FRAMEWORK FOR ANALYSING SECONDARY MATHEMATICS TEACHERS' DEVELOPMENT OF GEOMETRIC REASONING

Shikha Takker, Iresha Ratnayake, Craig Pournara, and Jill Adler

University of the Witwatersrand

Developing geometric reasoning is an important aim of school mathematics. Some influential theoretical frameworks have been used to map progress in geometric reasoning among learners and teachers internationally. We propose an alternate analytical framework, suitable for the context of our work, to analyse secondary mathematics teachers' development of geometric reasoning as they participate in content-focused professional development. We show how the emerging framework uses conceptual categories drawn from the existing literature on geometric reasoning and empirical levels mapped to analyse teachers' developing reasoning. We conclude by anticipating ways in which the framework can be used and extended.

INTRODUCTION

In the South African context, geometry is considered a difficult domain, with low learner and teacher achievement (Bowie, 2013). But the challenges posed by geometry education have also been reported internationally. In a review of research on geometry presented in the PME conferences from 2005 to 2015, Jones and Tzekaki (2016) observed low understanding of geometry subject matter among students and teachers. Research investigating teachers' geometric reasoning has reported teachers' struggles in (a) defining an angle (Silfverberg & Joutsenlahti, 2014), slope or gradient (Mudaly & Moore-Russo, 2011), quadrilaterals using necessary and sufficient conditions (Brunheira & da Ponte, 2016), and in (b) connecting properties with definitions to reason deductively (Chiang & Stacey, 2015).

Our interest is in developing geometric reasoning (henceforth GR) among secondary teachers who identify as being weak in geometry. We define GR as an understanding of the theoretical status of the different geometric attributes (GAs) such as definitions, properties, axioms, and theorems relevant to a geometric object; and organising them in a deductive argument. In the research reported in this paper, we worked with a unique group of teachers, who identified themselves as needing support in basic geometrical ideas, that is, GAs of lines, angles, and triangles. Most of these teachers work in under-resourced schools that cater for learners from relatively poor socio-economic backgrounds, with little access to digital technology for teaching and learning. An initial analysis of teachers' difficulties suggested that they (a) struggled to use the necessary and sufficient conditions when constructing definitions; (b) were aware of the propositions but could not easily recall them correctly or sequence them to solve a problem; and (c) faced difficulty in writing simple numerical proofs. A short research based professional development course (PD) was offered to support these

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teachers' GR. The research reported here contributes to the existing literature on developing GR by disaggregating levels appropriate for analysing teachers' basic understanding of GR. The research questions we address are:

1. What are the key elements of a trajectory for developing GR?
2. How can we use this trajectory to map progress in teachers' GR?

EXISTING FRAMEWORKS ON GEOMETRIC REASONING

Several influential frameworks have been used in geometry research to map learners' and teachers' GR, for example, Van Hiele levels (Crowley, 1987), Fischbein's theory of figural concepts (1993), Kuzniak's Geometric spaces (2018), and de Villiers (1995). The key ideas from the existing frameworks, relevant to the reported research, are discussed here.

In extensive reviews of the literature in geometry education, Jones and Tzekaki (2016) and Sinclair et al. (2016) have noted the use of Van Hiele levels to analyse teachers' and learners' reasoning. Despite criticisms about the discreteness of these levels, they are useful in identifying shifts in attention from visual to discursive reasoning. de Villiers (1995) concluded that most learners entering high school in South Africa are at the early Van Hiele levels and Bowie (2013) reported similar results for pre-service teachers' knowledge of definitions. Van Hiele levels are a hierarchical and fine-grained classification of the processes involved in visualisation noted by Duval, namely, (a) visual identification, (b) visual identification with discursive elements such as properties, and (c) discursive tools supporting visualisation. The link between the visual and discursive is key to developing GR. In the theory of figural concepts, Fischbein (1993) acknowledges the tensions that learners face between the conceptual and figural components of a geometric figure. The "turn" in drawing attention from visual aspects to the conceptual constraints of the geometric figure is significant for developing deductive reasoning. Kuzniak (2018) identifies reasoning based on perception, experimentation, and deduction as key to school geometric space.

The existing frameworks on GR drew our attention to the links between the visual and the discursive, and the need to move from perceptual to deductive reasoning. However, these frameworks were insufficient for mapping (a) the low levels of GR that our teachers brought to the PD setting; and (b) the growth in their GR through the PD. We learnt from the existing frameworks and our work in GR that it is important to:

1. identify a trajectory for developing GR, where progressive differentiation is characterised and supported through relevant tasks;
2. acknowledge the interplay of visual and/or verbal information along with the inferred information about the geometric figure in focus;
3. experience a geometric figure in multiple orientations and complexity;
4. focus on definitions in terms of necessary and sufficient properties; and

5. pay explicit attention to organising relevant properties to formulate an argument.

These learnings form the core of the analytical framework discussed below.

ANALYTICAL FRAMEWORK

We adapted the analytical framework, Mathematical Discourse in Instruction (MDI, see Adler & Ronda, 2015), which was initially developed for algebra and functions, and we also used the aforementioned frameworks on geometric reasoning to develop a working framework for investigating and supporting secondary teachers' GR.

Located within a socio-cultural orientation, MDI assumes that teaching and learning is goal directed, towards an *object of learning* (OoL) which could be a mathematical concept, process, or capability. The OoL is mediated through *exemplification* and accompanying *explanatory talk*. The selection and sequencing of example sets in and across lessons, along with the accompanying tasks, constitutes exemplification. Explanatory talk includes the naming of mathematical objects using symbols and words, and the criteria for their legitimation. These criteria are divided into non-mathematical (visual, positional, or everyday) and mathematical (local, partially, or fully general). *Learner participation* in mathematical activity is promoted through systematically varying examples and the accompanying tasks; linking of representations; and an explicit focus on the accompanying discursive tools.

From the geometric education research, we draw on Duval's (1998) characterisation of GR in terms of its *form* which includes verbal (identified as discursive and conceptual in the existing literature) and visual components, and the *organisation* of propositions (or attributes) to form a logical argument. He suggests that reasoning through propositions distinguishes naïve apprehensions from mathematical behaviour in geometry. GR is defined as understanding the theoretical status of propositions and sequencing them to form arguments to reach conclusions. We designed the PD to promote and support this kind of GR among teachers with explicit attention to its form and organisation, using the mediational tools identified in MDI. The conceptual categories for Promoting Geometric Reasoning (PGR) emerged from the networking of MDI with the topic-specific literature on GR (see Figure 1).

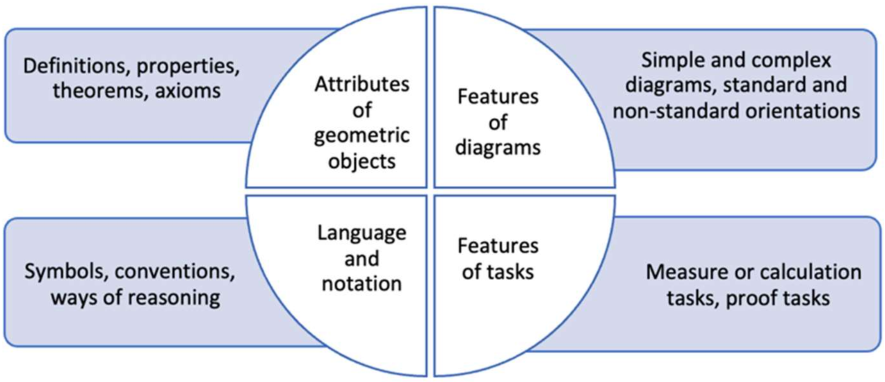


Figure 1: Conceptual categories in PGR

METHODOLOGY

We worked with a group of 10 junior secondary maths teachers (teaching Grades 8 and 9) from eight schools in Johannesburg, South Africa. This group is relatively unique in that the teachers identified themselves as needing support to improve their knowledge of basic geometric ideas, that is, attributes of angles, lines, and triangles. In the South African school curriculum, while geometry learning is compulsory, it has not received as much attention in teaching and research as arithmetic, algebra, and functions.

We designed a short PD course dealing with lines, angles, and triangles. Due to the COVID-19 pandemic, the implementation involved five online and four in-person sessions over three months in mid-2021. The course began with a pre-assessment, with simpler tasks, to identify teachers' challenges. We identified that teachers struggled to use relevant GAs to solve problems and formulate an argument. The course tasks were designed to promote progression in complexity and hence demand, for instance, moving from simple figures in standard orientation to complex figures in multiple orientations, and from numerical measures to proof tasks. Teachers worked on these tasks individually and in groups during the course. We use 10 teachers' responses to nineteen tasks of which six tasks were offered in the form of pre-assessment, ten during the course, and three towards the end to map their developing GR.

RESULTS

The results section is organised around the two research questions and the use of the analytical framework to capture teachers' developing reasoning.

Elements of our framework on geometric reasoning

We argue that three elements are important in mapping and promoting GR. These are (a) identification of relevant GA based on how the given information is interpreted; (b) the accuracy of mathematical statements and reasons, and their structure as a mathematical argument; and (c) connections that are made between the given verbal and visual information. We explain these elements using a task (see Figure 2) and its accompanying explanation.

You will use algebra to answer the questions.

a) Determine the value of x. Give reasons.
b) Determine the value of y. Give reasons.
c) Fill in the sizes of all the 8 angles on the diagram.

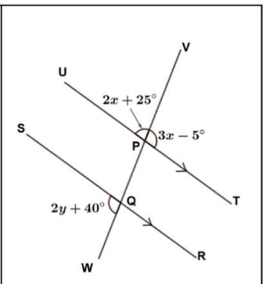


Figure 2: Algebra Task (T19)

The geometric figure includes a pair of parallel lines intersected by a transversal. Three angle measures are given as algebraic expressions. To begin solving the task, angles formed by the straight line UPT or alternate angles can be used. The properties of corresponding angles and vertically opposite angles can be used to find the measures of other angles. An awareness of these properties (angles formed and relations between them) of the geometric figures in focus (parallel lines intersected by a transversal) constitute the GAs of this task. One of the common errors among the teachers was overgeneralising these properties for a pair of non-parallel lines cut by a transversal. Thus, in teachers' justification, it was important to take note of the notation of parallel lines in the diagram, and to distinguish these properties and relations from those between angles formed by a pair of non-parallel lines and a transversal. Reading and interpreting information given in the diagram and as verbal statements is a part of the process of reasoning, we refer to this as "connecting verbal and visual". This task requires formulating and solving algebraic equations to find the measure of each angle along with accompanying justifications using GAs. We refer to forming algebraic equations as "mathematical statements" and the justification as "reasons". A complete argument contains a series of mathematical statements and reasons organised so that they either refer to the attributes of the geometric object or follow logically from one another. In summary, the three elements of GR are:

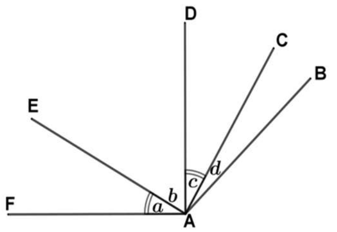
Geometric attributes: Properties of geometric figure in focus

Nature and quality of reasoning: Mathematical statement, reasons, flow of argument

Visual-Verbal connect: Relation between figural and verbal aspects

Framework to map teachers' developing geometric reasoning

We analysed teachers' responses to the tasks posed during the course using the three elements of GR. Analysis of two teachers' responses is presented in Table 1.

<table>
<tr>
<td>T7: In the given diagram, angles a and c are equal, b is twice a and d is half of c. All the angles add up to 135°.
a) Find the values of a, b, c, and d.
b) Name all the right angles in the figure.

</td>
<td>Task analysis
Geometric figure with angles in different orientation.
Relation between angle measure is given as verbal statements.
Visual supports the position of angles.</td>
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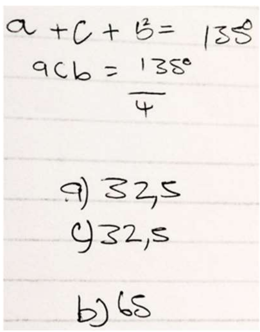
<tr>
<td>Ben's response

</td>
<td>Response analysis
Identifies angles but misses relation between angle measures.
Difficulty in connecting verbal and visual information.
Incorrect mathematical statement with no reasons.</td>
</tr>
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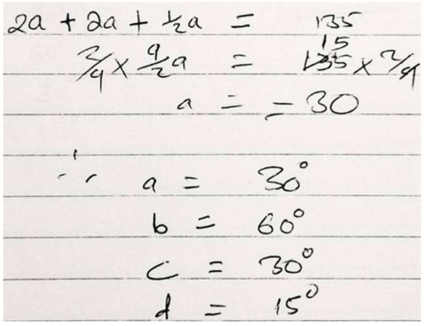
Adam's response		Identifies all relevant GA based on given information. Uses given verbal and visual information to infer GA. Formulates correct mathematical statements without explicit reasons.

Table 1: Analysis of two teachers' responses to T7

Since our interest is in tracking development of teachers' GR, we need a framework to capture levels in their reasoning. In Table 2, we present the analytical framework with progressive levels to differentiate teachers' developing reasoning.

Geometric attributes	*Nature and quality of reasoning*	*Visual-Verbal connect*
L1: Does not identify relevant geometric attributes. L2: Identify some of the relevant GA based on given information. L3: Identify all relevant GA based on given information. L4: Use relevant GA to solve the task.	L1: Does not formulate mathematical statements (MS). L2: Formulates incorrect MS. L3: Formulates correct MS. L4: Partial reasoning using MS and reasons. L5: Complete reasoning with MS and reasons. L6: Uses MS and reasons in a deductive argument.	L1: Uses only visual clues to identify GA. L2: Struggles to connect visual and verbal information. L3: Uses given verbal information and interprets visual notations to infer GA.

Table 2: Analytical framework - Levels in Geometric Reasoning

Using the analytical framework, we map levels in two teachers' (Adam and Tony) developing GR. Table 3 is a summary of the levels for numerical measure, algebraic measure, and proof tasks in early (T1, T7, T8) and late PD sessions (T14, T17, T18).

	Numerical measure		Algebraic measure		Proof	
	T1	T17	T7	T18	T8	T14
Adam	L2,3,3	L4,5,3	L2-3,3,2	L4,5,3	L3,3,3	L4,6,3
Tony	L2,0,2	L4,4,3	L2-3,3,2	L3,4,3	L2,4,2	L4,3,3

Table 3: Summary of two teachers' responses to six tasks

Table 3 shows noticeable changes in teachers' GR. The qualitative changes in teachers' reasoning become evident through the description of these levels. We are currently in the process of investigating the progress in each teacher's reasoning.

CONCLUSIONS AND DISCUSSION

The paper proposes an analytical framework to analyse secondary teachers' development of GR, who are weak in their geometry knowledge. The analytical framework for mapping the development of teachers' GR has been developed using the existing literature and empirical work with teachers. The categories of GR are informed by the research literature in geometry education. The descriptive levels within each category are empirically derived from teachers' responses to tasks. While the levels might vary depending on the contexts where the framework is used, the potential of the framework lies in offering conceptual categories, drawn from a synthesis of research on GR, which are fairly generalisable.

At the beginning of the paper, we raised two research questions about the development of a framework and its use in capturing developing teachers' GR. We have shown that such a framework can be developed using the existing literature and empirical work with teachers, with descriptive levels of progressive GR. Further, we have indicated how the framework can be used to map teachers' changing GR. We anticipate that this framework can be used to analyse learners' and teachers' reasoning for complex tasks and for mapping development of GR.

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References

Adler, J., & Ronda, E. (2015). A framework for describing mathematics discourse in instruction and interpreting differences in teaching. *African Journal of Research in Mathematics, Science and Technology Education, 19*(3), 237-254.

Bowie, L. (2013). T*he constitution of school geometry in the Mathematics National Curriculum Statement and two Grade 10 geometry textbooks in South Africa*. PhD Manuscript. University of the Witwatersrand, Johannesburg.

Brunheira, L., & da Ponte, J. P. (2016). Prospective teachers work on defining quadrilaterals through an exploratory approach. *Didactica Mathematicae, 38*, 33-56.

Chiang, P. C., & Stacey, K. (2015). Geometric concepts of two-dimensional shapes by primary school teachers in Taiwan. In K. Beswick, T. Muir, & J. Fielding-Wells (Eds.), *Proceedings of PME39* (Vol. 2, pp. 161-168). PME.

Crowley, M. L. (1987). The van Hiele model of the development of geometric thought. *Learning and teaching geometry*, K-12, 1-16.

de Villiers, M. (1995). The handling of geometry definitions in school textbooks. *Pythagoras, 38*, 3-4.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana, & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century. An ICMI study* (pp. 37-52). Kluwer.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics. 24*(2), 139-162.

Jones, K., & Tzekaki, M. (2016). Research on the teaching and learning of geometry. In A. Gutiérrez, G. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education: The journey continues* (pp. 109-149). Sense.

Kuzniak, A. (2018). Thinking about the teaching of geometry through the lens of the theory of Geometric Working Spaces. In P. Herbst et al. (Eds.), *International perspectives on the teaching and learning of geometry in secondary schools,* ICME-13 Monographs (pp. 5-21). Springer.

Mudaly, V., & Moore-Russo, D. (2011). South African teachers' conceptualisations of gradient: a study of historically disadvantaged teachers in an Advanced Certificate in Education programme. *Pythagoras, 32*(1), 1-8.

Silfverberg, H., & Joutsenlahti, J. (2014). Prospective teachers' conceptions about a plane angle and the context dependency of the conceptions. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME38 and PME-NA36* (Vol. 5, pp. 185-192). PME.

Sinclair, N., Bussi, M. G. B., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: An ICME-13 survey team report. *ZDM-Mathematics Education*, *48*(5), 691-719.