HOW PRIMARY SCHOOL STUDENTS PERFORM MULTIPLICATIVE STRUCTURE PROBLEMS WITH NATURAL AND RATIONAL NUMBERS

Cristina Zorrilla[1], Ceneida Fernández[1], María C. Cañadas[2], and Pedro Ivars[1]

[1]University of Alicante, [2]University of Granada

This paper is part of a larger study focuses on a teaching experiment (pre-test, instruction, post-test) that aims to analyse primary school students' features on the transition from natural to rational numbers when solving multiplicative structure problems. Here, we analysed 61 6th graders responses to nine multiplicative structure problems with natural numbers and fractions (pre-test). We analysed students' performance and strategies. Results showed differences in students' performance considering the numerical set, indicating difficulties in identifying the problem structure's invariance. The most used strategy was the algorithm in both correct and incorrect answers. Results suggest that specific instruction is needed to help students focus on the problem structure invariance when the numerical set changes.

INTRODUCTION

In the 1980s, Bell et al. (1981) analysed 12-16-year-old students' difficulties when solving multiplicative structure problems with rational numbers. Some difficulties observed were related to the choice of the operation for solving the problem. Later, other studies (e.g., Fischbein et al., 1985; Levain, 1992) highlighted the existence of implicit models for operations (e.g., multiplication leads to a larger number and division leads to a smaller number) and students' difficulties in identifying the problem structure when the numerical set changed. These difficulties were also documented in the National Assessment of Educational Progress (NAEP) from the United States, where only 27% of 9-10-year-old students chose the correct answer in the multiple-choice question "Jim has 3/4 of a yard of string which he wishes to divide into pieces, each 1/8 of a yard long. How many pieces will he have?" (National Center for Education Statistics, 2003). These difficulties persisted in later grades since only 55% of 13-14-year-old students solved the question correctly.

More recent studies have focused on students' strategies used to solve multiplicative structure problems (e.g., Cheeseman & Downton, 2021; Downton, 2009; Empson & Levi, 2011; Hulbert et al., 2017; Ivars & Fernández, 2016; Mulligan, 1992). It has been shown that students' thinking evolves from strategies that do not lead to the correct answer to additive strategies, such as counting and repeated addition, that lead to correct answers. Later, multiplicative strategies such as the algorithm appear. Strategies leading to an incorrect answer also include the algorithm since students commonly identify an incorrect algorithm to use, which may be due to the lack of

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understanding of the situation and the relationship between the quantities involved (Hulbert et al., 2017).

Given these previous results, specific instruction in primary education focused on identifying the mathematical structure of the problem independently of the numerical set involved (natural or rational numbers) is necessary. With this regard, this paper is part of a larger study whose objective is to identify characteristics of the transition from natural to rational numbers when primary school students solve multiplicative structure problems. For this purpose, we have designed a teaching experiment that focuses primary school students' attention on identifying the mathematical structure of the problem independently of the numerical set involved. The teaching experiment consists of a pre-test, an instruction and a post-test. In this paper, we focus on the results of the pre-test. Its objective is to examine how sixth graders (11-12 years old) solve multiplicative structure problems with natural and rational numbers, and the strategies they use. This information will be used in the design of the instruction.

THEORETICAL BACKGROUND

Mathematically, we focus on the isomorphism of measures problems whose structure is a proportion between two measure spaces, each including two quantities (Vergnaud, 1981). In these problems, if one of the quantities is reduced to 1, three types of problems arise depending on which of the other three quantities is the unknown (Greer, 1992): (a) multiplication, whose unknown quantity is the total quantity; (b) partitive division, whose unknown quantity is the quantity per group; and (c) measurement division, whose unknown quantity is the number of groups.

Although different authors have pinpointed students' strategies to solve these kinds of problems (see above), we use the strategies identified by Empson and Levi (2011). These strategies develop from a basic way of thinking (*represents each group*) to a more sophisticated way of thinking (*multiplicative strategies*):

- Represents each group. Students represent all the quantities (symbolically or with drawings) and then count, add or subtract to get the answer. It includes *direct modeling* (the quantities are represented with a drawing) and *repeated addition* (the quantities are represented by mathematical symbols).
- Grouping and combining strategies. Students represent the "necessary" quantities, i.e., they group quantities additively until they get "friendlier amounts" to operate with (Empson & Levi, 2011, p. 57). Usually these "friendlier amounts" are natural numbers.
- Multiplicative strategies. Students form groupings, which are linked multiplicatively.

Considering the objective of this paper, the research questions are: How do sixth graders (11-12 years old) solve multiplication, partitive division and measurement division problems with natural numbers and fractions? What strategies do they use to solve these types of problems?

METHOD

Participants and instrument

The participants were 61 sixth graders (11-12 years old) from a Spanish primary school. According to the Spanish curriculum, the students had been introduced to multiplication and division algorithms with natural numbers in previous grades. They had also been introduced to the multiplication algorithm with fractions. Nevertheless, they had informal strategies to solve multiplication, partitive division and measurement division problems without the algorithm.

The pre-test consisted of nine problems (Table 1): three multiplication (M), three partitive division (PD) and three measurement division (MD) problems. Furthermore, considering our objective, we varied the numerical sets in each type of problem: a problem with natural numbers (N), a problem with a proper fraction (Q1) and a problem with two proper fractions (Q2). The pre-test was solved individually during a 50-minute session. The participants had to justify their answers, and they could not use electronic devices.

Characteristics of the problem	Statement	Structure
M-N	My grandmother uses 2 cups of flour when she makes a tray of cookies. If she wants to bake 8 trays of cookies, how many cups of flour will she need?	8 × 2 = 16
PD-N	We bought 20 yoghurts at the weekly shopping trip. If they came grouped in 5 packages with the same number of yoghurts, how many yoghurts are in each package?	5 × 4 = 20
MD-N	A baker made 24 *Easter cakes* and packed them in boxes of 4 cakes, how many boxes did he need?	6 × 4 = 24
M-Q1	At Marcos' birthday party there was lemon soda. If there were 3 bottles left at the end of the party and each bottle contained 2/3 of a litre, how many litres of lemon soda were left over?	3 × 2/3 = 2
PD-Q1	At Marcos' birthday party, there were 12 sandwiches left over. These sandwiches take up 3/4 of a tray. If all the sandwiches had the same size, how many sandwiches were there on the tray?	3/4 × 16 = 12
MD-Q1	My mother has made 2 litres of orange juice. If she has distributed the 2 litres in cups of 1/4 litres of capacity, how many cups has she filled?	8 × 1/4 = 2

M-Q2	Diego has picked oranges from his vegetable garden. To store them, he has used boxes of 3 kilos. If he finally counted 1/4 of a box, how many kilos of oranges has Diego picked?	1/4 × 3 = 3/4
PD-Q2	Roberto has used 3/4 of a kilo of clay to make figures. If he has made 6 identical figures, how much clay has he used for each figure?	6 × 1/8 = 3/4
MD-Q2	We have a bottle of 1/2 of a litre of perfume, and we want to distribute it in little bottles of 1/10 of a litre. How many little bottles do we need?	5 × 1/10 = 1/2

Table 1: Problems of the pre-test

Analysis

The analysis was performed in two phases. In the first phase, we analysed the correctness of students' responses. For each problem, we coded with "1" the students' correct responses and with "0" the students' incorrect responses (Table 2). In the second phase, we focused on the students' strategies used. We initially based on the strategies proposed by Empson and Levi (2011), and then we performed an inductive analysis from our data that allowed us to refine these categories. In what follows, we describe the final category system, with some examples in Table 2:

- Representing each group. This category includes direct modelling and repeated addition/subtraction.
- Additive grouping and combining strategies. Students form groupings, which are linked additively.
- Multiplicative strategies. Students establish multiplicative relationships.
  - Multiplicative grouping and combining strategies. Students form groupings linked multiplicatively.
  - Algorithm. Students use the multiplication algorithm in multiplication problems and the division algorithm in division problems.
  - Inverse algorithm. Students use the multiplication algorithm in division problems, looking for the value whose product is the total quantity.
  - Equivalent fraction. Students look for an equivalent fraction whose number of parts is equivalent to the parts of the unit of measure (MD problem) or whose numerator is equivalent to the number of groups (PD problem).
- Unidentified strategies. Answers without explaining the procedure or procedures without sense.

- Blank answers.

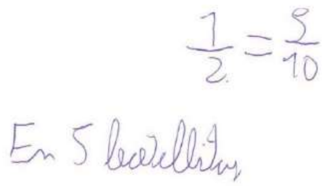
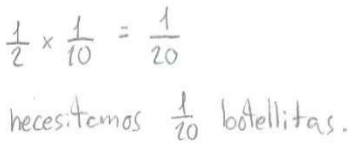
Correctness code	Strategy category	Example: Student answer	Description
1	Multiplicative strategy: Equivalent fraction		The student looks for an equivalent fraction, whose number of parts is equivalent to the parts of the unit of measure (1/10)
0	Algorithm		The student uses a multiplication algorithm incorrectly

Table 2: Examples of the analysis performed (MD-Q2 problem)

RESULTS

Students' performances in each type of problem and numerical set

Table 3 shows the percentage of students' correct responses considering each type of problem and the numerical set.

	N	Q1	Q2	Total
M	91.8	68.8	44.3	68.3
PD	91.8	41.0	24.6	52.5
MD	98.4	47.5	49.2	65.0
Total	94.0	52.4	39.3	61.9

Table 3: Percentage of correct responses

Students provided more correct responses in multiplication problems (68.3%) than in division ones (58.8%). Furthermore, they provided more correct responses in measurement division than in partitive division problems (65% and 52.5%, respectively). According to the numerical sets involved, students were more successful in problems with natural numbers (94%) than with fractions (45.9%).

Students' strategies in each type of problem and numerical set

Table 4 shows the percentages of students' use of each strategy and the percentage of correct (C) and incorrect (I) responses in each strategy. More than 80% of students used the *algorithm* in the three types of problems with natural numbers. Regarding problems with fractions, the use of the *algorithm* was also the most representative strategy. Nevertheless, this was the strategy that also led to more incorrect responses.

In the multiplication problem with a proper fraction, *representing each group* was also used, while in the problem with two proper fractions students used *multiplicative grouping and combining strategies*. In the partitive division problem with a proper fraction, they used *multiplicative grouping and combining strategies,* and in the problem with two proper fractions, they used *equivalent fraction*. In measurement division problems with proper fractions, *multiplicative grouping and combining strategies*, the *inverse algorithm* and *representing each group* were observed.

	N	Q1	Q2

Key

Green	Representing each group	Dark blue	Algorithm
Red	Additive grouping and combining strategies	Yellow	Inverse algorithm
Orange	Multiplicative grouping and combining strategies	Purple	Equivalent fraction
Light blue	Unidentified strategies	Brown	Blank answers

Table 4: Percentages of students' strategies and the type of answer

DISCUSSION AND CONCLUSION

In this paper, we examined how sixth graders solve multiplicative structure problems with natural and rational numbers, and the strategies they use. Our results show that students provided more correct responses in problems with natural numbers (94%); this percentage dropped remarkably when natural numbers were replaced by fractions (52.4% with one proper fraction and 39.3% with two proper fractions). Therefore, these results show that students did not recognise the invariance of the mathematical structure of the problem independently of the numerical set involved, showing the same difficulties obtained in previous studies some decades ago (Levain, 1992).

Regarding students' strategies, the use of the algorithm was the most common strategy in multiplicative structure problems with natural numbers and proper fractions. Nevertheless, in problems with fractions, the use of the algorithm led students to incorrect answers. This result was identified both in multiplication problems (in which students were introduced to the multiplication algorithm with fractions) and in division problems (in which students were not introduced to the division algorithm with fractions although they used the algorithm by converting the fractions to decimal numbers). Our results have also shown that students used other strategies different to the algorithm to solve multiplication and division problems with fractions and that these strategies allowed students to get correct responses to these problems.

Considering the results obtained in the pre-test, the next step in our research is to design the instruction aimed at focusing primary school students' attention on identifying the invariance of the problem structure when the numerical set changes. Theoretically, this instruction will be based on developing students' relational thinking, that is, on developing flexible strategies based on properties and relations between quantities (Empson & Levi, 2011) and on teaching with variation (e.g., Sun, 2019). In other words, we will provide a systematic variation of problems and quantities that will allow students to observe that the structure of the problem remains invariable.

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