# UNDERSTANDING THE ‚AUXILIARY TASK' CONCEPTUALLY – DISCRETE VERSUS CONTINUOUS CARDINAL OBJECTS

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*For conceptually understanding the ‚Auxiliary Task', learners have to understand the compensation process. Yet, since the strategy is highly complex compared to other mental calculation strategies, an important question is how the conceptual understanding of the strategy can be fostered and for this purpose, ordinal as well as cardinal learning environments were developed and evaluated in a design-based-study (which is part of the mixed-methods MaG-Project). Prior analyzes showed that especially the cardinal learning environment leads to more thorough conceptual discourses. In this paper, qualitative insights into the use of specific forms of cardinal representation – discrete versus continuous – and its interpretations by four 11-year-old German primary school learners' will be given.*

## STARTING POINTS AND THEORETICAL BACKGROUNDS

### The ‚Auxiliary Task' and its relevance as a mental calculation strategy

In the last decades, there has been a shift in the perception of the importance of mental calculation strategies: Mental calculation strategies are not seen as mere pre-steps for the full algorithms anymore but have an important role in the emergence of flexible calculation processes (Heinze, Marschick & Lipowsky 2009). At the same time, a problem of over-emphasizing specific mental calculation strategies is visible: Students tend to use the HundredsTensUnits(HTU)-strategy, where they calculate in an order being structured by the hundreds, tens and units of the first and the second number, or the Stepwise-strategy, where they calculate by dividing the second number into hundreds, tens and units (see Selter 2001). Different and more complex mental calculation strategies like the ‚Auxiliary Task' are mostly not activated by learners. The ‚Auxiliary Task' differs from the HTU- and Stepwise-strategy insofar as that learners have to utilize compensation rules and have to see specific numerical properties before using the strategy, leading to a so-called ‚analytical noticing' (Threlfall 2002): When calculating 332 – 118 for example, learners using the ‚Auxiliary Task' have to recognize the proximity of the 118 to 120, thus modifying the second number by rounding it up to 120 through adding + 2 and compensating the modification at the interim result by adding back what was taken away too much (+ 2) (see figure 1).



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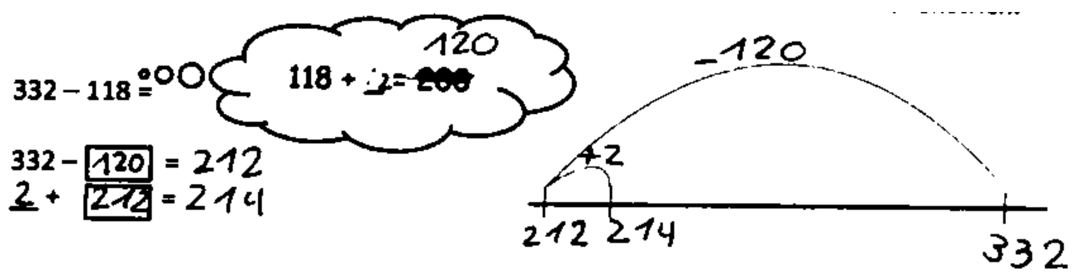



Figure 1: The ‚Auxiliary Task' as sequenced task and non-numbered line (Kuzu 2021).

Thus, the ‚Auxilary Task' is complex in terms of processing and mental noticing since the learners have to see the option to modify and compensate, but it also mediates a crucially different view on numbers: They have to be perceived as flexible, modificable objects, where one is allowed to change them, *if* every modification is compensated (equally) by adding the modification amount to the (interim) result – a different view on numbers which is fostering (pre-)algebraic aspects like the perception of indeterminant, flexible parameters (see Kuzu & Nührenbörger 2021).

**Cardinal versus ordinal ways of teaching the ‚Auxiliary Task'**

From a didactical standpoint, the teaching of the ‚Auxiliary Task' thus seems to be of high relevance in the transition from primary to secondary school, but there is a need for explorative research especially concerning the design-related question on *how* to teach the ‚Auxiliary Task' conceptually (see ibid.): In most learning environments, the conceptual understanding is fostered by using ordinal representations (f.e. through the use of non-numbered lines, see figure 1) and only very few learning environments do utilize cardinal means of representation, although conceptual aspects – like the compensation process – can be represented through cardinal manipulatives in a more meaning-related way, for example when taking-away and putting-back an equal amount of objects (see Britt & Irwin 2011). This leads to a specific, design-related research gap: The development and evaluation of a learning environment utilizing a cardinal representation of the ‚Auxiliary Task'. Prior analyzes conducted in Kuzu & Nührenbörger (2021) indicate specific hurdles on the conceptual as well as linguistic level: Interpreting and explaining a cardinally represented compensational process is highly complex due to the sequence of steps, which have to be visualized in a coherent, intuitive and relational way, but it is worthwhile to do so since interestingly, the cardinal representation led to more and thorough conceptual discourses. In comparison, the ordinal representation led to a faster transition to procedural discourses (with non-viable notions not being discussed as much as with the cardinal representation) (see ibid.). What is yet unclear is the effect of using *different* cardinal representations of the ‚Auxiliary Task' – discrete versus continuous – since both forms of representing a cardinal amount are of relevance in primary school (see Greer 1992). This is the research question to be focussed in this paper: *How do learners interpret discrete versus continous ways of cardinally representing conceptual facets of the ‚Auxiliary Task' in the context of the designed learning environment?*

## METHODS OF THE LEARNING-PROCESS STUDY

**Research context and data corpus of the study.** The research question was pursued in a design-based-study (see Prediger, Gravemeijer & Confrey 2015) that was part of the larger mixed-methods project MaG. The aim of the study was to develop a learning environment fostering the conceptual understanding of the ‚Auxiliary Task' for all four arithmetics and to generate local theories about the effects of the design principles and design elements by analyzing students' learning processes (see ibid.). In groups of 2-3 learners and with an iterative research design, a learning environment consisting of two 60 minutes sessions was developed and conducted. The data corpus consisted of n = 18 learners from age 11-14 and at the end of the second iteration, a total of 520 minutes of video material was cumulated (the learners being analyzed in this paper were 11 years old). The use of a continuous cardinal representation was a design element of the learning-environment from the first iteration, whereas the discrete representation was an adaption made for the second iteration.

**Methods for qualitative data analysis**. The transcripts were analyzed with respect to students' epistemological processes when interpreting and explaining the ‚Auxiliary Task' with cardinal manipulatives and representation. For this purpose, two analytical steps were followed: In the first step, a turn-by-turn interpretative analysis was conducted, in which the researcher analyzed students' utterances and interactions being based on the research questions of the study. These analyzes were discussed in teams of researchers and the main aim was to get carfully reflected insights into students' processes of interpretation (Schütte, Friesen & Jung 2019). In a second, complementary step, an epistemological analysis was conducted for deepening the analysis in specific transcript sections with so-called epistemological triangles (Steinbring 2006). In this paper, mainly the turn-by-turn analyzes will be focussed and sign-related utterances will be interpreted verbally without depicting the full epistemological triangles.

**The design of the learning environment**. The learning environment was designed based on two design principles: 1. Fostering of a conceptual understanding through a content-and-language-integrated approach preceding procedural calculation and the 2. Fostering of generalization processes through demanding verbal explanations. For the research question of this paper, especially the first design principle is of relevance since the use of different cardinal representations for fostering the conceptual understanding is focussed (see figure 2).

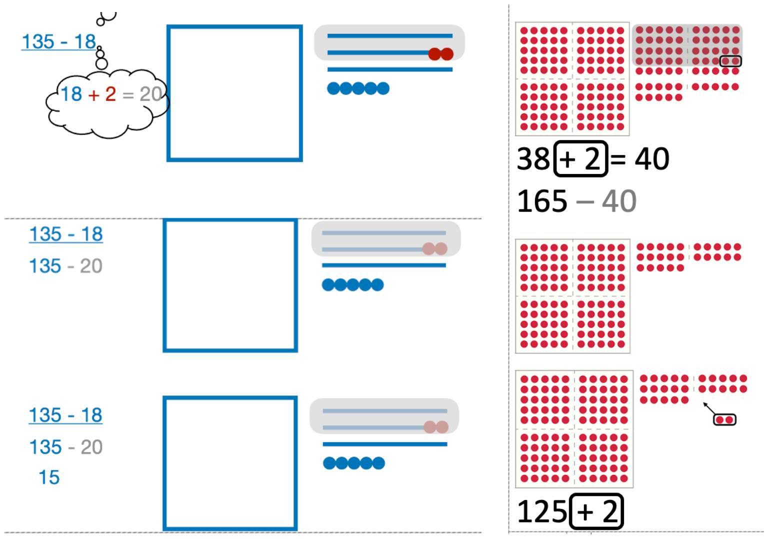



Figure 2: The continuous-cardinal representation from iteration 1 (left side) versus the discrete-cardinal representation from iteration 2 (right side).

On the left side of figure 2, a mainly continuous cardinal representation with continous tens and hundreds and discrete ones being based on Oehl (1962) is visible. The representation focussed the rounding-up of the second number and three colors were used: Blue for the task, red for the added number and grey for taking away the rounded-up number. On the right side, a discrete cardinal form of representation, where tens, hundreds and ones are discretely depicted, is visible and was used with a similar color coding (red for the task, a black rectangle for the added number and grey for taking away the rounded-up number). The cardinal objects in both variants are visible in an iconized form on the worksheet, but they were also available in form of manipulatives and for discussing the meaning of the ‚Auxiliary Task', an enactive approach with manipulatives was the first step to re-understand the fictive student Max' use of the ‚Auxiliary Task'.

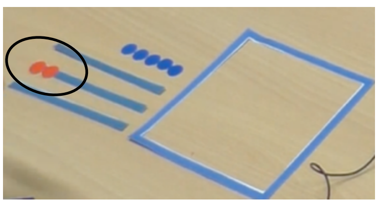
## EMPIRICAL INSIGHTS INTO STUDENTS' INTERPRETATIONS

### Sequence 1: Insights into the interpretation of the continous cardinal material

The students S1 and S2 try to explain the ‚Auxiliary Task' for the task 135 – 18 with continuous cardinal objects. The task sheet with the iconic representation of Max' procedure (see figure 2) is laid visibily on the table. They have available the continious cardinal material consisting of a hundreds-square, tens-lines and ones-dots.

2 I Okay, sorry- let's put it on the table again as it was earlier so that it looks as in Max' picture *[the Interviewer had nudged the objects by mistake]*.

3 S1 Yes *[puts two ones-dots on the right side of the second tens-stripe]*



4 I And then, and the we have learned not to relocate the material *[smiling]*. Yes, like that, thank you.

5 S1: Two he takes on it. But then he takes away

these two *[pointing at the two red dots]*. And the twenty, he takes away also.

| | | |
|---|---|---|
| 6 | I: | Aha. |

*[Turn 7-13: Organizational discourse. Continuation in Turn 14.]*

| | | |
|---|---|---|
| 14 | S2 | So, he also takes away these two? *[pointing at the last two blue dots]* |
| 15 | S1 | No. He has, one, these are five. Then he has seven *[pointing at the five blue dots first and then on the two red dots]*. And then, when he as eighteen, plus two makes twenty. Then he takes away the twenty. And then it is ten minus two I believe. |

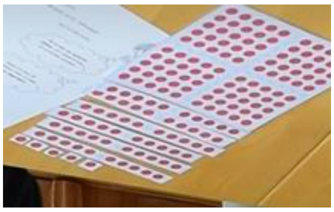
From Turn 3 to 19, the learners S1 and S2 try to interpret the continuous cardinal material. In Turn 3, they put it on the table as it was depicted on the task sheet (see figure 2). After that, in Turn 5, a first individual interpretation of the cardinal material becoms visible: S1 interprets the two blue-tens lines with the two red dots on it, as visible on the picture in Turn 5 (the grey box being not visible very good), as twenty, where *"two he takes on it… then takes away these two. And the twenty, he takes away also"*. It seems that she infers a double-subtraction process, a non-related tie between the two red ones-dots and the two blue tens-lines since she uses a paratactic structure in her sentence and verbalizes the sequence with the language means "then" and "and", indicating non-related processes instead of interpreting the two red dots (the rounding amount) as an integral part of the twenty (18 + 2 = 20), the rounded-up number. S2 seems to be irritated and asks in Turn 14, if he (possibly the fictive student Max from the task) *also* takes away the red ones-dots. S1 at first neglects the presumption from S2 in Turn 15 and hints at the total number of dots on the table (five blue dots and two red dots), but after that she gives a similar answer to her explanation from Turn 5: That it is eighteen plus two, which makes twenty to take away, *and then* ten minus two. Especially the last part of her utterance, with the emphasis of a last following step ("and then"), where she describes the taking-away of *another* two (10 – 2) beside of the two being taken away within the twenty, shows again a double-subtractional notion.

From an epistemological perspective, the mathematical signs being interpreted here are mainly the continuous cardinal representation on the task sheet, the objects on the table and the numerical representation. Especially the objects on the table are discussed and interpreted in a mathematically non-viable way, but S1s interpretation seems to be rooted in the ambiguity being related to the material-design: The placement of two ones-dots on a continuous tens-line seems not self-explanatory and leads to a perturbation (irritation): Instead of interpreting the two red ones-dots as part of the twenty, thus as being part of the whole of the second number, a distinct interpretation of the ones-dots and tens-lines becomes fostered.

**Sequence 2: Insights into the interpretation of the discrete cardinal material**

The students S3 and S4 try to explain the way the ‚Auxiliary Task' is used for calculating the task 165 – 38 with the given discrete cardinal material. While doing so, they try to explain the ‚Auxiliary Task' (which was demonstrated by the fictive student

Max, see figure 2). They have ones-dots, stripes of tens-dots and squares of hundreds-dots (consisting of ten stripes of tens-dots). The iconic representation of Max' procedure (see figure 2) is visibly beside the amount on the table.



| | | |
|---|---|---|
| 21 | I | Which number did you now put on the table? |
| 22 | S3 | Ehm these *[points at the amount on the table]* |
| 23 | S4 | 165 *[puts the laste ones-dots to the discrete amount]* |
| 24 | S3 | Yes. |
| 25 | I | Okay, very good. So, what does Max do now? How does he proceed? |
| 26 | S3 | 38 plus 2 *[takes stripe of tens into his hand]* |
| 27 | S4 | That is 40. Well- I'll just make it like this, plus two *[puts two ones-dots on the right side of the hundreds-dots]*- although, more like under it *[indicates to put the ones-dots beside the other ones-dots under the amount]* |
| 28 | S3 | Let's just do the result. |
| 29 | S4 | Well okay. Then it is 40. |
| 30 | S3 | And then 40 minus *[has four stripes of tens-dots in his hand and holds them next to the stripes of tens-dots on the table]* five *[looks at the five ones-dots]* now this comes away *[takes away four of the stripes of tens-dots and pushes the ones up right under the hundreds-dots]* Plus… these two *[puts two ones to the ones under the 165]* that makes hundred- *[5 seconds]* 117. |

In sequence 2, a similar interactional process to sequence 1 is visible: The learners S3 and S4 discuss the meaning of the ‚Auxiliary Task' with cardinal means, but S3 and S4 discuss a discrete representation here (see figure 2). Being asked what they put down on the table in Turn 21 by the Interviewer, S3 deictically points at the cardinal representation of 165, which S4 verbalizes and finalizes in Turn 22 by putting down the last ones-dots. In Turn 26 then, after being asked what the fictive student Max may have thought in Turn 25 by the Interviewer, S3 and S4 begin to verbalize their interpretation: S3 verbalizes the numerical task (38 plus 2) but at the same time takes stripes of tens-dots into his hands. S4 then finishes S3s task in Turn 27 by saying *"that is 40"*, but more importantly, S4 here also adds two ones-dots on the right side of the material on the table, an action matching the rounding amount of *"plus two"* in the task (38 + 2). The numerical utterance thus is accompanied by the analoguous enactive action of putting down the matching number of discrete cardinal material. From Turn 28 to 30 then, especially in Turn 30, S3 shows again enactively, what he seems to have meant in Turn 26: By holding four stripes of tens-dots, which he took into his hand already in Turn 26, beside of four stripes of tens-dots already on the table, he indicates first, how many stripes have to be taken away (minus 40), and then takes away these. After that, he pushes the ones-dots up so that there is no gap, whereafter he finalizes his calculation.

From an epistemological viewpoint, the mathematical signs being interpreted here seem less ambiguous when compared to the signs from sequence 1: The ones-dots and stripes of tens-dots as well as squares of hundreds-dots are interpreted by S3 and S4 in a more coherent and relational way, visible by the utterances being accompanied by parallel and matching enactive action (see Turn 27 and 30). It seems that S3s and S4s interpretation does not differ from the task as S1s interpretation in sequence 1: The two ones being put down on the table in Turn 26 and 27 are not taken away twice but once. The extra ones-dots thus seem to be interpreted by the learners as part of a viable compensative thinking being analoguous to the cardinal and numerical representation on the task sheet. Generally, the learners seem not to be perturbed through the cardinal material and representation if compared to the interaction in sequence 1.

## DISCUSSION OF RESULTS AND LIMITATIONS

With regard of the research question of this paper, the analysis of both sequences shows an important difference in the interpretation of the ‚Auxiliary Task' with discrete versus continuous cardinal material: The continuous cardinal objects seem to be ambiguous in terms of their meaning since putting two dots onto the continuous tens-line leads to a non-relational, non-integral interpretion of the rounding amount (+2) and the rounded-up number (18 + 2 = 20), resulting in a non-viable interpretation of the ‚Auxiliary Task' as a double-subtraction. In contrast, with the discrete continuous material, the rounding amount and the rounded-up number seem to be interpreted in a more integral way by verbalizing the compensation process more directly in an unequivocal way and by accompanying it with analoguous enactive actions (see sequence 2). This hypothesis can be verified by broadening the analysis to all n = 18 learners: The learners from iteration 2, where the discrete cardinal material was used, seem to understand the compensation process more viably than the learners from iteration 1, where a lot of sign-related irritations could be reconstructed in the qualitative analyzes. An important local theory for designing a learning environment, which is utilizing a cardinal approach to explaining the ‚Auxiliary Task', thus is that the use of discrete material may lead to a more relational, more viable interpretation of the compensation process due to lesser sign-related ambiguities. For the continuous material, at least a redesign of the double layered objects with hidden versus visible elements seems to be necessary, for example by using shortened tens-lines, where the ones-dots are not *on* the line, hiding a part beneath it, but *beside* it, but then another ambiguity would occurs: Tens-lines with a shorter and a "normal" length. This leads to the conclusion that the use of discrete objects seems to be more adequate for explaining the ‚Auxiliary Task' with cardinal means.

What the analyzes do not show is if a discrete representation is better than a continuous representation generally: The insights are local, meaning they are closely connected to the designed learning-environment about the ‚Auxiliary Task'.

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