PERSPECTIVES OF PROFESSORS IN MATHEMATICS EDUCATION ON FRUIT SALAD ALGEBRA – A COMPARISON BETWEEN TAIWAN AND GERMANY

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This study investigated intercultural differences in experts' perspectives on the appropriateness of the use of mathematical representations. A total of 19 Taiwanese and 12 German experts (professors in mathematics education) participated in an online survey to evaluate the use of representations in a text vignette illustrating a classroom situation in which the teacher used fruit salad algebra. Through content analysis, this study revealed that the majority of German experts criticized the inappropriateness of the representations for causing letter as object misconceptions, whereas the perspectives of Taiwanese experts were dominated by practical concerns related to whether the use of the representations was a waste of time or could help students successfully perform mathematical operations on the symbols.

INTRODUCTION

The use of representations, as both something to be taught and something to aid learning (Cai & Wang, 2006), has long been a key issue in mathematics education. However, are perspectives regarding the appropriateness of the use of representations universal across cultures? Teachers' perspectives on good or proper mathematics teaching have been investigated in cross-cultural studies, and the findings have revealed that the definitions of good or proper mathematics teaching vary widely between cultures (e.g., Bryan et al., 2007). As those responsible for training teachers, the perspectives held by professors in mathematics education usually reflect the features of expected instructional quality in their cultures. Therefore, this study examines how perspectives of professors (experts) in our international mathematics education community on the appropriateness of representations used in mathematics educational contexts can vary between different cultures, with Taiwan and Germany representing an East Asian and a Western country. Specifically, we focus on how experts in mathematics education in these two countries evaluate the use of representations in a classroom situation where the teacher uses fruit salad algebra.

THEORETICAL BACKGROUND

Appropriate use of representations

In mathematics, representations cannot be understood in isolation; they are embedded in a wider structure where meanings and conventions are established and where they complement and richly relate to one another. The appropriate use of representations

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can help students successfully construct mathematical concepts and procedures, whereas their inappropriate use hinders student learning and gives rise to misconceptions (Goldin & Shteingold, 2001).

One well-known student misconception caused by the improper use of representations (so-called "fruit salad algebra") is the *letter as object* misconception, (Küchemann, 1981). An example of fruit salad algebra is the use of images of apples (as the referent of a) and bananas (as the referent of b) to represent $2a + 3b$ (Chick, 2009), where a letter is reinforced to be regarded as an object rather than as an unknown or variable. Küchemann (1981) indicated that this reduction in the meaning of the letters from something abstract to something concrete and "real" allowed many students, who had problems with variables, to successfully deal with symbols. However, this reduction hindered subsequent learning when it became essential to substitute numbers for letters, to execute further operations (e.g., multiplication), or to form a relationship between variables. One famous example is students representing the statement *six times as many students as professors* as 6S = P rather than 6P = S in the mathematical problem formulated by Clement (1982). In Chick's (2009) study, few teachers were aware that fruit salad algebra is inappropriate in the first place, and more than 70% of that study's teachers indicated interest in using such a representation model in the future. Is this a problem of teacher education? Or can this phenomenon also be seen among experts in mathematics education? Despite the long existing (Western) literature on the problems of fruit salad algebra, it is not clear whether a critical stance on such use of representations in algebra can be seen as a consensus among scholars in our intercultural mathematics education community.

Differences in mathematics education in East Asian and the Western cultures

Different identities in mathematics education deeply rooted in East Asian and the Western cultures have been identified in the literature. For example, Leung (2001) pointed out the dichotomy of emphasizing the final product versus the learning process in these cultures. The emphasis on the product in East Asia aligns with the findings of Pratt et al. (1999) that the aim of learning is to get foundational knowledge including factual knowledge, principles, and procedures, which is usually the content in the examinations. A series of studies empirically investigated common as well as distinct characteristics of effective teaching in different cultures (e.g., Bryan et al., 2007; Wang & Cai, 2007). East Asian and Western teachers were reported to emphasise the structure vs pragmatic aspects of mathematics, respectively. Teachers from both cultures valued learning processes that start from the concrete and move on to the abstract. Nonetheless, teachers from East Asia viewed concrete representations as only an initial crutch the student uses to gain facility with abstract mathematical concepts and skills while Western teachers viewed them as a thinking tool that students may continuously use.

Studies have further investigated the features of ideal mathematics pedagogy in East Asia after decades of Western influence on education (e.g., Hsieh et al., 2017). The

findings revealed that some of the factors endorsed by teachers and students are rooted in traditional Chinese educational culture (e.g., conceptual connection and meaning in handling teaching materials) and some are influenced by Western culture (e.g., concrete representations and those grounded in everyday life). In addition, some factors rooted in traditional culture were looked upon less favourably (e.g., an emphasis on speed and challenge in problem solving).

Studies comparing expert perspectives of characteristics on good mathematics teaching in different cultures are scarce. The Teacher Education and Development Study in Mathematics (TEDS-M) reported some similarities and differences between cultures not only for teachers but also for professors. For example, both Taiwanese and German professors who trained mathematics teachers valued active-learning approaches more than teacher-directed approaches in the teaching and learning of mathematics. However, regarding the nature of mathematics, approximately 60% of Taiwanese educators believed equally in the primacy of both, the process-of-inquiry and the rules-and-procedures aspect, and approximately 40% believed that the process-of-inquiry aspect is the more important of the two. In Germany, most educators believed in the primacy of the process-of-inquiry aspect (Wang & Hsieh, 2014).

RESEARCH METHOD

Instrument

This research report is part of the findings of a Taiwanese-German cooperative research project (TaiGer noticing) to explore the perspectives of experts on characteristics of good mathematics teaching, and teacher professional noticing (Dreher et al., 2021).

The teacher T introduces *combining like terms* in the unit *linear equations in two unknowns.* T takes out picture cards and puts them on the blackboard (as shown on the right).

T: Today we will learn what like terms are and what to do with them by using a real-life example. These are the candies I just bought. If you would sort them, how would you do that?
S: I would put the lollipops together and put the gummy bears together.
T: Very good. The candies that look alike are of the same sort and can be combined. And it's the same with combining like terms. Now, we have $3x+2y+2x+x+y$ [writing on the blackboard]. Each term of this expression corresponds to a candy card. So, we combine the like terms corresponding to the lollipops, this makes $6x$. We also combine the like terms corresponding to the the gummy bears, we get $3y$ [moves the candy cards as shown]. That is, we add the coefficients of like terms, $3 + 2 + 1 = 6$ and $2 + 1 = 3$.

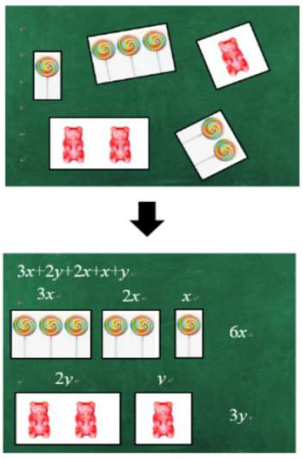

Figure 1: Vignette on the use of representations

The research team designed 18 vignettes, in which classroom situations and something not meeting our expectations of good teaching were depicted (breach-experiment), to elicit the experts' perspectives on a certain aspect of instructional quality and the teachers' noticing. In this report, we focus on a vignette representing a classroom situation where the teacher uses fruit salad algebra (Figure 1). The team members from

both countries agreed that the visual representation used by the teacher in the vignette was inappropriate; the picture cards used to represent the concept of combining like terms were unable to properly represent the nature of variables. The experts were prompted to evaluate the situation by the open-ended question 'Please evaluate the teacher's use of representations in this situation and give reasons for your answer.'

Sample and data collection

The survey was conducted online in the native language of the survey respondents. We recruited professors who are (1) active in mathematics education research and (2) active in preparing future secondary mathematics teachers. We aimed for a sample of 15 experts in each country. With the assumption of a participation rate of at least 50%, a random sample of 30 professors out of the full list of those meet these criteria was contacted in Germany. The criteria yielded a list of only 32 professors in Taiwan and thus all of them were contacted. In the end, 19 professors from 10 universities in Taiwan and 12 professors from 10 universities in Germany worked on the targeted vignette. Some experts had experience as school teachers (Taiwan: 14, Germany: 12) and some had experience as researchers in mathematics (Taiwan: 5, Germany: 4).

Data analysis

All the responses were translated into English, and the materials for coding were presented in both English and the native languages of the experts to facilitate the process of coding and discussion (for details see Dreher et al., 2021). The experts' evaluations were analysed mainly regarding two aspects, which were (1) whether they see the teacher's use of representations was inappropriate and (2) in which way it was inappropriate. With regard to the second, the expert responses were coded using a combination of top-down and bottom-up processes. The inappropriateness of the use of representations as seen by the research team (the picture cards used to represent the concept of combining like terms were unable to properly represent the nature of variables) made up one code, and other codes were determined through inductive analysis of the responses. The research team members analysed the responses in a first round to inductively extract possible codes within each culture, and then we determined the final coding scheme in a cross-cultural discussion. In addition to the original reason for inappropriateness, one major other reason emerged in the coding: some experts criticized that the representations used were limited in its functions and possibilities for extension. In the second round, first within-country coding was performed. All the responses were coded by two coders, and the discrepancies were resolved through discussion. Subsequently, a discussion was held between the Taiwanese and German researchers to reach a consensus.

RESULTS

In total, 63% of the Taiwanese experts and 92% of the German experts mentioned that the teacher's use of representations, as displayed in Figure 1, was *inappropriate* (Table 1). The Taiwanese and German experts provided different reasons why. Among those

evaluating the use of representations negatively, most of the German experts (73%) indicated the inappropriate use originally set by the research team, whereas only 33% of Taiwanese experts indicated this inappropriateness. These experts mentioned that using cards depicting candies to represent variables as displayed in Figure 1 is not appropriate and could cause *letter as object* misconceptions in students. The following answers are typical examples from both cultures.

> TW2: This is an old issue; although the teacher was teaching the combination of like terms, did the picture actually represent an object or a number? The teacher did not clearly know about the misconception that his approach could have possibly caused.
>
> GER3: […] there is the impressive example in which interviewees have translated the situation 'For each professor there are six students' into the equation P = 6S. Behind this is the substantial misconception that a variable like "P" is an abbreviation for an object like "professor". In fact, however, variables do not stand for objects, but for NUMBERS of objects. Exactly this misconception is however provoked by the situation described: "x stands for lollipop", so "3x stands for 3 lollipops". Insofar, the object "lollipop" is a very bad representation of the variable x. […]

	Taiwan (N=19)	Germany (N=12)
Evaluated the use of representations as being inappropriate	12(63%)	11(92%)
Reasons for inappropriateness		
Indicated the inappropriateness of representing variables by objects	4(33%)	8(73%)
Indicated that the representations used were limited in its extension or functions	9(75%)	3(27%)

Table 1: Frequencies and percentages of experts in each case

Among those evaluating the use of representations negatively, most of the Taiwanese experts (75%) indicated that the use of the representations was limited. Their views fell mainly into three categories. Specifically, three experts said that the representation limits the coefficients to only natural numbers (TW5). Two experts mentioned that the representation did not suggest that you cannot combine xs and ys in terms of addition because lollipops and gummy bears can, in real life, be combined as candies (TW15). All of these concerns focused on whether students could proceed the operations correctly when doing future tasks. Another three Taiwanese experts mentioned that the students were cognitively mature enough to grasp the concepts without this (basic) card representation, which may indicate that the extra effort of making or using cards was not necessary (TW19). Three German experts also indicated limitations of the representation with one falling into each of the three categories.

TW5: Teacher T analogised different terms to the different categories regarding lollipops and gummy bears and used the comparison between symbolic representations and picture cards (connecting different representations and contexts) to establish the meaning of the addition of the terms. This was helpful for understanding and applying the concept of combining like terms. However, difficulties would occur when directly applying such representations to certain special cases, such as the subtraction of terms with negative coefficients or with positive coefficients that are not integers. […]

TW15: 1. Using the representations of lollipops and gummy bears was well intended, but the meaning of the lollipops and gummy bears corresponding to x and y was not dealt with in the lesson. 2. The students could also have said that a total of nine candies could be eaten without distinguishing between lollipops and gummy bears. […]

TW19: If the lesson was for students aged 12 years and older, I personally think that this representation is probably excessive. If the students have already achieved the cognitive competence for [understanding] *variables* (a symbol representing a value), the teacher can directly manipulate the symbol. I would guess that if students had not yet acquired the ability to manage the multiplication, addition, and subtraction of abstract symbols, it would be futile to switch to using representations of cards and flower counters. […]

The perspectives of the Taiwanese experts revealed that, regarding the use of the representations, they did not address the nature of the letters for being unknowns or variables as much as the German experts did. However, given the high content knowledge of Taiwanese experts, we doubt whether they missed the point that the representation does not properly represent the nature of variables. So even if Taiwanese experts understood the *letter-as-object* misconception, they were more concerned with 'practical' factors, such as whether students could perform the operations correctly in the future and also with the efficient use of time in the class. These may reflect educational beliefs rooted in traditional Chinese culture: Students are expected to grasp the body of mathematics knowledge correctly and perform well in examinations, and thus, whether the representations can help them succeed in solving problems in the future is essential (Leung, 2001). Mathematics instruction should be fast paced to cover the demanding curriculum, and sufficient exercises should be provided for examination preparation; wasting time doing something viewed as unnecessary is thus undesirable.

Notably, as many as 68% of the 19 Taiwanese experts mentioned some positive aspects of the use of the representation, whereas only 2 German experts did so. Some of their perspectives aligned with the function of the reduction indicated by Küchemann (1981): They mentioned that the use of cards was natural and reasonable for student cognition regarding the analogy between the categorisation of concrete objects in real life and the operations performed on the symbols and that this helped students quickly learn how to perform the operations (TW1 and TW13). This phenomenon again reflected the "practical" issue aforementioned. The experts saw the advantages of quickly obtaining

the skills to perform operations correctly even if they may understand the potential disadvantages of causing misunderstanding or hindering future learning. Some experts mentioned that the graphical representation was helpful because of its characteristics of being visualizable and manipulable (TW7). The perspectives of these experts are in line with findings regarding the Western influence on the East Asian use of concrete representations (Hsieh et al., 2017; Wang & Cai, 2007).

> TW1: The teacher introduced like terms by classifying real-life objects (candy) as a representation, and the students could immediately relate to the situation.
>
> TW13: Appropriate. [The teacher] could use concrete objects to represent the abstract x. However […]
>
> TW7: The use of manipulative iconic representations to link to symbolic representations was good […]

CONCLUSION AND DISCUSSION

This study has its limitation that the findings based on the expert evaluation on one vignette regarding the use representation may not be generalizable. However, using this vignette representing a classroom situation that used fruit salad algebra, this research report illustrated how experts' perspectives on the appropriateness of the use of representations can be different in East Asian and Western cultures. The large majority of German experts paid attention to whether the representation reflects the nature of the mathematical concept. From their perspectives, an appropriate representation of letters should reflect the nature of unknowns or variables, and they indicated that using cards of candies to represent algebraic terms could result in a "letter as object" misconception raised by Küchemann (1981). Although Taiwanese experts may recognise that the "letter as object" misconception could arise from the use of cards depicting candies, what dominated most of their perspectives on the appropriateness of the representation use were practical issues in mathematics instruction. Some experts considered the representations to be improper because the use of cards did not match student cognition and resulted in an inefficient use of time in mathematics class, and some indicated that the use of cards depicting candy would lead students to only perceive coefficients as natural (rather than rational or real) numbers and fail to grasp the illicitness of combining different letters. These could result in students being unsuccessful regarding the operations on symbols. Hsieh et al. (2017) explored teachers' perspectives on effective mathematics teaching using a questionnaire without specific teaching situations and revealed that Taiwanese teachers focused on conceptual understanding and meaningful learning over skill preparation and performance in evaluation. The findings of Taiwanese experts' perspectives in this research report seem to be in contrast to that of Hsieh et al. This phenomenon may indicate a gap between conceptions aroused without a specific classroom situation and the dominated perspectives revealed in situational practice.

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