THE USE OF RATIO AND RATE CONCEPTS BY STUDENTS IN PRIMARY AND SECONDARY SCHOOL

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This paper aims to analyse how primary and secondary school students use the concepts of ratio and rate when solving a ratio comparison problem. 954 primary and secondary school students (11-16 years old) solved a ratio comparison problem that involves four questions designed following the Reflection on the Activity-Effect Relationship mechanism. Students' answers were inductively analysed generating categories in relation to students' use of these concepts and the difficulties they revealed. Results have shown that a large number of students seems not to have the concept of ratio available during and at the end of secondary education, presenting difficulties not only with the identification of the multiplicative relationship between the extensive quantities but also with the norming techniques and with the referent.

THEORETICAL AND EMPIRICAL BACKGROUND

Quantity has been defined as an attribute of an object, which is expressed by an ordered pair, formed by a number and a magnitude unit, for example, two meters. Two types can be distinguished: extensive and intensive quantities. Extensive quantities, such as mass or length, can be measured directly while intensive quantities, such as density or speed, cannot (Schwartz, 1988).

Ratio or internalized ratio is defined as the result of comparing two quantities multiplicatively in a particular situation (Thompson, 1994). For example, in the problem "a car travels 70km in 1h, if it is driven for 5h, how many kilometres has travelled?", the internalized ratio is each iteration "70km in 1h", "140km in 2 hours", ..., so the internalized ratio is the particular ratio for each iteration. When a ratio is conceived beyond a particular situation, a constant ratio is obtained for any situation, called interiorized ratio or rate (Thompson, 1994). In the example, the rate 70km/1h is understood as a new quantity (intensive quantity) that measures the attribute speed, valid for any situation in which the relationship between quantities remains constant. Understanding the concept of rate implies understanding that extensive quantities can vary and still maintain the same relationship. That is, the quantity of kilometres and the quantity of hours (extensive quantities) can vary and the speed (intensive quantity) can stay the same (Simon & Placa, 2012).

Both ratios and rates can be established in ratio comparison problems which are situations where two ratios are given and should be compared. In these problems, students have to identify the multiplicative relationship between quantities that can be equal or unequal, and use norming techniques to favour the comparison between ratios

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(Castillo & Fernández, 2021). Norming describes the process of reconceptualising a system in relation to some fixed unit or standard (Lamon, 1994).

Previous studies have focused on ratio comparison problems showing students' success levels, strategies, misconceptions and the effect of some variables of the problem on students' strategies (Alatorre & Figueras, 2005; Nunes et al., 2003; Yeong et al., 2018). Nunes et al. (2003) showed that primary school students have difficulties solving ratio comparison problems that involve intensive quantities since students have to face two challenges: thinking in terms of proportional relations and understanding the connection between the intensive quantity and the two extensive quantities. Castillo and Fernández (2021) showed that these difficulties persisted also during the secondary education (12-16 years old students). Johnson (2015) conducted a study focused on investigating secondary school students' quantification of ratio and rate as relationships between quantities. She proposed the "change in covarying quantities framework" that shows the operations of comparison (extensive quantities) and coordination (intensive quantities) containing three levels of reasoning each one. This author claimed that the question how students shift from the operation of comparison to the operation of coordination needs further investigation.

As previous studies have shown, primary and secondary school students have difficulties with the concept of rate (intensive quantities). We are developing a cross-sectional study embedded in this line of research. It is focused on examining how primary (6th grade – 11 years old) and secondary school students (from 7th to 10th grade – 12-16 years old) construct the rate concept. For this purpose, we use a characterization of the Reflection on the Activity-Effect Relationship mechanism elaborated from the Reflective Abstraction of Piaget (Simon et al., 2004; Tzur & Simon, 2004).

From this perspective, two stages have been identified in the development of a concept: participatory and anticipatory. The participatory stage starts when a perturbation happens. In this stage, a new concept is abstracted, but it is provisional since it has been built from a single situation. This stage is divided into three phases: projection, reflection type-I and reflection type-II. In the projection phase, students compare what happens when they apply an available concept from a known situation in the proposed one, called generative situation. This comparison leads students to reorganize what they know about both situations (reflection type-I phase) and the new concept (more advanced than the available one) is built, but it is considered only for the generative situation. The reflection type-II phase occurs in the transition between the participatory and anticipatory stages. In this phase, a new situation, different from the generative one but of the same type, is proposed. When students observe that the generative and the new situations are of the same type, the developed concept is rearranged again to add this new situation, making the concept even more complex. Finally, when students can apply the developed concept in situations different from the generative one, they have reached the anticipatory stage, what it means that the concept is no longer provisional.

Three types of tasks related to this process were identified (Tzur, 1999). Initial tasks that involve concepts that students have. Reflective tasks (related to the participatory stage) that seek to cause perturbations to start the construction of the new concept. Anticipatory tasks (related to the anticipatory stage) that students can solve using the new concept that they have developed in the reflective tasks.

This paper is part of the cross-sectional study mentioned before and aims to answer the research question: how do primary and secondary school students use ratio and rate concepts when they solve a ratio comparison problem?

METHOD

Participants and instrument

Participants were 954 primary and secondary school students from 6th grade (n=161), 7th grade (n=188), 8th grade (n=240), 9th grade (n=229) and 10th grade (n=136). There was approximately the same number of boys and girls in each grade, and students were from diverse socio-economic backgrounds.

Participants solved the following problem: Melania's coach tells her that for each 20 meters, she should take 5 seconds to be able to qualify. a) If Melania has covered 250 meters in 60 seconds, has she qualified? b) Melania is competing against Cristina who has covered 300 meters in 70 seconds. What is the speed of each one? Who is faster? c) If Melania runs twice as many meters in twice as many seconds, would her speed change or be the same? Why? If her speed changes, what would this speed be? d) Propose three cases in which the speed would be the same as Cristina's speed (300 meters in 70 seconds). Justify your answer.

This problem was designed taking into account the Reflection on the Activity-Effect Relationship mechanism and the three type of tasks. Question a) is an initial task since the use of the ratio concept is involved and it is considered as an available concept to the students. Question b) is considered a reflective task (reflection type-I) because it implies to identify the ratio as the intensive quantity "speed" (rate). Question c) is considered a reflective task (reflection type-II) because it proposes a different situation from the generative one (question b) but of the same type. In this situation, students can realize that, although the extensive quantities change, the rate (speed) is the same than in question b). At this point, the rate concept should have been built for this particular problem and it should be available. Finally, question d) is considered an anticipatory task because it implies the use of the rate concept in situations different from those in which it was conceived.

Analysis

Three researchers analysed individually a subset of students' answers for the four questions, generating categories. Agreements and disagreements were discussed until an agreement was reached with the final categories. Later, the rest of students' answers were analysed using these categories. If an answer was not fit with the categories generated, it was discussed and a new category was generated.

Four categories emerged in question a): (i) students who did not identify the extensive quantities or they did not identify a multiplicative relationship between them (category A1); (ii) students who identified the extensive quantities and the multiplicative relationship between them, but they had difficulties with the norming techniques to obtain the ratios to be compared (category A2); (iii) students who obtained the ratios correctly using a norming technique but they had difficulties with the referent comparing the ratios (category A3); and (iv) students who obtained and compared the ratios correctly, identifying the inequality of ratios (category A4). The same categories were identified in question b) (categories as B1, B2, B3 and B4, respectively). In question b), a new category was identified: students who compared the ratios correctly, using the speed (ratio m/s) (category B5). In Figure 1, in the category B4, the student compared the ratios 250/60 and 300/70 identifying equivalent fractions. In the category B5, the student compared the same ratios with the quotient, obtaining the speed and specifying the units.

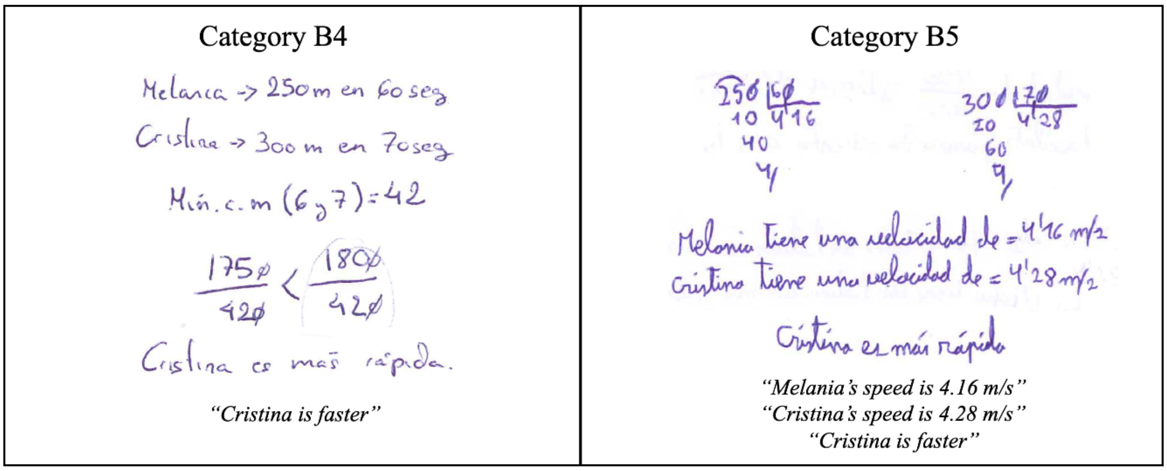

Figure 1: Examples of the categories B4 and B5

In question c), three categories emerged: (i) students who did not identify the variation of the extensive quantities neither the ratio's constancy (category C1); (ii) students who identified the variation but not the constancy (category C2); and (iii) students who identified the variation and the constancy (category C3). In C3, two subcategories were distinguished: students who answered without using numerical relationships (C3A) and students who made operations (C3B). The difference between them is exemplified in Figure 2. In C3A, the student explained that the speed is the same because Melania runs twice as many meters in twice as seconds while in C3B, the student justified the answer multiplying both quantities by 2 and dividing them to check if the speed was equal.

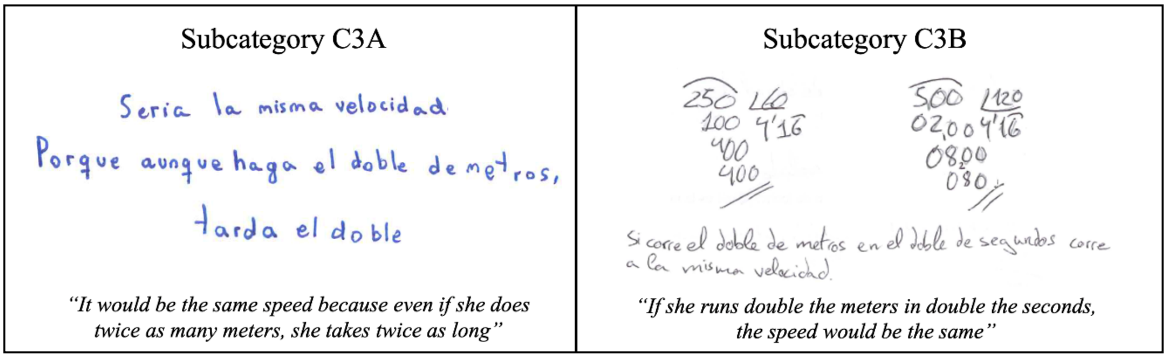

Figure 2: Examples of the subcategories C3A and C3B

In question d), three categories were identified: (i) students who did not identify the ratio’s constancy in other situations (category D1); (ii) students who identified the ratio’s constancy in other situations multiplying the extensive quantities by the same number (category D2); and students who identified the ratio’s constancy in other situations using the speed (ratio m/s) (category D3). Figure 3 shows examples of categories D2 and D3. In the category D2, the student multiplied both meters and seconds of Cristina by 2, 3 and 4, obtaining three situations where her speed is the same. In the category D3, the student multiplied the speed calculated in question b) by three random amounts of seconds, obtaining the respective meters.

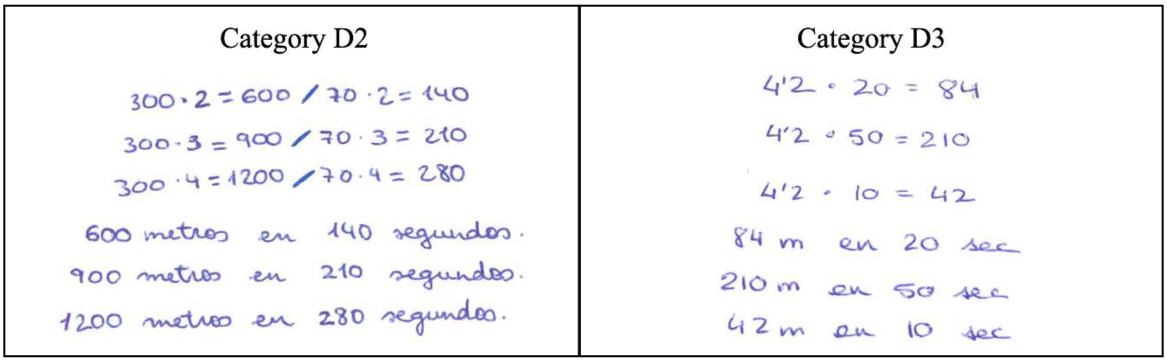

Figure 3: Examples of the categories D2 and D3

RESULTS

Tables 1, 2, 3 and 4 show the percentages of answers in each category by grade in questions a), b), c) and d), respectively.

Category	6th	7th	8th	9th	10th	Total
Blank answers	16.77	6.38	11.67	9.17	7.35	10.27
A1	44.10	51.06	39.58	33.62	43.38	41.72
A2	4.96	8.51	8.75	7.86	6.62	7.55
A3	7.45	8.51	13.75	19.65	15.44	13.31
A4	26.72	25.54	26.25	29.70	27.21	27.15

Table 1: Percentage of answers in each category by grade in question a)

Category	6th	7th	8th	9th	10th	Total
Blank answers	21.74	12.23	20.42	17.47	19.11	18.13
B1	52.18	62.23	46.25	41.05	44.85	48.95
B2	4.97	3.72	6.67	4.80	8.82	5.66
B3	8.07	11.17	6.67	10.48	6.62	8.70
B4	11.80	6.91	7.91	10.48	5.89	8.70
B5	1.24	3.74	12.08	15.72	14.71	9.86

Table 2: Percentage of answers in each category by grade in question b)

In questions a) and b), more than 40% of the students did not identify the extensive quantities or the multiplicative relationship between them (category A1 and B1). Other difficulties were related with the norming techniques or with the referent in the comparison between ratios. Furthermore, less than 30% of the students compared the ratios correctly (categories A4 and B4, B5). Percentages in each category remains similar along the grades. So, a large number of students seems not to have the concept of ratio available along and at the end of secondary education.

Comparing the percentages of correct answers in questions a) (category A4) and b) (categories B4 and B5), students revealed more difficulties in question b) that asks for the intensive quantity (speed) (we added this question as a perturbation). From the group of students who were able to compare the ratios in question b), some of them compared the ratios correctly (B4) but not using the speed (ratio m/s). These students did not observe differences between the known situation (question a) and the generative one (question b), answering equally in both; and others identified the ratio m/s as an intensive quantity (B5). These last students seemed to reorganize what they know about both situations (reflection type-I phase) and the new concept (more advanced than the available one – identifying the speed as a new quantity) is built, but it is considered only for this generative situation.

Category	6th	7th	8th	9th	10th	Total
Blank answers	28.57	23.40	19.58	17.47	28.68	22.63
C1	27.33	22.87	20.83	18.34	12.50	20.55
C2	12.42	15.43	9.58	7.86	11.03	11.01
C3A	13.66	12.23	24.58	24.45	25.00	20.34
C3B	18.02	26.07	25.43	31.88	22.79	25.47

Table 3: Percentage of answers in each category by grade in question c)

Category	6th	7th	8th	9th	10th	Total
Blank answers	56.52	57.98	54.58	53.28	50.74	54.71
D1	22.36	21.28	14.58	14.41	13.97	17.10
D2	21.12	20.21	30.84	31.00	33.82	27.56
D3	0.00	0.53	0.00	1.31	1.47	0.63

Table 4: Percentage of answers in each category by grade in question d)

In question c), 45.81% of the students identified the variation of the extensive quantities and the ratio's (speed) constancy (C3). Therefore, it seems that these students had the rate concept available for this particular problem. Some of them used the concept of rate without using numerical relationships (C3A) and others checked speeds numerically (C3B). However, only 28.19% of the students were able to identify the ratio's constancy in other situations in question d) (D2 and D3; anticipatory task).

DISCUSSION AND CONCLUSIONS

Our study focuses on how primary and secondary school students use the concepts of ratio and rate when solving a ratio comparison problem. Our results have shown that a large number of students seems not to have the concept of ratio available during and at the end of secondary education. These results coincide with those obtained by Nunes et al. (2003) with primary school students and by Castillo and Fernández (2021) with secondary school students. However, for the construction of the interiorized ratio (rate concept), it is fundamental the ratio concept since rate is defined as "reflectively abstracted constant ratio" (Thompson, 1994, p.192).

Yeong et al. (2018) explained that the base of students' misconceptions of ratios is that they do not understand ratio as a relationship between quantities. Our results are in line with this explanation since more than 40% of the students did not identify the extensive quantities or the multiplicative relationship between them (categories A1 and B1). However, other difficulties appeared linked to the norming techniques and the identification of the referent in the comparison. So, it seems that not only the identification of the multiplicative relationship between the extensive quantities is a key issue (Nunes et al., 2003; Thompson, 1994; Yeong et al., 2018) but also other elements such as the norming techniques and the referent in a comparison.

The students who identified the speed (ratio m/s) (category B5) seem to understand this ratio as a new quantity. However, to understand the speed as intensive quantity (rate) it is necessary to identify that the extensive quantities can vary but still maintain the same relationship. A little more of the 40% of students were able to identify it in question c) but few students were able to use the concept of rate in other situations. Therefore, the construction of the rate concept is complex.

Our next step is to identify students' profiles since it could allow us to identify how students move from one stage to another of the Reflection on the Activity-Effect

Relationship mechanism. This identification can also give us information about key elements in the construction of the rate concept.

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