LINEAR ALGEBRA PROOFS: WAYS OF UNDERSTANDING AND WAYS OF THINKING IN THE FORMAL WORLD

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Research on teaching and learning of proofs in linear algebra is scarce. To help fill this gap, we interviewed six students taking a second linear algebra course and examined some of their coursework that they handed in. In this paper, we examined students' ways of understanding and ways of thinking (Harel, 2008) in the formal world (Tall, 2008) of mathematical thinking from their statements as they unpacked a particular linear algebra proof. The results show that students were able to unpack and adjust the proof formally in a second course and reacted positively.

BACKGROUND AND THEORETICAL PERSPECTIVES

Linear algebra is an important topic for many mathematics majors. In a survey paper by Stewart, Andrews-Larson, and Zandieh (2019), the authors summarized some advances in many areas of linear algebra education (e.g., span, linear independence, eigenvectors, and eigenvalues). These studies highlight students' thought processes and difficulties while making sense of these concepts. The authors also identified areas needing more research and revealed some gaps in the literature. For example, research on how students make sense of linear algebra proofs is scarce. Research on topics in second courses of linear algebra, which contain more abstract content, is also desperately needed. The Linear Algebra Curriculum Study Group (LACSG) recommended that "at least one second course in matrix theory/linear algebra should be a high priority for every mathematics curriculum" (Carlson, Johnson, Lay, & Porter, 1993, p. 45). The LACSG 2.0 recommends that mathematics departments offer a variety of second courses (e.g., numerical linear algebra) and include wider topics (Stewart et al., 2022).

Recognizing the wealth of studies in proof in mathematics education literature, in this paper, we focus our attention explicitly on linear algebra proofs (e.g., Stewart & Thomas, 2019; Britton & Henderson, 2009; Hannah, 2017; Uhlig, 2002; Malek & Movshovitz-Hadar, 2011). Stewart and Thomas (2019) aimed to uncover linear algebra students' perceptions of proofs in a first course. The results revealed that many students expressed their need for understanding. Both Hannah (2017) and Britton and Henderson (2009) agreed that the number of new definitions which linear algebra students must learn to begin writing proofs is overwhelming and makes learning proofs more difficult. Malek and Movshovitz- Hadar (2011) employed one-on-one workshops to determine the effect of using their Transparent Pseudo Proofs (TPPs) in teaching first-year linear algebra proofs. Their results showed that, for non-algorithmic proofs, students who learned using the TPPs wrote more in-depth and satisfactory answers

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than students who learned proofs traditionally. For algorithmic proofs, both groups of students performed equally. Likewise, Uhlig (2002) developed a novel approach compared to the traditional Definition, Lemma, Proof, Theorem, Proof, Corollary (DLPTPC) to teach linear algebra proofs. His technique includes asking the following questions: "What happens if? Why does it happen? How do different cases occur? What is true here?" (p. 338). He believed after exploring these questions deeply for one specific subject, we can collect the gained knowledge in 'Theorems'. "Such a WWHWT sequence of presentation quickly leads students to understand, construct, reason through, enjoy, and actually demand 'salient point' type proofs" (p. 338).

As part of the framework of three worlds of mathematical thinking, Tall (2008) asserted that the formal world of mathematical thinking, which is based on formal definitions and proofs, "reverses the sequence of construction of meaning from definitions based on known objects to formal concepts based on set theoretical definitions" (p. 7). Harel (2008) introduced the notion of a mental act as actions such as interpreting, conjecturing, proving, justifying, and problem solving, which are not necessarily unique to mathematics. Harel (2008) also defined the notion of a way of understanding as "a particular cognitive product of a mental act carried out by an individual" (p. 269), and a way of thinking as "a cognitive characteristic of a mental act" (p. 269). In Harel's (2008, p. 269) view:

> when analyzing students' mathematical behavior in terms of ways of understanding and ways of thinking, one begins with, and fixes, a mental act under consideration, looks at a class of its products (i.e., ways of understanding associated with it), and attempts to determine common cognitive properties among these ways of understanding. Any property found is a way of thinking associated with the mental act.

Harel asserts that the ability to reason abstractly, generalize, structure, visualize, and reason logically comes under the umbrella of ways of thinking. In terms of proofs, Harel (2008) claims that many students depend on the authority of the teacher or the textbook, namely the "*authoritative proof scheme"* (p. 271), others may rely on examples and visual tools, namely "*empirical proof scheme"* (p. 271). In his view, "proof schemes are ways of thinking associated with the proving act" (p. 271), and a proof is a way of understanding. Employing the above theories, the overarching research question for this project is: What are the ways of understanding and ways of thinking necessary for grasping linear algebra proofs in the formal world?

METHOD

This case study is part of a larger study on linear algebra proofs. The first named author was teaching a second course which was highly theoretical and proof-based, and selected the textbook, *Linear Algebra Done Right* by Sheldon Axler (2015) for this course. Abstract Linear Algebra course is the only second course in linear algebra offered at this mathematics department. The course is also slash-listed, meaning that graduate students can also take it since many do not have an adequate background in linear algebra and often benefit from taking this course. A second course in linear algebra usually attracts mathematics majors primarily. However, because of the increasing importance of linear algebra in business and industry, some computer science, meteorology, and physics majors (to name a few) also take this course. All students had taken a first course in linear algebra and at least one more advanced course, such as abstract algebra or analysis.

Theorem	Theorem Description	Proof
Pythagorean (p. 170)	Suppose u and v are orthogonal vectors in V. Then $\|\|u+v\|\|^2 = \|\|u\|\|^2 + \|\|v\|\|^2$.	Proof We have $\\|u+v\\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle = \\|u\\|^2 + \\|v\\|^2$, as desired.
Complex Spectral (p. 218)	Suppose **F** = C and $T \in L(V)$. Then the following are equivalent: (a) *T* is normal. (b) *V* has an orthonormal basis consisting of eigenvectors of *T*. (c) *T* has a diagonal matrix with respect to some orthonormal basis of *V*.	Proof First suppose (c) holds, so T has a diagonal matrix with respect to some orthonormal basis of V. The matrix of T^* (with respect to the same basis) is obtained by taking the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix. Any two diagonal matrices commute; thus T commutes with T^*, which means that T is normal. In other words, (a) holds. Now suppose (a) holds, so T is normal. By Schur's Theorem (6.38), there is an orthonormal basis $e_1, \dots, e_n$ of V with respect to which T has an upper-triangular matrix. Thus we can write **7.25** $\mathcal{M}(T, (e_1, \dots, e_n)) = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix}$. We will show that this matrix is actually a diagonal matrix. We see from the matrix above that $\\|Te_1\\|^2 = \lvert a_{1,1}\rvert^2$ and $\\|T^*e_1\\|^2 = \lvert a_{1,1}\rvert^2 + \lvert a_{1,2}\rvert^2 + \dots + \lvert a_{1,n}\rvert^2$. Because T is normal, $\\|Te_1\\| = \\|T^*e_1\\|$ (see 7.20). Thus the two equations above imply that all entries in the first row of the matrix in 7.25, except possibly the first entry $a_{1,1}$, equal 0. Now from 7.25 we see that $\\|Te_2\\|^2 = \lvert a_{2,2}\rvert^2$ (because $a_{1,2} = 0$, as we showed in the paragraph above) and $\\|T^*e_2\\|^2 = \lvert a_{2,2}\rvert^2 + \lvert a_{2,3}\rvert^2 + \dots + \lvert a_{2,n}\rvert^2$. Because T is normal, $\\|Te_2\\| = \\|T^*e_2\\|$. Thus the two equations above imply that all entries in the second row of the matrix in 7.25, except possibly the diagonal entry $a_{2,2}$, equal 0. Continuing in this fashion, we see that all the nondiagonal entries in the matrix 7.25 equal 0. Thus (c) holds. ■

Table 1: The Pythagorean and Complex Spectral theorems and proofs (Axler, 2015).

The course covered the following topics: Vector spaces and their properties (including special Vector Spaces such as Isomorphic Vector Spaces and Invertibility), subspaces, span, and linear independence, bases, dimension, linear maps, polynomials, eigenvalues, eigenvectors, and invariant subspaces, inner product spaces/ operators on inner product spaces (The Spectral Theorem, Self-Adjoint, and Normal Operators, etc.), and trace and determinant. The course was taught as a mixture of lectures and tasks assigned in groups. The lecturer (first named author) engaged the students in a variety of activities, including evaluating proofs for clarity, elegance, and other criteria. On occasions, students were given pieces of a proof on paper to reassemble. Students also came to the front of the class and presented their own proofs or explained an existing one. Some homework assignments included unpacking a proof in their own words and sometimes coming up with different proofs and presenting them to the class. The interviews with six volunteers from this course took about 40-45 minutes. They were audio-recorded and later transcribed. A sample of the interview questions was: Which of the following proofs are convincing to you and why (Pythagorean theorem; Gram-Schmidt procedure; Characterization of Isometries, Complex and Real Spectral Theorems)? What is the purpose of the proofs in linear algebra? Describe the nature of the proofs in linear algebra. Is there a difference between linear algebra proofs and abstract algebra or real analysis proofs? How can we best teach linear algebra proof to enhance your learning experiences? Open coding by Strauss and Corbin (1998) was performed to analyse the data. In this paper, we analyse students' responses to the question of which of the given five proofs was most convincing and briefly show a glimpse into students' responses on the nature of linear algebra proofs. We also examined students' responses to a homework that asked them to: Study the proof of the Complex Spectral Theorem and fill out the gaps (missing steps or theorems).

RESULTS

Five out of six students mentioned that the proof of the Pythagorean Theorem (see Table 1) was the most convincing. Their common reason was that it "follows from definition", "used properties of inner products", "really simple and easy to understand", "no words only symbols", "the proof looks clean", "it's familiar. You know what the start and end are going to be". For example, Student 1 (S1) said:

> S1: I felt like it followed directly from the definition; only needed like one or two definitions to work through that proof, that's what made it more convincing to me, no words, all just equals, equals, equals, which is a pretty clear contrast from spectral theorems, where it's a lot of explaining.

Student 4 (S4) found the proof for the Gram-Schmidt theorem more convincing. Among the five proofs presented, students also made remarks about the Complex Spectral Theorem and found the proof convincing.

Normally, to prove three equivalent statements, we show (a) implies (b) implies (c) implies (a). To prove the Complex Spectral Theorem, Axler (2015) stated that the equivalence of (b) and (c) is easy and only focused on proving (c) implies (a) and (a) implies (c). To prove (c) implies (a), following the definitions, the proof naturally emerged (Table 1). However, the other direction, (a) implies (c), required more work. Student 1 (S1) did not hand in his homework. During the interview, he mentioned: "my understanding of them was not as great as it could have been… it's not that I distrusted, didn't trust the proofs, but I was that it was less like obvious or implicit, I guess".

Student 2 (S2) seemed to find the proof complete and wrote, "Are there any holes in this proof…? It looks pretty complete to me." However, during the interview, when she was shown this proof again, she showed some concerns:

> S2: I also think is a little confusing. I get lost personally where we show that the matrix is the diagonal matrix. I think that that's something I understand from experience with other, like, I understand why that's true, but not because of the way that the proof is presented here.

It seems that S2 had some conflicts with the way proof was written. We noticed that Student 3 (S3) translated everything symbolically for the proof of (c) implies to (a) (see figure 1). The theorem and its proof appeared after all the definitions and their symbolic representations were established and explained in the textbook prior to the proof. Here is the statement of the main definition: “An operator on an inner product space is called normal if it commutes its adjoint” (Axler, 2015, p. 212). Namely, T is normal if $TT^{*}=T^{*}T$.

During the interview, Student 3 (S3) mentioned:

> S3: I like his complex spectral. I think the only reason why I can read through it so quickly now is just because I have it practically memorized. But I like the way he goes through it. It’s very satisfying proof because it uses these two not exactly like, these two different ways of looking at this transformation or, and it provides a lot of, I guess, inside of like really why that ends up working side of, I’d almost say that this one my second favourite one of the bunch. It uses Schur’s theorem, which isn’t super intuitive, right off the top of your head.

Student 4 suggested some changes to (a) implies (c) part of the proof to make it clearer. She mentioned: “It would have been better to reaffirm that T^{*} is conjugate transpose matrix of T. She also wrote: “It would have been nice to see an ‘updated’ M(T) after…so that we can visualize that”. During the interview, she said:

> S4: I really like the arguments that’s given in the complex spectral theorem proof, um, as to particularly why there is a diagonal matrix with respect to the Schur’s, the normal basis that I, I found very clever but also relatively easy to follow. Um, and so I was, um, I guess that’s the main part of the proof. The proof of a and b is it’s straightforward. But then that last, the last part of the proof, I, I think I was able to follow it and, and understand how we got the diagonal matrix based on, on those basic assumptions. I thought that was pretty cool.

She added that “I’m better at remembering like the symbols that go with it because the words to me are easier to put on. So, I like ones with symbols and words”. In some ways, Student 5 went ahead and performed some of the ideas that Student 4 suggested (figure 1, RHS). He unpacked part of Axler’s (c) implies (a) section by displaying the matrices. He showed both M(T) and $M(T^{*})$ matrices which were similar to the work shown by S3 (LHS), who wrote: $M(T)\,M(T^{*}) = M(T^{*})\,M(T)$. Student 5 also tried to visualize the vectors by expanding them. He did not make any comments about this proof during his interview. Similarly, Student 6 (S6) wrote and included all the definitions and theorems that were mentioned but not shown in the text.

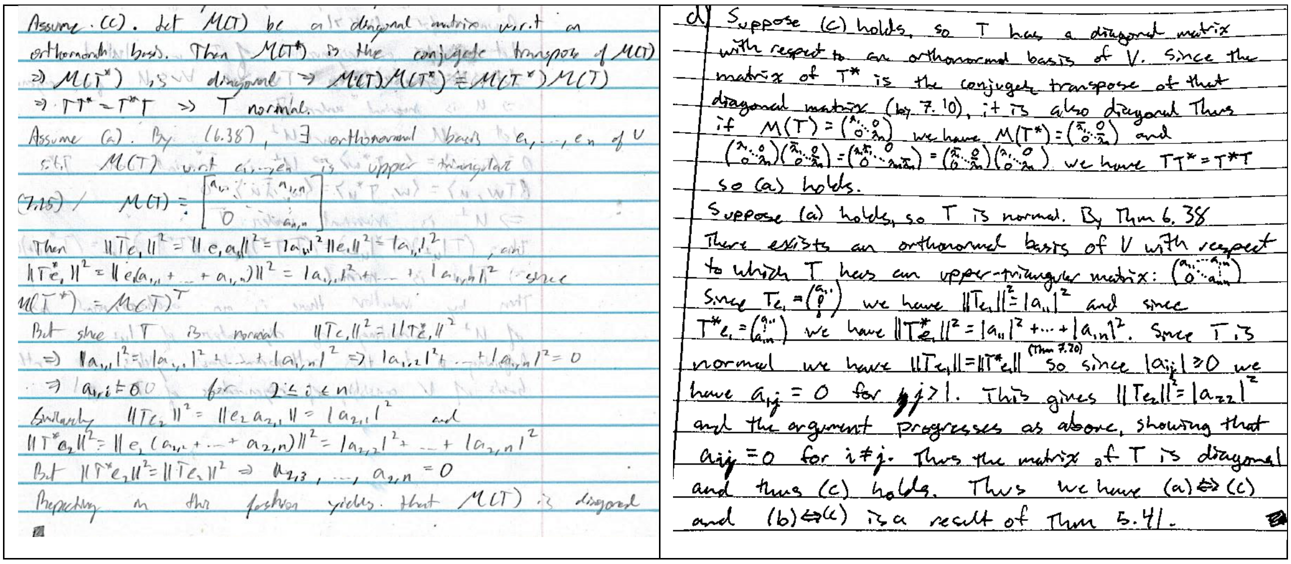

Figure 1: S3 (LHS) and S5's (RHS) works on Complex Spectral Theorem.

The analysis of the data also revealed that students believed that linear algebra proofs are different from other pure mathematics subjects such as analysis, algebra, and topology. The most common themes among their responses were: Connections to other concepts, many definitions, and theorems, unique, self-contained subjects and definitions, and conceptually difficult. For example, Student 3 noted the structure of linear algebra proofs and the ability to progressively reach the destination without worrying about small things along the way.

S3: It's like a lot of structure with a lot of linear algebra stuff. I enjoyed like being able to really not have to worry about the fact that or not having to worry about any type of convergence or doing epsilon delta proofs like you would an analysis. They'd get kind of messy, and you're just trying to almost like the little, the little thing that makes everything fall. It didn't feel the same with linear algebra. It felt much more like you're progressively getting to your destination rather than how can I find the one little key that or one little like modification to this Delta or Epsilon to make this work.

DISCUSSION AND CONCLUDING REMARKS

Results of this small case study revealed that students' ways of understanding and ways of thinking of dealing with the proof of the Complex Spectral Theorem had some common characteristics. All four students (S3, S4, S5, & S6) were content in working in the formal world and presented their arguments in the most general form. Their responses during the interviews indicated that they enjoyed the proof, and it was satisfying to learn it. There was no sign that they accepted the proof because they blindly trusted the textbook. However, both S1 and S2 appeared to place the responsibility of understanding a proof, in this case, the act of analysing a proof, on themselves. S1 mentioned that he still trusted the proofs even though his "understanding of them was not as great as it could have been". S2 also claimed that she did not find any holes in the proof, but she later said that it confused her. These conflicts seem to indicate that they simply trust the proof and they do not consider the book or teacher at fault for their confusion. According to Harel (2008, p. 286):

> …a way of understanding should not be treated by teachers as an absolute universal entity shared by all students, for it is inevitable that each individual student is likely to process an idiosyncratic way of understanding that depends on her or his experience and background.

Results also showed that the proof of the Pythagorean Theorem was most convincing for most students in this study. Their ways of thinking on this were almost identical. This was unexpected from the researchers' standpoint since we spent a considerable amount of time on the Complex Spectral Theorem in class, and in some ways, it is the climax of many fun previous results.

Although the literature reveals some insights on students' thought processes in first courses in linear algebra, studies on second courses and unravelling students' ways of understanding and ways of thinking in the formal world need more careful attention.

While employing Harel's (2008) framework in this study showed a glimpse into students' ways of understanding and ways of thinking, it is not the aim of this small study to make any concrete conclusions. From the teaching point of view, we believe that evaluating and constantly reviewing proofs in groups during the class helped the students to think critically about any written proofs. In Harel's view, "the goal of the teacher should be to promote interactions among students so that their necessarily different ways of understanding become compatible with each other and with that of the mathematical community" (p. 286). As Tall (2008, p. 15) asserts:

> …as mathematicians we begin to appreciate the purity and logic of the formal approach, but as human beings we should recognise the cognitive journey through embodiment and symbolism that enabled us to reach this viewpoint and helps us sustain it.

As we continue this new terrain of research, we plan to develop the theoretical framework further. Our future work will also include collecting more data from students and making recommendations for teaching linear algebra proof in seconds courses.

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