-) - v_1\| \leq \|f_1(t_1-) - v_2\|. \end{cases} .$$

A parametric representation of the extended graph of f is a continuous, non-decreasing, surjective function

$$(r,u)\colon [0,1] \to \Gamma(f), \qquad (r,u)(0) = (0, f(0)),\; (r,u)(1) = (T, f(T)).$$

Let $\Pi(f)$ denote the set of all parametric representations of f.

For $f_1, f_2 \in D([0,T],V)$, we define

$$d_M(f_1,f_2) := \inf\Big\{|r_1 - r_2|_\infty \vee \|u_1 - u_2\|_\infty \colon (r_i,u_i) \in \Pi(f_i),\, i = 1,2\Big\}.$$

The mapping d_M is called strong M_1 metric on $D([0,T],V)$. This topology was introduced by Skorokhod in his seminal paper [Sko56]. The extensive analysis of M_1-topology in the finite dimensional setting can be found in [Whi02]. For a generalization to Banach and Hilbert spaces, see [PR15].

Remark 6.4. *The approximations Z^n defined in* (6.12), (6.13) *and* (6.14) *are absolutely continuous and converge to Z a.s. in the (strong) M_1 topology in $D([0,T],\mathbb{R})$.*

The second main result of this chapter is the following theorem:

Theorem 6.5. *Let $T > 0$ and let $Z^n \to Z$ in probability in $D([0,T],\mathbb{R};d_M)$ as $n \to \infty$, and let Z_n, $n \geq 1$, be absolutely continuous. Then the classical solutions u^n driven by Z^n converge to u determined in Theorem 6.3 in probability in $D([0,T],H^\theta(\mathbb{R}_+);d_M)$ as $n \to \infty$.*

Proof. Similarly to the convergence in the uniform topology and in the standard Skorokhod metric J_1, convergence of a sequence of functions in the metric d_M can be described by quantifying the oscillation of the functions. For $v, v_1, v_2 \in V$

the distance from v to the segment $[\![v_1, v_2]\!]$ is defined by

$$M(v_1, v, v_2) := \inf_{\alpha \in [0,1]} \|v - (\alpha v_1 + (1-\alpha) v_2)\|.$$

Define for $f \in D([0,T]; V)$ and $\delta > 0$ the oscillation function by

$$M(f;\delta) := \sup \Big\{ M\Big(f(t_1), f(t), f(t_2)\Big) : 0 \le t_1 < t < t_2 \le T \text{ and } t_2 - t_1 \le \delta \Big\}.$$

Let $T > 0$, and let $\{Z^n\}_{n \ge 1}$ be a sequence of absolutely continuous functions, such that $Z_n \to Z$ a.s. in M_1-topology on $[0,T]$. Since

$$\lim_{A \to \infty} \mathbb{P}\Big(\sup_{t \in [0,T]} |\Delta Z(t)| > A \Big) = 0$$

from now on we assume that the jumps of Z are bounded by some constant $A > 0$. Furthermore due to the M_1-convergence we can assume that for n large enough

$$\sup_{s \in [0,T]} |Z^n(s)| \le \sup_{s \in [0,T]} |Z(s)| + 1. \tag{6.15}$$

Let u^n be classical continuous solutions to (6.1). From [PR15], Theorem 3.2, we know it is sufficient to show that

(i) for every $t \in [0,T]$ we have $\|u^n(t,\cdot) - u(t,\cdot)\|_{H^\theta(\mathbb{R}_+)} \xrightarrow{\mathbb{P}} 0$, $n \to \infty$, and

(ii) for every $\varepsilon > 0$ the oscillation function $M(u^n, \delta)$ obeys

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}(M(u^n, \delta) \ge \varepsilon) = 0.$$

1. Neumann case ($\theta < -\frac{1}{2}$):

First note that we can extend the solutions $u(t,\cdot)$ and $u^n(t,\cdot)$ to $\mathbb{R}$, simply by extending $G_N(s,x)$ defined in (6.10) to a function $G_N \colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ in the following way:

$$G_N(x,s) := \frac{1}{\sqrt{\pi s}} \mathrm{e}^{-\frac{(\mu s - x)^2}{4s}} + \frac{\mu}{2} \operatorname{erfc}\Big(\frac{|x| - \mu s}{2\sqrt{s}} \Big)$$

and

$$\mathcal{F} G_N(\cdot, t-s)(\xi) = 2\mathrm{e}^{-(\xi^2 + \mathrm{i}\mu\xi)(t-s)} + \frac{\mu}{2} \mathcal{F} \operatorname{erfc}\Big(\frac{|\cdot| - \mu(t-s)}{2\sqrt{(t-s)}} \Big)(\xi).$$

If we can show that (i) and (ii) hold in $H^\theta(\mathbb{R})$ for these explicit extensions, then the result follows easily for $H^\theta(\mathbb{R}_+)$.

Linearity of the integral allows us to split both u and u^n into two parts and consider each separately. So, for (i), we see that for every $t \in [0,T]$

$$\|u^n(t,\cdot) - u(t,\cdot)\|_{H^\theta(\mathbb{R}_+)} \le \|u_1^n(t,\cdot) - u_1(t,\cdot)\|_{H^\theta(\mathbb{R}_+)} + \|u_2^n(t,\cdot) - u_2(t,\cdot)\|_{H^\theta(\mathbb{R}_+)},$$

where

$$\begin{aligned} u_1(t,x) &= \int_0^t \frac{1}{\sqrt{\pi s}} \mathrm{e}^{-\frac{(\mu(t-s)-x)^2}{4(t-s)}} \,\mathrm{d}Z(s), \\ u_2(t,x) &= \int_0^t \frac{\mu}{2} \operatorname{erfc}\Big(\frac{x-\mu(t-s)}{2\sqrt{(t-s)}}\Big) \,\mathrm{d}Z(s), \end{aligned}$$

and u_1^n and u_2^n are defined analogously.

To estimate the convergence $u_1^n \to u_1$, we integrate by parts to obtain

$$\begin{aligned} &\|u_1^n(t,\cdot) - u_1(t,\cdot)\|^2_{H^\theta(\mathbb{R}_+)} \\ &\qquad \le 4 \int_{-\infty}^{\infty} (1+\xi^2)^\theta \Big| \int_0^t \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)} \,\mathrm{d}(Z^n(s) - Z(s)) \Big|^2 \mathrm{d}\xi \\ &\qquad \le 4 \int_{-\infty}^{\infty} (1+\xi^2)^\theta \Big| Z^n(t) - Z(t) - Z^n(0) \\ &\qquad\qquad + \int_0^t \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)} (\xi^2 + \mathrm{i}\mu\xi)(Z^n(s) - Z(s)) \,\mathrm{d}s \Big|^2 \mathrm{d}\xi \\ &\qquad \le 8\Big(|Z^n(t) - Z(t)|^2 + |Z^n(0)|^2\Big) \int_{-\infty}^{\infty} (1+\xi^2)^\theta \,\mathrm{d}\xi \\ &\qquad + 8 \int_{-\infty}^{\infty} (1+\xi^2)^\theta \Big| \int_0^t \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)} (\xi^2 + \mathrm{i}\mu\xi)(Z^n(s) - Z(s)) \,\mathrm{d}s \Big|^2 \mathrm{d}\xi. \end{aligned}$$

Since $\{Z^n\}_{n\ge1}$ converge to Z in M_1 on $[0,T]$ and Z is stochastically continuous, it follows that $Z^n(t) \to Z(t)$ for any $t \in [0,T]$ in probability, so that the first summand vanishes in probability as $n \to \infty$.

To estimate the second summand, we apply the Hölder inequality:

$$\begin{aligned} &\Big| \int_0^t \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)} (\xi^2 + \mathrm{i}\mu\xi)(Z^n(s) - Z(s)) \,\mathrm{d}s \Big| \\ &\le (\xi^2 + |\mu||\xi|) \int_0^t \mathrm{e}^{-\xi^2(t-s)} |Z^n(s) - Z(s)| \,\mathrm{d}s \\ &\le (\xi^2 + |\mu||\xi|)|\xi|^{-2/p} p^{-1/p} \Big(\int_0^t p\xi^2 \mathrm{e}^{-p\xi^2(t-s)} \,\mathrm{d}s \Big)^{1/p} \Big(\int_0^t |Z^n(s) - Z(s)|^q \,\mathrm{d}s \Big)^{1/q} \\ &\le (\xi^2 + |\mu||\xi|)|\xi|^{-2/p} \Big(1 - \mathrm{e}^{-p\xi^2 t}\Big) \Big(\int_0^t |Z^n(s) - Z(s)|^q \,\mathrm{d}s \Big)^{1/q} \end{aligned}$$

and choose $p > 1$ such that $1 - \frac{1}{p} < \frac{2|\theta|-1}{4}$ to get

$$\int_{-\infty}^{\infty} (1+\xi^2)^\theta (\xi^2 + |\mu||\xi|)^2 |\xi|^{-4/p} \Big(1 - \mathrm{e}^{-p\xi^2 t}\Big)^2 \,\mathrm{d}\xi =: C(\theta, p, \mu, t) < \infty.$$

Finally we note that the estimate (6.15) and the boundedness of jumps of Z imply that for any $q > 1$ and n large enough there are $C_1, C_2 > 0$ such that

$$\mathbf{E} \int_0^t |Z^n(s) - Z(s)|^q \,\mathrm{d}s \le C_1 + C_2 \mathbf{E} \sup_{s\in[0,T]} |Z(t)|^q < \infty.$$

Hence, the dominated convergence theorem yields

$$\mathbf{E} \int_0^t |Z^n(s) - Z(s)|^q \,\mathrm{d}s \to 0.$$

To estimate the difference $\|u_2^n(t,\cdot) - u_2(t,\cdot)\|^2_{H^\theta(\mathbb{R}_+)}$, we integrate by parts again. Note that for $0 \le s < t \le T$ and $x \in \mathbb{R}$, $\frac{|x|-\mu(t-s)}{2\sqrt{(t-s)}} \ge -\frac{|\mu|}{2}\sqrt{T}$. Since $x \mapsto \operatorname{erfc}(|x|)$ is integrable, so is $x \mapsto \operatorname{erfc}\Big(\frac{|x|-\mu(t-s)}{2\sqrt{(t-s)}}\Big)$, and we can estimate the Fourier transform in the following way:

$$\sup_{\xi\in\mathbb{R}} \Big|\mathcal{F} \operatorname{erfc}\Big(\frac{|\cdot| - \mu(t-s)}{2\sqrt{(t-s)}}\Big)(\xi)\Big| \le \frac{1}{\sqrt{2\pi}} \Big\| \operatorname{erfc}\Big(\frac{|\cdot| - \mu(t-s)}{2\sqrt{(t-s)}}\Big)\Big\|_{L_1(\mathbb{R})} \le C(T),$$

where $C(T)$ is a constant that only depends on T. It follows, that

$$\begin{aligned}\Big|\Big[(Z^n(s) - Z(s))\mathcal{F}\frac{\mu}{2} \operatorname{erfc}\Big(\frac{|x| - \mu(t-s)}{2\sqrt{(t-s)}}\Big)\Big]_{s=0}^{s=t}\Big|^2 \\ \le 2\Big(|Z^n(t) - Z(t)|^2 + |Z^n(0)|^2\Big)(C(T))^2.\end{aligned}$$

Furthermore, for the derivative of the error function we see:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} \operatorname{erfc}\Big(\frac{|x| - \mu(t-s)}{2\sqrt{(t-s)}}\Big) &= -\frac{1}{\sqrt{\pi}}\Big(\frac{\mu}{2\sqrt{(t-s)}} + \frac{|x|}{2(t-s)^{\frac{3}{2}}}\Big)\mathrm{e}^{-\frac{(|x|-\mu(t-s))^2}{4(t-s)}} \\ &= -\frac{1}{\sqrt{\pi}}\Big(\frac{\mu}{\sqrt{(t-s)}} + \frac{|x| - \mu(t-s)}{2(t-s)^{\frac{3}{2}}}\Big)\mathrm{e}^{-\frac{(|x|-\mu(t-s))^2}{4(t-s)}}.\end{aligned}$$

Obviously, $x \mapsto \mathrm{e}^{-\frac{(|x|-\mu(t-s))^2}{4(t-s)}}$ is integrable for every $s \in [0, t)$, so we can estimate

$$\sup_{\xi\in\mathbb{R}} \Big|\mathcal{F}\Big(\frac{\mu}{2}\frac{\mu}{\sqrt{\pi(t-s)}}\mathrm{e}^{-\frac{(|\cdot|-\mu(t-s))^2}{4(t-s)}}\Big)(\xi)\Big| \le \frac{C_1}{\sqrt{t-s}},$$

for some constant C_1, that does not depend on ξ or s. For the second term we make a simple substitution to see

$$\int_0^\infty \frac{x-\mu(t-s)}{2(t-s)^{\frac{3}{2}}} \mathrm{e}^{-\frac{(x-\mu(t-s))^2}{4(t-s)}} \,\mathrm{d}x = \int_{\frac{-\mu(t-s)}{2\sqrt{t-s}}}^\infty \frac{2y}{\sqrt{t-s}} \mathrm{e}^{y^2} \,\mathrm{d}y \\ \leq \int_{-\infty}^\infty \frac{2|y|}{\sqrt{t-s}} \mathrm{e}^{y^2} \,\mathrm{d}y \leq \frac{C_2}{\sqrt{t-s}},$$

with C_2 being another constant. Eventually this yields for some $C_3 > 0$

$$\begin{aligned} &\|u_2^n(t,\cdot) - u_2(t,\cdot)\|^2_{H^\theta(\mathbb{R}_+)} \\ &\quad \leq 8\Big(|Z^n(t) - Z(t)|^2 + |Z^n(0)|^2\Big)(C(T))^2 \int_{-\infty}^\infty (1+\xi^2)^\theta \,\mathrm{d}\xi \\ &\quad + 8 \int_{-\infty}^\infty (1+\xi^2)^\theta \,\mathrm{d}\xi \cdot \Big(\int_0^t \frac{C_3}{\sqrt{(t-s)}} |(Z^n(s) - Z(s))| \,\mathrm{d}s \Big)^2. \end{aligned}$$

Since $\int_0^t (t-s)^{-p/2} \,\mathrm{d}s < \infty$ for any $1 < p < 2$ we can use the Hölder inequality to get the anticipated convergence.

Now, we turn to condition (ii). We will only look at the first summand of G_N here. The term containing u_2^n can be treated similarly. For any $0 \leq t_1 \leq t \leq t_2 \leq T$, $|t_2 - t_1| \leq \delta$ and any $\alpha \in [0,1]$ we estimate

$$\begin{aligned} &|M(u_1^n(t_1), u_1^n(t), u_1^n(t_2))|^2 \\ &\qquad \leq \int_{-\infty}^\infty (1+\xi^2)^\theta \; |\mathcal{F}(u^n(t) - \alpha u^n(t_1) - (1-\alpha) u^n(t_2))|^2 \,\mathrm{d}\xi, \end{aligned}$$

where

$$\begin{aligned} &|\mathcal{F}(u^n(t) - \alpha u^n(t_1) - (1-\alpha) u^n(t_2))| \\ =&\Big| \int_0^t \mathrm{e}^{-(\xi^2 + \mathrm{i}\mu\xi)(t-s)} \,\mathrm{d}Z^n(s) \\ &\quad - \alpha \int_0^{t_1} \mathrm{e}^{-(\xi^2 + \mathrm{i}\mu\xi)(t_1-s)} \,\mathrm{d}Z^n(s) - (1-\alpha) \int_0^{t_2} \mathrm{e}^{-(\xi^2 + \mathrm{i}\mu\xi)(t_2-s)} \,\mathrm{d}Z^n(s) \Big|, \end{aligned}$$

and hence

$$
\begin{aligned}
&|\mathcal{F}(u^n(t) - \alpha u^n(t_1) - (1-\alpha)u^n(t_2))| \\
&\le |Z^n(t) - \alpha Z^n(t_1) - (1-\alpha)Z^n(t_2)| \\
&+ \alpha\Big|\int_0^{t_1} (\mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)} - \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t_1-s)})(\xi^2+\mathrm{i}\mu\xi)Z^n(s)\,\mathrm{d}s\Big| \\
&+ (1-\alpha)\Big|\int_0^{t} (\mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)} - \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t_2-s)})(\xi^2+\mathrm{i}\mu\xi)Z^n(s)\,\mathrm{d}s\Big| \\
&+ \alpha\Big|\int_{t_1}^{t} \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)}(\xi^2+\mathrm{i}\mu\xi)Z^n(s)\,\mathrm{d}s\Big| \\
&+ (1-\alpha)\Big|\int_{t}^{t_2} \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t_2-s)}(\xi^2+\mathrm{i}\mu\xi)Z^n(s)\,\mathrm{d}s\Big| \\
&= |Z^n(t) - \alpha Z^n(t_1) - (1-\alpha)Z^n(t_2)| + \alpha I_1 + (1-\alpha)I_2 + \alpha I_3 + (1-\alpha)I_4.
\end{aligned}
$$

Because of the M_1-convergence of Z^n, for any $\varepsilon > 0$ there is α, such that for $n \to \infty$

$$|Z^n(t) - \alpha Z^n(t_1) - (1-\alpha)Z^n(t_2)| < \varepsilon.$$

We estimate the first integral as

$$I_1 \le \sup_{s\in[0,T]} |Z^n(s)| \cdot (\xi^2 + |\mu||\xi|) \int_0^{t_1} |\mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t_1-s)} - \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)}|\,\mathrm{d}s,$$

and

$$
\begin{aligned}
&\int_0^{t_1} |\mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t_1-s)} - \mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)}|\,\mathrm{d}s \\
&= \int_0^{t_1} \mathrm{e}^{-\xi^2 r}|\mathrm{e}^{\mathrm{i}\mu\xi r} - \mathrm{e}^{\mathrm{i}\mu\xi(t-t_1+r)} + \mathrm{e}^{\mathrm{i}\mu\xi(t-t_1+r)} - \mathrm{e}^{-\xi^2(t-t_1)}\mathrm{e}^{\mathrm{i}\mu\xi(t-t_1+r)}|\,\mathrm{d}r \\
&\le \int_0^{t_1} \mathrm{e}^{-\xi^2 r}|1 - \mathrm{e}^{\mathrm{i}\mu\xi(t-t_1)}|\,\mathrm{d}r + \int_0^{t_1} \mathrm{e}^{-\xi^2 r}|1 - \mathrm{e}^{-\xi^2(t-t_1)}|\,\mathrm{d}r \\
&\le (|\mu\xi\delta| \wedge 2) \int_0^{t_1} \mathrm{e}^{-\xi^2 r}\,\mathrm{d}r + (1 - \mathrm{e}^{-\xi^2\delta}) \int_0^{t_1} \mathrm{e}^{-\xi^2 r}\,\mathrm{d}r \\
&= (|\mu\xi\delta| \wedge 2)\xi^{-2}(1 - \mathrm{e}^{-\xi^2 t_1}) + (1 - \mathrm{e}^{-\xi^2\delta})\xi^{-2}(1 - \mathrm{e}^{-\xi^2 t_1}).
\end{aligned}
$$

Finally

$$\begin{aligned}
&\int_{-\infty}^{\infty} (1+\xi^2)^\theta (\xi^2 + |\mu||\xi|)^2 \xi^{-2} (|\mu\xi\delta| \wedge 2)^2 (1-\mathrm{e}^{-\xi^2 t_1})^2 \,\mathrm{d}\xi \\
&= \mu^2\delta^2 \int_{|\xi|\le 2/|\mu|\delta} (1+\xi^2)^\theta (\xi^2 + |\mu||\xi|)^2 \xi^{-2} (1-\mathrm{e}^{-\xi^2 t_1})^2 \,\mathrm{d}\xi \\
&+ 4 \int_{|\xi|>2/|\mu|\delta} (1+\xi^2)^\theta (\xi^2 + |\mu||\xi|)^2 \xi^{-4} \,\mathrm{d}\xi \le C\delta^{2|\theta|-1} \to 0, \quad \delta \to 0.
\end{aligned}$$

The terms I_2, I_3, I_4 are estimated analogously.

2. Dirichlet case ($\theta < -\frac{3}{2}$):
In the Dirichlet case the Fourier transform of G_D has the explicit form

$$\mathcal{F}G_D(\cdot, t-s)(\xi) = -(\mu - 2\mathrm{i}\xi)\mathrm{e}^{-(\xi^2+\mathrm{i}\mu\xi)(t-s)}.$$

Obviously, the only difference to the first summand in the Neumann case is the factor $-(\mu - 2\mathrm{i}\xi)$. But since in this case $\theta < -\frac{3}{2}$, we only have to note, that the term

$$|(\mu - 2\mathrm{i}\xi)|^2 (1+\xi^2)^\theta \tag{6.16}$$

plays the same role as $(1+\xi^2)^\theta$ for $\theta < -\frac{1}{2}$ in the Neumann case. Consequently, the proof in the Dirichlet case essentially repeats the steps of the Neumann case. □

Finally we note that away of the boundary $x = 0$, the solution $(t, x) \mapsto u(t, x)$ is a smooth function. Thus the following theorem holds.

Theorem 6.6. *Let $T > 0$ and let $Z^n \to Z$ in probability in $D([0,T], \mathbb{R}; d_M)$ as $n \to \infty$, and let Z^n, $n \ge 1$, be absolutely continuous. Then for any $x > 0$*

$$\sup_{t\in[0,T]} |u^n(t,x) - u(t,x)| \to 0$$

in probability, as $n \to \infty$.

Proof. We consider the case of Neumann boundary conditions.

It is easy to see that for $x > 0$ and $t > 0$ the function

$$\begin{aligned}
G_N(t,x) &= \frac{1}{\sqrt{\pi t}} \mathrm{e}^{-\frac{(x-\mu t)^2}{4t}} + \frac{\mu}{2} \operatorname{erfc}\Big(\frac{x-\mu t}{2\sqrt{t}}\Big), \quad t > 0, \\
G_N(0,x) &= \lim_{t\downarrow 0} G_N(t,x) = 0,
\end{aligned}$$

is absolutely continuous, and its time derivative equals

$$\begin{aligned}\dot{G}_N(t,x) =& \frac{\mathrm{d}}{\mathrm{d}t} G_N(t,x) = \frac{x^2 - t(2+t\mu^2)}{4\sqrt{\pi}t^{5/2}} \mathrm{e}^{-\frac{(x-t\mu)^2}{4t}} + \frac{\mu}{\sqrt{\pi}}\Big(\frac{\mu}{4\sqrt{t}} + \frac{x}{4t^{\frac{3}{2}}}\Big)\mathrm{e}^{-\frac{(x-\mu t)^2}{4t}} \\ =& \frac{x^2 + \mu t x - 2t}{4\sqrt{\pi}t^{5/2}} \mathrm{e}^{-\frac{(x-\mu t)^2}{4t}}\end{aligned}$$

and is also continuous, $\dot{G}_N(0,x) = 0$, $\sup_{t\geq 0} |\dot{G}(t,x)| \leq M(x) < \infty$. Thus integration by parts yields

$$\begin{aligned}u(t,x) &= \int_0^t G_N(t-s,x)\,\mathrm{d}Z(s) = \int_0^t \dot{G}_N(t-s,x) Z(s)\,\mathrm{d}s, \\ u^n(t,x) &= \int_0^t G_N(t-s,x)\,\mathrm{d}Z^n(s) = Z^n(0) + \int_0^t \dot{G}_N(t-s,x) Z^n(s)\,\mathrm{d}s.\end{aligned}$$

Thus,

$$\begin{aligned}\sup_{t\in[0,T]} |u(t,x) - u^n(t,x)| &\leq |Z^n(0)| + \sup_{t\in[0,T]} \int_0^t |\dot{G}_N(t-s,x)| \cdot |Z(s) - Z^n(s)|\,\mathrm{d}s \\ &\leq |Z^n(0)| + M(x) \int_0^T |Z(s) - Z^n(s)|\,\mathrm{d}s,\end{aligned}$$

which converges to 0 in probability due to convergence $Z^n(t) \to Z(t)$ in M_1 in probability. $\square$

Appendix

In the following we write $L(X,Y)$ for the space of all bounded, linear operators from X to Y. For $p \geq 1$ and $U \in \mathbb{R}^d$, $L_p(U)$ denotes the Banach space of (equivalence classes of) all complex-valued, measurable functions in U, such that

$$\|f\|_p = \left(\int_U |f(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

In Chapters 5 and 6 we work with the so-called fractional Sobolev (or Bessel-potential) spaces

$$H_p^\theta(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) \colon \mathcal{F}^{-1}(1+\xi^2)^{\frac{\theta}{2}}(\mathcal{F}f)(\xi) \in L_p(\mathbb{R}^d) \},$$

where $\mathcal{S}'$ denotes the space of tempered distributions and $\mathcal{F}$ the Fourier transform on $\mathcal{S}'$.

Sobolev spaces have been studied in numerous books, e.g. in [HT08, Tri92]. However, a lot of the results presented there are either valid for $\theta > 0$, or they are proved in a much broader generality and the proofs rely heavily on the more complex theory of function spaces. The aim of this appendix is to direct proofs specifically for the properties of $H^\theta(\mathbb{R}_+)$ that we needed for this thesis. Lemmas A.1 and A.2 have been published in [HP19].

For our main result of Chapter 6 on differential equations with noise on the boundary (Theorem 6.3) we are using that $H^\theta(\mathbb{R}_+)$ is a separable Hilbert space. For $\theta > 0$ this result follows, e.g. from [HT08, Proposition 3.39]. In [Tri92, Theorem 4.5.5], an explicit extension operator is given for a more general class of function spaces. In the following lemma we will prove the minimality of the norm of the extension directly.

Lemma A.1. *Let $\theta \in \mathbb{R}$.*

(i) For every $f \in H^\theta(\mathbb{R}_+)$, there is a unique extension $\operatorname{ext} f$ *to $\mathbb{R}$ such that*

$$\|f\|_{H^\theta(\mathbb{R}_+)} = \|\operatorname{ext} f\|_{\theta,2}.$$

(ii) The operator $\operatorname{ext}\colon H^\theta(\mathbb{R}_+) \to H^\theta(\mathbb{R})$ *is bounded and linear.*

Proof. We start by showing the existence of the extension for fixed $f \in H^\theta(\mathbb{R}_+)$. Let E be the subset of $H^\theta(\mathbb{R})$, containing all extensions of f to $\mathbb{R}$. Let

$$\delta := \|f\|_{H^\theta(\mathbb{R}_+)} = \inf_{g\in E} \|g\|_{\theta,2}.$$

For $g, h \in E$ we have by the parallelogram law that

$$\|g-h\|^2_{\theta,2} = 2\|g\|^2_{\theta,2} + 2\|h\|^2_{\theta,2} - 4\Big\|\frac{g+h}{2}\Big\|^2_{\theta,2}.$$

Since for every $\varphi \in \mathcal{D}(\mathbb{R}_+)$

$$\Big(\frac{g+h}{2}\Big)(\varphi) = \frac{1}{2}(g(\varphi)+h(\varphi)) = \frac{1}{2}(f(\varphi)+f(\varphi)) = f(\varphi),$$

$\frac{g+h}{2} \in E$ and $\|\frac{g+h}{2}\|^2_{\theta,2} \geq \delta^2$. So we get

$$\|g-h\|^2_{\theta,2} \leq 2\|g\|^2_{\theta,2} + 2\|h\|^2_{\theta,2} - 4\delta^2. \tag{A.17}$$

Now, there is a sequence $(g_n)_{n\in\mathbb{N}}$ in E with $\lim_{n\to\infty}\|g_n\|_{\theta,2} = \delta$. Then by (A.17) we have

$$\|g_n - g_m\|^2_{\theta,2} \leq 2\|g_n\|^2_{\theta,2} + 2\|g_m\|^2_{\theta,2} - 4\delta^2 \to 0, \quad n, m \to \infty.$$

That means, (g_n) is a Cauchy sequence and since $H^\theta(\mathbb{R})$ is complete, there is $\widetilde{f} \in H^\theta(\mathbb{R})$ with $\widetilde{f} = \lim_{n\to\infty} g_n$ in $H^\theta(\mathbb{R})$. Since $H^\theta(\mathbb{R})$ is a subspace of $\mathcal{S}'(\mathbb{R})$, this especially implies, that for $\varphi \in \mathcal{D}(\mathbb{R}_+)$,

$$\widetilde{f}(\varphi) = \lim_{n\to\infty} g_n(\varphi) = \lim_{n\to\infty} f(\varphi) = f(\varphi).$$

Thus $\widetilde{f} \in E$ and, due to the continuity of the norm, $\|\widetilde{f}\|_{\theta,2} = \lim_{n\to\infty}\|g_n\|_{\theta,2} = \delta$. The uniqueness (in the sense of equivalent classes) follows easily from (A.17). Let $g \in E$ be another element with $\|g\|_{\theta,2} = \delta$. Then

$$\|g - \widetilde{f}\|^2_{\theta,2} \leq 2\|g\|^2_{\theta,2} + 2\|\widetilde{f}\|^2_{\theta,2} - 4\delta^2 = 0.$$

For (ii) we only have to prove the linearity of the operator. Boundedness directly follows from the definition.

To show linearity we define $M := \{f \in H^\theta(\mathbb{R})\colon f|_{\mathbb{R}_+} = 0\}$ and its orthogonal complement $M^\perp = \{g \in H^\theta(\mathbb{R})\colon \langle g, f\rangle_{\theta,2} = 0 \text{ for all } f \in M\}$. First we show, that for every $h \in H^\theta(\mathbb{R}_+)$, $\mathrm{ext}(h) \in M^\perp$.

Indeed, it is enough to show, that $\langle h, m\rangle_{\theta,2} = 0$ for all $m \in M$ with $\|m\|_{\theta,2} = 1$. Note that, since $m|_{\mathbb{R}_+} = 0$, $\mathrm{ext}(h) - \alpha m$ is another extension of h for any $\alpha \in \mathbb{C}$. Due to its definition, the norm of $\mathrm{ext}(h)$ is minimal in comparison to every other extension and therefore

$$\begin{aligned}
&\langle \operatorname{ext} h, \operatorname{ext} h\rangle_{\theta,2} \\
&\qquad \le \langle \operatorname{ext} h - \alpha m, \operatorname{ext} h - \alpha m\rangle_{\theta,2} \\
&\qquad = \langle \operatorname{ext} h, \operatorname{ext} h\rangle_{\theta,2} + |\alpha|^2 \langle m, m\rangle_{\theta,2} - \overline{\alpha}\langle \operatorname{ext} h, m\rangle_{\theta,2} - \alpha\langle m, \operatorname{ext} h\rangle_{\theta,2}.
\end{aligned}$$

Choosing $\alpha = \langle \operatorname{ext} h, m\rangle_{\theta,2}$ we get

$$0 \le |\alpha|^2 - \overline{\alpha}\alpha - \alpha\overline{\alpha} = -|\alpha|^2 = -|\langle \mathrm{ext}(h), m\rangle_{\theta,2}|^2$$

and hence

$$\langle \mathrm{ext}(h), m\rangle_{\theta,2} = 0. \tag{A.18}$$

Let now $f, g \in H^\theta(\mathbb{R}_+)$ and $\lambda \in \mathbb{C}$. We have to show that

$$\| \mathrm{ext}(f + \lambda g) - \mathrm{ext}(f) - \lambda \,\mathrm{ext}(g)\|_{\theta,2} = 0.$$

It is easy to see, that $(\mathrm{ext}(f) + \lambda \,\mathrm{ext}(g))|_{\mathbb{R}_+} = f + \lambda g$ and thus

$$m := \mathrm{ext}(f + \lambda g) - \mathrm{ext}(f) - \lambda \,\mathrm{ext}(g) \in M.$$

By (A.18) we have

$$\begin{aligned}
0 =& \langle \mathrm{ext}(f + \lambda g), m\rangle_{\theta,2} - \langle \mathrm{ext}(f), m\rangle_{\theta,2} - \lambda\langle \mathrm{ext}(g), m\rangle_{\theta,2} \\
=& \langle m, m\rangle_{\theta,2} \\
=& \| \mathrm{ext}(f + \lambda g) - \mathrm{ext}(f) - \lambda \,\mathrm{ext}(g)\|^2_{\theta,2}.
\end{aligned}$$

$\square$

Lemma A.1 and the properties of $H^\theta(\mathbb{R})$ imply that $H^\theta(\mathbb{R}_+)$ is a complete, separable Hilbert space with

$$\|f\|^2_{H^\theta(\mathbb{R}_+)} = \langle f, f\rangle_{H^\theta(\mathbb{R}_+)} = \langle \operatorname{ext} f, \operatorname{ext} f\rangle_{\theta,2}.$$

Eventually we show denseness of test functions in $H^\theta(\mathbb{R}_+)$ for negative θ. The idea of the proof follows the argument of Lemma 1.11.1 in [LM72].

Lemma A.2. *If $\theta < \frac{1}{2}$, then $\mathcal{D}(\mathbb{R}_+)$ is dense in $H^\theta(\mathbb{R}_+)$.*

Proof. Denote by $\mathcal{D}_0(\mathbb{R})$ the subspace of all $\varphi \in \mathcal{D}(\mathbb{R})$ with $\varphi = 0$ in a neighbourhood of 0, that is:

$$\mathcal{D}_0(\mathbb{R}) := \{\varphi \in \mathcal{D}(\mathbb{R})\colon \exists r > 0 \text{ such that } \forall x \in B_r(0) \quad \varphi(x) = 0\}.$$

Obviously it is enough to show that $\mathcal{D}_0(\mathbb{R}) \subset H^\theta(\mathbb{R})$ is dense if $\theta < \frac{1}{2}$, because then for any $h \in H^\theta(\mathbb{R}_+)$ there is a sequence $\{d_n\}_{n\in\mathbb{N}} \subset \mathcal{D}_0(\mathbb{R})$ that approximates the extension of h in $H^\theta(\mathbb{R})$. Since for every $n \in \mathbb{N}$, $d_n = 0$ in a neighbourhood of zero, the restriction $d_n|_{\mathbb{R}_+}$ is in $\mathcal{D}(\mathbb{R}_+)$ and $\{d_n|_{\mathbb{R}_+}\}_{n\in\mathbb{N}}$ approximates h in $H^\theta(\mathbb{R}_+)$.

Let $N\colon H^\theta(\mathbb{R}) \to \mathbb{C}$ be a continuous linear functional. A consequence of the Hahn-Banach Theorem states, that we only have to show, that if N vanishes on $\mathcal{D}_0(\mathbb{R})$, it also vanishes on the whole space $H^\theta(\mathbb{R})$ (see [Bre11], Corollary 1.8 and Remark 5). According to the Riesz representation theorem there is a unique element $h_N \in H^\theta(\mathbb{R})$, such that for every $u \in H^\theta(\mathbb{R})$

$$N(u) = \langle u, h_N\rangle_{H^\theta(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^\theta \overline{\mathcal{F}(h_N)(\xi)} \mathcal{F}(u)(\xi)\, \mathrm{d}\xi.$$

Let now $N(u) = 0$ for all $u \in \mathcal{D}_0(\mathbb{R})$. We can interpret the function $f(\xi) = \mathcal{F}(1 + |\xi|^2)^\theta \overline{\mathcal{F}(h_N)}(\xi)$ as an element of $\mathcal{S}'(\mathbb{R})$ and get by the definition of $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R})$ that

$$(f, u) = ((1 + |\xi|^2)^\theta \overline{\mathcal{F}(h_N)}, \mathcal{F}u) = N(u) = 0 \quad \forall u \in \mathcal{D}_0(\mathbb{R}).$$

This, however, means that $\operatorname{supp} f = \{0\}$, from which follows (see [HT08], Theorem 2.31), that

$$f = \sum_{j\le m} c_j D^j \delta_0$$

for some $m \in \mathbb{N}_0$ and $c_j \in \mathbb{C}$, $j = 0, \dots, m$, where δ_0 is the δ-distribution.

Hence, we know

$$(1 + |\xi|^2)^\theta \overline{\mathcal{F}(h_N)(\xi)} = (1 + |\xi|^2)^{-\frac{\theta}{2}} (\mathcal{F}^{-1} f)(\xi) = (1 + |\xi|^2)^{-\frac{\theta}{2}} \sum_{j=0}^{m} c_j (-i)^j \xi^j \underbrace{\mathcal{F}^{-1}\delta_0}_{=1},$$

and since $h_N \in H^\theta(\mathbb{R})$, $(1+|\xi|^2)^{\frac{\theta}{2}}\overline{\mathcal{F}(h_N)(\xi)} \in L^2(\mathbb{R})$, so that

$$\int_{\mathbb{R}} \frac{\left|\sum_{j=0}^{m} c_j(-i)^j \xi^j\right|^2}{(1+|\xi|^2)^\theta} \,\mathrm{d}\xi < \infty.$$

For $\theta < \frac{1}{2}$ this is only possible, if $c_j = 0$ for $j = 0, \dots, m$, and so $h_N \equiv 0$ and therefore $N \equiv 0$ on $H^\theta(\mathbb{R})$. $\square$

An embedding theorem used in Chapter 5 is [RS96, Theorem 4.4.3.2], which deals with the products of distributions and functions in Triebel-Lizorkin and Besov spaces. Since fractional Sobolev spaces are a special case of Triebel-Lizorkin spaces we can deduce the following theorem from Theorem 4.4.3.2:

Theorem A.3. *Assume that $\theta_1, \theta_2 \in \mathbb{R}$ and $p_1, p_2 \in \mathbb{R}_+$ satisfy the following conditions:*

$$\begin{aligned}
&\theta_1 < 0 < \theta_2\\
&\theta_1 + \theta_2 > 0\\
&\theta_1 + \theta_2 > \frac{n}{p_1} + \frac{n}{p_2} - n\\
&\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}\\
&\theta_2 > \frac{n}{p_2}.
\end{aligned}$$

Then there is a constant $C > 0$, so that

$$\|f \cdot g\|_{H^{\theta_1}_{p_1}} \leq C\|f\|_{H^{\theta_1}_{p_1}} \cdot \|g\|_{H^{\theta_2}_{p_2}} \tag{A.19}$$

for all $f \in H^{\theta_1}_{p_1}$ and $g \in H^{\theta_2}_{p_2}$.

Note that there is a close connection between fractional Sobolev spaces and weighted L_2-spaces, since $H^s_2(\mathbb{R})$ can be understood as $\mathcal{F}L_2(\mathbb{R}, w_s)$, where $w_s(\xi) = (1+\xi^2)^{\frac{s}{2}}$. Considering this, the Interpolation Theorem of Stein-Weiss (see [BL76, Theorem 5.4.1]) implies the following result:

Theorem A.4. *Let $0 < \vartheta < 1$ and T be a linear operator with*

$$\begin{aligned}
&T \in L(H^s_2, H^{s_1}_2) \textit{ and}\\
&T \in L(H^s_2, H^{s_2}_2).
\end{aligned}$$

Then for $\tilde{s} = (1-\vartheta)s_1 + \vartheta s_2$,

$$T \in L(H_2^s, H_2^{\tilde{s}}),$$

and we can estimate the operator norm of T *in the following way:*

$$\|T\|_{L(H_2^s, H_2^{\tilde{s}})} \leq \|T\|^{1-\vartheta}_{L(H_2^s, H_2^{s_1})} \|T\|^{\vartheta}_{L(H_2^s, H_2^{s_2})}.$$

To prove the existence of mild solutions in Chapter 5, we also need some basic integral inequalities, namely Young's inequality for convolution integrals, which states that

$$\Big\| \int_{\mathbb{R}^d} g(\cdot - y) f(y) \, \mathrm{d}y \Big\|_{L_q(\mathbb{R}^d)} \leq \|g\|_{L_r(\mathbb{R}^d)} \cdot \|f\|_{L_p(\mathbb{R}^d)}, \tag{A.20}$$

for $1 \leq r < \infty$, $1 \leq p \leq q$ with $\frac{1}{r} + \frac{1}{p} = 1 + \frac{1}{q}$, and $f \in L_p(\mathbb{R}^d)$, $g \in L_r(\mathbb{R}^d)$, and the following corollary of Hölder's inequality, which is sometimes called Minkowski's inequality for integrals and states that for two measure spaces (S_1, σ_1), (S_2, σ_2) and a measurable function $f : S_1 \times S_2 \to \mathbb{C}$,

$$\left(\int_{S_2} \left| \int_{S_1} f(x,y) \sigma_1(\mathrm{d}x) \right|^p \sigma_2(\mathrm{d}y) \right)^{\frac{1}{p}} \leq \int_{S_1} \left(\int_{S_2} |f(x,y)|^p \sigma_2(\mathrm{d}y) \right)^{\frac{1}{p}} \sigma_1(\mathrm{d}x), \tag{A.21}$$

see for example [Gar07, Corollary 5.4.2].

Index of Notation

Acronyms

a.s. almost surely

càdlàg right continuous with left limits (continu á droite, limitè á gauche), page 8

cf. compare (confer)

e.g. for example (exempli gratia)

i.e. that is (id est)

PRM Poisson random measure, page 11

r.h.s. right hand side

s.t. such that

SDE stochastic differential equation

SPDE stochastic partial differential equation

w.r.t. with regard to

Symbols

∇f gradient of f, page 16

$\mathbb{1}_{[a,b)}(\cdot)$ indicator function

$\langle f, g\rangle_{\theta,2}$ scalar product in $H^\theta(\mathbb{R})$, page 17

$\langle f, g\rangle_2$ scalar product in $L^2(\mathbb{R}^d)$, page 16

$\langle f, g\rangle_{H^\theta(\mathbb{R}_+)}$ scalar product in $H^\theta(\mathbb{R}_+)$, page 17

$\langle \cdot, \cdot\rangle_{\mathcal{H}}$ scalar product in the Hilbert space $\mathcal{H}$, page 18

$\|f\|$ supremum norm for bounded function f, page 16

$\|f\|_2$ norm in $L^2(\mathbb{R}^d)$, page 16

$\|f\|_{\theta,2}$ norm in $H^\theta(\mathbb{R})$, page 17

$\|g\|_{H^\theta(\mathbb{R}_+)}$ norm in $H^\theta(\mathbb{R}_+)$, page 17

$\|\cdot\|_\lambda$ norm in $\mathcal{M}^2$, page 73

$\|A\|_{L(E,F)}$ $\sup\{\|Ax\|_F \colon x \in E \text{ with } \|x\|_E \leq 1\}$, page 18

$\|\cdot\|_E$ norm in the Banach space E, page 18

$\langle M\rangle_t$, page 7

$\langle M, N\rangle_t$ angle bracket, page 7

$[X,Y]_t$ quadratic co-variation of the processes X and Y, page 9

$[X]_t$ quadratic variation of the process X, page 9

B_D Dirichlet boundary operator, page 81

B_N Neumann boundary operator, page 81

$\mathcal{B}(\mathbb{R}^d)$ Borel σ-algebra

C placeholder for constant bigger than zero; the value may vary from line to line, page 7

C^n n times continuously differentiable functions, page 16

C_b^n n times continuously differentiable bounded functions with bounded derivatives, page 16

C_c^n n times continuously differentiable functions with compact support, page 16

$D([0,T],V)$ space of V-valued càdlàg functions, page 91

$D(A)$ domain of the operator A, page 18

ΔX_t $\Delta X_t = X_t - X_{t-}$

Df Jacobian matrix of f, page 16

$d_M(f_1,f_2)$ M_1 metric on $D([0,T],V)$, page 92

$\mathcal{D}(\mathbb{R}_+)$ space of infinitely differentiable functions, page 85

$\mathbf{e}^{\varphi(\cdot;r,z)}(x)$ exponential mapping, page 27

$\operatorname{erfc}(x)$, page 85

$\mathbb{E}$ expected value

$\mathcal{F}\varphi$ Fourier transform of φ, page 16

$H^\theta(\mathbb{R})$ fractional Sobolev space, page 17

$H^\theta(\mathbb{R}_+)$, page 17

Id identity operator, page 18

$L^2(\mathbb{R}^d)$, page 16

$L^2(\langle M\rangle$ Hilbert space of predictable processes that are integrable w.r.t. M, page 8

$L(E,F)$ space of all bounded, linear operators from E to F, page 18

$\mathcal{M}^2$ $\mathcal{M}^2(0,T;H^\theta)$, page 73

$\tilde{N}$ compensated Poisson random measure, page 11

N Poisson random measure, page 11

$\mathbb{N}$ $\{1,2,3,\dots\}$

ν Lévy measure of the Lévy process, page 11

$(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ filtered probability space, satisfying the usual conditions

$\mathbb{R}^m_*$ $\mathbb{R}^m_* = \mathbb{R}^m\backslash\{0\}$, page 59

$\rho(A)$ resolvent set of the operator A, page 19

$\mathbb{R}$ real numbers

$\mathbb{R}_+$ positive real numbers (without zero)

$\mathcal{S}(\mathbb{R}^d)$ Schwartz space, page 16

$\mathcal{S}'(\mathbb{R}^d)$ space of tempered distributions, page 16

X_{t-} $X_{t-} = \lim_{s\to t, s<t} X_s$

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