THE CONNECTION BETWEEN MATHEMATICS AND OTHER FIELDS: MATHEMATICIANS' AND TEACHERS' VIEWS

Anna Hoffmann and Ruhama Even

Weizmann Institute of Science

This study investigated: (1) what secondary school teachers, who participated in an academic program that included applied mathematics, learned about the connections between mathematics and other fields, and how this knowledge contributed to their teaching, and (2) what mathematicians, who taught in that program, wanted to teach teachers about those connections. Data source included interviews with five research mathematicians and 14 teachers. Analysis revealed that the mathematicians wished to teach teachers about the contribution of mathematics to other fields as well as the reciprocal contribution of other fields to mathematics. Yet, the teachers enriched their knowledge only about the former, and used their new knowledge to raise students' interest and motivation to learn mathematics, but not for doing mathematical work.

INTRODUCTION

Applied mathematics, which links between mathematics and other fields (e.g., physics, computer science, engineering, economics and biology) is an integral and essential part of the discipline of mathematics. Its central role in the discipline is reflected in the growing number of areas of applied mathematics research conducted by mathematicians at prominent research universities around the world (e.g., Department of Mathematics at ETH Zurich, 2022; Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 2022; School of Mathematical Sciences of Fudan University, 2016; University of California, Berkeley, n.d.). These include, for example, spectral and dynamical problems of quantum mechanics, population genetics, image processing and medical imaging, mathematical finance and quantitative risk management.

An important characteristic of applied mathematics is that the interactions between mathematics and other fields are often bi-directional. One direction is from mathematics to other fields, denoting the contribution of mathematics to solving problems in various fields; the other direction is from other fields to mathematics, denoting the contribution of other fields to the development of mathematics, as explained in the following: "This interaction is often bi-directional: mathematical concepts and techniques are used to model and solve concrete problems in other fields. Reciprocally, scientific progress raises new mathematical problems, and motivates the development of new mathematical concepts and tools" (Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 2022).

In contrast to the central role that applied mathematics has in the discipline of mathematics, the professional education and development of mathematics teachers

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rarely include opportunities to learn about this key aspect of the discipline (Cai et al., 2014; Greefrath & Vorhölter, 2016; Novotná, 2019; Schmidt et al., 2008). For example, Schmidt et al. (2008), who examined the structure of secondary mathematics teacher preparation programs in six countries from three continents (Bulgaria, Taiwan, Germany, South Korea, Mexico, and the US), found that applied mathematics courses were not included in the academic mathematics content courses that prospective teachers were required to study. Novotná (2019) reported similar results regarding the Czech Republic, and Greefrath & Vorhölter (2016) revealed that mathematical modelling is not a compulsory content in teacher education programmes at universities in German-speaking countries.

With the current broad consensus regarding the importance of promoting applications and mathematical modelling in schools (e.g., Galbraith et al., 2007; Kaiser, 2020), the vast attention given to mathematical literacy and the relevance of mathematics to real life (e.g., COMAP & SIAM, 2019; PISA, 2018), and the growing interest in STEM education (e.g., Li et al., 2020; Maass et al., 2019), the need to attend to this deficiency in the professional education and development of mathematics teachers is further enhanced.

Our study addresses this issue by examining what secondary mathematics teachers may learn about the connections between mathematics and other fields in an academic program that comprises a focus on applied mathematics. Furthermore, as learning is shaped by teaching, we also examine what mathematicians that teach in such a program wish to teach teachers about these connections. The research questions are:

1. What mathematicians, who teach in an academic program for secondary teachers that includes applied mathematics, want to teach teachers about the connections between mathematics and other fields?
2. What teachers, who participate in an academic program that includes applied mathematics, learn about the connections between mathematics and other fields and how this knowledge contributes to their teaching?

METHODS

Setting and Participants

The study was situated in a unique master's program for practicing secondary school mathematics teachers, in which academic-level mathematics and research in mathematics education are the main components. The mathematics component comprised eight courses, designed and taught by research mathematicians. Four of these courses dealt with topics in the school curriculum at an advanced level: algebra, analysis, geometry, and probability and statistics. Three courses dealt with topics in applied mathematics, presenting modern use and application of mathematics in computer science, natural sciences, social sciences and everyday technologies. One course appraised the history and philosophy of mathematics. In addition, a final project was carried out under the guidance of a mathematician.

Five of the seven mathematicians who taught in the program participated in the study (M1, M2 … M5). They taught all the mathematics courses in the program but two: algebra and the use of mathematics in computer science. All were prominent research mathematicians. The research interests of three of them involved applied mathematics. The teachers participating in the study were 14 program graduates (T1, T2 … T14). All held a bachelor's degree in mathematics or in a mathematics-related field before starting the program. Their teaching experience ranged from 5 to 23 years.

Data Source and Analysis

The main data source included individual semi-structured in-depth interviews with the mathematicians and the teachers. These interviews were conducted as part of a comprehensive research program that examine the relevance and contribution of academic mathematics studies to secondary school mathematics teachers' knowledge about the discipline of mathematics, and how that knowledge contribute to their teaching (Hoffmann & Even, 2021, 2018). The mathematicians were asked about their teaching goals in the program, first in general and then specifically regarding what mathematics is. Correspondingly, the teachers were asked whether they learned something new about what mathematics is from their mathematical studies in the program, and if they did, whether that knowledge contributed to their teaching. Additionally, the teachers were presented with eight phrases that appear in the literature in relation to characteristics of the discipline of mathematics (e.g., mathematical definitions, thinking in mathematics, formal presentation in mathematics, the connection between mathematics and other disciplines), and were asked to choose three for which their mathematical studies in the program enriched their knowledge, and to describe what they learned.

For the purpose of this study, the interviews were analyzed qualitatively in an iterative and comparative process, aiming to identify what, if at all, the mathematicians wished to teach teachers about the connections between mathematics and other fields, what the teachers learned, and how this new knowledge contributed to their teaching.

FINDINGS

Mathematicians

All five participating mathematicians expressed in their interviews a wish to enrich teachers' knowledge about the connections between mathematics and other disciplines. No differences were found in this regard between the two mathematicians who taught applied mathematics courses and the three that taught the other mathematics courses. For example (interviewer denoted by I):

5	I:	I'd be happy if you could elaborate on your goals. You say that you sat down and thought about what this program should be. Could you elaborate?
6	M1	…we, at least I and some of my colleagues, felt that there is a need to show the connection between mathematics and other disciplines and everyday world.

The mathematicians referred to the connection between mathematics and other disciplines as bi-directional, reciprocal contributions. One direction is from mathematics to other fields, when mathematics is used to solve problems in other fields. The other direction is from other fields to mathematics, when work on solving problems in other fields raises new mathematical problems which then promotes the development of mathematical concepts and methods and thus advance mathematics.

When referring to the contribution of mathematics to other fields, the mathematicians stressed that they wished to expand teachers' knowledge about the practical worth of mathematics. At times they associated it with what they viewed as deficiencies in the contents of the high-school curriculum. For example,

> One of the things that... bothers me very much in high school mathematics... is that the mathematics that is taught in high school is not related to life at all. It usually viewed by students as an annoying exercise that is meant to upset them, and it doesn't look like something that has any value... That is, they teach students calculus, for example, for no reason... That is, why is there a derivative? and why is there an integral?... Where on earth did it come from? …And why did they develop it?... Quite a few of the mathematics teachers... don't know that a derivative is related to speed, that is, acceleration... there is a kind of thinking here that mathematics is a philosophical field that has nothing to do with science, and nothing to do with technology, and nothing to do with anything... I think that it is simply unacceptable that they would talk about derivatives and wouldn't know why Newton developed it. (M4)

The mathematicians emphasized that they would like to show teachers that mathematics is not just a theoretical science with no connection to reality. Instead, it can be used to develop the world. For example,

> Mathematics is a tool to describe the world around us... Moreover, you can use it too. What does it mean? After describing the world, it can be used to make a better world. (M3)

They explained that the usefulness of mathematics is conveyed through its contribution to solving problems in various disciplines and diverse areas of life, such as physics, medicine, economics, biology, engineering, geography, computer science, communication, navigation, etc. Often, they drew on examples from their own teaching in the program. For example, M5 reported that he presents in his course contemporary uses of mathematics, such as, GPS, search engines, encryption, robotic movements; and M4 described the emphasis she puts in her course on the power of mathematical models in solving problems from different disciplines:

> I start the discussion with an applied problem, and in fact, I also define for them this field of applied mathematics, which is building models. How to build a mathematical model for a problem... So, I think it gives them this beautiful connection that I keep emphasizing in the course, that mathematical language is a language that can be used to describe lots and lots of different problems in the same language. Once you have the mathematical tools, then you can answer questions from different disciplines. (M4)

When referring to the contribution of other fields to mathematics, the mathematicians associated it with the idea that the development of the discipline of mathematics lies in mathematical work related to questions. They stressed that questions may originate in mathematics as well as in other disciplines. Referring to mathematical work on questions originated in other disciplines, the mathematicians emphasized that work on such questions is central not only to the development of other disciplines, but also to the development of the discipline of mathematics itself:

> A great many of the developments in mathematics had real motivation... questions from life, not from mathematics, and the mathematical way helped solve them, and *then they developed mathematics* [emphasis added]. (M4)

Exemplifying the contribution of work on such questions to the development of mathematics, the mathematicians mentioned, for instance: calculating area and volume that contributed to the development of calculus; navigation that accelerated the development of geometry and trigonometry; and computer science, electricity, and electronics that promoted the development of graph theory. For example,

> Graph theory is an example of a mathematical field that has developed on its own, from… mathematical questions... but also from practical questions of applied mathematics. Because graph theory is the internet, graph theory plays a role more or less in all fields of science today. So, graph theory has very much developed in recent years. (M4)

The mathematicians further stressed the dialectic relationship between the development of the discipline of mathematics and that of other disciplines:

> This cycle that there is a problem that starts with something real and then goes through many degrees of abstraction and becomes something completely theoretical, and at the end, it goes back to something real, happens a lot in mathematics. (M1)

Teachers

Twelve of the 14 participating teachers reported in their interviews that academic mathematics studies contributed to their knowledge about the connection between mathematics and other disciplines; 10 specifically chose the phrase "the connection between mathematics and other disciplines" when asked to select topics for which their mathematical studies in the program enriched their knowledge. For example, T12 picked up the card with this phrase and repeated things she mentioned before:

> The connection between mathematics and other disciplines. In this regard this program gave me a lot. Physics, philosophy, in computers... as I said, nature, science... It enabled me to see that everything in nature can be organized in a mathematical way… (T12)

All 12 teachers connected their new knowledge regarding the connection between mathematics and other disciplines to two of the applied mathematics courses: mathematical applications in natural sciences, and mathematical applications in social sciences and everyday technologies. Three other mathematics courses – probability and statistics, application of mathematics in computer science, and the history and

philosophy of mathematics – as well as the final project were also mentioned by some teachers.

In contrast to the mathematicians, none of the teachers referred to the connection between mathematics and other fields as bi-directional. Instead, they described a uni-directional contribution only: from mathematics to other fields. For example,

> The mathematical applications course was an eye-opening course… Image compression, how it is expressed mathematically... I have never seen how there is at all a connection between mathematics and these things... Google search, how it works… and a medical problem that you can find how to model it mathematically. And how mathematics can help not only to develop thinking and the mind and to enjoy mathematics but… to discover its use. And I tell about it to my students as a motivation. (T7)

Similarly, T2 described how the course in probability and statistics enriched her knowledge regarding daily life usage of mathematics.

> Let me give you an example, say, [lecturer's] probability course. I remember his first two lessons. He sent us home to look in the paper for all sorts of things of probability... say, what the poverty index is in the country, how they are calculated, and to check their correctness… I felt that it was ... something that is very connected to us. Like, where in our lives we find this connection to mathematics…
>
> Last year I gave it to the 11th grade, middle-level track. I asked them to look for all kinds of statistics and how they were obtained. And they brought it to class. After that, I talked with them about the role of statistics in our lives… At the middle-level track, there is always this question: "What does it give me?" and "Where does it accompany me in my life?" (T2)

Nine teachers of the 12 teachers who reported on new knowledge regarding the contribution of mathematics to other disciplines, reported also that this new knowledge was relevant to their teaching work. All explained that it helped them increase students' interest and raise their motivation to learn mathematics, as was illustrated above in T7's and T2's quotations.

CONCLUSION

Applied mathematics, a key aspect of the discipline of mathematics, entails bi-directional interactions between mathematics and other fields (Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 2022). One direction is from mathematics to other fields, when mathematics is used to answer questions in other fields. The other direction is from other fields to mathematics when work on problems in other fields promotes the development of new mathematical concepts, tools, questions and theories.

The literature suggests that the professional education and development of mathematics teachers rarely include opportunities to learn about applied mathematics (Cai et al., 2014; Greefrath & Vorhölter, 2016; Novotná, 2019; Schmidt et al., 2008). Situated in a professional development program for practicing secondary school mathematics teachers that comprises a focus on applied mathematics, our study

provides important information regarding the potential contribution of such programs to teacher knowledge and practice related to the connections between mathematics and other fields.

Our findings suggest that the participating mathematicians aimed to enrich teachers' knowledge about the connection between mathematics and other disciplines, and the participating teachers considerably advanced their knowledge on one aspect of this connection, namely, the contribution of mathematics to other fields. Correspondingly, the teachers acquired a rich repertoire of contemporary examples of authentic use of mathematics to solve important problems in various areas of life, which they then used in their teaching to raise students' interest and motivation to learn mathematics. Yet, such use in teaching mainly involved informing students about fascinating uses of mathematics in the real world without actually doing any mathematical work. This result might be connected to factors, such as, not having teaching materials on which the teachers could draw in order to incorporate their newly acquired knowledge with the school mathematics curriculum, lack of support from their work environment for doing so, and more.

Additionally, in contrast to the participating mathematicians' wish for teachers to learn about the bi-directional connection between mathematics and other fields, the teachers did not mention in their interviews the contribution of other fields to mathematics. This result appears to be in line with the way applied mathematics is often dealt with in mathematics education, emphasizing the use of mathematics for solving problems in real-world contexts (e.g., Cai et al., 2014; Kaiser, 2020). As the other direction has been central to the development of mathematics, and continues to be so today, a question then arises whether this aspect needs to be more explicitly incorporated into the professional education and preparation of mathematics teachers.

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