DESIGNING PROBLEMS INTRODUCING THE CONCEPT OF NUMERICAL INTEGRATION IN AN INQURY-BASED SETTING

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Research literature argues for the benefits of inquiry-based approaches to provide opportunities for in-depth understanding of mathematics. This paper studies the design of mathematical problems for the purpose of introducing the concept of numerical integration in an inquiry-based setting. We present a series of six developmental stages (represent, refocus, area, accumulate, approximate, and refine) indicating a natural trajectory for students to follow when inquiring on the concept of numerical integration before any formal introduction to the topic. Further, we present a sequence of three problems illustrating how the developmental stages can be applied in problem design.

INTRODUCTION

The work presented in this paper concerns the design of problems in an inquiry-based setting in calculus, and is part of a larger research project focusing on how inquiry approaches in calculus in a Norwegian continuing professional development (CPD) program can support teachers', students' and teacher educators' development of mathematical competencies and inquiring habits of mind. In addition to the mastering of procedural skills, it is important to provide calculus students opportunities to develop deep understanding of context and connections between concepts (Hall, 2010; Sofronas et al., 2011). Mathematics education literature strongly speaks in favour of inquiry-based approaches for in-depth mathematical learning and critical application of knowledge, and such approaches have been promoted in educational policy and mathematics curriculum documents across the world (Artigue & Blomhøj, 2013; Dorier & Maass, 2014). Inquiry nurtures the critical, creative, and reflective mind, and encourages students to engage in mathematics in ways that mathematicians do (Artigue & Blomhøj, 2013; Dorier & Maass, 2014) by using mathematical key ideas to wonder, explore, discuss, justify, interpret, and collaborate with others on mathematical problems. Hence, knowledge on how problems given to students are designed, is crucial for facilitating inquiry-based learning (Artigue & Blomhøj, 2013; Cai, 2010), and is the focus of this paper.

One important element to consider in problem design is how central mathematical properties must be understood and used to solve the problem (Lithner, 2017), and the mathematical topic of this paper is numerical integration. Students' understanding of integration should be given special attention, as the topic is central in calculus and has a broad area of use in the real world (Jones, 2013; Sofronas et al., 2011). Calculus students are expected to make sense of integrals as limits of Riemann sums, develop

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and implement numerical algorithms to calculate integrals, interpret the meaning of integrals in different situations, and use integration to solve problems. By informally approaching the concept of numerical integration through inquiry, students are provided opportunities to develop an in-depth understanding of the topic. This paper asks the following question: *How can problems be designed for the purpose of introducing the concept of numerical integration in an inquiry-based setting*?

INQUIRY-BASED PROBLEMS IN MATHEMATICS

Inquiry-based problems encourage exploration, discussion, the posing of questions, and evaluation. Inquiry is built on the idea of exploring something unknown or challenging, but requires that this can be approached through building on existing knowledge (Artigue & Blomhøj, 2013). Inquiry-based problems must therefore have a delicate balance between creating challenges for the students and enabling them to make sense of the challenges by accumulating their knowledge. A skewness towards the known can contribute to degrade the problem to uncritical rote learning and repetition of known procedures and algorithms. Lithner (2017) suggests not providing pre-decided strategies or procedures in the problem text. The argument is that if a solution method is given, or already known by the students, they often uncritically apply it. Similar arguments given by Schoenfeld (1985) and Cai (2010) suggest that reducing the number of specific algorithms and techniques increases the potential for exploration. On the other side, too many unknown elements may hinder new learning and meaning making as the students do not have the prerequisite knowledge to approach the problem. The importance of providing students the opportunity to solve the problems and justify their solutions through building on what they know, is emphasized in the literature (Artigue & Blomhøj, 2013; Lithner, 2017; Schoenfeld, 1985).

Mathematical inquiry is therefore closely linked to problem solving, emphasising multiple solutions (Cai, 2010). Such problems invite students to develop an ownership to their solution methods (Cai, 2010; Schoenfeld, 2012). They should facilitate reflection beyond concrete situations (Goldin, 2010; Schoenfeld, 2012), for example on what happens if we change some problem criteria or introduce higher number situations. Such reflection can be approached through designing “the most elementary, generic example” (Goldin, 2010, p. 248) or through sequences of problems with increasing complexity (Schoenfeld, 2012). From an inquiry perspective, these approaches enable the students to accumulate their knowledge (Artigue & Blomhøj, 2013).

INTRODUCING THE CONCEPT OF NUMERICAL INTEGRATION

Introducing integration in a way that helps students develop in-depth understanding of the concept, is challenging (Orton, 1983). Research suggests emphasizing a variety of representations and interpretations and the connections between them (Hall, 2010; Orton, 1983; Sofronas et al., 2011) to facilitate deeper understanding of the topic.

Using different representations might also help students trace their own solution processes and ideas when working with mathematical problems (Goldin, 2010).

Orton (1983) suggests introducing integration through active use of visual *representations*. For a real valued function of a single variable, the visual representation of an integral is the area between the function and the x-axis in a given interval. Thus, *moving focus* away from the function itself, and to the area under the graph as well as the interpretation of this area, can be considered a key "discovery" when approaching numerical integration through inquiring on visual representations. Calculating this *area* is a rather simple process if the function is linear, and can be considered what Goldin (2010) calls the most elementary example. If the function is a curve or a piecewise function, one might have to split the area into smaller subareas and sum up these areas – introducing *accumulation* to the process and providing the increasing complexity that Schoenfeld (2012) suggests. Approaching integration as an area under a curve or as accumulation of "bits" (Jones, 2013; Sofronas et al., 2011) requires an understanding of covariation between x and $f(x)$, the ability to imagine or visualize the "bits", and an understanding of why an area can give information on another quantity (Thompson & Silverman, 2008). Such understanding, and moreover understanding the connection between area and accumulation, can build deep understanding of the concept of numerical integration and for critical application of this understanding.

To encourage reflection beyond one concrete situation (cf. Goldin, 2010; Schoenfeld, 2012), problems for introducing the concept of numerical integration should stimulate wondering on how to approach situations where the areas cannot be calculated exactly (for example if there are no "simple" geometrical shapes that can be used or if there are too many different shapes). This introduces the idea of *approximation*. Combining this with inquiry on higher number situations (Goldin, 2010), increasing the number of accumulations, can stimulate reflection on how the interval breadth affect the accuracy and efficiency of the approximation, *refining* the approach.

DESIGNING PROBLEMS FOR INTRODUCING THE CONCEPT OF NUMERICAL INTEGRATION IN AN INQUIRY-BASED SETTING

When the aim is to introduce a new concept, the problem does not have to be very difficult (Lithner, 2017). One idea is to create a sequence of problems with increasing complexity (Schoenfeld, 2012) to balance what is known with what is unknown. This approach helps the students develop ways to approach general "find the area under the graph"-problems, reflect on when these approaches become inefficient, and propose ways to tackle this new obstacle. The goal of the problem should be to facilitate students' construction of some aspects of the concept (Lithner, 2017) and reflection beyond one concrete example (Goldin, 2010; Schoenfeld, 2012). This calls for problems that separate from strict prescriptions and encourage informal discoveries through inquiry.

Based on the presented theories, and empirical observations from introducing numerical integration in a CPD calculus course, we suggest that the following developmental stages should be emphasized, in this order, when designing problems for an inquiry-based introduction to the concept of numerical integration:

Stage	Description
Represent	Actively using and combining representations of the problem. This includes extracting information from the problem, translating from text to graphical, geometrical, or algebraic representations and moving between representations.
Refocus	Move focus from the function to the geometrical shape between the function and the x-axis in a given interval.
Area	Discovering that the area of the geometrical shape can be calculated to solve the problem. Inquiring on *why* it is interesting to calculate the area.
Accumulate	Finding ways to calculate the area through dividing it into subareas and accumulating the areas of these “bits”.
Approximate	Reflecting on how to tackle the problem if the area cannot be calculated exactly. Inquiring on effective ways to approximate the area.
Refine	Reflecting on how to improve the approximation.

Table 1: Developmental stages for an inquiry-based introduction to the concept of numerical integration.

These developmental stages follow a natural progression, balancing the known and unknown, from expected knowledge (drawing a graph, understand what a function is) to the mathematical objective (develop an understanding of the concept of numerical integration through accumulation of areas of repeated geometrical shapes with small interval breadth, and inquire on why this can provide useful information on a quantity other than area).

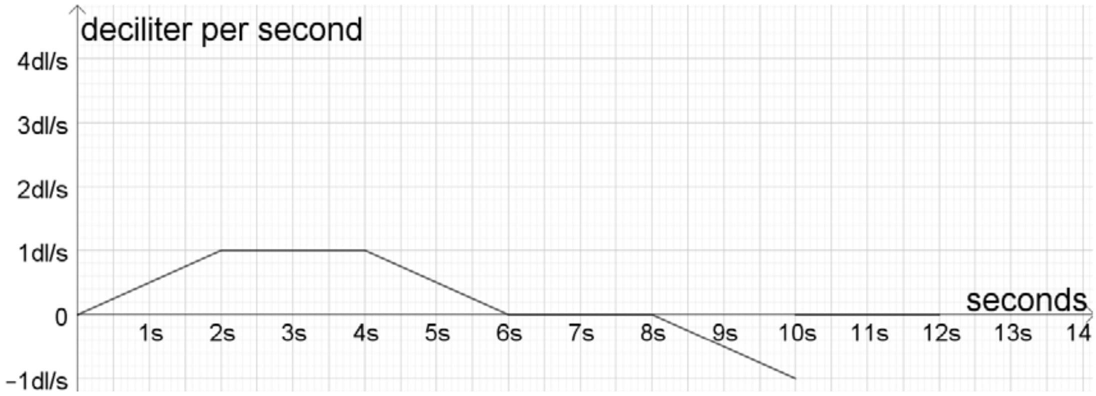
EXAMPLE: A SERIES OF THREE PROBLEMS

The following sequence of three problems for introducing the concept of numerical integration is designed based on the developmental stages and inquiry-based criteria presented above. Naalsund, Bråtalien, & Skogholt (2022) present a short transcript and analysis based on one student group’s collaboration when working with problem 2.

Problem 1 (A car trip)
A family is on a car trip driving with a constant velocity of 50 km/h.

- Choose some time between 0 and 5 hours. Calculate the distance travelled by the family at this time.
- Plot the velocity as a function of time for t between 0 and 5 hours. Discuss if you could have used this graph to find the distance you calculated above.

Problem 2 (Filling a bottle with water)
Anne wants to fill an empty bottle with exactly 3 dl of water. The figure below shows how much water is entering or leaving the bottle at time t. Discuss if Anne is successful.



Problem 3 (Pedal to the metal)
Kåre is driving a car and steps hard on the gas pedal at the time *t=0*. The acceleration of the car is shown as a function of time in the graph below. Estimate the velocity Kåre is driving at after 4 seconds. Write done the assumptions you make if any. Discuss how your estimate can be improved.

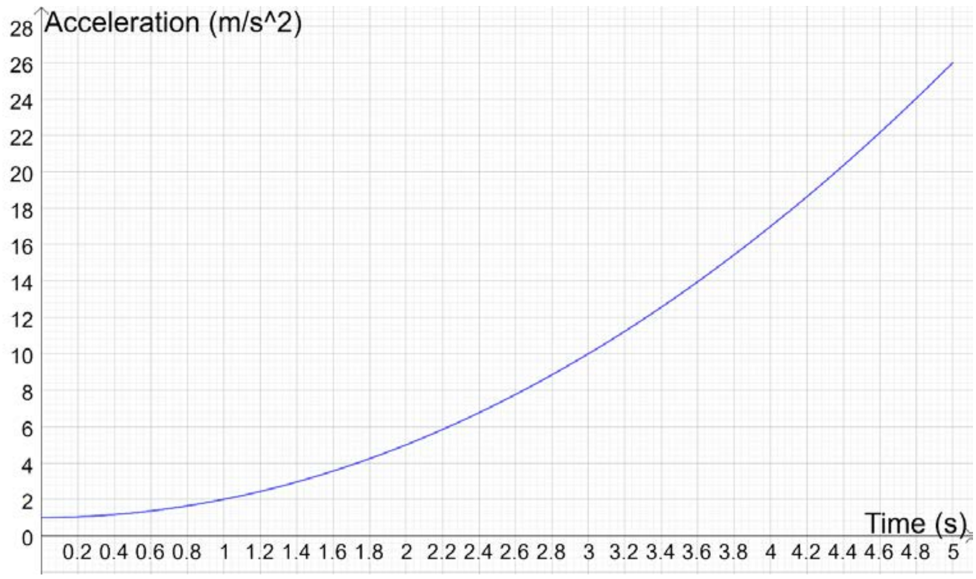


Figure 1: The sequence of problems.

Understanding the concept of integration consists of a network of smaller units (Hall, 2010; Jones, 2013) such as, but not limited to, ideas of area, limits, functions, algebraic operations, geometry, accumulation, and covariation. Problem 1 asks the students to calculate a distance using both algebraic and graphical representations. Asking the students to consider if the distance could be calculated from a velocity-time graph encourages them to combine graphical, algebraic (distance = time · velocity), and geometrical (area = length · breadth) representations to refocus from the graph itself and discover that the area under a graph can have a physical interpretation and hence be used to solve problems involving other quantities than area. This process includes the three first developmental stages of represent, refocus, and area. As the area of interest is a rectangle, the problem can be considered an example of the most elementary example (Goldin 2010).

Problem 2 also encourages the students to refocus and consider the area and its physical interpretation. The graph is piecewise linear, which naturally probes the students to split the area into some combination of triangles, rectangles, and trapezoids. This adds the developmental stage accumulate to the students' inquiry. The combination of positive and negative areas, as well as the graph having segments with increasing and decreasing (but positive) rates of change encourage reflection on the physical interpretation of both the graph and the area.

Problem 3 introduces the developmental stages approximate and refine. The function was chosen so that the area could not be computed exactly, and hence stimulating reflection on approximations to the area. In contrast with problem 2, problem 3 omits any mention of an initial condition to prompt a discussion of the interpretation of the definite integrals as the net change in the quantity considered. A variety of approaches are possible (Cai, 2010), and in our experience the students will consider approximations similar to the trapezoidal rule, the midpoint rule, as well as lower and upper sums with rectangles, even if none of these formal approaches to numerical integration are mentioned or have been formally introduced. The problem therefore invites the students to develop an ownership to their solution methods (Cai, 2010; Schoenfeld, 2012) and a sound foundation for learning about formal approaches. By also asking the students to discuss how their solution can be improved, the problem includes reflection beyond the concrete example (Goldin, 2010; Schoenfeld, 2012), i.e., on higher number cases (Goldin, 2010), encouraging the students to reflect on their solution and introducing the developmental stage refine. The students might discover that it is in principle simple to refine the approximation (by subdividing the x-axis into smaller intervals), but that the computation of the areas becomes an issue. This can be used as motivation for introducing formal integration procedures and even programming.

The three problems presented in this paper include few prescriptions, increasing the potential for exploration (Cai, 2010; Schoenfeld, 1985), but they also provide necessary support for the students to follow the trajectory of the developmental stages presented. Opening the problem by not providing a prescription makes the solution method itself a part of the unknown, engaging the student to wonder, explore and reflect in the process of constructing both strategies and solutions. Such inquiry will include several ideas being what Artigue & Blomhøj (2013) refer to as unknowns to the students. Students might struggle understanding integration as accumulation (Orton, 1983; Thompson & Silverman, 2008), as deep understanding of this requires the ability to visualize the “bits” that should be accumulated and a complex understanding of the area as representing something other than an area (Thompson & Silverman, 2008). To help students approach these unknowns, the problems use units on the axes and rates of change that are considered to be known for the students.

In this paper, we have presented and argued for the benefits of a progression through six developmental stages when designing problems for introducing the concept of numerical integration in an inquiry-based setting. We argue that an informal approach such as inquiry, holding back formal notation, symbols, prescriptions, and methods, allows the students to discover these stages themselves and reflect on obstacles they meet in their inquiry. The three problems we have presented follow the developmental stages, and these problems together with questioning, exploration, discussion, and evaluation that inquiry entails, can provide opportunities for connecting representations and interpretations (Hall, 2010; Orton, 1983; Sofronas et al., 2011), and reflection on the central questions of how and why an accumulation of areas can be used to represent another quantity than an area (Thompson & Silverman, 2010). This can prepare the students for formal methods of numerical integration, such as Riemann sums, where the idea of area and accumulation is combined.

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