STUDENTS' EXPLORATION OF TANGIBLE GEOMETRIC MODELS: FOCUS ON SHIFTS OF ATTENTION

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This empirical study applies the analytical apparatus of Mason's shifts of attention theory to investigate why and how using physical models of different scales can facilitate learning of (spatial) geometry. In the presented case study, six high school students learned the properties of icosahedron by constructing and exploring physical models. Shifts in the focus and structures of attention were associated with multimodal perception and collaborative physical actions of students with and through the models. Models of different scales landed students different affordances for exploration, facilitating noticing of invariant scale-free features of a geometric object and influencing the dynamic of student collaboration.

INTRODUCTION AND THEORETICAL FRAMEWORK

According to Goldenberg et al. (1998), geometry is "an ideal intellectual territory within which to perform experiments, develop visually based reasoning styles, learn to search for invariants, and use these and other reasoning styles to spawn constructive arguments" (p. 5). This claim concurs both with Freudenthal's view on geometry as "one of the best opportunities which exist to learn how to mathematize reality" (Freudenthal, 1973, p. 407) and with tenets of embodied design for mathematics instruction (Abrahamson et al., 2020) supporting primacy of students' enactment of conceptually oriented movement forms and gradual formalization of gestures and actions in disciplinary formats. Embodied learning is rooted in an ecological approach in cognitive psychology (Gibson, 1986/2015), capitalizing on organism-environment relations. In particular, Gibson conceived perception as an active, embodied process in which we notice optical invariances of the object under the movement of the source of light, movement of the observer, movement of an observer's head, and manipulations and local transformations of the object itself. Students facing tasks in realistic 3D contexts can be introduced to the language of geometry, its objects and constructions (Doorman et al., 2020). They conduct mathematical modeling of their experiential world and then are invited to use informal strategies (horizontal mathematization) and further develop them into normative forms and practices of mathematics (vertical mathematization) (Gravemeijer, 1998). Several scholars suggested that mathematical modeling of geometric figures should take into account four distinct perceptual systems of the figure(s): (a) as physical navigation of macrospace (objects more than 50 times the size of an individual); (b) as capturing an object in mesospace (0.5 to 50 times); (c) as constructions of small objects in microspace (less than 0.5 times); and (d) as descriptions and manipulations of small objects in microspace (e.g. Herbst et al., 2017). Still, why and how physical models of different scales can facilitate learning of

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(spatial) geometry remains an open question. This empirical study seeks to provide an answer using the analytic apparatus of Mason's shifts of attention theory.

Learning as shifts of attention

Mason (2010) claims that learning is a transformation of attention involving "shifts in the form as well as the focus of attention" (p. 24). Thus, to characterize learning, Mason considers *what* is attended to and *how* the objects are attended to. Per Mason (2008), there are five different forms or *structures of attention.* One may *hold the wholes* without focusing on particularities or *discern details* among the rest of the elements of the attended object. From there, one may *recognize relationships* between discerned elements and even *perceive properties* by actively searching for additional elements fitting the relationship. The ultimate structure of attention is *reasoning based on perceived properties*. The shifts in attention structures are not necessarily sequential, and one may return to *holding the whole* to reassess the situation.

In Mason's works (2008, 2010), these theoretical constructs were suggested for use in teachers' education. In more recent studies, the theory of shifts of attention was applied as an analytical framework to study students' problem-solving efforts (Palatnik & Koichu, 2015) and assess individual changes in children's communication and conceptualization of arithmetical tasks (Voutsina et al., 2019). Palatnik and Sigler (2019) suggested that shifts in form and focus of attention can also be applied to analyzing geometric tasks and activities, particularly when introducing an auxiliary element is necessary. The current study expands the application of shifts of attention as an analytical framework for investigating spatial geometry learning. In this report, the analytical lens of shifts of attention is applied to collaborative geometric activity in which students explore tangible models of a geometric object on different scales.

Research questions

When students study 3D geometrical objects by exploring physical models, which shifts of attention do they experience? What role do physical features of the models (i.e., their relative size and their orientation in space) play in the process of student exploration?

METHOD

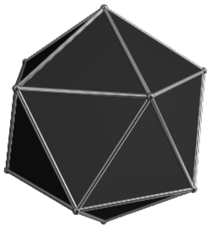
Context

This study is a part of a research project *Learning Geometry as Negotiating Perspectival Complementarities* studying activities that foster conceptually productive discursive and pragmatic tension between differing perspectives on sensorial features of shared displays of geometric objects (Benally et al., 2022). A distinctive feature of the empirical context of the current study is that students explore the same geometric objects at different scales. In one of the tasks, students are given a 2D diagram and written instruction (see Figure 1) to construct an icosahedron—a polyhedron whose exterior is composed of twenty equilateral triangular faces. They have to build relatively small as well as human-scale models using wooden rods and silicone joints.

Once both models are built, students are asked questions concerning the icosahedron's properties, for example, "How many vertices and edges does an icosahedron have?", "How many parallel edges?", "If the icosahedron were standing on its triangular base and filled halfway up with water, what would be the water's surface shape?"

Your team has to construct two three-dimensional models (one large, one small) of a geometric solid, a polyhedron.
The polyhedron has the following properties:

- All the faces are congruent equilateral triangles.
- The same number of edges converges at each vertex.



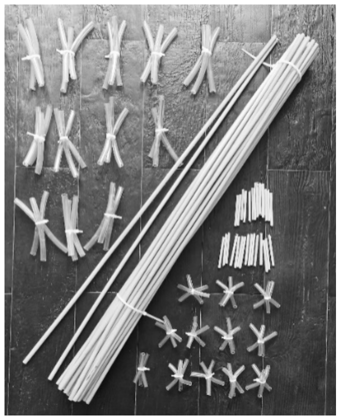


Figure 1: The icosahedron construction task and materials.

Data collection and analysis

The case presented in the paper provides an account of an outdoor implementation of the activity with a group of six tenth-grade students. This activity was a part of an enrichment program for the students at the beginning of their first year in the new high school. This case was chosen from the data collected (14 cases)for two reasons: First, the way this group constructed the models and answered the questions was typical of this activity. Second, the students were more verbal than other groups, making indications of their attention shifts more distinguishable.

The activity was video-audio recorded. To analyze the data, we combined multimodal analysis of students' interactions (Abrahamson et al., 2020) with microgenetic analysis of shifts of attention (Voutsina et al., 2019) in the following way. We prepared a complete transcription of the activity, overlaid with a description of students' actions, gestures, and movements. The resulting protocol was divided into episodes (i.e., construction of the large-scale model, construction of the hand-held model, the answer to the first question, etc.) In each episode, we looked for the indicators of the shifts in *focus* (what is attended to) and *structures of attention* (how it is attended to). Marking the objects directly mentioned in the conversation, the direction of the gestures and gaze (where available) helped us identify the focus of attention. To identify shifts in structures of attention, we, following Gibson's approach, interpreted changes in students' movement in space, manipulations with and local transformations of the object itself (for instance, change in the model's orientation). Particular attention was paid to the actions, gestures, and utterances that preceded students' advancements in the task. At the subsequent analysis stage, we compared how students interacted within the team and with models in different episodes. Due to the page limitation, we focus here on two episodes: finding the number of vertices and the number of edges.

FINDINGS

a b c d

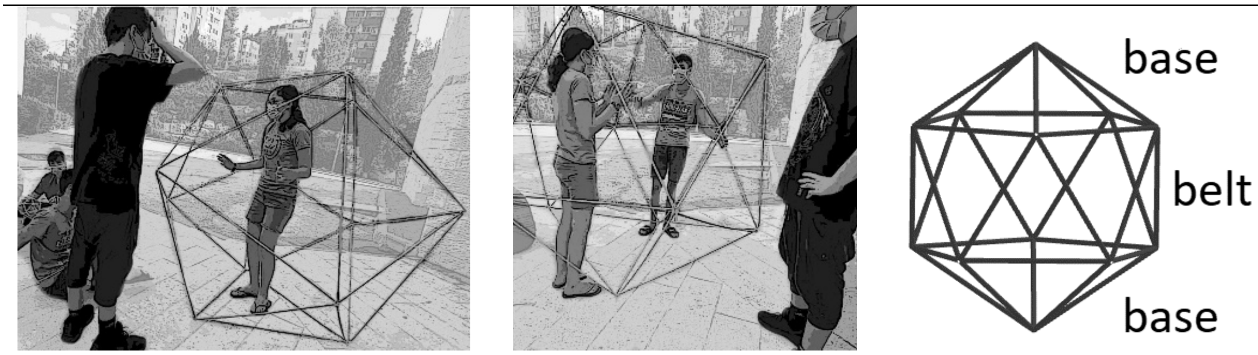
Figure 2: (a) Students discuss a small-scale model resting on a triangular face; (b) the student examines a small-scale model while holding it on a vertex; (c-d) different points of view on the same model.

Having constructed both the large and small models, the students used the small model to find the number of vertices (Participants are referred to by the color of their t-shirts; transcription translated from Hebrew by the author).

Grey	[holds a model on its vertex, starts counting] One, two, three, four, five. [touches an upper vertex, touches two additional vertices]
Black	How many vertices? [Reaches out for the model and touches it]. We already counted (them). It is a number of joints.
Yellow	[takes the model, starts to count by touching silicon joints] One, two.
Grey	Twelve. Times five. Sixty.
Yellow	How (it can be) twelve times five? How (it can be) sixty? [looks at the model].
Grey	[tries to take a model from Yellow] Ah, vertices… Twelve. Put it (the model) like this [tries to orient the model on the vertex]
Yellow	Give it (the model) to me for a moment. I know what I'm doing [takes the model away from Gray].
Grey	But, but…It's… Ohhh…
Yellow	[starts counting the joints from two facing her]. One, two. [continues counting] One, two, three…ten. It is twelve! [puts a model on a floor to write an answer]
Grey	[takes a model and tilts it on a vertex] Look [addressing Yellow] at it this way. [Starts counting from a lower basis] One, two…
Yellow	There are twelve!!!

In this episode, Gray and Yellow answer the question by counting the silicon joints of the small-scale model. The small size of the model allowed the students to group around it. The model became the *focus* of their joint attention. The model's size also enables students to simultaneously grasp most of its features, *holding the whole*. Both students physically touched the joints while counting vertices which helped them to *discern* these relevant *details* of the model (separating it from edges and faces). Both students were successful in counting 12 vertices. However, the ways of counting were qualitatively different. By orienting the model in a particular way (Figure 2 b, d), Grey *recognized a relationship* between several groups of vertices of the icosahedron. Grey's persistence to explain his point of view and his frustration when he was denied the explanation can indicate that he *perceived* this orientation as *the property* of an icosahedron, and it served as the *base for his reasoning*. Yellow was also successful in her attempt to count the vertices, which she separated into two groups of two and ten and did not see the value in the alternative orientation of the model in space. The next episode will demonstrate that Gray's unappreciated know-how of holding an icosahedron on its vertex will help answer how many edges an icosahedron has while Yellow's attempt will fail.

Yellow	How many edges are there?
Black	Okay, that's tricky because they're shared. (i.e., each edge is shared by two triangles).
Blue	I'll put a finger [on the first edge, to Yellow help her monitor the count].
Orange	You just count the sticks.
Yellow	I'll go to the big one (i.e., the large-scale model).
Black	The big one is just nicer.



a b c

Figure 3: (a) students' problem-solving attempt inside and outside a human-scale model (standing on a triangular face); (b) having tilted the structure onto a vertex, the students soon arrive at a critical breakthrough; (c) partition of an icosahedron.

Three of the six students rose and walked over to the nearby large-scale model. This larger model is advantageous for counting because its edges are more perceptually distinct. A model's greater size, while availing perceptual acuteness, may come with a price that its figural elements in question (the to-be-counted edges) are never all in one's arm's reach—you cannot directly touch or gesture to each edge as you tally it. Thus, using structures of attention terminology, greater size afforded students easier *holding the wholes* while impairing *discerning detail* by a sense of touch. To overcome this, Yellow entered inside the model, where all edges are within her reach (Figure 3a).

Still, when you are inside an object, part of it is always behind you, so you might lose track of your count. Indeed, Yellow's initial attempts to count failed.

The excerpt below demonstrates two phases of student solution. During the first phase, Yellow attempted to use some of the icosahedron properties that the team discovered during construction, yet again she failed to develop a systematic approach. During the second, Grey received an opportunity to demonstrate that his strategy of putting the model on a vertex has an advantage. In seconds, his physical action facilitated the restructuring of Yellow's attention leading her to a correct solution.

Yellow	There are five from each vertex. One should be subtracted. Then there are four. Two should be subtracted here. It's three. It doesn't work that way…3, 4, 5 [sits on the floor, inside the model, frustrated]. I can't count this. [Stands up]. How many sticks did we use [during the construction stage]? Three and another three, and another three, and another three, it's 12…
Grey	Let's do it as we did with (inaudible) [Stands up]
Yellow	[referring to triangular faces] …another three, 15, another three…
Black	We need a formula for this…
Gray	I'm tilting it. [starts tilting the model]
Yellow	No, no, no, no! eighteen…No! Why?
Gray	To make it like this (standing on the vertex). It will be easier to count like that [holds the model on the vertex] (Figure 3b). 1, 2, 3, 4, 5 [counts the edges diverging from the upper base vertex by pointing at them]; 1, 2, 3, 4, 5 [counts the edges diverging from the lower base vertex]
Yellow	[turns inside the model and counts the edges of a lower base pentagon by pointing at them] 1, 2, 3, 4, 5.
Grey	Look, the base is ten.
Yellow	[counts the middle section] 1, 2, 3, 4... Where did I start? (to Grey) Put your hand here. [continues to count silently] ... [raises arms to the upper base] ten, [lowers arms to the lower base] ten, [makes a circular breaststroke movement with both hands indicating a middle part] ten, ...thirty!

At the beginning of the episode, Yellow *discerned* relevant *details* of the model and even *recognized* the *relationship* between them: five edges meeting at the vertex, three edges forming each of the triangular faces. Each of these relationships has the potential to become *the property* leading students to a correct solution. However, these properties were not useful for Yellow's approach of direct counting. While standing inside the model and reassessing the situation, she cannot *hold the whole*. From a mathematical point of view, it does not matter how the icosahedron is positioned in space—the polyhedron's mathematical properties remain the same. However, in a material gravitational world, the model lay on one of its triangular faces, making it difficult to perceive certain structural symmetries.

When Grey tilted the model onto a vertex (Figure 3b), he restructured Yellow's attention. Previously this action enabled Grey to count the vertices, and now it helped Yellow to perceive an icosahedron as tripartite: two opposing "bases" and a connecting "belt" (Figure 3c). Grey also gave Yellow a hand (literally) in counting edges in a "belt." New structures of attention facilitated counting, and three aggregating gestures summarized the *perceived property* that there are ten edges in each of three groups.

DISCUSSION

The first research question raised by this study was on shifts in focus and form of students' attention when studying 3D geometrical objects by exploring physical models. By moving in space, changing points of view, and modifying a physical object (Gibson, 2015), the students experienced shifts in focus (small and large model, three distinct parts of the model, vertices, edges, groups of edges) and structures of attention. All five theoretical structures of attention and shifts between them (Mason, 2008) were documented in two episodes. Note that shifts in the structures of attention were associated with vision and touch, proprioception, and physical actions of students with and through the models. For instance, tilting the model on its vertex allowed students to structure their seeing of the icosahedron into three visible sets. We reported this case as indicative since this action helped students answer questions about vertices and edges or explain their solution to their peers in all the cases we possess.

The second question was on the role of physical features of the models in the process of student exploration. Models of different scales landed students different affordances (Gibson, 2015) for inquiry. For instance, in most cases, at least one student entered a human-scale model to examine the features of the polyhedron from within (as Yellow did). The activity enabled students to ground conceptions of the geometric figure simultaneously as objects in mesospace and macrospace (c.f. Herbst et al., 2017), providing more opportunities for possible shifts in focus and structures of attention and thus learning (Mason, 2008, 2010). Each model served as a physical attractor with different affordances for and constraints on the action; accordingly, students reorganized around these affordances and constraints. For instance, the small-scale model centered the group's multimodal interactions, but its size could not accommodate students' form of inquiry; the apparent availability of a larger model catalyzed splitting the group, which, in turn, juggled students' social roles.

The case study findings highlight the pedagogical potential of using different scales 3D models in spatial geometry instruction. First, the students experienced construction and informal exploration of polyhedron models producing a multitude of perspectives and collaborative insights on their features. Their efforts combined collaborative actions, gestures (indexing and iconic), and speech to indicate and highlight models' properties. The fluency with which students moved from one model to another—both physically and inferentially—suggests they noticed invariant scale-free features of a geometric object. Then, students' shifts of attention were multimodally grounded in their senses and converged to a gradual disciplinary formalization of the polyhedron's concept (c.f. Abrahamson et al.,2020, embodied design for mathematics instruction).

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