STUDENTS' KNOWLEDGE ABOUT PROOF AND HANDLING PROOF

Femke Sporn, Daniel Sommerhoff, Aiso Heinze

Leibniz Institute for Science and Mathematics Education, Kiel, Germany

Research has highlighted that students' have problems regarding mathematical proof. In part, these have been connected to deficits in their knowledge about proof and handling proof. However, empirical data on students' knowledge about proof and handling proof throughout secondary education is so far missing. Further, it is unclear if there is a connection between concept- and action-oriented knowledge about proof and handling proof. To address these gaps, an empirical study was conducted, investigating the knowledge about proof and handling proof concerning proof principles of N = 456 students in grade 8 to 11. Results indicate that (i) only concept-oriented knowledge significantly increases throughout secondary education and (ii) there is only little relation between concept- and action-oriented knowledge.

INTRODUCTION

Proofs play a central role in mathematics as a proving scientific discipline (Mariotti, 2006) and are thus an important part of mathematics education. Although there is a particular focus on proofs at the university level, proof and handling proofs are also established and important goals in secondary mathematics education (e.g., CCSSI, 2010). Thus, learners in secondary school are expected to build up an individual understanding of proof (cf. Sporn et al., 2021) throughout secondary education. One aspect of this individual understanding of proof is the availability of sufficient *knowledge about proof and handling proof*. Such knowledge about proof and handling proof has been investigated in prior research and includes, for example, knowledge about valid and invalid arguments in mathematical proofs or about different proof methods. In particular, it was found that learners often have deficits regarding knowledge about proof and handling proof, which was identified as a possible explanation for difficulties in handling proof (e.g., Harel & Sowder, 1998; Healy & Hoyles, 2000; Heinze & Reiss, 2003; Reiss et al., 2001). Accordingly, this knowledge plays an important role in the context of proof and is of great interest for research. However, empirical data on students' knowledge about proof and handling proof throughout secondary education is so far missing. In this context, knowledge about proof and handling proof can be focused (i) in relation to proofs or generic action situations in general without reference to a specific action situation (cf. Andersen, 2018), so-called *concept-oriented knowledge*, and (ii) in relation to a specific mathematical action situation, for example the construction or validation of a specific proof (cf. Healy & Hoyles, 1998; Heinze & Reiss, 2003), so-called *action-oriented knowledge* (cf. Sporn et al., 2021). Furthermore, it is not clear how concept- and action-oriented knowledge about proof and handling proof are related. While it makes sense

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 27-34). PME.

that both are closely related and go hand in hand, it is also conceivable that concept-oriented knowledge does not automatically transfer to action-oriented knowledge and vice versa.

Data regarding both of these aspects, that is knowledge about proof and handling proof throughout secondary education and the relation of concept- and action-oriented knowledge, are relevant to better understand how students' reported difficulties with proof can be addressed and how specific learning opportunities in secondary education can be structured and designed. This paper thus presents results of an empirical study examining concept- and action-oriented knowledge about proof and handling proof by learners in grade 8 to 11 from German secondary schools.

THEORETICAL BACKGROUND

Mathematical Proof in Secondary Education

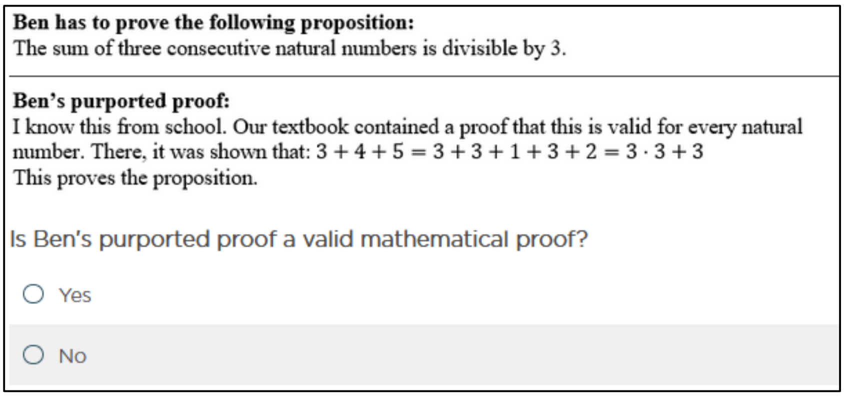
That proofs and proof-related activities (e.g., constructing or validating proofs) play a role in secondary mathematics education is reflected in their implementation in standard documents worldwide (e.g., CCSSI, 2010). Research on mathematical proof has repeatedly shown that learners at different stages of their mathematics education often have severe problems with mathematical proof. For example, Healy & Hoyles (2000) showed that even high attaining students have problems to correctly validate proofs and suggest that difficulties can (at least partially) be explained by different proof schemes (Harel & Sowder, 1998). Insufficient and differing knowledge about acceptance criteria for validating mathematical proofs (Sommerhoff & Ufer, 2019) and insufficient methodological knowledge (Heinze & Reiss, 2003) were highlighted as further explanations for learners' difficulties in this context. Overall, studies suggest that learners form a certain individual understanding of proof throughout their school mathematics education, which in specific ways can both facilitate and impede an exploration of mathematical proof (Sporn et al., 2021).

Knowledge about Proof and Handling Proof

Varying socio-mathematical norms in different mathematical communities and settings can lead to the lack of a universal acceptance of a correct proof (Inglis et al., 2013). Thus, specifying exactly when a general proof is to be considered a correct mathematical proof is highly non-trivial (cf. Sommerhoff & Ufer, 2019). However, from a *disciplinary perspective* on proofs and handling proof (i.e., focusing on the discipline of mathematics as a whole), criteria that an (ideal) mathematical proof must fulfil to be considered as correct can be defined. Such criteria, which are valid independently of a specific proof, can be summarized as *proof principles* (Besides these proof principles, proof methods, proof functions, and proof presentation can also be described from a disciplinary perspective; see Sporn et al., 2021).

From an *individual-psychological perspective* on proofs and handling proof (i.e., focusing on individual learners, cf. Sporn et al., 2021), knowing these proof principles can be seen as an important facet of individuals' *knowledge about proof and handling* *proof*, one aspect of an individual's understanding of proof. For example, Reiss et al. (2001) analyzed secondary students' proofs, suggesting that their knowledge about proof and handling proof concerning proof principles was insufficient, which was an obstacle for their construction of proofs. Heinze und Reiss (2003) give further indications for the importance of knowledge about proof and handling proof concerning proof principles for success in dealing with mathematical proofs by focusing on three important principles that they combine under the umbrella term "methodological knowledge".

While learners' *knowledge about proof and handling proof concerning proof principles* can (at least partially) explain their difficulties with proofs, prior research indicates that it is not only necessary to distinguish between knowledge and non-knowledge. It is additionally needed to distinguish whether (i) this knowledge about proof and handling proof is required in general contexts, meaning without reference to a specific proof and proving activity (e.g., Andersen, 2018) or whether (ii) this knowledge is required in a specific mathematical action situation, for example, in the context of the construction or validation of a specific proof (e.g., Healy & Hoyles, 1998; Heinze & Reiss, 2003). Here we refer to (i) *concept-oriented* and (ii) *action-oriented knowledge about proof and handling proof* (cf. Sporn et al., 2021). For example, concept-oriented knowledge about proof and handling proof concerning proof principles can be assessed using the statement *"Mathematical proofs that use the statement to be proved as a premise are particularly elegant."* and thus without reference to a specific proof and proving activity. In contrast, Figure 1 shows an example item, highlighting how knowledge about proof and handling proof concerning proof principles can be assessed with an action-oriented focus in the context of proof validation.


Ben has to prove the following proposition:
The sum of three consecutive natural numbers is divisible by 3.

Ben's purported proof:
I know this from school. Our textbook contained a proof that this is valid for every natural number. There, it was shown that: $3 + 4 + 5 = 3 + 3 + 1 + 3 + 2 = 3 \cdot 3 + 3$
This proves the proposition.

Is Ben's purported proof a valid mathematical proof?

- Yes
- No

Figure 1: Example item for action-oriented knowledge concerning proof principles.

In order to achieve the proof related goals in secondary education, it appears necessary that learners acquire sufficient knowledge about proof and handling proof (both concept- and action-oriented), otherwise problems may arise (cf. Healy & Hoyles, 1998; Heinze & Reiss, 2003). Further, German secondary education curricula include proof and handling proof in most grades, thus providing corresponding learning opportunities. Assuming that these learning opportunities occur in appropriate quality and quantity and that learners use them properly, students' knowledge about proof and handling proof can be expected to increase throughout their secondary school education.

RESEARCH QUESTIONS

The present paper sets out to gain a better understanding of (i) students' individual knowledge about proof and handling proof throughout secondary school education as well as (ii) about the connection of concept- and action-oriented knowledge about proof and handling proof. For this, a quasi-longitudinal study investigating concept- and action-oriented knowledge about proof and handling proof concerning proof principles of students in grades 8 to 11 is presented. The following research questions were focused:

> (RQ1) How can students' concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles be characterized throughout their secondary school education?
>
> (RQ2) How do students' concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles relate to each other in different grades?

Regarding RQ1, it was expected that students' knowledge concerning proof principles – both concept- and action-oriented – would increase (mostly monotonically) during the course of schooling, as mathematical proof is a learning goal throughout secondary education and according learning opportunities exist throughout multiple grades. Regarding RQ2, it was expected that concept-oriented knowledge about proof and handling proof serve as a basis for action-oriented knowledge about proof and handling proof, resulting in an at least medium correlation between concept- and action-oriented knowledge.

METHOD

To answer these questions, $N = 456$ (183 m, 263 f, 10 d) students in grade 8 to 11 ($N_8 = 139$; $N_9 = 122$; $N_{10} = 72$; $N_{11} = 123$) from eight German secondary schools were surveyed in an online study at the end of the school year 2020/2021. As part of the survey, students were questioned regarding various aspects of their individual understanding of proof and regarding more general information about the participants, for example demographic data. In this paper, we only consider the demographic data as well as data on concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles.

To assess concept-oriented knowledge about proof and handling proof concerning proof principles, students were asked to evaluate 18 statements (see example item above) on a 6-point Likert scale ("Not true at all" to "Totally true"). Each item was evaluated based on the disciplinary perspective on proof and handling proof (i.e., regarding an ideal concept of mathematics/proof). Internal consistency was acceptable with $\alpha = .75$. Action-oriented knowledge about proof and handling proof concerning proof principles was assessed using a task format focusing on the activity of proof validation, based on the tasks used by Healy & Hoyles (2000). Students were presented six purported proofs, all of which contained different errors regarding proof principles. They were asked to judge each purported proof as to whether it is a valid mathematical proof (dichotomous answer). Figure 1 shows an example item with a proof that includes higher authority as an invalid argument. The six purported proofs addressed different mathematical contents, which were known to all participants at the time of the survey. The internal consistency was poor, with $\alpha = .60$. The 18 statements corresponding to concept-oriented knowledge were combined to a mean score (M_{coK}; possible values for each statement ranged from 1 to 6). For action-oriented knowledge, the judgements of the 6 purported proof were combined to a mean score (M_{aoK}; possible values for each proof: 0 or 1). To allow a better comparison with concept-oriented knowledge, M_{aoK} was rescaled to a possible range from 1 to 6. To answer RQ1, we conducted two independent linear regressions with either M_{coK} or M_{aoK} as dependent variables and grade as independent variable. To answer RQ2, correlations between M_{coK} and M_{aoK} were calculated for each grade as well as based on the whole sample.

RESULTS

Table 1 shows descriptive statistics for concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles for each grade. Data indicate that concept-oriented knowledge increases with grade, while action-oriented knowledge remains mostly stable across grades with its lowest value at the end of 10th grade. Overall, the standard deviations for action-oriented knowledge appear to be higher than for concept-oriented knowledge for all grades.

Grade	**Concept-oriented Knowledge**		**Action-oriented Knowledge**	
	M_{coK}	*SD*	M_{aoK}	*SD*
8	3.69	0.58	3.24	1.55
9	3.75	0.59	3.44	1.84
10	3.91	0.65	3.04	1.61
11	3.99	0.67	3.12	1.62

Note: Possible values ranged from 1 to 6.

Table 1: Descriptive statistics for concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles for grade 8 to 11.

Students' Concept-oriented and Action-oriented Knowledge about Proof and Handling Proof Concerning Proof Principles (RQ1)

The linear regression for concept-oriented knowledge showed a significant positive impact of grade ($\beta = .193$, $p < .001$), while the regression for action-oriented knowledge showed an insignificant impact ($\beta = -.048$, $p = .393$). Similarly, the regression model for concept-oriented knowledge is significantly better than a Null-Model ($p_{\text{coK}} < .001$) while the model for action-oriented knowledge does not differ significantly ($p_{\text{aoK}} = .393$). However, the variance explained by grade is rather small in both linear models ($R^2_{\text{coK}} = .037$, $R^2_{\text{aoK}} = .002$).

Relation of Students' Concept-oriented and Action-oriented Knowledge about Proof and Handling Proof Concerning Proof Principles (RQ2)

Correlations between M_{coK} and M_{aoK} in total showed a significant positive, weak correlation (r = .16, p = .004). Table 2 shows corresponding correlations for each grade. Here, only for 9th grade a significant positive, weak correlation was found, while the correlations are insignificant for all other classes.

Grade	**8**	**9**	**10**	**11**	**Total**
$\boldsymbol{r_{\mathrm{Pearson}}}$ **($\boldsymbol{p}$)**	.14 (.190)	**.22 (.044)**	.12 (.399)	.20 (.051)	**.16 (.004)**

Note: Significant results are highlighted in bold.

Table 2: Correlations between concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles

DISCUSSION & OUTLOOK

The present study investigated students' knowledge about proof and handling proof concerning proof principles by learners in grade 8 to 11, distinguishing between concept- and action-oriented foci on this knowledge. Descriptive data reveal medium mean scores for students‘ knowledge about proof and handling proof, both for the concept- and action-oriented focus. While this does not appear too detrimental, the problems pointed out in previous research (e.g., Healy & Hoyles, 2000) suggest that this amount of knowledge is not sufficient to handle proofs sufficiently well. Further, the results show that students' concept-oriented knowledge about proof and handling proof concerning proof principles increases throughout secondary education (RQ1). It thus seems, that the existing learning opportunities during this period have a positive effect and are used by learners at least in some way. However, with an increase of about half a standard deviation in four years of mathematical schooling, this effect does not seem particularly large. Thus, the quality and quantity of learning opportunities regarding mathematical proof need to be further investigated. The expected significant increase of students' action-oriented knowledge about proof and handling proof concerning proof principles throughout secondary education could not be confirmed (RQ1). One reason for this unexpected finding and difference to concept-oriented knowledge may be that learning opportunities for mathematical proof in secondary education do not include sufficient opportunities for students to engage in proofs themselves (Hemmi, 2006) or possibly require a rather imitating style of reasoning (Lithner, 2008). Another reason may be, that the result is an artefact of the items used to measure action-oriented knowledge in this study, which focus on proof validation. While validating proofs as an action situation may be easier than constructing proofs, it may be an activity that occurs less explicit in school and thus participants were less familiar with it.

Although the expected positive relation between concept- and action-oriented knowledge about proof and handling proof concerning proof principles could be confirmed (RQ2), the correlation is quite weak and about the size that would be expected for most types of cognitive variables. The result may indicate that while concept-oriented knowledge about proof and handling proof is available, the corresponding knowledge is not available in specific action situations or can only insufficiently be used. The result can be backed up by an observation by Healy and Hoyles (1998): Students knew in principle that an empirical proof was not sufficient to construct an acceptable proof, that is they had the corresponding concept-oriented knowledge about proof and handling proof. Still, many students constructed empirical proofs when put in a corresponding action situation, as they had no alternative means to construct an acceptable proof themselves. From a research perspective, the fact that concept- and action-oriented knowledge about proof and handling proof only show a weak correlation highlights the relevance of distinguishing between both foci and confirms the assumption by Sporn et al. (2021).

While the results appear plausible and give first empirical data on students' knowledge about proof and handling proof in grade 8 to 11, the presented study also has some limitations that should be considered: First, data refers to a quasi-longitudinal study. While this allows for a good overview, a longitudinal study may help to characterize the individual development of knowledge about proof and handling proof. Second, the low Cronbach's α of the action-oriented knowledge about proof and handling proof may limit the reliability of the results and require future research. However, the scale for action-oriented knowledge about proof and handling proof concerning proof principles includes only 6 items which refer to different mathematical topics and different proof principles, so that a particularly high Cronbach's α was not expected. Third, all interpretations of the results that are connected to learning opportunities rely solely on standards and curricula in Germany. However, we did not gather specific data about the actual learning opportunities of the participants during their secondary education. Future research is needed in this regard, especially focusing on quantity, quality and variance of these opportunity in the different grades and between teachers and schools.

Results show that students acquire an increasing amount of concept-oriented knowledge about proof and handling proof concerning proof principles throughout their secondary education, which should be celebrated. However, results from this study in conjunction with prior research indicating students' difficulties (e.g., Healy & Hoyles, 1998; Heinze & Reiss, 2003) also highlight that existing learning opportunities should be improved so that (i) more concept-oriented knowledge about proof and handling proof is acquired and (ii) also action-oriented knowledge is acquired, a current deficit clearly pointed out by the results. Moreover, comparisons between the examined grades suggest that it may be particularly preferable to provide support between the end of 9th grade and the end of 10th grade to prevent a drop-off in action-oriented knowledge about proof and handling proof at this point.

Overall, results give a first and important empirical impression of students' knowledge about proof and handling proof throughout secondary education. However, future research should more explicitly consider (i) a real (and not quasi-) longitudinal development of students' knowledge, (ii) include further aspects of knowledge about proof and handling proof (such as proof methods, proof functions, and proof presentation), and (iii) also include further aspects that may be relevant in the development in knowledge about proof and handling proof (such as beliefs, and learning orientations). Combining these points will lead to a more comprehensive overview that can be basis for further support and curricular adjustments.

References

Andersen, L. E. (2018). Acceptable gaps in mathematical proofs. *Synthese*, *197*(1), 233–247.

CCSSI. (2010). *Common core state standards for mathematics*. National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, E. Kaput, E. Dubinsky, & T. Dick (Eds.), *Research in collegiate mathematics education* (pp. 234–283). AMS.

Healy, L., & Hoyles, C. (1998). *Justifying and proving in school mathematics: Technical report on the nationwide survey*. Institute of Education, University of London.

Healy, L., & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education*, *21*(4), 396.

Heinze, A., & Reiss, K. (2003). Reasoning and proof: Methodological knowledge as a component of proof competence. In M. A. Mariotti (Ed.), *Proceedings of the third conference of the European society for research in mathematics education* (Vol. 4, pp. 1–10).

Hemmi, K. (2006). *Approaching proof in a community of mathematical practice*. Department of Mathematics, Stockholm University.

Inglis, M., Mejia-Ramos, J. P., Weber, K., & Alcock, L. (2013). On Mathematicians' Different Standards When Evaluating Elementary Proofs. *Topics in Cognitive Science*, *5*(2), 270–282.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, *67*(3), 255–276.

Mariotti, M. A. (2006). Proof and proving in mathematics education. *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future*, 173–204.

Reiss, K., Klieme, E., & Heinze, A. (2001). Prerequisites for the understanding of proofs in the geometry classroom. In M. van de Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 97–104).

Sommerhoff, D., & Ufer, S. (2019). Acceptance criteria for validating mathematical proofs used by school students, university students, and mathematicians in the context of teaching. *ZDM*, *51*(5), 717–730.

Sporn, F., Sommerhoff, D., & Heinze, A. (2021). Beginning university mathematics students' proof understanding. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 102–110).