MAKING REALISTIC ASSUMPTIONS IN MATHEMATICAL MODELLING

Janina Krawitz[1], Jonas Kanefke[1], Stanislaw Schukajlow[1], Katrin Rakoczy[2]

[1]University of Münster, Germany, [2]University of Gießen, Germany

Making realistic assumptions is an important part of solving open modelling problems and also a potential source of errors. But little is known about the difficulties that result from the openness of modelling problems and how they can be addressed in interventions. Here, we focus on two central solution steps that are necessary for making assumptions: noticing the openness and estimating the missing quantities. In a qualitative study with four ninth graders, we asked students to solve a modelling problem after informing them about the openness of the problem. We identified barriers that expand the two-step model (e.g., trouble integrating assumptions into the model). In addition, informing students about the openness of the problem improved their solution to the problem at hand but did not help them solve subsequent problems.

INTRODUCTION

Mathematics can help people solve problems from every day or professional life. These problems typically do not contain all of the information required to obtain a solution. To replace missing values and simplify the situation, it is often necessary to make assumptions so that a mathematical model can be set up and used to solve the problem. Hence, specific skills (e.g., estimation skills) are needed, and mathematics classrooms should foster these skills to prepare students to apply their mathematical knowledge in order to solve real-world problems. Galbraith and Stillman (2001) highlighted the importance of making assumptions as a genuine but underrated aspect of successful modelling and stressed the need for systematic research in this area. This need was recently recalled (Schukajlow et al., 2021) and is addressed in the present study. We analyzed (1) the difficulties students experience when making assumptions to solve open modelling problems and (2) how information about the openness of the problem helps them overcome these difficulties. Our findings contribute to a better understanding of the process of making assumptions and the kinds of information that might help students overcome their difficulties with regard to making assumptions.

THEORETICAL BACKROUND AND RESEARCH QUESTIONS

Making assumptions

Making an assumption means proposing that a statement is temporally true as a productive basis for subsequent activities (Djepaxhija et al., 2015). Assumptions are necessary to solve open problems because important aspects of the problem situation are not specified, and additional information is needed. Assumptions specify the missing information and help the problem solver find a solution under the restrictive conditions that come along with making assumptions. Two broad types of assumptions

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 59-66). PME.

can be distinguished: Non-numerical and numerical assumptions. Non-numerical assumptions refer to assumptions about situational conditions, whereas numerical assumptions refer to assumptions about missing quantities. Both types require realistic considerations and extra-mathematical knowledge, but in order to make numerical assumptions, estimation skills may also be necessary (Chang et al., 2020). Estimations, which are rough calculations or judgments, can refer to different objects, including measurements (e.g., estimating length, height, or weight) and numerosity (e.g., estimating the quantity of objects) (Hogan & Brezinski, 2003). A number of studies indicate that estimation skills are difficult for students to acquire, and students often fail to estimate measurements with the appropriate accuracy (Jones et al., 2012).

Mathematical modelling competence and making assumptions

Mathematical modelling refers to the use of mathematics to solve real-world problems (Niss et al., 2007). The key aspect of modelling is that a real-world problem must be converted into a mathematical model that allows mathematical procedures to be applied to solve the problem. The mathematical result needs to be interpreted and validated with regard to the initial real-world situation. Thus, modelling can be considered a cyclic process that begins and ends in reality and passes through the mathematical domain. In mathematics classrooms, modelling problems are used to foster students' modelling competence. Figure 1 presents an example of a modelling problem.

Speaker
Maria bought the *Ultimate Ears BOOM* Speaker for 149.95 €. It has 360° sound with deep and precise bass. The speaker is 18.4 cm high.
Maria looks for a box with a cover for her speaker. On the web, she found a beautiful box. It is 14 cm wide, 10 cm high, and 14 cm deep.
Will the speaker fit in the box?

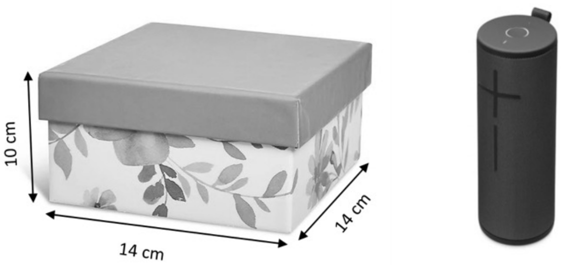


Figure 1: Modelling problem that requires assumptions to be made.

A characteristic feature of modelling problems is their openness as they often do not include all of the necessary information. To solve open modelling problems, two different solution steps are necessary (Krawitz et al., 2018): First, students need to notice the openness of the problem, and second, they have to estimate the missing quantities. For example, in the Speaker problem (Figure 1), students need to notice that the diameter of the speaker has to be taken into account and replace the missing quantity with an estimate (e.g., about 5 cm because, in the picture, the diameter is about one fourth of the height). Prior modelling research has shown that many students have trouble understanding, structuring, and simplifying the information given in modelling problems (Krawitz et al., 2021). Some of these challenges might result from the openness of modelling problems and the cognitive demands of making assumptions (Ärlebäck, 2009). An impressive body of research on word problems has demonstrated that students tend to neglect the realistic context of the problems, including the necessity of making assumptions, even if this leads to unrealistic responses (Verschaffel et al., 2000). In the Speaker problem, for example, an unrealistic response would be to ignore the fact that the diameter of the speaker has to be taken into account, calculate the diagonal of the box ($d = \sqrt{(14^2 + 14^2) + 10^2} = 22.18$), and conclude that the speaker fits because the speaker is shorter than the length of the diagonal. One potential reason for students' unrealistic responses is that they fail to notice the openness of the problem (Krawitz et al., 2018). In several interventions, researchers have tried to help students notice the openness, for example, by informing the students that the problems are tricky and cannot be solved in a straightforward way or by adding pictures to the problems (Dewolf et al., 2013), with little to no success. Students' restricted beliefs about word problems were found to be a reason for their difficulties (Djepaxhija et al., 2015). This finding indicates that the difficulties are persistent and hard to change. Initial indications for difficulties in noticing the openness of modelling problems came from a study conducted by Chang et al. (2020) where the failure to notice the openness was found to be a major barrier, whereas estimation skills seemed to play a minor role.

PRESENT STUDY AND RESEARCH QUESTIONS

The present study was conducted within the framework of the Open Modelling Problems in Self-Regulated Teaching (OModA) project, which is aimed at investigating cognitive, strategic, and affective conditions for the teaching and learning of open modelling problems. The research questions in the present study were:

RQ 1: What difficulties do students experience with respect to making assumptions when they solve open modelling problems?

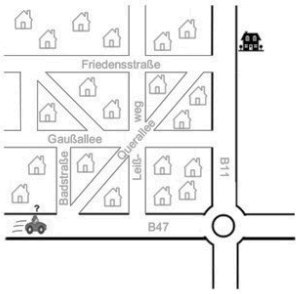
RQ 2: How does providing information about the openness of the problems help students overcome these difficulties?

METHOD

Participants and Data Collection

The sample involved four ninth graders (one female, all 16 years old) from two high-track schools (German Gymnasium). The students participated voluntarily in the study. Three of the participants were high achievers in mathematics (excellent grades), and one of them was an average achiever (average grades). In the following, the participants are referred to with pseudonyms. One of the participants (Andreas) stated that he had prior experience with open modelling problems, whereas the others did not. We used a qualitative approach to gather information on the underlying reasons for students' difficulties with open modelling problems and conducted individual sessions. The sessions consisted of three stages: problem solving, stimulated-recall interview, and semi-structured interview. In the problem-solving stage, participants were first given an open modelling problem (Shortcut Route Problem, Table 1) without information about the openness of the problem, a subsequent problem (Speaker Problem, Figure 1) with information about the openness ("To solve the problem, you must estimate the diameter of the speaker"), and finally another problem without such information (Tree Problem, Table 1).

Shortcut Route Problem: Mrs. Mai drives home on route B 47 and is running late. Fortunately, there is little traffic on the streets at night. She will soon come to the junction where the Street named Querallee branches off to the left. From there it would be another 1.5 km on B47 straight ahead, and from the roundabout another 2 km after turning left on B11 until she is home. Is the drive through the residential area worth it for Mrs. Mai so that she can get home earlier?



Tree Problem: Freshly planted trees are not yet rooted in the earth and need help attaching for the first few years. Support poles are often used to help. One end of the pole is hammered obliquely into the ground. A distance of 1.25 m from the tree is maintained so that the pole does not damage the roots of the fresh tree. The other end of the pole is tied to the tree with a rope at a height of 1.5 m. What is the length of the pole?

Table 1: Open modelling problems used in the study.

A quantitative pilot study with 143 students revealed that students rarely make realistic assumptions when solving these open modelling problems (percentage of solutions with realistic assumptions: 4.1% (Abbreviation problem), 3.2% (Speaker problem), 0.8% (Tree problem)).

Data Analysis

The video material was transcribed and sequenced. Sequences of the stimulated recall interviews were assigned to the related problem-solving sequences in order to collect more information about students' assumption-making processes. The sequences were categorized using qualitative content analysis (Mayring, 2014). In the coding process, noticing the openness and making assumptions were used as the main categories, and subcategories were inductively identified. Thereby, different types of assumptions (situational assumptions, numerical assumptions), purposes of assumption-making (simplify the situation, estimate missing quantities, interpret the result), and difficulties that could be attributed to the openness (noticing the openness, recognizing the possibility and necessity of making assumptions, integrating assumptions into the mathematical model) were distinguished. For example, the sequence "What is the diameter of the speaker? I would say, about as large as my water bottle. […] Okay, it is about 7 cm." Was paraphrased as "Estimated the length of the diameter of the speaker (7 cm)," and this was coded as a realistic numerical assumption.

FINDINGS

We analyzed students' difficulties that could be attributed to the openness of the problems. Table 2 gives an overview of the categories developed in the coding process.

Difficulties with:	Description
Noticing the openness	Not noticing the openness and consequently not making assumptions
Recognizing that assumptions might need to be made	Noticing the openness but not recognizing that making assumptions is a way to deal with it
Recognizing the need to make assumptions	Noticing the openness but thinking that it is not necessary to make assumptions
Integrating assumptions into the mathematical model	Not being able to set up an appropriate mathematical model that takes the missing quantities into account

Table 2. Overview of the difficulties that were attributed to the openness of the problem.

To answer the first research question, we analyzed students' solution processes for the first open modelling problem (Shortcut Route problem). Two of the participants (Tabea and Niklas) did not make any assumptions. Both calculated the distance without taking into account the different speed limits for the routes. Tabea did not notice the openness of the problem, whereas Niklas commented that he thought about the speed limits in his solution process but thought they were not important for the solution. Andreas directly recognized the need to make assumptions in the Shortcut Route problem. He made situational assumptions in order to simplify the real-world situation ("under the assumption that the street is perpendicular to the junction") and to specify his estimations ("because there are houses next to the road, the car has to look for pedestrians and cannot drive 100 km/h"). On this basis, he made realistic numerical assumptions about the speed limits (main road: 80 km/h; housing area: 30 km/h) and also defined situational requirements that did not need to be considered ("the speed while turning at the junction can be ignored"). Further, he used his assumptions to calculate the time that was needed to take the shortcut and to take the main road and completed the process by providing a realistic answer to the problem ("It is not worth it because of the speed limits"). In Christian's solution process, it was not clear at what point he noticed the openness of the problem. Christian did not make any assumptions and calculated the distances of both routes without considering the different speed limits. But his answer to the problem shows that he was aware of the fact that he neglected to consider this aspect in his solution ("The way through the housing area would be shorter but not necessarily faster"). His way of dealing with the openness of the problem was to acknowledge that his answer might not be valid. For Christian, noticing the openness did not lead him to make assumptions. Thus, simply noticing the openness is not enough for students to also recognize the need to make assumptions.

To address the second research question, we analyzed students' solution processes after they were given information about the openness of the problem (Speaker problem). We found that informing the students that a quantity was missing helped all participants in our study notice the openness. Two of four participants, Christian and Andreas, made assumptions about the missing quantity (here, the diameter of the speaker) and used their estimates to set up a mathematical model. One participant, Tabea, did not estimate the length of the diameter but took this quantity into account when interpreting her result ("It depends on the width of the speaker […] the maximum width would be 1.4 cm. I think this is too narrow."). Niklas also noticed that the diameter of the speaker was important but did not know how to use this information to solve the problem. Instead of estimating the diameter, he ended his solution process by simply guessing that the speaker would not fit into the box. His solution process exemplifies that integrating the missing quantities into a mathematical model can also be a barrier, in particular if the mathematical model becomes more complex when the additional information is included, as was the case for the Speaker problem.

To find out if the information also helps students notice openness while solving additional open modelling problems, the participants were given a third open modelling problem (Tree problem) without any information about the openness of the problem. None of the four participants noticed the openness of the problem. All of them neglected the fact that an assumption had to be made about the additional length of the support pole needed to fasten it to the ground in order to obtain a realistic solution (see Christian's solution in Figure 1). Consequently, the participants did not transfer their experience with the previous open modelling problem to the next one.

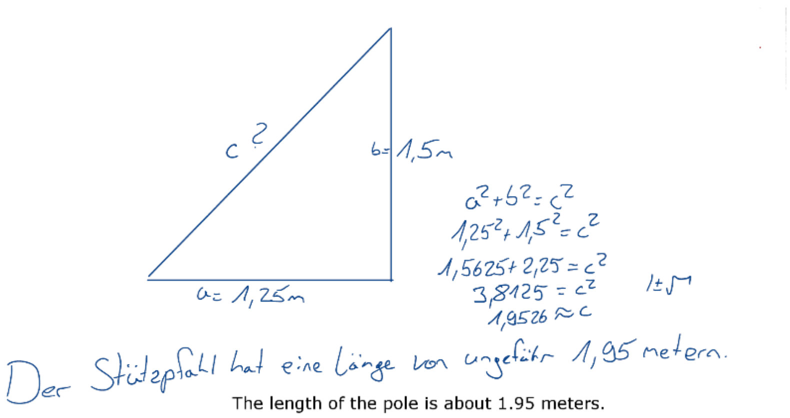


Figure 1. Christian's solution to the Tree problem.

Interestingly, Andreas and Bettina assigned the best value to their unrealistic solution:

Interviewer: Which of your solutions did you find the best?

Andreas: Best means that it is correct. Therefore, I would say the last one [Tree problem]. Because this is the one that really is correct. With the other, you have a greater inaccuracy because of the estimation.

In this excerpt, Andreas, who had previous experience with open modelling problems and was able to solve the Shortcut Route problem and the Speaker problem by making assumptions, states that he believes that his realistic solutions, which included assumptions, were less correct than his last unrealistic solution. He thinks the realistic solutions were less accurate due to estimation errors.

SUMMARY AND DISCUSSION

In line with previous research (Chang et al., 2020), noticing the openness of problems was revealed as a key difficulty. Further, noticing the openness did not automatically result in making assumptions. We identified three difficulties that prevented students from making assumptions after noticing the openness. First, making assumptions was not assumed to be necessary. Second, strategies or knowledge about how to deal with open problems were missing. Third, it was difficult to set up a mathematical model that took the missing quantities into account. Hence, our findings expand on the proposed two-step model for solving open modelling problems involving the steps of noticing the openness and estimating the missing quantities (Krawitz et al., 2018). These additional barriers should be taken into account in future studies investigating the role that making assumptions plays in mathematical modelling.

Contrary to studies that have revealed students' difficulties with estimation tasks (Jones et al., 2012), estimating the missing quantities did not hinder problem solving. Maybe the problems did not challenge our participants' estimation skills, or perhaps they failed at earlier stages in their solution processes so that we could not detect these difficulties.

Further, students' difficulties with noticing the openness could be overcome by providing information. However, the information helped only for the problem at hand, but it did not help students notice the openness of subsequent problems. Similar to research findings on word problems (Dewolf et al., 2013), students' difficulties with noticing the openness of a modelling problem seem to be persistent. Future studies should examine how the difficulties identified in the present study can be addressed in teaching methods. Students' restricted beliefs about word problems, in particular, the belief that every problem has a single numerical answer, were also found in our data and may have prevented students from making assumptions (Djepaxhija et al., 2015).

On a theoretical level, our study contributes to a better understanding of the process of solving open modelling problems and the challenges that are induced by the openness. Our findings provide a basis for developing teaching methods that address these difficulties in future research. A practical implication might be to provide more learning opportunities to deal with open problems in class so that students can acquire the knowledge and strategies that are necessary to deal with open modelling problems.

Acknowledgments: This study was financially supported by the German Research Foundation (GZs: RA 1940/2-1 and SCHU 2629/5-1).

REFERENCES

Ärlebäck, J. B. (2009). On the use of realistic fermi problems for introducing mathematical modelling in school. *The Mathematics Enthusiast*, *6*(3), 331–364. https://doi.org/10.1007/978-1-4419-0561-1_52

Chang, Y.-P., Krawitz, J., Schukajlow, S., & Yang, K.-L. (2020). Comparing German and Taiwanese secondary school students' knowledge in solving mathematical modelling tasks requiring their assumptions. *ZDM Mathematics Education*, *52*, 59–72. https://doi.org/10.1007/s11858-019-01090-4

Dewolf, T., Van Dooren, W., Cimen, E., & Verschaffel, L. (2013). The impact of illustrations and warnings on solving mathematical word problems realistically. *The Journal of Experimental Education*, *82*(1), 103–120. https://doi.org/10.1080/00220973.2012.745468

Djepaxhija, B., Vos, P., & Fuglestad, A. (2015). Exploring grade 9 students' assumption making when mathematizing. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 848–854). ERME.

Galbraith, P., & Stillman, G. (2001). Assumptions and Context: Pursuing their Role in Modelling Activity. In J. F. Matos, W. Blum, K. Houston, & S. P. Carreira (Eds.), *Modelling and Mathematics Education* (pp. 300–310). Woodhead Publishing. https://doi.org/10.1533/9780857099655.5.300

Hogan, T. P., & Brezinski, K. L. (2003). Quantitative Estimation: One, Two, or Three Abilities? *Mathematical Thinking and Learning*, *5*(4), 259–280. https://doi.org/10.1207/S15327833MTL0504_02

Jones, M. G., Gardner, G. E., Taylor, A. R., Forrester, J. H., & Andre, T. (2012). Students' Accuracy of Measurement Estimation: Context, Units, and Logical Thinking. *School Science and Mathematics*, *112*(3), 171–178. https://doi.org/10.1111/j.1949-8594.2011.00130.x

Krawitz, J., Chang, Y.-P., Yang, K.-L., & Schukajlow, S. (2021). The role of reading comprehension in mathematical modelling: Improving the construction of a real model and interest in Germany and Taiwan. *Educational Studies in Mathematics*, *Online-Vorveröffentlichung*. https://doi.org/10.1007/s10649-021-10058-9

Krawitz, J., Schukajlow, S., & Van Dooren, W. (2018). Unrealistic responses to realistic problems with missing information: what are important barriers? *Educational Psychology*, *38*(10), 1221–1238. https://doi.org/10.1080/01443410.2018.1502413

Mayring, P. (2014). Qualitative content analysis: Theoretical foundation, basic procedures and software solution. Beltz.

Niss, M., Blum, W., & Galbraith, P. L. (2007). Introduction. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education: the 14th ICMI Study* (pp. 1–32). Springer.

Schukajlow, S., Kaiser, G., & Stillman, G. (2021). Modeling from a cognitive perspective: theoretical considerations and empirical contributions. *Mathematical Thinking and Learning*. https://doi.org/https://doi.org/10.1080/10986065.2021.2012631

Verschaffel, L., Greer, B., & De Corte, E. (Eds.). (2000). *Making sense of word problems*. Lisse, The Netherlands: Swets & Zeitlinger.